Volume 29, Number 1 January 2021
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS, LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)
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A new techniques applied to Volterra–Fredholm integral equations with discontinuous kernel

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Abstract

The purpose of this paper is to establish the general solution of a Volterra–Fredholm integral equation with discontinuous kernel in a Banach space. Banach's fixed point theorem is used to prove the existence and uniqueness of the solution. By using separation of variables method, the problem is reduced to a Volterra integral equations of the second kind with continuous kernel. Normality and continuity of the integral operator are also discussed.

Mathematics Subject Classification(2010): 45L05; 46B45; 65R20.

Key–Words: Banach space, Volterra–Fredholm integral equation, Separation of variables method.

1 Introduction

It is well-known that the integral equations govern many mathematical models of various phenomena in physics, economy, biology, engineering, even in mathematics and other fields of science. The illustrative examples of such models can be found in the literature, (see, e.g., [5, 6, 9, 11, 12, 14, 18, 20]). Many problems of mathematical physics, applied mathematics, and engineering are reduced to Volterra–Fredholm integral equations, see [1, 2].

Analytical solutions of integral equations, either do not exist or it's hard to compute. Eventual an exact solution is computable, the required calculations may be tedious, or the resulting solution may be difficult to interpret. Due to this, it is required to obtain an efficient numerical solution. There are numerous studies in literature concerning the numerical solution of integral equations such as [4, 8, 10, 13, 16, 17, 21].

In this present paper, the existence and uniqueness solution of the Eq. (1) are discussed and proved in the space $L_2(\Omega) \times C[0, T]$, $0 \leq T < 1$. Moreover, the normality and continuity of the integral operator are obtained. A numerical method is used to translate the Volterra–Fredholm integral equation (1) to a Volterra integral equations of the second kind with continuous kernel,

The outline of the paper is as follows: Sect. 1 is the introduction; In Sect. 2, the existence of a unique solution of the Volterra–Fredholm integral equation is discussed and proved using Picard's method and Banach's fixed point method. Sect. 3, include the general solution of the Volterra–Fredholm integral equation by applying the method of separation of variables. A brief conclusion is presented in Sect. 4.

Consider the following linear Volterra–Fredholm integral equation:

$$\mu\psi(x,t) - \lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\psi(y,\tau)\mathrm{d}y\mathrm{d}\tau - \lambda\int_0^t F(t,\tau)\psi(x,\tau)\mathrm{d}\tau = g(x,t), \qquad (1)$$
$$(x=\bar{x}(x_1,x_1,\ldots,x_n), \quad y=\bar{y}(y_1,y_1,\ldots,y_n)),$$

where μ is a constant, defined the kind of integral equation, λ is constant, may be complex and has many physical meaning. The function $\psi(x,t)$ is unknown in the Banach space $L_2(\Omega)\times C[0,T]$, $0\leq T<1$, where Ω is the domain of integration with respect to position and the time $t\in[0,T]$ and it called the potential function of the Volterra–Fredholm integral equation. The kernels of time $\Phi(t,\tau)$, $F(t,\tau)$ are continuous in $C[0,T]$ and the known function $g(x,t)$ is continuous in the space $L_2(\Omega)\times C[0,T]$, $0\leq t\leq T$. In addition the kernel of position $k(|x-y|)$ is discontinuous function.

2 The existence of a unique solution of the Volterra–Fredholm integral equation

In this paper, for discussing the existence and uniqueness of the solution of Eq. (1), we assume the following conditions:

(i) The kernel of position $k(|x-y|)\in L_2([\Omega]\times[\Omega])$, $x,y\in[\Omega]$ satisfies the discontinuity condition:

$$\left\{\int_\Omega\int_\Omega k^2(|x-y|)\mathrm{d}x\mathrm{d}y\right\}^{\frac{1}{2}} = k^*, \quad k^* \textit{ is constant.}$$

(ii) The kernels of time $\Phi(t,\tau)$, $F(t,\tau)\in C[0,T]$ and satisfies $|\Phi(t,\tau)|\leq M_1$, $|F(t,\tau)|\leq M_2$, s.t M_1, M_2 are constants, $\forall t,\tau\in[0,T]$.

(iii) The given function $g(x,t)$ with its partial derivatives with respect to the position and time is continuous in the space $L_2(\Omega)\times C[0,T]$, $0\leq\tau\leq T<1$ and its norm is defined as,

$$\|g(x,t)\| = \max_{0\leq t\leq T}\int_0^t\left(\int_\Omega g^2(x,\tau)\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}\tau = N, \ N \textit{ is a constant.}$$

Theorem 1. *If the conditions (i)–(iii) are satisfied, then Eq. (1) has a unique solution* $\psi(x,t)$ *in the Banach space* $L_2(\Omega)\times C[0,T]$, $0\le T<1$, *under the condition,*

$$|\lambda|<\frac{|\mu|}{M_1k^*+M_2T}.$$

Proof. To prove the existence of a unique solution of Eq. (1) we use the successive approximations method (*Picard's method*), or we can used Banach's fixed point theorem.

2.1 Picard's method

We assume the solution of Eq. (1) takes the form:

$$\psi(x,t)=\lim_{n\to\infty}\psi_n(x,t),$$

where

$$\psi_n(x,t)=\sum_{i=0}^{n}H_i(x,t),\quad t\in[0,T],\qquad n=1,2,\dots$$

where the functions $G_i(x,t)$, $i=0,1,\dots,n$ are continuous functions of the form:

$$\left.\begin{aligned} H_n(x,t)&=\psi_n(x,t)-\psi_{n-1}(x,t),\\ H_0(x,t)&=g(x,t)\end{aligned}\right\}. \tag{2}$$

□

Now we should prove the following lemmas:

Lemma 1. *The series* $\sum_{i=0}^{n}H_i(x,t)$ *is uniformly convergent to a continuous solution function* $\psi(x,t)$.

Proof. We construct the sequences,

$$\begin{aligned}\mu\psi_n(x,t)&=g(x,t)+\lambda\int_0^t\int_\Omega\Phi(t,\tau)k(|x-y|)\psi_{n-1}(y,\tau)\mathrm{d}y\mathrm{d}\tau+\lambda\int_0^tF(t,\tau)\psi_{n-1}(x,\tau)\mathrm{d}\tau,\\ \psi_0(x,t)&=g(x,t).\end{aligned}$$

Then, we get

$$\begin{aligned}\psi_n(x,t)-\psi_{n-1}(x,t)=&\frac{\lambda}{\mu}\int_0^t\int_\Omega\Phi(t,\tau)k(|x-y|)(\psi_{n-1}(y,\tau)-\psi_{n-2}(y,\tau))\mathrm{d}y\mathrm{d}\tau\\ &+\frac{\lambda}{\mu}\int_0^tF(t,\tau)(\psi_{n-1}(x,\tau)-\psi_{n-2}(x,\tau))\mathrm{d}\tau.\end{aligned}$$

From Eq. (2), then, we have

$$H_n(x,t) = |\gamma| \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|) H_{n-1}(y,\tau) \mathrm{d}y \mathrm{d}\tau + |\gamma| \int_0^t F(t,\tau) H_{n-1}(x,\tau) \mathrm{d}\tau; \qquad \gamma = \frac{\lambda}{\mu},$$

using the properties of the norm, we obtain

$$\|H_n(x,t)\| \leq |\gamma| \left\| \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|) H_{n-1}(y,\tau) \mathrm{d}y \mathrm{d}\tau \right\| + |\gamma| \left\| \int_0^t F(t,\tau) H_{n-1}(x,\tau) \mathrm{d}\tau \right\|. \tag{3}$$

For $n = 1$, the formula (3) yields

$$\|H_1(x,t)\| \leq |\gamma| \left\| \int_0^t \int_\Omega \Phi(t,\tau) k(|x-y|) H_0(y,\tau) \mathrm{d}y \mathrm{d}\tau \right\| + |\gamma| \left\| \int_0^t F(t,\tau) H_0(x,\tau) \mathrm{d}\tau \right\|,$$

by applying Cauchy–Schwarz inequality and using the condition (ii) we get

$$\begin{aligned} \|H_1(x,t)\| \leq & |\gamma| M_1 \left\| \left(\int_\Omega |k(|x-y|)|^2 \mathrm{d}y \right)^{\frac{1}{2}} . \max_{0 \leq t \leq T} \left| \int_0^t \left(\int_\Omega |H_0(y,\tau)|^2 \mathrm{d}y \right)^{\frac{1}{2}} \mathrm{d}\tau \right| \right\| \\ & + |\gamma| M_2 \int_0^t \|H_0(x,\tau)\| \, \mathrm{d}\tau, \end{aligned}$$

using the conditions (i) and (iii), we have

$$\|H_1(x,t)\| \leq |\gamma| M_1 k^* N + |\gamma| M_2 N \|t\|, \tag{4}$$

where $\max_{0 \leq t \leq T} |t| = T$, so that formula (4) becomes

$$\|H_1(x,t)\| \leq |\gamma| N (M_1 k^* + M_2 T),$$

by induction, we get

$$\|H_n(x,t)\| \leq \beta^n N; \qquad \beta = |\gamma| (M_1 k^* + M_2 T) < 1; \quad n = 1, 2, \ldots.$$

Since

$$|\lambda| < \frac{|\mu|}{M_1 k^* + M_2 T},$$

this leads us to say that the sequence $\psi_n(x,t)$ has a convergent solution. So that, for $n \to \infty$, we have

$$\psi(x,t) = \sum_{i=0}^{\infty} H_i(x,t). \tag{5}$$

The above formula represents an infinite convergence series. ☐

Lemma 2. *The function $\psi(x,t)$ of the series* (5) *represents an unique solution of Eq.* (1).

Proof. To show that $\psi(x,t)$ is the only solution of Eq. (1), we assume the existence of another solution $\varphi(x,t)$ of Eq. (1), then we obtain

$$\begin{aligned}\mu[\psi(x,t)-\varphi(x,t)] =&\lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)[\psi(y,\tau)-\varphi(y,\tau)]\mathrm{d}y\mathrm{d}\tau\\ &+\lambda\int_0^t F(t,\tau)[\psi(x,\tau)-\varphi(x,\tau)]\mathrm{d}\tau,\end{aligned}$$

which leads us to the following

$$\begin{aligned}\|\psi(x,t)-\varphi(x,t)\| =&|\gamma|\left\|\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)(\psi(y,\tau)-\varphi(y,\tau))\mathrm{d}y\mathrm{d}\tau\right\|\\ &+|\gamma|\left\|\int_0^t F(t,\tau)(\psi(x,\tau)-\varphi(x,\tau))\mathrm{d}\tau\right\|,\end{aligned}$$

by applying the Cauchy–Schwarz inequality and using the conditions (i) and (ii), we get

$$\begin{aligned}\|\psi(x,t)-\varphi(x,t)\| &\le |\gamma|M_1k^*\int_0^t\int_\Omega \|\psi(y,\tau)-\varphi(y,\tau)\|\mathrm{d}y\mathrm{d}\tau\\ &+|\gamma|M_2\int_0^t \|\psi(x,\tau)-\varphi(x,\tau)\|\mathrm{d}\tau,\\ &\le \beta\|\psi(x,t)-\varphi(x,t)\|, \qquad \beta=|\gamma|M_1k^*+|\gamma|M_2T<1.\end{aligned}\tag{6}$$

The formula (6) can be adapted as,

$$(1-\beta)\|\psi(x,t)-\varphi(x,t)\|\le 0.$$

Since $\beta<1$, so that $\psi(x,t)=\varphi(x,t)$, that is the solution is unique. □

2.2 Banach's fixed point theorem

When the Picard's method fails to prove the existence of a unique solution for the homogeneous integral equations or for the integral equations of the first kind, we must use Banach's fixed point theorem. For this, we write the formula (1) in the integral operator form:

$$\begin{aligned}(\overline{U}\psi)(x,t) &= \frac{1}{\mu}g(x,t)+(U\psi)(x,t),\\ (U\psi)(x,t) &= \frac{\lambda}{\mu}\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\psi(y,\tau)\mathrm{d}y\mathrm{d}\tau+\frac{\lambda}{\mu}\int_0^t F(t,\tau)\psi(x,\tau)\mathrm{d}\tau.\end{aligned}\tag{7}$$

To prove the existence of a unique solution of Eq. (1), using Banach's fixed point theorem, we must prove the normality and continuity of the integral operator (7).

(a) For the normality, we use Eq. (7) to get

$$\|(U\psi)(x,t)\| = \left|\frac{\lambda}{\mu}\right|\left\|\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\psi(y,\tau)\mathrm{d}y\mathrm{d}\tau\right\|+\left|\frac{\lambda}{\mu}\right|\left\|\int_0^t F(t,\tau)\psi(x,\tau)\mathrm{d}\tau\right\|;\quad \mu\neq 0.$$

Using the condition (ii), then applying Cauchy–Schwarz inequality, we get

$$\begin{aligned}\|(U\psi)(x,t)\| \leq \left|\frac{\lambda}{\mu}\right| M_1 \left\| \left(\int_\Omega |k(|x-y|)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \cdot \max_{0\leq t\leq T} \left| \int_0^t \left(\int_\Omega |H_0(y,\tau)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \mathrm{d}\tau \right| \right\| \\ + \left|\frac{\lambda}{\mu}\right| M_2 \left\| \int_0^t \|H_0(x,\tau)\| \mathrm{d}\tau \right\|,\end{aligned}$$

using the condition (i), we obtain

$$\|(U\psi)(x,t)\| \leq \left|\frac{\lambda}{\mu}\right| (M_1 k^* + M_2 T)\|\psi(x,t)\|,$$

since

$$\|(U\psi)(x,t)\| \leq \beta\|\psi(x,t)\|; \qquad \beta = \left|\frac{\lambda}{\mu}\right| (M_1 k^* + M_2 T) < 1,$$

where

$$|\lambda| < \frac{|\mu|}{M_1 k^* + M_2 T}.$$

Therefore, the integral operator U has a normality, which leads immediately after using the condition (iii) to the normality of the operator $\overline{U}$.

(b) For the continuity, we suppose the two potential functions $\psi_1(x,t)$ and $\psi_2(x,t)$ in the space $L_2(\Omega) \times C[0,T]$ are satisfied Eq. (7), then

$$\begin{aligned}(\overline{U}\psi_1)(x,t) &= \frac{1}{\mu}g(x,t) + \frac{\lambda}{\mu}\int_0^t \int_\Omega \Phi(t,\tau)k(|x-y|)\psi_1(y,\tau)\mathrm{d}y\mathrm{d}\tau + \frac{\lambda}{\mu}\int_0^t F(t,\tau)\psi_1(x,\tau)\mathrm{d}\tau, \\ (\overline{U}\psi_2)(x,t) &= \frac{1}{\mu}g(x,t) + \frac{\lambda}{\mu}\int_0^t \int_\Omega \Phi(t,\tau)k(|x-y|)\psi_2(y,\tau)\mathrm{d}y\mathrm{d}\tau + \frac{\lambda}{\mu}\int_0^t F(t,\tau)\psi_2(x,\tau)\mathrm{d}\tau,\end{aligned} \tag{8}$$

Using equations (8), we get

$$\begin{aligned}\overline{U}[\psi_1(x,t) - \psi_2(x,t)] =& \frac{\lambda}{\mu}\int_0^t \int_\Omega \Phi(t,\tau)k(|x-y|)[\psi_1(y,\tau) - \psi_2(y,\tau)]\mathrm{d}y\mathrm{d}\tau \\ &+ \frac{\lambda}{\mu}\int_0^t F(t,\tau)[\psi_1(x,\tau) - \psi_2(x,\tau)]\mathrm{d}\tau.\end{aligned}$$

using the condition (ii) and applying the Cauchy–Schwarz inequality we get ,

$$\begin{aligned}\|\overline{U}[\psi_1(x,t) - \psi_2(x,t)]\| \leq \left|\frac{\lambda}{\mu}\right| M_1 \left\| \left(\int_\Omega |k(|x-y|)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \right. \\ \left. \cdot \max_{0\leq t\leq T} \left| \int_0^t \left(\int_\Omega |\psi_1(y,\tau) - \psi_2(y,\tau)|^2 \mathrm{d}y\right)^{\frac{1}{2}} \mathrm{d}\tau \right| \right\| + \left|\frac{\lambda}{\mu}\right| M_2 \left\| \int_0^t |\psi_1(x,\tau) - \psi_2(x,\tau)| \mathrm{d}\tau \right\|.\end{aligned}$$

By using the condition (i), the last inequality becomes,

$$\|\overline{U}[\psi_1(x,t)-\psi_2(x,t)]\| \leq \left|\frac{\lambda}{\mu}\right| (M_1k^* + M_2T)\|\psi_1(x,t)-\psi_2(x,t)\|,$$

hence, we have

$$\|\overline{U}[\psi_1(x,t)-\psi_2(x,t)]\| \leq \beta\|\psi_1(x,t)-\psi_2(x,t)\|; \qquad \beta = \left|\frac{\lambda}{\mu}\right| (M_1k^* + M_2T) < 1, \tag{9}$$

with

$$|\lambda| < \frac{|\mu|}{(M_1k^* + M_2T)}.$$

Inequality (9) leads us to the continuity of the integral operator $\overline{U}$. So that, $\overline{U}$ is a contraction operator. Therefore by Banach's fixed point theorem, there is an unique fixed point $\psi(x,t)$, which is the solution of the linear mixed integral equation (1).

3 Separation of variables method

To obtain the general solution of Eq. (1), we do the following:
For $t = 0$, the formula (1) becomes

$$\mu\psi(x,0) = g(x,0). \tag{10}$$

Then, seek the solution of equation (1) in the form:

$$\psi(x,t) = \sum_{n=1}^{\infty} c_n(t)\psi_n(x),$$

in this aim, we write

$$\psi(x,t) = \psi_0(x,t) + \psi_1(x,t), \tag{11}$$

where $\psi_0(x,t)$, $\psi_1(x,t)$ are called, respectively, the complementary and particularly solution of (1). Using Eq. (11) in Eq. (1), we get

$$\mu\psi_k(x,t) - \lambda\int_0^t\int_\Omega \Phi(t,\tau)k(|x-y|)\psi_k(y,\tau)\mathrm{d}y\mathrm{d}\tau - \lambda\int_0^t F(t,\tau)\psi_k(x,\tau)\mathrm{d}\tau = \delta_k g(x,t);\ k = 0,1, \tag{12}$$

also, for Eq. (10), we have

$$\mu\psi_k(x,0) = \delta_k g(x,0), \tag{13}$$

where,

$$\delta_k = \left\{ \begin{array}{ll} 0; & k=0 \\ 1; & k=1 \end{array} \right. .$$

From the two Eqs. (12), (13), we get

$$\begin{aligned} \mu[\psi_k(x,t)-\psi_k(x,0)] &- \lambda \int_0^t \int_\Omega \Phi(t,\tau)k(|x-y|)\psi_k(y,\tau)\mathrm{d}y\mathrm{d}\tau \\ &- \lambda \int_0^t F(t,\tau)\psi_k(x,\tau)\mathrm{d}\tau = \delta_k[g(x,t)-g(x,0)]. \end{aligned} \tag{14}$$

Now, we can represent the solution of (11) in the series form

$$\psi_k(x,t) = \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(t)\psi_{2n}(x) + c_{2n-1}^{(k)}(t)\psi_{2n-1}(x) \right), \tag{15}$$

where $\psi_{2n}(x),\ \psi_{2n-1}(x)$ are the even and odd functions respectively.
Using Eq. (15) in Eq. (14), we obtain

$$\begin{aligned} &\mu \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(t) - c_{2n}^{(k)}(0) \right) \psi_{2n}(x) + \mu \sum_{n=1}^{\infty} \left(c_{2n-1}^{(k)}(t) - c_{2n-1}^{(k)}(0) \right) \psi_{2n-1}(x) \\ &- \lambda \int_0^t \int_\Omega \Phi(t,\tau)k(|x-y|) \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(\tau)\psi_{2n}(y) + c_{2n-1}^{(k)}(\tau)\psi_{2n-1}(y) \right) \mathrm{d}y\mathrm{d}\tau \\ &- \lambda \int_0^t F(t,\tau) \sum_{n=1}^{\infty} \left(c_{2n}^{(k)}(\tau)\psi_{2n}(x) + c_{2n-1}^{(k)}(\tau)\psi_{2n-1}(x) \right) \mathrm{d}\tau = \delta_k[g(x,t)-g(x,0)]. \end{aligned} \tag{16}$$

Taking $k=0$, in Eq. (14), yields

$$\begin{aligned} &\mu \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(t) - c_{2n}^{(0)}(0) \right) \psi_{2n}(x) + \mu \sum_{n=1}^{\infty} \left(c_{2n-1}^{(0)}(t) - c_{2n-1}^{(0)}(0) \right) \psi_{2n-1}(x) \\ &- \lambda \int_0^t \int_\Omega \Phi(t,\tau)k(|x-y|) \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(\tau)\psi_{2n}(y) + c_{2n-1}^{(0)}(\tau)\psi_{2n-1}(y) \right) \mathrm{d}y\mathrm{d}\tau \\ &- \lambda \int_0^t F(t,\tau) \sum_{n=1}^{\infty} \left(c_{2n}^{(0)}(\tau)\psi_{2n}(x) + c_{2n-1}^{(0)}(\tau)\psi_{2n-1}(x) \right) \mathrm{d}y\mathrm{d}\tau = 0. \end{aligned} \tag{17}$$

Theorem 2. *(see [3, 19]). For a symmetric and positive kernel of Fredholm integral term of Eq.* (1)*, the integral operator,*

$$(K\psi_n)(x) = \int_\Omega k(|x-y|)\psi_n(y)\mathrm{d}y,$$

through the time interval $0 \le t \le T < 1$ *is compact and self–adjoint operator. So, we may write* $(K\psi_n)(x) = \alpha_n \psi_n(x)$*, where* α_n *and* $\psi_n(x)$ *are the eigenvalues and the eigenfunctions of the integral operator, respectively.*

In view of theorem 2, and Eq. (17), we arrive to the following

$$\begin{aligned}
&\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(0)}(t)-c_{2n}^{(0)}(0)\right)\psi_{2n}(x)+\mu\sum_{n=1}^{\infty}\left(c_{2n-1}^{(0)}(t)-c_{2n-1}^{(0)}(0)\right)\psi_{2n-1}(x)\\
&-\lambda\int_0^t\Phi(t,\tau)\sum_{n=1}^{\infty}\left(\alpha_{2n}c_{2n}^{(0)}(\tau)\psi_{2n}(x)+\alpha_{2n-1}c_{2n-1}^{(0)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}\tau\\
&-\lambda\int_0^t F(t,\tau)\sum_{n=1}^{\infty}\left(c_{2n}^{(0)}(\tau)\psi_{2n}(x)+c_{2n-1}^{(0)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}y\mathrm{d}\tau=0.
\end{aligned}$$

Separating the odd and even terms, we obtain

$$c_{2n}^{(0)}(t)-\gamma\int_0^t(\alpha_{2n}\Phi(t,\tau)+F(t,\tau))c_{2n}^{(0)}(\tau)\mathrm{d}\tau=c_{2n}^{(0)}(0);\quad \gamma=\frac{\lambda}{\mu}, \tag{18}$$

and,

$$c_{2n-1}^{(0)}(t)-\gamma\int_0^t(\alpha_{2n-1}\Phi(t,\tau)+F(t,\tau))c_{2n-1}^{(0)}(\tau)\mathrm{d}\tau=c_{2n-1}^{(0)}(0), \tag{19}$$

the two Eqs. (18) and (19) give the same results for even and odd functions, so it is suffice to study the following equation,

$$c_{n}^{(0)}(t)-\gamma\int_0^t(\alpha_{n}\Phi(t,\tau)+F(t,\tau))c_{n}^{(0)}(\tau)\mathrm{d}\tau=c_{n}^{(0)}(0);\quad \gamma=\frac{\lambda}{\mu}, \tag{20}$$

where $c_n^{(0)}(0)$ is constant will be determined.
Also, taking $k=1$ in formula (16), we obtain

$$\begin{aligned}
&\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(t)-c_{2n}^{(1)}(0)\right)\psi_{2n}(x)+\mu\sum_{n=1}^{\infty}\left(c_{2n-1}^{(1)}(t)-c_{2n-1}^{(1)}(0)\right)\psi_{2n-1}(x)\\
&-\lambda\int_0^t\int_\Omega\Phi(t,\tau)k(|x-y|)\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(\tau)\psi_{2n}(y)+c_{2n-1}^{(1)}(\tau)\psi_{2n-1}(y)\right)\mathrm{d}y\mathrm{d}\tau\\
&-\lambda\int_0^t F(t,\tau)\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(\tau)\psi_{2n}(x)+c_{2n-1}^{(1)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}y\mathrm{d}\tau=[g(x,t)-g(x,0)].
\end{aligned} \tag{21}$$

Using theorem 2 in Eq. (21), to have

$$\begin{aligned}
&\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(t)-c_{2n}^{(1)}(0)\right)\psi_{2n}(x)+\mu\sum_{n=1}^{\infty}\left(c_{2n-1}^{(1)}(t)-c_{2n-1}^{(1)}(0)\right)\psi_{2n-1}(x)\\
&-\lambda\int_0^t\Phi(t,\tau)\sum_{n=1}^{\infty}\left(\alpha_{2n}c_{2n}^{(1)}(\tau)\psi_{2n}(x)+\alpha_{2n-1}c_{2n-1}^{(1)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}\tau\\
&-\lambda\int_0^t F(t,\tau)\sum_{n=1}^{\infty}\left(c_{2n}^{(1)}(\tau)\psi_{2n}(x)+c_{2n-1}^{(1)}(\tau)\psi_{2n-1}(x)\right)\mathrm{d}y\mathrm{d}\tau\\
&=\sum_{n=1}^{\infty}a_{2n}\psi_{2n}(x)[g(x,t)-g(x,0)],
\end{aligned} \tag{22}$$

where,

$$1=\sum_{n=1}^{\infty}a_{2n}\psi_{2n}(x),$$

the formula (22) can be separated to the following equations

$$\begin{aligned}c_{2n}^{(1)}(t)-\gamma\int_0^t(\alpha_{2n}\Phi(t,\tau)+F(t,\tau))c_{2n}^{(1)}(\tau)\mathrm{d}\tau&=\frac{1}{\mu}a_{2n}[g(x,t)-g(x,0)]+c_{2n}^{(1)}(0);\ \gamma=\frac{\lambda}{\mu},\\ c_{2n-1}^{(1)}(t)-\gamma\int_0^t(\alpha_{2n-1}\Phi(t,\tau)+F(t,\tau))c_{2n-1}^{(1)}(\tau)\mathrm{d}\tau&=c_{2n-1}^{(1)}(0).\end{aligned}\tag{23}$$

Eqs. (20) and (23) represent Volterra integral equations of the second kind that have the same continuous kernel $\Phi(t,\tau)\in C([0,T]\times[0,T])$, and each of them has a unique solution in the class $C[0,T]$ the books edited by Linz [15] and Burton [7] contain many different methods to solve the integral equations (20) and (23).

The values of $c_n^{(0)}(0)$, $c_{2n}^{(1)}(0)$ and $c_{2n-1}^{(1)}(0)$ can be obtained, we return to the equation (10), and we seek the solution of this equation in the form,

$$\psi(x,0)=\sum_{n=1}^{\infty}c_n(0)\psi_n(x).$$

Hence, in this respect, we write

$$\psi(x,t)=\psi_0(x,t)+\psi_1(x,t),\tag{24}$$

where $\psi_0(x,t)$ is a complementary solution while $\psi_1(x,t)$ is a particularly solution. So, from Eq. (10) we write

$$\mu\psi_k(x,0)=\delta_k g(x,0),\tag{25}$$

and expand the solution of equation (24) in the form

$$\psi_k(x,0)=\sum_{n=1}^{\infty}\left(c_{2n}^{(k)}(0)\psi_{2n}(x)+c_{2n-1}^{(k)}(0)\psi_{2n-1}(x)\right),\tag{26}$$

using Eq. (26) in Eq. (25), we obtain

$$\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(k)}(0)\psi_{2n}(x)+c_{2n-1}^{(k)}(0)\psi_{2n-1}(x)\right)=\delta_k g(x,0).\tag{27}$$

If we take $k=0$ in Eq. (27), we obtain

$$\mu\sum_{n=1}^{\infty}\left(c_{2n}^{(0)}(0)\psi_{2n}(x)+c_{2n-1}^{(0)}(0)\psi_{2n-1}(x)\right)=0,$$

equating the odd and even terms in both sides, we get

$$c_{2n}^{(0)}(0) = 0, \qquad c_{2n-1}^{(0)}(0) = 0,$$

then , we have

$$c_n^{(0)}(0) = 0.$$

Taking $k = 1$ in Eq. (27), we have

$$\mu \sum_{n=1}^{\infty} \Big(c_{2n}^{(1)}(0)\psi_{2n}(x) + c_{2n-1}^{(1)}(0)\psi_{2n-1}(x) \Big) = g(x, 0).$$

Equating both sides of the last equation, we get

$$c_{2n}^{(1)}(0) = 0, \qquad c_{2n-1}^{(1)}(0) = 0,$$

so, the last two formulas give us

$$c_n^{(1)}(0) = 0.$$

In view of Eqs. (20) and (23), the general solution of (1) can be adapted in the form

$$\psi_N(x, t) = \sum_{n=1}^{N} \left(c_n^{(0)}(t) + c_n^{(1)}(t) \right) \psi_n(x), \tag{28}$$

where $c_n^{(0)}(t)$ and $c_n^{(1)}(t)$ must satisfy the inequality

$$\sum_{n=1}^{N} \left| c_n^{(0)}(t) + c_n^{(1)}(t) \right| < \epsilon, \quad (N \to \infty,\ \epsilon \ll 1,\ 0 \le t \le T < 1). \tag{29}$$

Theorem 3. *If, for $t \in [0, T]$, the inequality (29) holds, the series (28) is uniformly convergent in the space $\ell_2(\Omega) \times C[0, T]$, $N \to \infty$. Hence the solution of the Volterra–Fredholm integral equation (1) can be obtained in a series form of (28).*

Theorem 4. *For the given functions $g(x, t) \in L_2(\Omega) \times C[0, T]$, $\Phi(t, \tau) \in C([0, T] \times [0, T])$, $k(x, y) \in C([\Omega] \times [\Omega])$, and under the condition (29), we have*

$$\|\psi(x, t) - \psi_N(x, t)\| \to 0 \quad as \quad N \to \infty,$$

where $\psi(x, t)$ represents the unique solution of Eq. (1) and the error takes the form:

$$E_N = \|\psi(x, t) - \psi_N(x, t)\|,$$

where

$$E_N \to 0 \quad as \quad N \to 1.$$

4 Conclusion and remarks

From the above results and discussion, the following may be concluded:

1. Equation (1) has a unique solution $\psi(x,t)$ in the space $L_2(\Omega) \times C[0,T]$, under some conditions.
2. The Volterra–Fredholm integral equation of the second kind, in time and position, after using separation of variables method leads to a Volterra integral equations of the second kind with continuous kernel.
3. Solutions of the Volterra integral equations can be obtained by numerical methods.

Acknowledgments

The authors are very grateful to Prof. Dr. M. A. Abdou, (Dep. of Maths. Faculty of Education, Alexandria University) for their help and suggestions in the process of numerical calculation. We are also very grateful to the reviewers for their constructive suggestions towards upgrading the quality of the manuscript.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

A COMPARATIVE STUDY OF THREE FORMS OF ENTROPY ON TRADE VALUES BETWEEN KOREA AND FOUR COUNTRIES

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Abstract. Recently Wood-Jang[21] studied some applications of the Choquet integral in the trading relationship that Korea shares with selected trading partners. In this study, we consider the fuzzy entropy and the Shannon entropy, in addition we also define the Choquet entropy on a fuzzy set and develop four fuzzy sets which are related to that of the Choquet expected utility $CEU(u(a))$ for the trade values of utility u from an act a on S. Using this data set, we calculate three forms of entropy on four fuzzy sets as in the Choquet expcetes utility for the trade values that exist between Korea and four trading partner countries. Furthermore, we provide comparisons with three forms of entropy on four fuzzy sets which are representative of the four trading partner countries analyzed in this study.

1. Introduction

Many researchers have studied the Choquet integrals with respect to a fuzzy measure of fuzzy sets or interval-valued fuzzy sets and their applications in [2,4-13,19,20,26]. There are several examples of such analysis, these include student evaluations, similarity measures, the examination of the Choquet expected utility, and various other forms of inequalities. Recently, by using Choquet integrals with respect to a fuzzy measure, Wood-Jang[20,21] studied applications of them . These include some applications of the Choquet integral by firstly examining the imprecise market premium functions, and then more recently the trade relationship that Korea shares with selected trading partners. By using fuzzy sets and Choquet integrals in [17], studies utilized the concept of Choquet integral expected utility and its related areas(see[12,13,19,21,26]). Note that Biswas [2] investigated a student's evaluation on the space of fuzzy sets which include data information from the students' respective classes.

Our first motivation was to build on our previous efforts by considering three forms of fuzzy entropy in [1,3,14,15,18,24,25], the Shannon entropy in [3,21,24,25], and the Choquet entropy which we define in this study. From this point of view, we provide a comparative study of three

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1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. Choquet integral, fuzzy entropy, Shannon entropy, Choquet entropy, fuzzy set.

forms of entropy on the level of trade that exists between Korea and four trading partners using data obtained from the WTO [23]. Our second motivation for conducting this study, was to provide a unique analysis of international trade flows using the Shannon entropy, fuzzy entropy, and Choquet entropy techniques.

In this study, we consider the fuzzy entropy and the Shannon entropy, in addition we provide a definition of the Choquet entropy on a fuzzy set and also develop four fuzzy sets which are related to the Choquet expected utility $CEU(u(a))$ for the trade values of utility u from an act a on S for the specified 2-digit HS product codes $(01 - 05)$ for animal product exports between Korea and selected trading partners for years 2010-2013 using date obtained from the WTO regional trade database[23]. Using this data set, we calculate three forms of entropy on four fuzzy sets as in the Choquet expected utility for the trade values that exist between Korea and four trading partner countries (New Zealand, USA, India, Turkey). Furthermore, we provide comparisons for three forms of entropy on four fuzzy sets for the four trading partner countries.

2. Three forms of entropy on fuzzy sets

Let X be a finite set of states of nature and $F(X)$ be the set of all fuzzy sets $A = \{(x, m_A(x)) \mid x \in X,\ m_A \longrightarrow [0,1] \text{ is a function}\}$. Recall that m_A is called a membership function of A.

Definition 2.1. ([2,4-13,19,20,26])

(1) A real-valued function μ on X is called a fuzzy measure if it satisfies

$$\begin{array}{ll} \text{(i)} & \mu(\emptyset) = 0,\ \mu(X) = 1, \\ \text{(ii)} & A \subset B \Rightarrow \mu(A) \leq \mu(B), \end{array} \tag{1}$$

where A, B are subsets of X.

(2) The Choquet integrals with respect to a fuzzy measure μ of $A \in F(X)$ is defined by

$$(C)\int m_A d\mu = \int_0^1 \mu(\{x \in X | m_A(x) \geq \alpha\}) d\alpha. \tag{2}$$

Remark 2.1. *Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite set. It is well known that the discrete Choquet integral with respect to a fuzzy measure μ is followings.*

$$(C)\int m_A d\mu = \sum_{i=1}^{n} m_A(x_{(i)}) \left[\mu(E^{(i)}) - \mu(E^{(i+1)})\right], \tag{3}$$

where $(\cdot)$ indicates a permutation on $\{1, 2, \cdots, n\}$ such that

$$m_A(x_{(1)}) \leq m_A(x_{(2)} \leq \cdots \leq m_A(x_{(n)}), \tag{4}$$

$E^{(i)} = \{x \in X | m_A(x) \geq m_A(x^{(i)}\}$ *for* $i = 1, 2, \cdots, n$ *and let* $E^{n+1} = \emptyset$ (see [4-7,9,10,11,13]).

By using the Choquet integral, we consider the Choquet expected utility $CEU(u(a))$ of utility $u(a)$ from an act a as follows. Note that in economics, the utility function is an important concept that measures preferences over a set of goods and services. Utility is measured in units called utils, which represent the welfare or satisfaction of a consumer from consuming a certain number of goods. Here, we assume that an act is a function from S to X, where S is a finite set of states of nature.

Definition 2.2. ([5]) Let $u : X \longrightarrow [0,1]$ be a utility and a be an act from S to X. The full version of the Choquet expected utility is mentioned above so we can used the CEU abbreviation here. with respect to a fuzzy measure μ of utility u from act a is defined by

$$CEU(u(a)) = (C)\int u(a(s))d\mu(s). \tag{5}$$

Now, we introduce three forms of entropy on a fuzzy set as follows. Firstly, the Shannon entropy which was first developed by Shannon [24]. The mathematical communication theory has been utilized to measure the fuzziness in a fuzzy set or system [25]. According to Shannon, the information source is a person or a device that produces messages, using the average minimum amount of information.

Definition 2.3. ([3,22,24,25]) Let $A \in F(X)$ and $X = \{x_1, x_2, \cdots, x_n\}$ be finite set. The Shannon entropy on A is defined by

$$E_S(A) = -\sum_{i=1}^{n} m_A(x_i)\log m_A(x_i). \tag{6}$$

Secondly, the fuzzy entropy on a fuzzy set is used to express the mathematical values of the fuzziness of fuzzy sets. The concept of entropy, the basic subject of information theory and telecommunication, is a measure of fuzziness in fuzzy sets. Luca-Termini [18] note that Fuzzy entropy $D(A)$ can be represented by the Shannon function as follows

$$D(A) = k\sum_{i=1}^{n} s(m_A(x_i)),$$

where $s(x) = -x\log x - (1-x)\log(1-x)$ is the Shannon function. When we put $X = \{x_1, x_2, \cdots, x_n\}$, we consider the followin fuzzy entropy $E_F(A)$ which is the fuzzy entropy $D(A)$ with $k = -\frac{1}{n}$.

Definition 2.4. ([1,3,14,15,18,24,25]) Let $A \in F(X)$ and $X = \{x_1, x_2, \cdots, x_n\}$ be finite set. The fuzzy entropy on A is defined by

$$E_F(A) = -\frac{1}{n}\sum_{i=1}^{n}[m_A(x_i)\log m_A(x_i) + (1-m_A(x_i))\log(1-m_A(x_i))]. \tag{7}$$

Thirdly, we define the Choquet entropy on a fuzzy set and compare the Choquet entropy to another two forms of entropy, this helps to demonstrate the role of the Choquet entropy through the trading relationship that exists between Korea and four of its trading partners in the next section.

Definition 2.5. Let $A \in F(X)$ and X be a set. The fuzzy entropy on A is defined by

$$E_C(A) = 1 - (C)\int m_A(x)d\mu(x). \tag{8}$$

Note that if $X = \{x_1, x_2, \cdots, x_n\}$ is finite set, then we get

$$E_C(A) = 1 - \sum_{i=1}^{n} m_A(x_{(i)})[\mu(A_{(i)}) - \mu(A_{(i+1)})], \tag{9}$$

where $(\cdot)$ indicates a permutation on $\{1, 2, \cdots, n\}$ such that

$$m_A(x_{(1)}) \le m_A(x_{(2)}) \le \cdots \le m_A(x_{(n)}) \tag{10}$$

and $A_{(i)} = i, 2, \cdots, n$ and $A_{(n+1)} = \emptyset$.

3. Constructions of four fuzzy sets

In this section, we create four fuzzy sets, the USA-fuzzy set U, the NZ-fuzzy set N, IN-fuzzy set I, TR-fuzzy set T for the CEU of the trade values that exist between Korea and four countries. Now, we consider the CEU of a utility on a set of trade values (in USD) that represent the trading relationship that Korea shares with selected trading partners(i.e. Korea-USA, Korea-New Zealand, Korea-India, and Korea-Turkey). In [21], we examined these respective trading relationships by incorporating a clearly defined set of Harmonized System (HS) product code categories (i.e. HS Codes $i = 1, 2, 3, 4, 5$) for each individual year that is under review (i.e. $2010, 2011, 2012, 2013$). We note that the product code definitions have been provided by the UN Comtrade's online database and the relevant categories are defined as follows(see [21]):

1. Live animals; animal products.
2. Meat and edible meat offal.
3. Fish and crustaceans, mollusks and other aquatic invertebrates.
4. Dairy produce; birds' eggs; natural honey; edible products of animal origin, not elsewhere specified or included.
5. Products of animal origin, not elsewhere specified or included.

Denote that HSPC=HS Product Code, s=Year, $a(s)$=Trade Value, $u(a(s))$=the utility of $a(s)$, $CEU(u(a(s))$=the Choquet Expected Utility of $u(a)$ from a. By using the trade values in tables A_1 through to A_4 in the Appendix, we calculate the Choquet expected utility $CEU(u(a))$ for the set of trade values (in USD) that represent Korea's trading relationship with a particular country for years 2010, 2011, 2012, and 2013. Let $s_1 = 2010, s_2 = 2011, s_3 = 2012, s_4 = 2013$. Let $X = \{1, 2, 3, 4, 5\}$. Note that $(\cdot)$ indicates a permutation on $\{1, 2, 3, 4, 5\}$, such that

$$CEU(u(a_{(1)})) \leq CEU(u(a_{(2)}) \leq \cdots \leq CEU(u(a_{(5)}). \tag{11}$$

Then we denote that $a_{(i)} = a(s_{(i)})$ for $i = 1, 2, 3, 4, 5$ satisfy the equation (11). We define a fuzzy measure μ on X as follows.

$$\begin{aligned}&\mu(E^{(4)}) = \mu_1(\{a_{(4)}\}) = 0.1, \ \ \mu(E^{(3)}) = \mu_1(\{a_{(3)}, a_{(4)}\}) = 0.3,\\ &\mu(E^{(2)}) = \mu_1(\{a_{(2)}, a_{(3)}, a_{(4)}\}) = 0.6, \ \ \mu(E^{(1)}) = \mu_1(\{a_{(4)}, a_{(3)}, a_{(2)}, a_{(1)}\}) = 1,\end{aligned} \tag{12}$$

and if $a(s)$ is the trade value of s and $u(a) = \sqrt{\frac{a}{100141401}}$, then by using Definition 2.3, we obtain the following $CEU(u(a))$ as follows:

$$\begin{aligned} CEU(u(a)) &= \sum_{i=1}^{4} u(a(s^{(i)})) \left(\mu(E^{(i)}) - (\mu(E^{(i+1)})\right) \\ &= 0.4u(a(s^{(1)})) + 0.3u(a(s^{(2)})) + 0.2u(a(s^{(3)})) + 0.1u(a(s^{(4)})). \end{aligned} \tag{13}$$

By using (5), we obtained the four tables $A_1 \sim A_4$ (see [17]). If we take $m_Y(i) = CEU(u(a(s_i)))$, then we develop four fuzzy sets $Y : \{1, 2, 3, 4, 5\} \to [0, 1]$ by $Y = \{(i, m_X(i))|i = 1, 2, 3, 4, 5\}$. Here, Y is one of USA-fuzzy set U, NZ-fuzzy set N, IN-fuzzy set I, and TR-fuzzy set T defined by

$$U = \{(1, 0.05664), \ (2, 0.04483), \ (3, 0.93879), \ (4, 0.20821), \ (5, 0.04858)\} \tag{14}$$

$$N = \{(1, 0.00533), \ (2, 0.00000), \ (3, 0.78873), \ (4, 0.15976), \ (5, 0.01557)\} \tag{15}$$

$$I = \{(1, 0.00154), \ (2, 0.00000), \ (3, 0.04570), \ (4, 0.00000), \ (5, 0.00000)\} \tag{16}$$

$$T = \{(1, 0.00264), \ (2, 0.00887), \ (3, 0.00368), \ (4, 0.00470), \ (5, 0.00000)\} \tag{17}$$

4. Calculate three forms of entropy on four fuzzy sets

In this section, we calculate three forms of entropy on four fuzzy sets U, N, I, T. Note that $0log0 = 1$. Firstly, from (6) and (14)-(17), we get Shannon entroys $E_S(U), E_S(N), E_S(T)$ on four fuzzy sets as follows.

$$\begin{aligned} E_S(U) = &-(0.05664 \log 0.05664 + 0.04483 \log 0.04483 + 0.93879 \log 0.93879 \\ &+ 0.20821 \log 0.20821 + 0.04858 \log 0.04858) = 0.83476, \end{aligned} \tag{18}$$

$$\begin{aligned} E_S(N) = &-(0.00533 \log 0.00533 + 0.00000 \log 0.00000 + 0.78873 \log 0.78878 \\ &+ 0.15978 \log 0.15978 + 0.01557 \log 0.01557) = 0.57291, \end{aligned} \tag{19}$$

$$\begin{aligned} E_S(I) = &-(0.00154 \log 0.00154 + 0.00000 \log 0.00000 + 0.04570 \log 0.04570 \\ &+ 0.00000 \log 0.00000 + 0.00000 \log 0.00000) = 0.15099, \end{aligned} \tag{20}$$

$$\begin{aligned} E_S(T) = &-(0.00264 \log 0.00264 + 0.00887 \log 0.00887 + 0.00368 \log 0.00368 \\ &+ 0.00470 \log 0.00470 + 0.00000 \log 0.00000) = 0.10340. \end{aligned} \tag{21}$$

Secondly, from (7) and (14)-(17), we get fuzzy entropys $E_F(U), E_F(N), E_F(I), E_F(T)$ on four fuzzy sets as follows.

$$\begin{aligned} E_F(U) = &-\frac{1}{5}(0.05664 \log 0.05664 + (1 - 0.05664) \log(1 - 0.05664) \\ &+ 0.04483 \log 0.04483 + (1 - 0.04483) \log(1 - 0.04483) \\ &+ 0.93879 \log 0.93879 + (1 - 0.93879) \log(1 - 0.93879) \\ &+ 0.20821 \log 0.20821 + (1 - 0.20821) \log(1 - 0.20821) \\ &+ 0.04858 \log 0.04858 + (1 - 0.04858) \log(1 - 0.04858)) = 0.26736, \end{aligned} \tag{22}$$

$$\begin{aligned} E_F(N) = &-\frac{1}{5}(0.00533 \log 0.00533 + (1 - 0.00533) \log(1 - 0.00533) \\ &+ 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000) \\ &+ 0.78873 \log 0.78878 + (1 - 0.78873) \log(1 - 0.78878) \\ &+ 0.15978 \log 0.15978 + (1 - 0.15978) \log(1 - 0.15978) \\ &+ 0.01557 \log 0.01557 + (1 - 0.01557) \log(1 - 0.01557)) = 0.21368, \end{aligned} \tag{23}$$

$$\begin{aligned} E_F(I) = &-\frac{1}{5}(0.00154 \log 0.00154 + (1 - 0.00154) \log(1 - 0.00154) \\ &+ 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000) \\ &+ 0.04570 \log 0.04570 + (1 - 0.04570) \log(1 - 0.04570) \\ &+ 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000) \\ &+ 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000)) = 0.03834, \end{aligned} \tag{24}$$

$$\begin{aligned} E_F(T) = &-\frac{1}{5}(0.00264 \log 0.00264 + (1 - 0.00264) \log(1 - 0.00264) \\ &+ 0.00887 \log 0.00887 + (1 - 0.00887) \log(1 - 0.00887) \\ &+ 0.00368 \log 0.00368 + (1 - 0.00368) \log(1 - 0.00368) \\ &+ 0.00470 \log 0.00470 + (1 - 0.00470) \log(1 - 0.00470) \\ &+ 0.00000 \log 0.00000 + (1 - 0.00000) \log(1 - 0.00000)) = 0.01447. \end{aligned} \tag{25}$$

Thirdly, from (9) and (14)-(17), we get Choquet entropys $E_C(U), E_C(N), E_C(I), E_C(T)$ on four fuzzy sets as follows.

$$\begin{aligned} E_C(U) &= 1-(0.04483\times 0.1+0.04858\times 0.1+0.05664\times 0.6 \\ &\quad +0.20821\times 0.1+0.93879\times 0.1)=0.536119, \end{aligned} \tag{26}$$

$$\begin{aligned} E_C(N) &= 1-(0.00000\times 0.1+0.00533\times 0.1+0.01557\times 0.6 \\ &\quad +0.15976\times 0.1+0.78873\times 0.1)=0.895276, \end{aligned} \tag{27}$$

$$\begin{aligned} E_C(I) &= 1-(0.00154\log 0.00154+0.00000\log 0.00000+0.04570\log 0.04570 \\ &\quad +0.00000\log 0.00000+0.00000\log 0.00000)=0.15099, \end{aligned} \tag{28}$$

$$\begin{aligned} E_C(T) &= 1-(0.00264\log 0.00264+0.00887\log 0.00887+0.00368\log 0.00368 \\ &\quad +0.00470\log 0.00470+0.00000\log 0.00000)=0.10340. \end{aligned} \tag{29}$$

Through investigations (18)-(29), we can compare the three forms of entropy on the four fuzzy sets U, N, I, T as follows.

(1) In this study, we found that the Shannon entropy had a range $[0,1]$ and the fuzzy entropy had a range $[0,0.5]$ and the Choquet entropy had a range $[0.5,1]$.

(2) The Shannon entropy E_S and the fuzzy entropy E_F have the same order of four countries in the trading relationship that Korea shares with selected trading partner as follows.

$$E_S(U)\geq E_S(N)\geq E_S(I)\geq E_S(T), \tag{30}$$

and

$$E_F(U)\geq E_F(N)\geq E_F(I)\geq E_F(T). \tag{31}$$

(3) The Choquet entropy has the following order of the trading relationship that Korea shares with selected trading partner.

$$E_C(U)\leq E_F(N)\leq E_F(I)\leq E_F(T). \tag{32}$$

From (2) and (3), we observe the order in which Choquet entropy was investigated is the opposite of the order in the other two forms.

5. Conclusions

According to the data analyzed in experiments (27) - (29), we can see that the Shannon entropy and the Fuzzy entropy contain the same order of results. However, the results for the Choquet entropy were very different to that of the Fuzzy and Shannon entropies, as the opposite order of results being obtained. This finding also suggests that the results for the Choquet entropy exhibit a much higher level of ambiguity across the four countries (New Zealand, India, the US, and Turkey) analyzed, particular those countries that have a smaller trading relationship with Korea.

From an economic perspective, the Shannon and Fuzzy entropies have provided scholars with a means of better understanding the scope and magnitude of the potential relationship that exists between particular entities, in this case the trading relationship between Korea and four trading partners. In an era where the development of stronger bilateral economic ties through trade, such an analysis provides a unique but timely portrayal.

Furthermore, the ambiguities present in the Choquet entropy findings highlight the important need to carry out additional research. Such efforts would help to establish a clearer understanding on the types of trading relationships present between Korea and the four countries selected for this study.

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6. Appendix

Table A1: The CEU for animal product exports between Korea and the USA for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU_{(i,USA)}(u(a))$
1	s_1	$286892 = a(s^{(1)})$	0.05352	0.05664
	s_2	$330299 = a(s^{(2)})$	0.05743	
	s_3	$358496 = a(s^{(3)})$	0.05983	
	s_4	$364918 = a(s^{(4)})$	0.06037	
2	s_1	$997539 = a(s^{(4)})$	0.09981	0.04483
	s_2	$376805 = a(s^{(3)})$	0.06034	
	s_3	$30005 = a(s^{(1)})$	0.01731	
	s_4	$272884 = a(s^{(2)})$	0.05220	
3	s_1	$74866073 = a(s^{(1)})$	0.86464	0.93879
	s_2	$95654573 = a(s^{(2)})$	0.97734	
	s_3	$100141401 = a(s^{(4)})$	1.00000	
	s_4	$99871717 = a(s^{(3)})$	0.99865	
4	s_1	$3722326 = a(s^{(1)})$	0.19280	0.20821
	s_2	$4323214 = a(s^{(2)})$	0.20778	
	s_3	$5016833 = a(s^{(4)})$	0.22382	
	s_4	$4910771 = a(s^{(3)})$	0.22145	
5	s_1	$235669 = a(s^{(2)})$	0.04851	0.04858
	s_2	$359747 = a(s^{(3)})$	0.05994	
	s_3	$101795 = a(s^{(1)})$	0.05994	
	s_4	$863858 = a(s^{(4)})$	0.09088	

Table A2: The CEU for animal product exports between Korea and New Zealand for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU_{(i,NZ)}(u(a))$
1	s_1	$6650 = a(s^{(4)})$	0.00815	0.00533
	s_2	$4497 = a(s^{(3)})$	0.00670	
	s_3	$1589 = a(s^{(1)})$	0.00398	
	s_4	$2779 = a(s^{(2)})$	0.00527	
2	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
3	s_1	$70759196 = a(s^{(2)})$	0.84059	0.78873
	s_2	$91263506 = a(s^{(4)})$	0.95464	
	s_3	$70763937 = a(s^{(3)})$	0.84062	
	s_4	$46632301 = a(s^{(1)})$	0.68240	
4	s_1	$165773 = a(s^{(3)})$	0.04069	0.15976
	s_2	$113751 = a(s^{(1)})$	0.03370	
	s_3	$148756 = a(s^{(2)})$	0.03854	
	s_4	$277350 = a(s^{(4)})$	0.05263	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.01557
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$218022 = a(s^{(3)})$	0.04666	
	s_4	$393025 = a(s^{(4)})$	0.00265	

Table A3: the CEU for Animal product expert between Korea and India for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU(u(a))$
1	s_1	$1050 = a(s^{(3)})$	0.00324	0.00264
	s_2	$1300 = a(s^{(4)})$	0.00360	
	s_3	$450 = a(s^{(1)})$	0.00212	
	s_4	$700 = a(s^{(2)})$	0.00264	
2	s_1	$35432 = a(s^{(3)})$	0.01881	0.00887
	s_2	$50639 = a(s^{(4)})$	0.02249	
	s_3	$2656 = a(s^{(1)})$	0.00515	
	s_4	$8230 = a(s^{(2)})$	0.00907	
3	s_1	$8695 = a(s^{(4)})$	0.009318	0.00368
	s_2	$5247 = a(s^{(3)})$	0.00724	
	s_3	$0 = a(s^{(1)})$	0.00000	
	s_4	$1865 = a(s^{(2)})$	0.00432	
4	s_1	$0 = a(s^{(1)})$	0.00000	0.00470
	s_2	$21614 = a(s^{(3)})$	0.01469	
	s_3	$30938 = a(s^{(4)})$	0.01758	
	s_4	$0 = a(s^{(2)})$	0.00000	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	

Table A4: The CEU for animal product exports between Korea and Turkey for years 2010-2013

HSPC	s	$a(s)$(USD)	$u(a(s))$	$CEU(u(a))$
1	s_1	$0 = a(s^{(1)})$	0.00000	0.00154
	s_2	$6900 = a(s^{(4)})$	0.00830	
	s_3	$150 = a(s^{(2)})$	0.00122	
	S_4	$300 = a(s^{(3)})$	0.00173	
2	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
3	s_1	$0 = a(s^{(1)})$	0.00000	0.04570
	s_2	$672952 = a(s^{(3)})$	0.08198	
	s_3	$2532837 = a(s^{(4)})$	0.15904	
	s_4	$199874 = a(s^{(2)})$	0.04468	
4	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	
5	s_1	$0 = a(s^{(1)})$	0.00000	0.00000
	s_2	$0 = a(s^{(2)})$	0.00000	
	s_3	$0 = a(s^{(3)})$	0.00000	
	s_4	$0 = a(s^{(4)})$	0.00000	

J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

QUADRATIC FUNCTIONAL INEQUALITY IN MODULAR SPACES AND ITS STABILITY

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Abstract. In this paper, we prove the generalized Hyers-Ulam stability for the following functional inequality

$\rho(f(x+y)+f(x-y)-2f(x)-2f(y)) \geq \rho(k[f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y)])$

in modular spaces without $\triangle_2$-conditions.

1. Introduction and preliminaries

In 1940, Ulam proposed the following stability problem (cf. [16]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [4] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [13] for linear mappings by considering an unbounded Cauchy difference, the latter of which has influenced many developments in the stability theory. This area is then referred to as the generalized Hyers-Ulam stability. A generalization of the Rassias' theorem was obtained by Găvruta [3] by replasing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

A problem that mathematicians has dealt with is "how to generalize the classical function space L^p". A first attempt was made by Birnhaum and Orlicz in 1931. The more abstract generalization was given by Nakano [11] in 1950 based on replacing the particular integral form of the functional by an abstract one that satisfies some good properties. This functional was called *modular*([1], [6], [7], [8], [9], [12], [15], [18]). This idea was refined and generalized by Musielak and Orlicz [10] in 1959.

Recently, Sadeghi [14] presented a fixed point method to prove the generalized Hyers-Ulam stability of functional equations in modular spaces with the $\triangle_2$-condition and Wongkum, Chaipunya, and Kumam [17] proved the fixed point theorem and the generalized Hyers-Ulam stability for quadratic mappings in a modular space whose modular is convex, lower semi-continuous but do not satisfy the $\triangle_2$-condition.

2010 *Mathematics Subject Classification.* 39B52, 39B72, 47H09.

Key words and phrases. Fixed point theorem, Hyers-Ulam stability, modular spaces.

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In this paper, we prove the generalized Hyers-Ulam stability for the following quadratic functional equation

$$\begin{aligned}(1.1)\qquad &\rho(f(x+y)+f(x-y)-2f(x)-2f(y))\\ &\geq \rho(k[f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y)])\end{aligned}$$

in modular spaces without $\triangle_2$-conditions by using a fixed point theorem.

Definition 1.1. Let X be a vector space over a field $\mathbb{K}(\mathbb{R}$, $\mathbb{C}$, or $\mathbb{N})$.
(1) A generalized functional $\rho : X \longrightarrow [0,\infty]$ is called *a modular* if
(M1) $\rho(x)=0$ if and only if $x=0$,
(M2) $\rho(\alpha x)=\rho(x)$ for every scalar α with $|\alpha|=1$, and
(M3) $\rho(z)\leq \rho(x)+\rho(y)$ whenever z is a convex combination of x and y.
(2) If (M3) is replaced by
(M4) $\rho(\alpha x+\beta y)\leq \alpha\rho(x)+\beta\rho(y)$
for all $x,y\in V$ and for all nonnegative real numbers α, β with $\alpha+\beta=1$, then we say that ρ is *convex.*

For any modular ρ on X, the modular space X_ρ is defined by

$$X_\rho=\{x\in X \mid \rho(\lambda x)\to 0 \ as\ \lambda\to 0\}$$

and the modular space X_ρ can be equipped with a norm called the Luxemburg norm, defined by

$$\|x\|_\rho=\inf\Big\{\lambda>0 \mid \rho\Big(\frac{x}{\lambda}\Big)\leq 1\Big\}.$$

Let X_ρ be a modular space and $\{x_n\}$ a sequence in X_ρ. Then (i) $\{x_n\}$ is called *ρ-Cauchy* if for any $\epsilon>0$, one has $\rho(x_n-x_m)<\epsilon$ for sufficiently large $m,n\in\mathbb{N}$, (ii) $\{x_n\}$ is called *ρ-convergent* to a point $x\in X_\rho$ if $\rho(x_n-x)\to 0$ as $n\to\infty$, and (iii) a subset K of X_ρ is called *ρ-complete* if each ρ-Cauchy sequence is ρ-convergent to a point in K.

A modular space X_ρ is said to *satisfy the $\triangle_2$-condition* if there exists $k\geq 2$ such that $X_\rho(2x)\leq kX_\rho(x)$ for all $x\in X$.

Example 1.2. ([9], [11], [12]) A convex function ζ defined on the interval $[0,\infty)$, nondecreasing and continuous, such that $\zeta(0)=0, \zeta(\alpha)>0$ for $\alpha>0$, $\zeta(\alpha)\to\infty$ as $\alpha\to\infty$, is called an Orlicz function. Let (Ω,Σ,μ) be a measure space and $L^0(\mu)$ the set of all measurable real valued (or complex valued) functions on Ω. Deine the Orlicz modular ρ_ζ on $L^0(\mu)$ by the formula $\rho_\zeta(f)=\int_\Omega \zeta(|f|)d\mu$. The associated modular space with respect to this modular is called an Orlicz space, and will be denoted by (L^ζ,Ω,μ) or briefly L^ζ. In other words,

$$L^\zeta=\{f\in L^0(\mu) \mid \rho_\zeta(\lambda f)<\infty \ \text{ for some }\ \lambda>0\}.$$

It is known that the Orlicz space L^ζ is ρ_ζ-complete. Moreover, $(L^\zeta, \|\cdot\|_{\rho_\zeta})$ is a Banach space, where the Luxemburg norm $\|\cdot\|_{\rho_\zeta}$ is defined as follows

$$\|f\|_{\rho_\zeta}=\inf\Big\{\lambda>0 \ \Big|\ \int_\Omega \zeta\Big(\frac{|f|}{\lambda}\Big)d\mu\leq 1\Big\}.$$

Further, if μ is the Lebesgue measure on $\mathbb{R}$ and $\zeta(t)=e^t-1$, then ρ_ζ does not satisfy the $\triangle_2$-condition.

For a modular space X_ρ, a nonempty subset C of X_ρ, and a mapping $T : C \longrightarrow C$, *the orbit of* T *at* $x \in C$ is the set

$$\mathbb{O}(x) = \{x, Tx, T^2x, \cdots\}.$$

If $\delta_\rho(x) = \sup\{\rho(u-v) \mid u, v \in \mathbb{O}(x)\} < \infty$, then one says that T has *a bounded orbit at* x.

Lemma 1.3. [5] *Let* X_ρ *be a modular space whose induced modular is lower semi-continuous and let* $C \subseteq X_\rho$ *be a* ρ*-complete subset. If* $T : C \longrightarrow C$ *is a* ρ*-contraction, that is, there is a constant* $L \in [0,1)$ *such that*

$$\rho(Tx - Ty) \le L\rho(x-y),\ \ \forall x, y \in C$$

and T *has a bounded orbit at a point* $x_0 \in C$*, then the sequence* $\{T^n x_0\}$ *is* ρ*-convergent to a point* $w \in C$*.*

For any modular ρ on X and any linear space V, we define a set $\mathbb{M}$

$$\mathbb{M} := \{g : V \longrightarrow X_\rho \mid g(0) = 0\}$$

and the generalized function $\widetilde{\rho}$ on $\mathbb{M}$ by for each $g \in \mathbb{M}$,

$$\widetilde{\rho}(g) := \inf\{c > 0 \mid \rho(g(x)) \le c\psi(x, 0),\ \forall x \in V\},$$

where $\psi : V^2 \longrightarrow [0,\infty)$ is a mapping. The proof of the following lemma is similar to the proof of Lemma 10 in [17].

Lemma 1.4. *Let* V *be a linear space,* X_ρ *a* ρ*-complete modular space where* ρ *is convex lower semi-continuous and* $f : V \longrightarrow X_\rho$ *a mapping with* $f(0) = 0$*. Let* $\psi : V^2 \longrightarrow [0,\infty)$ *be a mapping such that*

$$\psi(ax, ax) \le a^2 L\psi(x, x) \tag{1.2}$$

for all $x, y \in V$ *and some* a *and* L *with* $a \ge 2$ *and* $0 \le L < 1$*. Then we have the following :*
(1) $\mathbb{M}$ *is a linear space,*
(2) $\widetilde{\rho}$ *is a convex modular, and*
(3) $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$ *and* $\mathbb{M}_{\widetilde{\rho}}$ *is* $\widetilde{\rho}$*-complete, and*
(4) $\widetilde{\rho}$ *is lower semi-continuous.*

2. Solutions of (1.1)

In this section, we consider solutions of (1.1).

For any $f : V \longrightarrow X_\rho$, let

$$A_f(x, y) = k[f(ax+by) + f(ax-by) - 2a^2 f(x) - 2b^2 f(y)]$$

and

$$B_f(x, y) = f(x+y) + f(x-y) - 2f(x) - 2f(y).$$

Lemma 2.1. *Let* ρ *be a convex modular on* X *and* $f : V \longrightarrow X_\rho$ *an even mapping with* $f(0) = 0$*. Suppose that* $ka^2 \ge 1$ *and* $b^2 > a^2$*. Then* f *is a quadratic mapping if and only if* f *is a solution of (1.1).*

Proof. Since $k \neq 0$ and f is even, we have

$$f(ax) = a^2 f(x),\ f(bx) = b^2 f(x) \tag{2.1}$$

for all $x \in V$. Putting $y = ay$ in (1.1), by (2.1), we have

$$\begin{aligned} &\rho(f(x+ay) + f(x-ay) - 2f(x) - 2a^2 f(y)) \\ &\geq \rho(ka^2[f(x+by) + f(x-by) - 2f(x) - 2b^2 f(y)]) \end{aligned} \tag{2.2}$$

for all $x, y \in V$ and letting $y = \frac{y}{a}$ in (2.2), by (2.1), we have

$$\rho(B_f(x,y)) \geq \rho(ka^2[f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)]) \tag{2.3}$$

for all $x, y \in V$,where $p = \frac{b}{a}$. Since ρ is convex and $ka^2 \geq 1$, by (2.3),

$$\rho(B_f(x,y)) \geq ka^2 \rho(f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)) \tag{2.4}$$

for all $x, y \in V$. Letting $x = py$ in (2.3), by (2.1), we have

$$\begin{aligned} &\rho(f(px+y) + f(px-y) - 2p^2 f(x) - 2f(y)) \\ &\geq kb^2 \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \end{aligned} \tag{2.5}$$

for all $x, y \in V$, because ρ is convex and $b^2 > a^2$. Interchanging x and y in (2.5), we have

$$\rho(f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)) \geq kb^2 \rho(B_f(x,y)) \tag{2.6}$$

for all $x, y \in V$. By (M4), (2.4), and (2.6), we have

$$\begin{aligned} &\rho(f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)) \\ &\geq k^2 a^2 b^2 \rho(f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y)) \end{aligned} \tag{2.7}$$

for all $x, y \in V$. Since $k^2 a^2 b^2 > 1$, by (2.7) and (M1), we get

$$f(x+py) + f(x-py) - 2f(x) - 2p^2 f(y) = 0$$

for all $x, y \in V$ and hence f is a quadratic mapping. The converse is trivial. □

Theorem 2.2. *Let ρ be a convex modular on X and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $ka^2 \geq 2$ and $b^2 > a^2$. Then f is a quadratic mapping if and only if f is a solution of (1.1).*

Proof. By (1.1), we have

$$\begin{aligned} \rho(A_{f_o}(x,y)) &\leq \frac{1}{2}\rho(A_f(x,y)) + \frac{1}{2}\rho(A_f(-x,-y)) \\ &\leq \frac{1}{2}\rho(B_f(x,y)) + \frac{1}{2}\rho(B_f(-x,-y)) \\ &\leq \frac{1}{2}\rho(2B_{f_o}(x,y)) + \frac{1}{2}\rho(2B_{f_e}(x,y)) \end{aligned} \tag{2.8}$$

for all $x, y \in V$ and similarly, we have

$$\rho(A_{f_e}(x,y)) \leq \frac{1}{2}\rho(2B_{f_o}(x,y)) + \frac{1}{2}\rho(2B_{f_e}(x,y)) \tag{2.9}$$

for all $x, y \in V$. Letting $x = 0$ in (2.8), by (M4), we have

$$\frac{1}{2}\rho(4f_o(y)) \geq \rho(2kb^2 f_o(y)) \geq \frac{kb^2}{2}\rho(4f_o(y)) \tag{2.10}$$

for all $y \in V$, because ρ is convex and $kb^2 > 2$. Since $kb^2 > 1$, by (2.10) and (M1), we have $f_o(y) = 0$ for all $y \in V$ and hence by (2.9), we have

$$\rho(A_{f_e}(x, y)) \leq \rho(2B_{f_e}(x, y)) \tag{2.11}$$

for all $x, y \in V$. Since $ka^2 \geq 2$ and $b^2 > a^2$, by Lemma 2.1 and (2.11), $2f_e$ is a quadratic mapping and since $f_o(x) = 0$ for all $x \in X$, f is a quadratic mapping. ☐

For $k = 1$ in Theorem 2.2, we have the following corollary:

Corollary 2.3. *Let ρ be a convex modular on X and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $b^2 > a^2 \geq 2$. The f is quadratic if and only if*

$$\rho(B_f(x, y)) \geq \rho(f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y))$$

for all $x, y \in V$.

Corollary 2.4. *Let ρ be a convex modular on X and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $ka^2 \geq 2$ and $b^2 > a^2$. Then the following are equivalent*

(1) *f is quadratic,*
(2) *f satisfies (1.1), and*
(3) *f satisfies the following*

$$\rho(rB_f(x, y)) \geq \rho(rA_f(x, y))$$

for all $x, y \in V$ and all real number r.

3. The generalized Hyers-Ulam stability for (1.1) in modular spaces

Throughout this section, we assume that every modular is convex and lower semicontinuous. In this section, we will prove the generalized Hyers-Ulam stability for (1.1).

Lemma 3.1. *Let ρ be a convex modular on X and t a real number with $2 \leq t$. Then*

$$\rho\Big(\frac{1}{t}x + \frac{1}{t}y\Big) \leq \frac{1}{t}\rho(x) + \frac{1}{t}\rho(y)$$

for all $x, y \in X$.

Proof. Since ρ is a convex modular on X, we have

$$\rho\Big(\frac{1}{t}x + \frac{1}{t}y\Big) \leq \frac{1}{t}\rho(x) + \Big(1 - \frac{1}{t}\Big)\rho\Big(\frac{1}{t-1}y\Big) \leq \frac{1}{t}\rho(x) + \frac{1}{t}\rho(y)$$

for all $x, y \in X$, because $2 \leq t$. ☐

Theorem 3.2. *Let ρ be a modular on X, V a linear space, X_ρ a ρ-complete modular space and $f : V \longrightarrow X_\rho$ a mapping with $f(0) = 0$. Suppose that $a \geq 2$, $k \geq a^2$, and $b^2 > a^2$. Let $\phi : V^2 \longrightarrow [0, \infty)$ be a mapping such that*

$$\phi(ax, ay) \leq a^2 L\phi(x, y) \tag{3.1}$$

for all $x, y \in V$ and some L with $0 < L < 1$ and

$$\rho(rA_f(x, y)) \leq \rho(rB_f(x, y)) + |r|\phi(x, y) \tag{3.2}$$

for all $x, y \in V$ and all real number r. Then there exists a unique quadratic mapping $Q : V \longrightarrow X_\rho$ such that

$$\rho\Big(Q(x) - \frac{1}{a^2}f(x)\Big) \leq \frac{1}{ka^4(1-L)}\phi(x, 0) \tag{3.3}$$

for all $x \in V$.

Proof. By Lemma 1.4, $\widetilde{\rho}$ is a lower semi-continuous convex modular on $\mathbb{M}_{\widetilde{\rho}}$, $\mathbb{M}_{\widetilde{\rho}} = \mathbb{M}$, and $\mathbb{M}_{\widetilde{\rho}}$ is $\widetilde{\rho}$-complete. Define $T : \mathbb{M}_{\widetilde{\rho}} \longrightarrow \mathbb{M}_{\widetilde{\rho}}$ by $Tg(x) = \dfrac{1}{a^2}g(ax)$ for all $g \in \mathbb{M}_{\widetilde{\rho}}$ and all $x \in V$. Let $g, h \in \mathbb{M}_{\widetilde{\rho}}$. Suppose that $\widetilde{\rho}(g-h) \le c$ for some nonnegative real number c. Then by (3.1), we have

$$\rho(Tg(x) - Th(x)) \le \frac{1}{a^2}\rho(g(ax) - h(ax)) \le Lc\phi(x, 0)$$

for all $x \in V$ and so $\widetilde{\rho}(Tg - Th) \le L\widetilde{\rho}(g - h)$. Hence T is a $\widetilde{\rho}$-contraction. Since $2k > 1$, by (3.2), for $r = 1$, we get

$$\rho\Big(f(ax) - a^2 f(x)\Big) \le \frac{1}{2k}\rho(2kf(ax) - 2ka^2 f(x)) \le \frac{1}{2k}\phi(x, 0) \tag{3.4}$$

for all $x \in X$. Since $a \ge 2$, by (3.4),

$$\rho(Tf(x)-f(x)) = \rho\Big(\frac{1}{a^2}f(ax)-f(x)\Big) \le \frac{1}{a^2}\rho(f(ax)-a^2 f(x)) \le \frac{1}{2ka^2}\phi(x, 0) \tag{3.5}$$

for all $x \in X$.

Now, we claim that T has a bounded orbit at $\frac{1}{a^2}f$. By Lemma 3.1 and (3.5), for any nonnegative integer n, we obtain

$$\begin{aligned}\rho\Big(\frac{1}{a}T^n f(x) - \frac{1}{a}f(x)\Big) &\le \frac{1}{a}\rho\Big(T^n f(x) - \frac{1}{a^2}f(ax)\Big) + \frac{1}{a}\rho\Big(\frac{1}{a^2}f(ax) - f(x)\Big)\\ &\le \frac{1}{a^2}\rho\Big(\frac{1}{a}T^{n-1}f(ax) - \frac{1}{a}f(ax)\Big) + \frac{1}{2ka^3}\phi(x, 0)\end{aligned}$$

for all $x \in V$ and by (3.1), we have

$$\rho\Big(\frac{1}{a}T^n f(x) - \frac{1}{a}f(x)\Big) \le \frac{1}{2ka^3}\sum_{i=0}^{n-1} L^i\phi(x, 0) \le \frac{1}{2ka^3(1-L)}\phi(x, 0) \tag{3.6}$$

for all $x \in V$ and all $n \in \mathbb{N}$. By Lemma 3.1 and (3.6), we get

$$\rho\Big(\frac{1}{a^2}T^n f(x) - \frac{1}{a^2}T^m f(x)\Big) = \rho\Big(\frac{1}{a^2}T^n f(x) - \frac{1}{a^2}T^m f(x)\Big) \le \frac{1}{ka^4(1-L)}\phi(x, 0) \tag{3.7}$$

for all $x \in V$ and all nonnegative integers n, m. Hence T has a bounded orbit at $\frac{1}{a^2}f$.

By Lemma 1.3, there is a $Q \in \mathbb{M}_{\widetilde{\rho}}$ such that $\{T^n\frac{1}{a^2}f\}$ is $\widetilde{\rho}$-convergent to Q. Since $\widetilde{\rho}$ is lower semi-continuous, we get

$$0 \le \widetilde{\rho}(TQ - Q) \le \liminf_{n\to\infty}\widetilde{\rho}\Big(TQ - T^{n+1}\frac{1}{a^2}f\Big) \le \liminf_{n\to\infty} L\widetilde{\rho}\Big(Q - T^n\frac{1}{a^2}f\Big) = 0$$

and hence Q is a fixed point of T in $\mathbb{M}_{\widetilde{\rho}}$. Since $a \ge 2$, there is a a natural number t with $k < a^{t-6}$ and $2kb^2 < a^{t-3}$ and hence we have

$$\begin{aligned}&\rho\Big(\frac{1}{a^t}\Big[A_Q(x, y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\Big]\Big)\\ &\le \frac{k}{a^t}\rho\Big(Q(ax + by) - \frac{1}{a^{2n+2}}f(a^{n+1}x + a^n by)\Big) + \frac{2k}{a^{t-2}}\rho\Big(Q(x) - \frac{1}{a^{2n+2}}f(a^n x)\Big)\\ &+ \frac{k}{a^t}\rho\Big(Q(ax - by) - \frac{1}{a^{2n+2}}f(a^{n+1}x - a^n by)\Big) + \frac{2kb^2}{a^t}\rho\Big(Q(y) - \frac{1}{a^{2n+2}}f(a^n y)\Big)\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the above inequality, we get

$$\lim_{n\to\infty}\rho\Big(\frac{1}{a^t}\Big[A_Q(x, y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\Big]\Big) = 0 \tag{3.8}$$

for all $x, y \in V$, because $\{\frac{1}{a^{2n+2}}f\}$ is $\widetilde{\rho}$-convergent to Q. Similarly, we have

$$\lim_{n\to\infty} \rho\Big(\frac{1}{a^t}\Big[B_Q(x,y) - \frac{1}{a^{2n+2}}B_f(a^n x, a^n y)\Big]\Big) = 0 \tag{3.9}$$

for all $x, y \in V$. Since $a^2 \le k$, by (3.2), we have

$$\begin{aligned}
&\rho\Big(\frac{1}{ka^{t+1}}A_Q(x,y)\Big)\\
&\le \frac{1}{a}\rho\Big(\frac{1}{ka^t}\Big[A_Q(x,y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\Big]\Big) + \frac{1}{a}\rho\Big(\frac{1}{a^{2n+t+4}}A_f(a^n x, a^n y)\Big)\\
&\le \frac{1}{a^3}\rho\Big(\frac{1}{a^t}\Big[A_Q(x,y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\Big]\Big) + \frac{1}{a}\rho\Big(\frac{1}{a^{2n+t+4}}B_f(a^n x, b^n y)\Big)\\
&\quad + \frac{1}{a^{2n+t+5}}\phi(a^n x, a^n y)\\
&\le \frac{1}{a^3}\rho\Big(\frac{1}{a^t}\Big[A_Q(x,y) - \frac{1}{a^{2n+2}}A_f(a^n x, a^n y)\Big]\Big) + \frac{1}{a^2}\rho\Big(\frac{1}{a^{t+1}}B_Q(x,y)\Big)\\
&\quad + \frac{1}{a^3}\rho\Big(\frac{1}{a^t}\Big[\frac{1}{a^{2n+2}}B_f(a^n x, a^n y) - B_Q(x,y)\Big]\Big) + \frac{L^n}{a^{t+5}}\phi(x,y)
\end{aligned}$$

for all $x, y \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, by (3.8) and (3.9), we get

$$\rho\Big(\frac{1}{ka^{t+1}}A_Q(x,y)\Big) \le \rho\Big(\frac{1}{a^{t+1}}B_Q(x,y)\Big)$$

for all $x, y \in V$. By Corollary 2.3, Q is a quadratic mapping. Moreover, since ρ is lower semi-continuous, by (3.7), we have (3.3). □

Corollary 3.3. *Let X and Y be normed spaces and ϵ, θ, and p real numbers with $\epsilon \ge 0$, $\theta \ge 0$, and $0 < p < 1$. Suppose that $a \ge 2$, $k \ge a^2$, and $b^2 > a^2$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\|A_f(x,y)\| \le \|B_f(x,y)\| + \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$$

for all $x, y \in X$. Then there is a quadratic mapping $Q : X \longrightarrow Y$ such that

$$\|Q(x) - f(x)\| \le \frac{1}{k(a^2 - a^{2p})}(\epsilon + \theta\|x\|^{2p})$$

for all $x \in X$.

Proof. Let $\rho(z) = \|z\|$ for all $y \in Y$ and $\phi(x,y) = \epsilon + \theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$ for all $x, y \in V$. Then ρ is a convex modular on a normed space Y, $Y = Y_\rho$, and $\phi(ax, ay) \le a^{2p}\phi(x,y)$ for all $x, y \in V$. By Theorem 3.2, we have the results. □

Using Example 1.1, we get the following example.

Example 3.4. Let θ, and p be real numbers with $\theta \ge 0$ and $0 < p < 1$. Suppose that $a \ge 2$, $k \ge a^2$, and $b^2 > a^2$. Let ζ be an Orlicz function and L^ζ the Orlicz space. Let $f : V \longrightarrow L^\zeta$ be a mapping such that $f(0) = 0$ and

$$\int_\Omega \zeta(rA_f(x,y))d\mu \le \int_\Omega \zeta(rB_f(x,y))d\mu + |r|\theta(\|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p)$$

for all $x, y \in X$ and all real number r. Then there is a quadratic mapping $Q : X \longrightarrow Y$ such that

$$\int_\Omega \zeta\Big(\Big|Q(x) - \frac{1}{a^2}f(x)\Big|\Big)d\mu \le \frac{\theta}{ka^2(a^2 - a^{2p})}\|x\|^{2p}$$

for all $x \in X$.

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Complex Multivariate Taylor's formula

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Abstract

We derive here a Taylor's formula with integral remainder in the several complex variables and we estimate its remainder.

2010 Mathematics Subject Classification : 32A05, 32A10, 32A99.
Key Words and phrases: Taylor's formula in several complex variables, remainder estimation.

1 Main Results

We need the following vector Taylor's formula:

Theorem 1 *(Shilov, [3], pp. 93-94) Let* $n \in \mathbb{N}$ *and* $f \in C^n([a,b], X)$*, where* $[a,b] \subset \mathbb{R}$ *and* $(X, \|\cdot\|)$ *is a Banach space. Then*

$$f(b) = f(a) + \sum_{i=1}^{n-1} \frac{(b-a)^i}{i!} f^{(i)}(a) + \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx. \quad (1)$$

The remainder here is the Riemann X*-valued integral (defined similar to numerical one) given by*

$$Q_{n-1} = \frac{1}{(n-1)!} \int_a^b (b-x)^{n-1} f^{(n)}(x)\, dx, \quad (2)$$

with the property:

$$\|Q_{n-1}\| \leq \max_{a \leq x \leq b} \left\| f^{(n)}(x) \right\| \frac{(b-a)^n}{n!}. \quad (3)$$

The derivatives above are defined similar to the numerical ones. We make

Remark 2 *Here* Q *is an open convex subset of* $\mathbb{C}^k$*,* $k \geq 2$*;* $z := (z_1, ..., z_k)$*,* $x_0 := (x_{01}, ..., x_{0k}) \in Q$*. Let* $f : Q \to \mathbb{C}$ *be a coordinate-wise holomorphic* *function. Then, by the famous Hartog's fundamental theorem ([1], [2]) f is jointly holomorphic and jointly continuous on Q. Let $n \in \mathbb{N}$. Each n^{th} order complex partial derivative is denoted by $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, ..., \alpha_k)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, ..., k$ and $|\alpha| := \sum_{i=1}^{k} \alpha_i = n$.*

Consider

$$g_z(t) := f(x_0 + t(z - x_0)), \quad 0 \le t \le 1. \tag{4}$$

Clearly it holds that $x_0 + t(z - x_0) \in Q$ and $g_z(t) \in \mathbb{C}$, $\forall\ t \in [0,1]$.

Then we derive

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i}\right)^j f\right](x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})), \tag{5}$$

for all $j = 0, 1, ..., n$.

Notice here that any mixed partials commute. We remind that $(\mathbb{C}, |\cdot|)$ is a Banach space. By Shilov's Theorem 1, about the Taylor's formula for Banach space valued functions, we obtain

Theorem 3 *It holds*

$$f(z_1, ..., z_k) = g_z(1) = \sum_{j=0}^{n-1} \frac{g_z^{(j)}(0)}{j!} + R_n(z, 0), \tag{6}$$

where

$$R_n(z, 0) = \frac{1}{(n-1)!} \int_0^1 (1-\theta)^{n-1} g_z^{(n)}(\theta) \, d\theta, \tag{7}$$

and notice that $g_z(0) = f(x_0)$.

We make

Remark 4 *Notice that (by (7)) we get*

$$|R_n(z, 0)| \le \left(\max_{0 \le \theta \le 1} \left|g_z^{(n)}(\theta)\right|\right) \frac{1}{n!}. \tag{8}$$

We also have for $j = 0, 1, ..., n$:

$$g_z^{(j)}(0) = \sum_{\substack{\alpha := (\alpha_1, ..., \alpha_k),\ \alpha_j \in \mathbb{Z}^+ \\ i = 1, ..., k;\ |\alpha| := \sum_{i=1}^{k} \alpha_i = j}} \left(\frac{j!}{\prod_{i=1}^{k} \alpha_i!}\right) \left(\prod_{i=1}^{k} (z_i - x_{0i})^{\alpha_i}\right) f_\alpha(x_0). \tag{9}$$

Furthermore it holds

$$g_z^{(n)}(\theta) = \sum_{\substack{\alpha:=(\alpha_1,\dots,\alpha_k),\ \alpha_j\in\mathbb{Z}^+\\ i=1,\dots,k;\ |\alpha|:=\sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k (z_i - x_{0i})^{\alpha_i} \right) f_\alpha (x_0 + \theta (z - x_0)), \tag{10}$$

$0 \le \theta \le 1$.

Another version of (6) is

$$f(z_1, \dots, z_k) = g_z(1) = \sum_{j=0}^n \frac{g_z^{(j)}(0)}{j!} + \overline{R_n}(z, 0), \tag{11}$$

where

$$\overline{R_n}(z, 0) = \frac{1}{(n-1)!} \int_0^1 (1-\theta)^{n-1} \left(g_z^{(n)}(\theta) - g_z^{(n)}(0) \right) d\theta. \tag{12}$$

Identities (6) and (11) are the multivariate complex Taylor's formula with integral remainders.

We give

Example 5 *Let $n = k = 2$. Then*

$$g_z(t) = f(x_{01} + t(z_1 - x_{01}), x_{02} + t(z_2 - x_{02})), \quad t \in [0, 1],$$

and

$$g_z'(t) = (z_1 - x_{01}) \frac{\partial f}{\partial x_1} (x_0 + t(z - x_0)) + (x_2 - x_{02}) \frac{\partial f}{\partial x_2} (x_0 + t(z - x_0)). \tag{13}$$

In addition,

$$\begin{aligned} g_z''(t) &= (z_1 - x_{01}) \left(\frac{\partial f}{\partial x_1} (x_0 + t(z - x_0)) \right)' + (x_2 - x_{02}) \left(\frac{\partial f}{\partial x_2} (x_0 + t(z - x_0)) \right)' \\ &= (z_1 - x_{01}) \left\{ (z_1 - x_{01}) \frac{\partial f^2}{\partial x_1^2} (*) + (z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1} (*) \right\} + \\ &\quad (z_2 - x_{02}) \left\{ (z_1 - x_{01}) \frac{\partial f^2}{\partial x_1 \partial x_2} (*) + (z_2 - x_{02}) \frac{\partial f^2}{\partial x_2^2} (*) \right\}. \end{aligned} \tag{14}$$

Hence,

$$\begin{aligned} g_z''(t) &= (z_1 - x_{01})^2 \frac{\partial f^2}{\partial x_1^2} (*) + (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_2 \partial x_1} (*) + \\ &\quad (z_1 - x_{01})(z_2 - x_{02}) \frac{\partial f^2}{\partial x_1 \partial x_2} (*) + (z_2 - x_{02})^2 \frac{\partial f^2}{\partial x_2^2} (*), \end{aligned} \tag{15}$$

where $* := x_0 + t(z - x_0)$.

Notice that $g_z(t), g_z'(t), g_z''(t) \in \mathbb{C}$.

We make

Remark 6 *We define*

$$\|f\|_{p,\overline{zx_0}} := \left(\int_0^1 |f(x_0 + \theta(z - x_0))|^p \, d\theta \right)^{\frac{1}{p}}, \quad p \geq 1, \tag{16}$$

where $\overline{zx_0}$ *denotes the segment* $\overline{zx_0} \subset Q$.

We also define

$$\|f\|_{\infty,\overline{zx_0}} := \max_{\theta \in [0,1]} |f(x_0 + \theta(z - x_0))|. \tag{17}$$

By (10) we obtain

$$\left| g_z^{(n)}(\theta) \right| \leq \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k),\ \alpha_j \in \mathbb{Z}^+ \\ i=1,\ldots,k;\ |\alpha| := \sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k |z_i - x_{0i}|^{\alpha_i} \right) |f_\alpha(x_0 + \theta(z - x_0))|, \tag{18}$$

$\forall\ \theta \in [0,1]$.

Therefore, by norm properties for $1 \leq p \leq \infty$, *it holds*

$$\left\| g_z^{(n)} \right\|_{p,\overline{zx_0}} \leq \sum_{\substack{\alpha := (\alpha_1, \ldots, \alpha_k),\ \alpha_j \in \mathbb{Z}^+ \\ i=1,\ldots,k;\ |\alpha| := \sum_{i=1}^k \alpha_i = n}} \left(\frac{n!}{\prod_{i=1}^k \alpha_i!} \right) \left(\prod_{i=1}^k |z_i - x_{0i}|^{\alpha_i} \right) \|f_\alpha\|_{p,\overline{zx_0}} \tag{19}$$

$$\leq \left(\sum_{i=1}^k |z_i - x_{0i}| \right)^n \|f_\alpha^*\|_{p,\overline{zx_0}},$$

where

$$\|f_\alpha^*\|_{p,\overline{zx_0}} := \max_{|\alpha|=n} \|f_\alpha\|_{p,\overline{zx_0}}, \tag{20}$$

for all $1 \leq p \leq \infty$.

That is

$$\left\| g_z^{(n)} \right\|_{p,\overline{zx_0}} \leq \left(\|z - x_0\|_{l_1} \right)^n \|f_\alpha^*\|_{p,\overline{zx_0}}, \tag{21}$$

for all $1 \leq p \leq \infty$.

Therefore by (8) we obtain

$$|R_n(z,0)| \leq \frac{\left(\|z - x_0\|_{l_1} \right)^n \|f_\alpha^*\|_{\infty,\overline{zx_0}}}{n!}. \tag{22}$$

Next, we put things together and we further estimate $R_n(z,0)$.

Theorem 7 *Here* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*. It holds*

$$|R_n(z,0)| \leq \min \left\{ \begin{array}{c} \frac{\left\| g_z^{(n)} \right\|_{\infty, \overline{zx_0}}}{n!}, \\ \\ \frac{\left\| g_z^{(n)} \right\|_{1, \overline{zx_0}}}{(n-1)!}, \\ \\ \frac{\left\| g_z^{(n)} \right\|_{p, \overline{zx_0}}}{(n-1)!(q(n-1)+1)^{\frac{1}{q}}} \end{array} \right\} \leq \tag{23}$$

$$\left(\|z - x_0\|_{l_1} \right)^n \min \left\{ \begin{array}{c} \frac{\|f_\alpha^*\|_{\infty, \overline{zx_0}}}{n!}, \\ \\ \frac{\|f_\alpha^*\|_{1, \overline{zx_0}}}{(n-1)!}, \\ \\ \frac{\|f_\alpha^*\|_{p, \overline{zx_0}}}{(n-1)!(q(n-1)+1)^{\frac{1}{q}}}. \end{array} \right. \tag{24}$$

Proof. Based on (7), Hölder's inequality and (21). ■

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On the Barnes-type multiple twisted q-Euler zeta function of the second kind

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Abstract :

In this paper we introduce the Barnes-type multiple twisted q-Euler numbers and polynomials of the second kind, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$. Using these numbers and polynomials, we construct the Barnes-type multiple twisted q-Euler zeta function of the second kind. Finally, we obtain the relations between these numbers and polynomials and Barnes-type multiple twisted q -Euler zeta function.

Key words : p-adic invariant integral on $\mathbb{Z}_p$, Euler numbers and polynomials of the second kind, q-Euler numbers and polynomials of the second kind, Barnes-type multiple twisted q-Euler numbers and polynomials of the second kind, Barnes-type multiple twisted q -Euler zeta function.

2000 Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Recently, Bernoulli numbers, Bernoulli polynomials, q-Bernoulli numbers, q-Bernoulli polynomials, the second kind Bernoulli number, the second kind Bernoulli polynomials, Euler numbers of the second kind , Euler polynomials of the second kind, Genocchi numbers, Genocchi polynomials, tangent numbers, tangent polynomials, and Bell polynomials were studied by many authors(see [1, 2, 3, 4, 9]). Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [5], by using Euler numbers E_j and polynomials $E_j(x)$ of the second kind, we investigated the alternating sums of powers of consecutive odd integers. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind Euler polynomials $E_n(x)$(see [6]). The outline of this paper is as follows. We introduce the Barnes-type multiple twisted q-Euler numbers and polynomials of the second kind, by using fermionic p-adic invariant integral on $\mathbb{Z}_p$. In Section 2, we construct the Barnes-type multiple twisted q-Euler zeta function of the second kind. Finally, we obtain the relations between these numbers and polynomials and Barnes-type multiple twisted q -Euler zeta function.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ of the function $g \in UD(\mathbb{Z}_p)$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x)(-1)^x, \text{ see [1, 2, 4].} \tag{1.1}$$

From (1.1), we note that

$$\int_{\mathbb{Z}_p} g(x+1)d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x)d\mu_{-1}(x) = 2g(0). \tag{1.2}$$

First, we introduced the second kind Euler numbers E_n. The second kind Euler numbers E_n are defined by the generating function:

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{1.3}$$

We introduce the second kind Euler polynomials $E_n(x)$ as follows:

$$\frac{2e^t}{e^{2t}+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{1.4}$$

In [5, 6], we studied the second kind Euler numbers E_n and polynomials $E_n(x)$ and investigate their properties.

2. Barnes-type multiple twisted q-Euler numbers and polynomials of the second kind

In this section, we assume that $w_1, \ldots, w_k \in \mathbb{Z}_p$ and $a_1, \ldots, a_k \in \mathbb{Z}$. Let $T_p = \cup_{N\geq 1} C_{p^N} = \lim_{N\to\infty} C_{p^N}$, where $C_{p^N} = \{\omega | \omega^{p^N} = 1\}$ is the cyclic group of order p^N. For $\omega \in T_p$, we denote by $\phi_\omega : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \longmapsto \omega^x$.

We construct the Barnes-type multiple twisted q-Euler polynomials of the second kind,

$$E_{n,\omega,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid x).$$

For $k \in \mathbb{N}$, we define Barnes-type multiple twisted q-Euler polynomials of the second kind as follows:

$$\begin{aligned}
&\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{x_1+\cdots+x_k} q^{a_1x_1+\cdots+a_kx_k} e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k) \\
&= \frac{2^k e^{kt}}{(\omega q^{a_1}e^{2w_1t}+1)(\omega q^{a_2}e^{2w_2t}+1)\cdots(\omega q^{a_k}e^{2w_kt}+1)} e^{xt} \\
&= \sum_{n=0}^{\infty} E_{n,\omega,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid x) \frac{t^n}{n!}.
\end{aligned} \tag{2.1}$$

In the special case, $x = 0$, $E_{n,\omega,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid 0) = E_{n,\omega,q}(w_1, \ldots, w_k; a_1, \ldots, a_k)$ are called the n-th Barnes-type multiple twisted q-Euler numbers of the second kind. By (2.1) and Taylor expansion of $e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t}$, we have the following theorem.

Theorem 1. For positive integers n and k, we have

$$\begin{aligned}
&E_{n,\omega,q}(w_1, \ldots, w_k; a_1, \ldots, a_k \mid x) \\
&= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{x_1+\cdots+x_k} q^{a_1x_1+\cdots+a_kx_k} (x+2w_1x_1+\cdots+2w_kx_k+k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).
\end{aligned}$$

In the case when $x = 0$ in (2.1), we have the following corollary.

Corollary 2. For positive integers n, we have

$$\begin{aligned}
&E_{n,\omega,q}(w_1, \ldots, w_k; a_1, \ldots, a_k) \\
&= \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k-\text{times}} \omega^{\sum_{i=1}^k x_i} q^{\sum_{i=1}^k a_ix_i} (2w_1x_1+\cdots+2w_kx_k+k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).
\end{aligned} \tag{2.2}$$

By Theorem 1 and (2.2), we obtain

$$E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)=\sum_{l=0}^{n}\binom{n}{l}E_{l,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k)x^{n-l}, \tag{2.3}$$

where $\binom{n}{k}$ is a binomial coefficient.

We define distribution relation of Barnes-type multiple twisted q-Euler polynomials of the second kind as follows: For $m\in\mathbb{N}$ with $m\equiv 1(\mod 2)$, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)\frac{t^n}{n!}\\&=\frac{2^k e^{kmt}}{(\omega^m q^{a_1 m}e^{2w_1mt}+1)(\omega^m q^{a_2 m}e^{2w_2mt}+1)\cdots(\omega^m q^{a_k m}e^{2w_kmt}+1)}\\&\quad\times\sum_{l_1,\dots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}\omega^{\sum_{i=1}^{k}l_i}q^{\sum_{i=1}^{k}a_il_i}e^{\left(\frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right)(mt)}.\end{aligned}$$

From the above equation, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}m^n\sum_{l_1,\dots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}\omega^{\sum_{i=1}^{k}l_i}q^{\sum_{i=1}^{k}a_il_i}\\&\quad\times E_{n,\omega^m,q^m}\left(w_1,\dots,w_k;a_1,\dots,a_k \mid \frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 3 (Distribution relation). For $m\in\mathbb{N}$ with $m\equiv 1(\mod 2)$, we have

$$\begin{aligned}&E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)\\&=m^n\sum_{l_1,\dots,l_k=0}^{m-1}(-1)^{l_1+\cdots+l_k}\omega^{\sum_{i=1}^{k}l_i}q^{\sum_{i=1}^{k}a_il_i}\\&\quad\times E_{n,\omega^m,q^m}\left(w_1,\dots,w_k;a_1,\dots,a_k \mid \frac{x+2w_1l_1+\cdots+2w_kl_k+k-mk}{m}\right).\end{aligned}$$

From (2.1), we derive

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k-\text{times}}\omega^{x_1+\cdots+x_k}q^{a_1x_1+\cdots+a_kx_k}e^{(x+2w_1x_1+\cdots+2w_kx_k+k)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k)\\&=2^k\sum_{m_1,\dots m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^{k}m_i}q^{\sum_{i=1}^{k}a_im_i}e^{(x+2w_1m_1+\cdots+2w_km_k+k)t}.\end{aligned} \tag{2.4}$$

From (2.2) and (2.4), we note that

$$\begin{aligned}&E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid x)\\&=2^k\sum_{m_1,\dots m_r=0}^{\infty}(-1)^{m_1+\cdots+m_r}q^{\sum_{i=1}^{k}a_im_i}(x+2w_1m_1+\cdots+2w_km_k+k)^n.\end{aligned} \tag{2.5}$$

By using binomial expansion and (2.1), we have the following addition theorem.

Theorem 4(Addition theorem). Barnes-type multiple twisted q-Euler polynomials of the second kind $E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid x)$ satisfies the following relation:

$$E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid x+y)=\sum_{l=0}^{n}\binom{n}{l}E_{l,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid x)y^{n-l}.$$

3. Barnes-type multiple twisted q-Euler zeta function of the second kind

In this section, we assume that $q\in\mathbb{C}$ with $|q|<1$ and the parameters $w_1,\dots,w_k$ are positive. Let ω be the p^N-th root of unity. By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the generating function of Barnes-type multiple twisted q-Euler polynomials of the second kind, $E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid x)$, we define Barnes-type multiple twisted q-Euler zeta function of the second kind. This function interpolates the Barnes-type multiple twisted q-Euler polynomials of the second kind at negative integers.

By (2.1), we obtain

$$\begin{aligned}F_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid x,t)&=\frac{2^ke^{kt}}{(\omega q^{a_1}e^{2w_1t}+1)\cdots(\omega q^{a_k}e^{2w_kt}+1)}e^{xt}\\&=\sum_{n=0}^{\infty}E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid x)\frac{t^n}{n!}.\end{aligned}\tag{3.1}$$

Hence, by (3.1), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid x)\frac{t^n}{n!}\\&=2^k\sum_{m_1,\dots m_r=0}^{\infty}(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^k m_i}q^{\sum_{i=1}^k a_im_i}e^{(x+2w_1m_1+\cdots+2w_km_k+k)t}.\end{aligned}$$

By applying derivative operator, $\frac{d^l}{dt^l}|_{t=0}$ to the above equation, we have

$$\begin{aligned}&E_{n,\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid x)\\&=2^k\sum_{m_1,\dots m_k=0}^{\infty}(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^k m_i}q^{\sum_{i=1}^k a_im_i}(x+2w_1m_1+\cdots+2w_km_k+k)^n.\end{aligned}\tag{3.2}$$

By (3.2), we define the Barnes-type multiple twisted q-Euler zeta function of the second kind $\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid s,x)$ as follows:

Definition 5. For $s,x\in\mathbb{C}$ with $Re(x)>0$, $a_1,\dots,a_k\in\mathbb{C}$, we define

$$\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid s,x)=2^k\sum_{m_1,\dots,m_k=0}^{\infty}\frac{(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^k m_i}q^{\sum_{i=1}^k a_im_i}}{(x+2w_1m_1+\cdots+2w_km_k+k)^s},\tag{3.3}$$

For $s=-l$ in (3.3) and using (3.2), we arrive at the following theorem.

Theorem 6. For positive integer l, we have

$$\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid -l,x)=E_l(w_1,\dots,w_k;a_1,\dots,a_k\mid x).$$

By (2.6), we define multiple twisted q-Euler zeta function of the second kind
$\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k\mid s)$ as follows:

Definition 7. For $s \in \mathbb{C}$, we define

$$\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid s) = 2^k \sum_{m_1,\dots,m_k=0}^{\infty} \frac{(-1)^{m_1+\cdots+m_k}\omega^{\sum_{i=1}^k m_i} q^{\sum_{i=1}^k a_i m_i}}{(2w_1m_1+\cdots+2w_km_k+k)^s}, \tag{3.4}$$

For $s=-l$ in (3.4) and using (2.5), we arrive at the following theorem.

Theorem 8. For positive integer l, we have

$$\zeta_{\omega,q}(w_1,\dots,w_k;a_1,\dots,a_k \mid -l) = E_l(w_1,\dots,w_k;a_1,\dots,a_k).$$

Acknowledgment

This work was supported by 2020 Hannam University Research Fund.

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Some Approximation Results of Kantorovich type operators

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April 26, 2019

In this manuscript, we investigate a variant of the operators define by Lupaş. We compute the rate of convergence for different class of functions. In section 3, the weighted approximation results are established. At the end, stated the problems for further research.

Keyword: Positive linear operators; Rate Convergence; Weighted approximation

2000 Mathematics Subject Classification: primary 41A25, 41A30, 41A36.

1 Introduction

In [1], Lupaş proposed to study the following sequence of linear and positive operators

$$P_n^{[0]}(f,x) = 2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right), \quad x \geq 0, \quad f:[0,\infty)\to\mathbb{R}, \tag{1.1}$$

where $(nx)_0 = 1$ and $(nx)_k = nx(nx+1)(nx+2)\dots(nx+k-1)$, $k \geq 1$.
We can consider that $P_n^{[0]}$, $n \geq 1$, are defined on E where $E = \bigcup\limits_{a>0} E_a$ and E_a is the subspace of all real valued continuous functions f on $[0,\infty)$ such as $\sup\limits_{x\geq 0}(\exp(-ax)|f(x)|) < \infty$. The space E_a is endowed with the norm $\|f\|_a = \sup\limits_{x\geq 0}(\exp(-ax)|f(x)|)$ with respect to which it becomes a Banach space.
In recent year, Patel and Mishra [2] generalized Jain operators type variant of the Lupaş operators defined as

$$P_n^{[\beta]}(f,x) = \sum_{k=0}^{\infty}\frac{(nx+k\beta)_k}{2^k k!}2^{-(nx+k\beta)}f\left(\frac{k}{n}\right), \quad x \geq 0, \quad f:[0,\infty)\to\mathbb{R}, \tag{1.2}$$

where $(nx+k\beta)_0 = 1$, $(nx+k\beta)_1 = nx$ and $(nx+k\beta)_k = nx(nx+k\beta+1)(nx+k\beta+2)\dots(nx+k\beta+k-1)$, $k \geq 2$.
We mention that $\beta = 0$, the operators $P_n^{[0]}$ reduce to Lupaş operators (1.1). In

[2], the authors have used following Lagrange's formula to define the operators (1.2):

$$\phi(z) = \phi(0) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[\frac{d^{k-1}}{dz^{k-1}} ((f(z)^k)\phi'(z) \right]_{z=0} \left(\frac{z}{f(z)} \right)^k . \qquad (1.3)$$

But, if we use following Lagrange's formula then the generalization of the operators (1.1) is written better way:

$$\phi(z) \left[1 - \frac{z}{f(z)} \frac{df(z)}{dz} \right]^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k}{dz^k} ((f(z)^k)\phi(z) \right]_{z=0} \left(\frac{z}{f(z)} \right)^k .$$

By choosing $\phi(z) = (1-z)^{-\alpha}$ and $f(z) = (1-z)^{\beta}$, we have

$$\begin{aligned} & (1-z)^{-\alpha} \left[1 - z\beta(1-z)^{-1}\right]^{-1} \\ = & \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha + k\beta)(\alpha + k\beta + 1) \dots (\alpha + k\beta + k - 1) \left(\frac{z}{(1-z)^{-\beta}} \right)^k . \end{aligned}$$

Taking $z = \frac{1}{2}$, we get

$$1 = (1-\beta) \sum_{k=0}^{\infty} \frac{1}{2^k k!} (\alpha + \beta k)_k 2^{-(\alpha+\beta k)}.$$

Now, we may define the operators as

$$P_n^{[\beta]}(f,x) = \sum_{k=0}^{\infty} \mathbf{p}_{\beta}(k, nx) f\left(\frac{k}{n}\right) \qquad (1.4)$$

where $\mathbf{p}_{\beta}(k, nx) = (1-\beta) \sum_{k=0}^{\infty} \frac{1}{2^k k!} (nx + \beta k)_k 2^{-(nx+\beta k)}$. where $(nx + \beta k)_0 = 1$ and $(nx + \beta k)_k = (nx + \beta k)(nx + \beta k + 2) \dots (nx + \beta k + k - 1)$, $k \geq 1$ and $0 \leq \dfrac{\beta + 1}{2} < 1$

The parameter β may depend on the natural number n. It is easy to see that for $\beta = 0$, the operators $P_n^{[\beta]}(f,x)$ reduces to Lupaş operator (1.1). We mention that, the operators (1.2) and (1.4) has no much difference as their moments are same. To calculate the moments of (1.4), we follows techniques developed in [2] and along the lines of [2], we have

$$\begin{aligned}
P_n^{[\beta]}(1,x) &= 1\\
P_n^{[\beta]}(t,x) &= \frac{x}{1-\beta}+\frac{2\beta}{n(1-\beta)^2}\\
P_n^{[\beta]}(t^2,x) &= \frac{x^2}{(1-\beta)^2}+\frac{2x(1+2\beta)}{n(1-\beta)^3}+\frac{6\left(\beta+\beta^2\right)}{n^2(1-\beta)^4}\\
P_n^{[\beta]}(t^3,x) &= \frac{x^3}{(1-\beta)^3}+\frac{6x^2(1+\beta)}{n(1-\beta)^4}+\frac{6x\left(1+6\beta+3\beta^2\right)}{n^2(1-\beta)^5}\\
&\quad+\frac{2\left(13\beta+34\beta^2+13\beta^3\right)}{n^3(1-\beta)^6}\\
P_n^{[\beta]}(t^4,x) &= \frac{x^4}{(1-\beta)^4}+\frac{4x^3(3+2\beta)}{n(1-\beta)^5}+\frac{36x^2\left(1+3\beta+\beta^2\right)}{n^2(1-\beta)^6}\\
&\quad+\frac{2x\left(13+146\beta+209\beta^2+52\beta^3\right)}{n^3(1-\beta)^7}+\frac{30\left(5\beta+23\beta^2+23\beta^3+5\beta^4\right)}{n^4(1-\beta)^8}.
\end{aligned}$$

In the present paper, we modify the operators defined by (1.4) into integral form in Kantorovich sense, see also G.G. Lorentz [3, Ch.II, p.30]. Actually, we replace $f\left(\frac{k}{n}\right)$ by an integral mean of $f(x)$ over a small interval around the point $\frac{k}{n}$ as follows

$$K_n^{[\beta]}(f,x)=n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}f(t)dt, \tag{1.5}$$

where $\mathbf{p}_\beta(k,nx)$ was as defined in (1.4) and f belongs to the class of local integrable functions defined on $[0,\infty)$.
The focus of the paper is to investigate these linear and positive operators. Section 2, provided results in connection with the rate of convergence for $K_n^{[\beta]}$ under different assumptions of the function f.

2 Approximation properties

For any integer $s\geq 0$, we denote by e_s the test function, $e_s(x)=x^s$, $x\geq 0$, and we also introduce the s-th order central moment of the operator $K_n^{[\beta]}$, that is

$$\Omega_{n,s}(x)=K_n^{[\beta]}(\psi_{x,s},x),\ \text{where } \psi_{x,s}(t)=(t-x)^s, x\geq 0, t\geq 0.$$

Lemma 1 *The operators $K_n^{[\beta]}$, $n\in\mathbf{N}$ defined by (1.5), verify*

1. $K_n^{[\beta]}(1,x)=1$;
2. $K_n^{[\beta]}(t,x)=\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}$;

3. $K_n^{[\beta]}(t^2,x) = \dfrac{x^2}{(1-\beta)^2} + \dfrac{x\left(3+2\beta+\beta^2\right)}{n(1-\beta)^3} + \dfrac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}$;

4. $$K_n^{[\beta]}(t^3,x) = \frac{x^3}{(1-\beta)^3} + \frac{3x^2\left(5+2\beta+\beta^2\right)}{2n(1-\beta)^4} + \frac{x\left(10+32\beta+15\beta^2+2\beta^3+\beta^4\right)}{n^2(1-\beta)^5} + \frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6}$$

5. $$\begin{aligned} K_n^{[\beta]}(t^4,x) = {} & \frac{x^4}{(1-\beta)^4} + \frac{2x^3\left(7+2\beta+\beta^2\right)}{n(1-\beta)^5} \\ & + \frac{2x^2\left(25+44\beta+18\beta^2+2\beta^3+\beta^4\right)}{n^2(1-\beta)^6} \\ & + \frac{x\left(43+326\beta+329\beta^2+116\beta^3+23\beta^4+2\beta^5+\beta^6\right)}{n^3(1-\beta)^7} \\ & + \frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}. \end{aligned}$$

Proof: Observe that $K_n^{[\beta]}(1,x) = P_n^{[\beta]}(1,x) = 1$.
Now,

$$\begin{aligned} K_n^{[\beta]}(t,x) &= n\sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n} t \; dt \\ &= \sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx)\frac{1+2k}{2n} = \frac{1}{2n}P_n^{[\beta]}(1,x) + P_n^{[\beta]}(t,x) \\ &= \frac{x}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} K_n^{[\beta]}(t^2,x) &= n\sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n} t^2 \; dt \\ &= \sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx)\frac{1+3k+3k^2}{3n^2} \\ &= \frac{1}{3n^2}P_n^{[\beta]}(1,x) + \frac{1}{n}P_n^{[\beta]}(t,x) + P_n^{[\beta]}(t^2,x) \\ &= \frac{x^2}{(1-\beta)^2} + \frac{x\left(3+2\beta+\beta^2\right)}{n(1-\beta)^3} + \frac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}. \end{aligned}$$

4

$$
\begin{aligned}
K_n^{[\beta]}(t^3,x) &= n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n} t^3 \;\; dt \\
&= \sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\left(\frac{1+4k+6k^2+4k^3}{4n^3}\right) \\
&= \frac{1}{4n^3}P_n^{[\beta]}(1,x)+\frac{1}{n^2}P_n^{[\beta]}(t,x)+\frac{6}{4n}P_n^{[\beta]}(t^2,x)+P_n^{[\beta]}(t^3,x) \\
&= \frac{x^3}{(1-\beta)^3}+\frac{3x^2\left(5+2\beta+\beta^2\right)}{2n(1-\beta)^4}+\frac{x\left(10+32\beta+15\beta^2+2\beta^3+\beta^4\right)}{n^2(1-\beta)^5} \\
&\quad +\frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6}
\end{aligned}
$$

and

$$
\begin{aligned}
K_n^{[\beta]}(t^4,x) &= n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n} t^4 \;\; dt \\
&= \sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\left(\frac{1}{5n^4}+\frac{k}{n^4}+\frac{2k^2}{n^4}+\frac{2k^3}{n^4}+\frac{k^4}{n^4}\right) \\
&= \frac{1}{5n^4}P_n^{[\beta]}(1,x)+\frac{1}{n^3}P_n^{[\beta]}(t,x)+\frac{2}{n^2}P_n^{[\beta]}(t^2,x) \\
&\quad +\frac{2}{n}P_n^{[\beta]}(t^3,x)+P_n^{[\beta]}(t^4,x) \\
&= \frac{x^4}{(1-\beta)^4}+\frac{2x^3\left(7+2\beta+\beta^2\right)}{n(1-\beta)^5}+\frac{2x^2\left(25+44\beta+18\beta^2+2\beta^3+\beta^4\right)}{n^2(1-\beta)^6} \\
&\quad +\frac{x\left(43+326\beta+329\beta^2+116\beta^3+23\beta^4+2\beta^5+\beta^6\right)}{n^3(1-\beta)^7} \\
&\quad +\frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}.
\end{aligned}
$$

5

Lemma 1 implies the following identities

$$
\begin{aligned}
\Omega_{n,0}(x) &= 1, \quad \Omega_{n,1}(x) = \frac{x\beta}{1-\beta} + \frac{(1+\beta)^2}{2n(1-\beta)^2}, && (2.1)\\
\Omega_{n,2}(x) &= \frac{x^2\beta^2}{(1-\beta)^2} + \frac{x\left(2+\beta+2\beta^2+\beta^3\right)}{n(1-\beta)^3} + \frac{1+20\beta+12\beta^2+2\beta^3+\beta^4}{3n^2(1-\beta)^4}, && (2.2)\\
\Omega_{n,3}(x) &= \frac{x^3\beta^3}{(1-\beta)^3} + \frac{3x^2\beta\left(4+\beta+2\beta^2+\beta^3\right)}{2n(1-\beta)^4}\\
&\quad + \frac{x\left(9+13\beta+23\beta^2+12\beta^3+2\beta^4+\beta^5\right)}{n^2(1-\beta)^5}\\
&\quad + \frac{1+142\beta+219\beta^2+96\beta^3+19\beta^4+2\beta^5+\beta^6}{4n^3(1-\beta)^6}, && (2.3)\\
\Omega_{n,4}(x) &= \frac{x^4\beta^4}{(1-\beta)^4} + \frac{2x^3\beta^2\left(6+\beta+2\beta^2+\beta^3\right)}{n(1-\beta)^5} && (2.4)\\
&\quad + \frac{2x^2\left(6+18\beta+25\beta^2+26\beta^3+12\beta^4+2\beta^5+\beta^6\right)}{n^2(1-\beta)^6}\\
&\quad + \frac{x\left(42+185\beta+252\beta^2+239\beta^3+100\beta^4+19\beta^5+2\beta^6+\beta^7\right)}{n^3(1-\beta)^7}\\
&\quad + \frac{1+1072\beta+3398\beta^2+2824\beta^3+900\beta^4+174\beta^5+28\beta^6+2\beta^7+\beta^8}{5n^4(1-\beta)^8}.
\end{aligned}
$$

Remark 1 *Since* $\beta \in [0,1)$, $(1-\beta)^2 \leq 1$ *and* $(1-\beta)^{-2} \leq (1-\beta)^{-3} \leq (1-\beta)^{-4}$, *we have*

$$
\Omega_{n,1}(x) \leq \frac{2xn\beta + (1+\beta)^2}{2n(1-\beta)^2} \qquad (2.5)
$$

and

$$
\begin{aligned}
\Omega_{n,2}(x) &\leq \frac{3n^2x^2\beta^2 + 3nx\left(2+\beta+2\beta^2+\beta^3\right) + 1 + 20\beta + 12\beta^2 + 2\beta^3 + \beta^4}{3n^2(1-\beta)^4}\\
&\leq \frac{3n^2x^2\beta^2 + 6nx\,(1+2\beta) + 1 + 35\beta}{3n^2(1-\beta)^4}. && (2.6)
\end{aligned}
$$

Also, using $\max\{1, x, x^2, x^3, x^4\} \leq (1 + x + x^2 + x^3 + x^4)$, *we have*

$$
\begin{aligned}
\Omega_{n,4}(x) &\leq \left(\frac{\beta^4}{(1-\beta)^8} + \frac{20}{n(1-\beta)^8} + \frac{180}{n^2(1-\beta)^8}\right.\\
&\quad \left. + \frac{840}{n^3(1-\beta)^8} + \frac{8400}{5n^4(1-\beta)^8}\right)(1 + x + x^2 + x^3 + x^4)\\
&\leq B_\beta(n)(1 + x + x^2 + x^3 + x^4), && (2.7)
\end{aligned}
$$

where

$$
B_\beta(n) = \frac{\beta^4}{(1-\beta)^8} + \frac{20}{n(1-\beta)^8} + \frac{180}{n^2(1-\beta)^8} + \frac{840}{n^3(1-\beta)^8} + \frac{8400}{5n^4(1-\beta)^8}. \qquad (2.8)
$$

Theorem 1 *Let $K_n^{[\beta_n]}$ be defined by (1.5) and $\beta_n \in [0,1)$ with $\beta_n \to 0$. Then for $f \in C[0,\infty)$ one has $\lim\limits_{n\to\infty} K_n^{[\beta_n]}(f,\cdot) = f$ uniformly on any compact $K \subset [0,\infty)$.*

Proof: By making use of Lemma 1, we have

$$\lim_{n\to\infty} K_n^{[\beta_n]}(t^j,x) = x^j, \text{ with } \beta_n \to 0$$

$j = 0,1,2$, uniformly on any compact $K \subset [0,\infty)$. Consequently, our assertion follows directly from the well-known theorem of Bohman-Korovkin.
Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions f on the interval $[0,\infty)$, endowed with the norm

$$\|f\| = \sup_{0\le x\le\infty} |f(x)|$$

For any $\delta > 0$, Peetre's K-functional is define by

$$K_2(f,\delta) = \inf_{g\in C_B^2[0,\infty)} \{\|f-g\| + \delta\|g''\|\},$$

where $C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By DeVore and Lorentz [4, P.177, Theorem 2.4], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2\left(f,\sqrt{\delta}\right), \tag{2.9}$$

where the second order modulus of smoothness of $g \in C_B[0,\infty)$ is defined as

$$\omega_2(g;\delta) = \sup_{0<h\le\delta}\sup_{x\ge} |g(x) - 2g(x+h) + g(x+2h)|, \qquad \delta > 0,$$

also usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega_1(g;\delta) = \sup_{0<h\le\delta}\sup_{x\ge 0} |g(x+h) - g(x)|, \qquad \delta > 0.$$

Theorem 2 *Let $K_n^{[\beta]}$ be defined by (1.5) and $\beta \in [0,1)$ then for each $x \ge 0$ the following inequality*

$$|K_n^{[\beta]}(f,x) - f(x)| \le \frac{4}{3}\omega_1\left(f; \frac{\sqrt{3n^2x^2\beta^2 + 6nx\,(1+2\beta) + 1 + 35\beta}}{3n(1-\beta)^2}\right)$$

holds.

Proof: Since $K_n^{[\beta]}(1,x) = 1$ and $\mathbf{p}_\beta(k,nx) \ge 0$, we can write

$$|K_n^{[\beta]}(f,x) - f(x)| \le n\sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx) \int_{k/n}^{(k+1)/n} |f(t) - f(x)|dt. \tag{2.10}$$

On the other hand

$$|f(t) - f(x)| \le \omega_1(f;|t-x|) \le (1 + \delta^{-2}(t-x)^2)\omega_1(f;\delta).$$

For $|t-x|<\delta$ the last increase is clear. For $|t-x|\geq\delta$, we use the following properties

$$\omega_1(f;\lambda\delta)\leq(1+\lambda)\omega_1(f;\delta)\leq(1+\lambda^2)\omega_1(f;\delta),$$

where one can choose $\lambda=\delta^{-1}|t-x|$. This way the relation (2.6) implies

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}(1+\delta^{-2}(x-t)^2)\omega_1(f;\delta)dt\\ &= \left(\Omega_{n,0}(x)+\delta^{-2}\Omega_{n,2}(x)\right)\omega_1(f;\delta)\\ &= \left(1+\delta^{-2}\left\{\frac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}\right\}\right)\omega_1(f;\delta).\end{aligned}$$

Choosing $\delta=\left(\dfrac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{n^2(1-\beta)^4}\right)^{1/2}$, we obtain the desired result.

Further, we estimate the rate of convergence for smooth functions.

Theorem 3 *Let $K_n^{[\beta]}$ be defined by (1.5) and $\beta\in[0,1)$. Then for $f\in C^1[0,\infty)$ and $a>0$ one has*

$$|K_n^{[\beta]}(f,x)-f(x)|\leq\frac{1}{2n(1-\beta)^2}\left(b_n\|f'\|_{C[0,a]}+c_n\omega_1\left(f';\frac{1}{\sqrt{n}}\right)\right),$$

where $b_n=2an\beta+(1+\beta)^2$ and

$$c_n=2\sqrt{n^2a^2\beta^2+2na\,(1+2\beta)+1+35\beta}\left(1+(1-\beta)^{-2}\sqrt{na^2\beta^2+2a\,(1+2\beta)+(1+35\beta)n^{-1}}\right).$$

Proof: We can write

$$f(x)-f(t)=(x-t)f'(x)+(x-t)(f'(\xi)-f'(x)),$$

where $\xi=\xi(t,x)$ is a point of the interval determinate by x and t. If we multiply both members of this inequality by $n\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}dt$ and sum over k , there follows

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq |f'(x)|\Omega_{n,1}(x)\\ &\quad+n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}|x-t|\cdot|f'(\xi)-f'(t)|dt\\ &\leq \left(\frac{2xn\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)\max_{x\in[0,a]}|f'(x)| \qquad (2.11)\\ &\quad+n\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}|x-t|(1+\delta^{-1}|t-x|)\omega_1(f';\delta)dt.\end{aligned}$$

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According to Cauchy's inequality, we have

$$\begin{aligned}
& n\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\int_{k/n}^{(k+1)/n}|x-t|dt \\
& \leq \sqrt{n}\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\left\{\int_{k/n}^{(k+1)/n}(x-t)^2dt\right\}^{1/2} \\
& \leq \sqrt{n}\left\{\left[\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\right]\left[\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\int_{k/n}^{(k+1)/n}(x-t)^2dt\right]\right\}^{1/2}.
\end{aligned}$$

Hence,

$$n\sum_{k=0}^{\infty}\mathbf{p}_{\beta}(k,nx)\int_{k/n}^{(k+1)/n}|x-t|dt \quad\leq\quad \sqrt{\Omega_{n,2}(x)}. \tag{2.12}$$

Using inequalities (2.12) in (2.11), we write

$$\begin{aligned}
|K_n^{[\beta]}(f,x)-f(x)| &\leq \left(\tfrac{2an\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)\|f'\|_{C[0,a]} \\
&+\sqrt{\Omega_{n,2}(x)}\left(1+\delta^{-1}\sqrt{\Omega_{n,2}(x)}\right)\omega_1(f';\delta)dt. \qquad (2.13)
\end{aligned}$$

Inserting $\delta=\dfrac{1}{\sqrt{n}}$ and using $\sqrt{\Omega_{n,2}(x)}\leq\dfrac{\sqrt{n^2a^2\beta^2+2na\,(1+2\beta)+1+35\beta}}{n(1-\beta)^2}$, $x\in[0,a]$, the proof of our theorem is complete.

Theorem 4 *Let $f\in C_B[0,\infty)$. Then for all $x\in[0,\infty)$ there exists a constant $A>0$ such that*

$$|K_n^{[\beta]}(f,x)-f(x)|\leq A\omega_2\left(f,\xi_n(x)\right)+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right),$$

where $\xi_n(x)=\frac{3n^2x^2\beta^2+6nx(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}+\left(\frac{2xn\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)^2.$

Proof: Consider the following operator

$$\hat{K}_n^{[\beta]}(f,x)=K_n^{[\beta]}(f,x)-f\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right)+f(x). \tag{2.14}$$

By the definition of the operators $\hat{K}_n^{[\beta]}$ and Lemma 1, we have

$$\hat{K}_n^{[\beta]}(t-x,x)=0.$$

Let $g\in C_B^2[0,\infty)$ and $x\in[0,\infty)$. By Taylor's formula of g, we get

$$g(t)-g(x)=(t-x)g'(x)+\int_x^t(t-u)g''(u)du,\quad t\in[0,\infty).$$

One may write

$$\begin{aligned}\hat{K}_n^{[\beta]}(g,x)-g(x) &= g'(x)\hat{K}_n^{[\beta]}(t-x,x)+\hat{K}_n^{[\beta]}\left(\int_x^t (t-u)g''(u)du,x\right)\\ &= \hat{K}_n^{[\beta]}\left(\int_x^t (t-u)g''(u)du,x\right)\\ &= K_n^{[\beta]}\left(\int_x^t (t-u)g''(u)du,x\right)\\ &\quad -\int_x^{\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}}\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}-u\right)du.\end{aligned}$$

Now, using the following inequalities

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2\|g''\| \tag{2.15}$$

and

$$\left|\int_x^{\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}}\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}-u\right)du\right| \leq \left[\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right]^2\|g''\|,$$

we reach to

$$\begin{aligned}|\hat{K}_n^{[\beta]}(g,x)-g(x)| &\leq \left\{K_n^{[\beta]}((t-x)^2,x)+\left[\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right]^2\right\}\|g''\|\\ &\leq \left\{\frac{3n^2x^2\beta^2+6nx\left(1+2\beta\right)+1+35\beta}{3n^2(1-\beta)^4}\right.\\ &\quad \left.+\left(\frac{2xn\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)^2\right\}\|g''\|. \end{aligned}\tag{2.16}$$

By means of the definitions of the operators $\hat{K}_n^{[\beta]}$ and $K_n^{[\beta]}$, we have

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq |\hat{K}_n^{[\beta]}(f-g,x)|+|(f-g)(x)|+\left|\hat{K}_n^{[\beta]}(g,x)-g(x)\right|\\ &\quad +\left|f\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right)-f(x)\right|\end{aligned}$$

and

$$\hat{K}_n^{[\beta]}(f,x)\leq |K_n^{[\beta]}(f,x)|+2\|f\|\leq \|f\|K_n^{[\beta]}(1,x)+2\|f\|=3\|f\|.$$

Thus, we may conclude that

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq 4\|f-g\|+|\hat{K}_n^{[\beta]}(g,x)-g(x)|\\ &\quad +\left|f\left(\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right)-f(x)\right|.\end{aligned}$$

In the light of inequality (2.16), one gets

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq 4\|f-g\| \\ &\quad+\left\{\frac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}\right. \\ &\quad\left.+\left(\frac{2xn\beta+(1+\beta)^2}{2n(1-\beta)^2}\right)^2\right\}\|g''\| \\ &\quad+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right).\end{aligned}$$

Therefore taking the infimum over all $g\in C_B[0,\infty)$ on the right-hand side of the last inequality and considering (2.9), we find that

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq 4K_2\left(f,\xi_n(x)\right)+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right) \\ &\leq 4C\omega_2\left(f,\xi_n(x)\right)+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right) \\ &\leq A\omega_2\left(f,\xi_n(x)\right)+\omega_1\left(f,\frac{x}{1-\beta}+\frac{(1+\beta)^2}{2n(1-\beta)^2}\right),\end{aligned}$$

which completes the proof.

Theorem 5 *Let* $0<\gamma\leq 1$, $\beta\in[0,1)$ *and* $f\in C_B[0,\infty)$. *Then if* $f\in Lip_M(\gamma)$, *that is, the inequality* $|f(t)-f(x)|\leq M|t-x|^\gamma, x,t\in[0,\infty)$ *holds, then for each* $x\in[0,\infty)$, *we have*

$$|K_n^{[\beta]}(f,x)-f(x)|\leq d_n^{\frac{\gamma}{2}}(x),$$

where $d_n(x)=\dfrac{3n^2x^2\beta^2+6nx\,(1+2\beta)+1+35\beta}{3n^2(1-\beta)^4}$ *and* $M>0$ *is a constant.*

Proof: Let $f\in C_B[0,\infty)\cap Lip_M(\gamma)$. By the linearity and monotonicity of the operators $K_n^{[\beta]}$, we get

$$\begin{aligned}|K_n^{[\beta]}(f,x)-f(x)| &\leq K_n^{[\beta]}(|f(t)-f(x)|,x) \\ &\leq MK_n^{[\beta]}(|t-x|^\gamma,x) \\ &= Mn\sum_{k=0}^{\infty}\mathbf{p}_\beta(k,nx)\int_{k/n}^{(k+1)/n}|t-x|^\gamma dt.\end{aligned}$$

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Now, applying the Hölder inequality two times successively with $p = \frac{2}{\gamma}$, $q = \frac{2}{2-\gamma}$, we obtain

$$\begin{aligned} |K_n^{[\beta]}(f,x) - f(x)| &\leq M \sum_{k=0}^{\infty} \mathbf{p}_\beta(k,nx) \left(n \int_{k/n}^{(k+1)/n} |t-x|^\gamma dt \right)^{\frac{\gamma}{2}} \\ &\leq M(\Omega_{n,2}(x))^{\frac{\gamma}{2}} \\ &\leq M \left(\frac{3n^2x^2\beta^2 + 6nx\,(1+2\beta) + 1 + 35\beta}{3n^2(1-\beta)^4} \right)^{\frac{\gamma}{2}} . \end{aligned}$$

This completes the proof.

3 Weighted approximation properties

Now, we introduce convergence properties of the operators $K_n^{[\beta]}$ via the weighted Korovkin type theorem given by Gadzhiev in [5, 6]. For this purpose, we recall some definitions and notations.
Let $\rho(x) = 1+x^2$ and $B_\rho[0,\infty)$ be the space of all functions having the property

$$|f(x)| \leq M_f \rho(x),$$

where $x \in [0,\infty)$ and M_f is a positive constant depending only on f. The set $B_\rho[0,\infty)$ is equipped with the norm

$$\|f\|_\rho = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}.$$

$C_\rho[0,\infty)$ denotes the space of all continuous functions belonging to $B_\rho[0,\infty)$. By $C_\rho^0[0,\infty)$, we denote the subspace of all functions $f \in C_\rho[0,\infty)$ for which

$$\lim_{x\to\infty} \frac{|f(x)|}{\rho(x)} < \infty.$$

Theorem 6 ([5, 6]) *Let $\{An\}$ be a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ and satisfying the conditions*

$$lim_{n\to\infty} \|A_n(t^v;x) - x^v\|_\rho = 0, v = 0,1,2.$$

Then for any function $f \in C_\rho^0[0,\infty)$,

$$\lim_{n\to\infty} \|A_n(f;\cdot) - f(\cdot)\|_\rho = 0.$$

Note that, a sequence of linear positive operators A_n acts from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ if and only if

$$\|A_n(\rho;x)\| \leq M_\rho,$$

where M_ρ is positive constant. This fact also given in [5, 6].

Theorem 7 *Let $\{K_n^{[\beta_n]}\}$ be the sequence of linear positive operators defined by (1.5) and $\beta_n \in [0,1)$ with $\beta_n \to 0$ as $n \to \infty$. Then for each $f \in C_\rho^0[0,\infty)$, we have*

$$\lim_{n\to\infty} \|K_n^{[\beta]}(f;x) - f(x)\|_\rho = 0.$$

Proof: Using Lemma 1, we may write

$$\begin{aligned}\sup_{x\in[0,\infty)} \frac{|K_n^{[\beta_n]}(\rho,x)|}{1+x^2} &\leq \frac{1}{(1-\beta_n)^2} + \frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3}\\ &\quad + \frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4} + 1.\end{aligned}$$

Since $\lim_{n\to\infty} \beta_n = 0$, , there exists a positive constant M^* such that

$$\frac{1}{(1-\beta_n)^2} + \frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3} + \frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4} \leq M^*$$

for each n. Hence, we get

$$\|K_n^{[\beta_n]}(\rho,x)\|_\rho \leq 1 + M^*,$$

which shows that $\{K_n^{[\beta_n]}\}$ is a sequence of positive linear operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$.
In order to complete the proof, it is enough to prove that the conditions of Theorem 6

$$\lim_{n\to\infty} \|K_n^{[\beta_n]}(t^v;x) - x^v\|_\rho = 0, v = 0,1,2$$

are satisfied. It is clear that

$$\lim_{n\to\infty} \|K_n^{[\beta_n]}(1;x) - 1\|_\rho = 0$$

By Lemma 1, we have

$$\begin{aligned}\|K_n^{[\beta_n]}(t;x) - x\|_\rho &= \sup_{x\in\infty} \left| \left(\frac{1}{1-\beta_n} - 1\right) \frac{x}{1+x^2} + \frac{(1+\beta_n)^2}{2n(1-\beta_n)^2} \frac{1}{1+x^2} \right|\\ &\leq \left| \frac{\beta_n}{1-\beta_n} + \frac{(1+\beta_n)^2}{2n(1-\beta_n)^2} \right|.\end{aligned}$$

Thus taking into consideration the conditions $\beta_n \to 0$ as $n \to \infty$, we can conclude that

$$\lim_{n\to\infty} \|K_n^{[\beta_n]}(t;x) - x\|_\rho = 0$$

Similarly, one gets

$$\begin{aligned}
&\|K_n^{[\beta_n]}(t^2;x)-x^2\|_\rho \\
= &\sup_{x\in\infty}\left|\left(\frac{1}{(1-\beta_n)^2}-1\right)\frac{x^2}{1+x^2}+\frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3}\frac{x}{1+x^2}\right. \\
&\left.+\frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4}\frac{1}{1+x^2}\right| \\
\leq &\sup_{x\in\infty}\left|\left(\frac{1}{(1-\beta_n)^2}-1\right)+\frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3}+\frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4}\right| \\
\leq &\sup_{x\in\infty}\left|\frac{2\beta_n-\beta_n^2}{(1-\beta_n)^2}+\frac{\left(3+2\beta_n+\beta_n^2\right)}{n(1-\beta_n)^3}+\frac{1+20\beta_n+12\beta_n^2+2\beta_n^3+\beta_n^4}{3n^2(1-\beta_n)^4}\right|
\end{aligned}$$

which leads to

$$\lim_{n\to\infty}\|K_n^{[\beta_n]}(t^2;x)-x^2\|_\rho=0 \text{ with } \beta_n\to 0.$$

Thus the proof is completed.
Now, we compute the order of approximation of the operators $K_n^{[\beta]}$ in terms of the weighted modulus of continuity $\Omega_2(f,\delta)$ (see[7]) defined by

$$\Omega_2(f,\delta)=\sup_{x\geq 0,0<h\leq\delta}\frac{|f(x+h)-f(x)|}{1+(x+h)^2},\qquad f\in C_\rho^0[0,\infty)$$

and has the following properties:

(a) $\Omega_2(f,\delta)$ is monotone increasing function of δ,

(b) $\lim_{\delta\to 0+}\Omega_2(f,\delta)=0$,

(c) for each $\lambda\in[0,\infty)$, $\Omega_2(f,\lambda\delta)\leq(1+\lambda)\Omega_2(f,\delta)$.

Theorem 8 *Let $\{K_n^{[\beta]}\}$ be the sequence of linear positive operators defined by (1.5). Then for each $f\in C_\rho^0[0,\infty)$, we have*

$$\sup_{0\leq x<\infty}\frac{|K_n^{[\beta]}(f;x)-f(x)|}{(1+x^2)^3}\leq C\Omega_2\left(f,(B_\beta(n))^{1/4}\right),$$

where C is positive constant and $B_\beta(n)$ is defined in (2.8).

Proof: For $x\geq 0$ and $t\geq 0$, by the definition of $\Omega_2(f,\delta)$ and the property (c), we may write

$$\begin{aligned}
|f(t)-f(x)| &\leq \left(1+\left(x+|t-x|^2\right)\right)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_n) \\
&\leq 2(1+x^2)(1+(t-x)^2)\left(1+\frac{|t-x|}{\delta_n}\right)\Omega_2(f,\delta_2).
\end{aligned}$$

By using the monotonicity of $K_n^{[\beta]}$ and the following inequality (see [8])

$$\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right) \le 2\left(1+\delta_n^2\right)\left(1+\frac{(t-x)^4}{\delta_n^4}\right),$$

one gets

$$\begin{aligned}
|K_n^{[\beta]}(f,x)-f(x)| &\le 2(1+x^2)K_n^{[\beta]}\left(\left(1+(t-x)^2\right)\left(1+\frac{|t-x|}{\delta_n}\right),x\right)\Omega_2(f,\delta_n)\\
&\le 4(1+x^2)(1+\delta_n^2)K_n^{[\beta]}\left(\left(1+\frac{(t-x)^4}{\delta_n^4}\right),x\right)\Omega_2(f,\delta_n)\\
&\le 4(1+x^2)(1+\delta_n^2)\left(1+\frac{1}{\delta_n^4}K_n^{[\beta]}((t-x)^4,x)\right)\Omega_2(f,\delta_n)\\
&\le C_1(1+x^2)\left(1+\frac{1}{\delta_n^4}K_n^{[\beta]}((t-x)^4,x)\right)\Omega_2(f,\delta_n),
\end{aligned}$$

with the help of the inequality (2.7) this inequality leads to

$$|K_n^{[\beta]}(f,x)-f(x)| \le C_1(1+x^2)\left(1+\frac{B_\beta(n)}{\delta_n^4}(1+x+x^2+x^3+x^4)\right)\Omega_2(f,\delta_n),$$

which gives the required result.

Remark 2 *In [9], the authors has consider the generalization of the operators (1.1) as*

$$P_n^{[0]}(f,a_n,b_n,x) = 2^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)_k}{2^k k!}f\left(\frac{k}{b_n}\right), \quad x\ge 0, \quad f:[0,\infty),\to\mathbb{R}, \tag{3.1}$$

where $\{a_n\}$, $\{b_n\}$ *are increasing and unbounded sequences of positive numbers such that*

$$\lim_{n\to\infty}\frac{1}{b_n}=0, \quad \frac{a_n}{b_n}=1+O\left(\frac{1}{b_n}\right).$$

They studied the convergence properties of these operators in weighted spaces of continuous functions on positive semi-axis. Also, A. Erençin and F Taşdelen [10] consider the generalization of the Kantorovich type operators $P_n^{[0]}(f,a_n,b_n,x)$ *given by (3.1) as follows:*

$$K_n^{[0]}(f,a_n,b_n,x) = b_n 2^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)_k}{2^k k!}\int_{k/b_n}^{(k+1)/b_n} f(t)dt, \tag{3.2}$$

where f *is an integrable function on* $[0,\infty)$ *and bounded on every compact subinterval of* $[0,\infty)$.

Motivated by the operators (3.1) *and* (3.2), *we generalize the operators* $P_n^{[\beta]}$ *and* $K_n^{[\beta]}$ *in following way*

$$P_n^{[\beta]}(f,a_n,b_n,x) = \sum_{k=0}^{\infty}\frac{2^{-(a_n x+k\beta)}(a_n x+k\beta)_k}{2^k k!}f\left(\frac{k}{b_n}\right), \quad x\ge 0, \quad f:[0,\infty)\to\mathbb{R}, \tag{3.3}$$

and

$$K_n^{[\beta]}(f,a_n,b_n,x)=b_n\sum_{k=0}^{\infty}\frac{2^{-(a_nx+k\beta)}(a_nx)_k}{2^k k!}\int_{k/b_n}^{(k+1)/b_n}f(t)dt \tag{3.4}$$

and extend the studies of the present article in a similar direction for the operators (3.3) *and* (3.4)*. The analysis is different so we may discuss that elsewhere.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Mittag-Leffler-Hyers-Ulam Stability of Linear Differential Equations using Fourier Transforms

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Abstract. In this paper, we are going to study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the general Linear Differential Equations of Higher order with constant coefficients using Fourier Transforms method. Moreover, the Mittag-Leffler-Hyers-Ulam stability constants of these differential equations are obtained. Some examples are given to illustrate the main results.

AMS Subject Classification: 34K20, 26D10, 34A40, 39B82.

Key Words and Phrases: Mittag-Leffler-Hyers-Ulam stability, Mittag-Leffler-Hyers-Ulam-Rassias stability, Linear Differential equation, Fourier Transforms.

1. Introduction

A classical question in the theory of functional equation is the following : "when is it true that a function which approximately satisfies a functional equation (g) must be close to an exact solution of (g)?" If the problem accepts a solution, we say that the equation (g) is stable.

A simulating and famous talk presented by Ulam [**40**] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range of talks before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. One of his question was that when is it true that a mapping that approximately satisfies a functional equation must be close to an exact solution of the equation? If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [**9**] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings on Banach spaces. In the course of time, the Theorem formulated by Hyers was generalized by Rassias Th. [**36**], Aoki [**4**], Bourgin [**6**] and J.M.Rassias [**29**] for additive mappings. Then a number of authors has studied the Ulam problem for various functional equations by different methods in [**2, 21, 30, 31, 32, 33, 34, 35**].

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

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has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', ...x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon\to 0} K(\epsilon) = 0$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability. Obloza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations [**25, 26**]. Thereafter, In 1998, C. Alsina and R. Ger [**3**] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved in [**3**] the following Theorem.

Theorem 1.1. *Assume that a differentiable function $f : I \to R$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open sub interval of $\mathbb{R}$. Then there exists a solution $g : I \to R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of C. Alsina and R. Ger [**3**] has been generalized by Takahasi [**39**]. They proved in [**39**] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [**11, 12, 13, 17, 18, 19, 20**]. Using the approach as in [**40**], Miura, Takahasi and Choda [**19**], Miura [**20**], Takahasi, Miura and Miyajima [**39**] and Miura, Jung and Takahasi are [**17**] proved that the Hyers-Ulam stability holds true for the differential equation $x' = \lambda x$, while Jung [**11**] proved a similar result for the differential equation $\phi(t)x'(t) = x$.

In 2006, S.M. Jung [**14**] investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients by using matrix method. In 2007, G. Wang, M. Zhou, L. Sun [**42**] studied the Hyers-Ulam stability of a class of first-order linear differential equations. I. A. Rus [**37**] discussed four types of Ulam stability: Ulam-Hyers stability, Generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and Generalized Ulam-Hyers-Rassias stability of the Ordinary Differential Equation $u'(t) = A(u(t)) + f(t, u(t)), t \in [a, b]$. In 2014, Q. H. Alqifiary and S. M. Jung [**5**] proved the Generalized Hyers-Ulam stability of linear differential equation by using the Laplace Transforms. These days the Hyers-Ulam stability of differential equations are investigated [**1, 7, 8, 15, 16, 22, 24, 27, 28, 43**] and the investigation is ongoing.

Recently, Vida Kalvandi, N. Eghbali and J.M. Rassias [**41**] studied the Mittag-Leffler-Hyers-Ulam stability of a fractional differential equation of second order. In this paper, with the help of Fourier Transforms, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation

$$x'(t) + l\ x(t) = 0 \tag{1.1}$$

and the non-homogeneous linear differential equation

$$x'(t) + l\ x(t) = r(t) \tag{1.2}$$

where l is a scalar, $x(t)$ and $r(t)$ are the continuously differentiable functions. Also, by using Fourier Transforms, we establish the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order homogeneous linear differential equation

$$x''(t) + l\ x'(t) + m\ x(t) = 0 \tag{1.3}$$

and the non-homogeneous second order differential equation

$$x''(t) + l\ x'(t) + m\ x(t) = r(t) \tag{1.4}$$

where l and m are scalars, $x(t)$ is a twice continuously differentiable function and $r(t)$ is a continuously differentiable function.

2. Preliminaries

In this section, we introduce some standard notations, Definitions and Theorems, it will be very useful to prove our main results.

Throughout this paper, $\mathbb{F}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{F}$ of exponential order if there exists a constants $A, B \in \mathbb{R}$ such that $|f(t)| \leq Ae^{tB}$ for all $t > 0$.

For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order. Let g denote the Fourier Transform of f so that

$$g(u) = \int_{-\infty}^{\infty} f(t)\ e^{-itu}\ dt.$$

Then, at points of continuity of f, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(u)\ e^{-ixu}\ du,$$

this is called the inverse Fourier transforms. The Fourier transform of f is denoted by $\mathcal{F}(\xi)$. We also introduce a notion, the convolution of two functions.

Definition 2.1. (Convolution). Given two functions f and g, both Lebesgue integrable on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t)\ g(x - t)\ dt$$

exists. This integral defines a function h on S called the convolution of f and g. We also write $h = f * g$ to denote this function.

Theorem 2.2. *The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of the Fourier transform of $f(x)$ and $g(x)$. That is,*

$$\mathcal{F}\{f(x) * g(x)\} = \mathcal{F}\{f(x)\}\ \mathcal{F}\{g(x)\} = F(s)\ G(s)$$

or

$$\mathcal{F}\left\{\int\limits_{-\infty}^{\infty} f(t)\ g(x-t)\ dt\right\} = F(s)\ G(s),$$

where $F(s)$ and $G(s)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively.

Definition 2.3. [**41**] The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+1)} z^k$$

where $z, \alpha \in \mathbb{C}$ and $Re(\alpha) > 0$. If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k} = e^z.$$

Definition 2.4. [**41**] The generalization of $E_\alpha(z)$ is defined as a function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k+\beta)} z^k$$

where $z, \alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.

Now, we give the definition of Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equations (1.1), (1.2), (1.3) and (1.4).

Definition 2.5. The linear differential equation (1.1) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l\ x(t)| \leq \epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \to \mathbb{F}$ satisfies the differential equation (1.1) such that $|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.1).

Definition 2.6. The linear differential equation (1.2) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l\ x(t) - r(t)| \leq \epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0, \infty) \to \mathbb{F}$ satisfies the differential equation (1.2) such that $|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.2).

Definition 2.7. The linear differential equation (1.3) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function satisfying

$$|x''(t) + l\ x'(t) + m\ x(t)| \leq \epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.3) such that $|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.3).

Definition 2.8. The linear differential equation (1.4) is said to have the Mittag-Leffler-Hyers-Ulam stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function satisfying

$$|x''(t) + l\ x'(t) + m\ x(t) - r(t)| \le \epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.4) such that $|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for the differential equation (1.4).

Definition 2.9. We say that the homogeneous linear differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function, if there exists $\phi : (0,\infty) \to (0,\infty)$ satisfies the inequality

$$|x'(t) + l\ x(t)| \le \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.1) such that $|x(t) - y(t)| \le K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.1).

Definition 2.10. We say that the non-homogeneous linear differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a continuously differentiable function, if there exists $\phi : (0,\infty) \to (0,\infty)$ satisfies the inequality

$$|x'(t) + l\ x(t) - r(t)| \le \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.2) such that $|x(t) - y(t)| \le K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.2).

Definition 2.11. We say that the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function, if there exists $\phi : (0,\infty) \to (0,\infty)$ satisfies the inequality

$$|x''(t) + l\ x'(t) + m\ x(t)| \le \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.3) such that $|x(t) - y(t)| \le K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.3).

Definition 2.12. We say that the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam-Rassias stability, if there exists a constant $K > 0$ with the following property: For every $\epsilon > 0$, let $x(t)$ be a twice continuously differentiable function, if there exists $\phi : (0,\infty) \to (0,\infty)$ satisfies the inequality

$$|x''(t) + l\ x'(t) + m\ x(t) - r(t)| \le \phi(t)\epsilon E_\alpha(t^\alpha),$$

where E_α is a Mittag-Leffler function, then there exists some $y : (0,\infty) \to \mathbb{F}$ satisfies the differential equation (1.4) such that $|x(t) - y(t)| \le K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for the equation (1.4).

3. Mittag-Leffler-Hyers-Ulam Stability

In the following theorems, we prove the Mittag-Leffler-Hyers-Ulam stability of the homogeneous and non-homogeneous linear differential equations (1.1), (1.2), (1.3) and (1.4). Firstly, we prove the Mittag-Leffler-Hyers-Ulam stability of first order homogeneous differential equation (1.1).

Theorem 3.1. *The differential equation* (1.1) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let l be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0,\infty) \to \mathbb{F}$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l\ x(t)| \le \epsilon E_\alpha(t^\alpha) \tag{3.1}$$

for all $t > 0$. We will prove that, there exists a solution $y : (0,\infty) \to \mathbb{F}$ satisfying the differential equation $y'(t) + l\ y(t) = 0$ such that

$$|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha)$$

for any $t > 0$. Let us define a function $p : (0,\infty) \to \mathbb{F}$ such that $p(t) =: x'(t) + l\ x(t)$ for each $t > 0$. In view of (3.1), we have $|p(t)| \le \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned}
\mathcal{F}\{p(t)\} &= \mathcal{F}\{x'(t) + l\ x(t)\}\\
P(\xi) &= \mathcal{F}\{x'(t)\} + l\ \mathcal{F}\{x(t)\} = -i\xi X(\xi) + l\ X(\xi) = (l - i\xi)X(\xi)\\
X(\xi) &= \frac{P(\xi)}{(l - i\xi)}.
\end{aligned}$$

Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)\ (l + i\xi)}{l^2 - \xi^2}. \tag{3.2}$$

Taking $Q(\xi) = \dfrac{1}{(l - i\xi)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l - i\xi)} \quad \Rightarrow \quad q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(l - i\xi)}\right\}.$$

Now, we set $y(t) = e^{-lt}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt}\ e^{ist}\ dt = \int_{-\infty}^{0} e^{-lt}\ e^{ist}\ dt + \int_{0}^{\infty} e^{-lt}\ e^{ist}\ dt = 0. \tag{3.3}$$

Now,

$$\mathcal{F}\{y'(t) + l\ y(t)\} = \mathcal{F}\{y'(t)\} + l\ \mathcal{F}\{y(t)\} = -i\xi Y(\xi) + l\ Y(\xi) = (l - i\xi)Y(\xi).$$

Then by using (3.3), we have $\mathcal{F}\{y'(t) + l\ y(t)\} = 0$, since $\mathcal{F}$ is one-to-one operator, thus $y'(t) + l\ y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.1). Then by using (3.2) and (3.3) we can obtain

$$\begin{aligned}\mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} = X(\xi) - Y(\xi) = \frac{P(\xi)\ (l + i\xi)}{l^2 - \xi^2} = P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\}\\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}.\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \le |p(t)|\ \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right| \le K\epsilon E_\alpha(t^\alpha).$$

Where $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$ exists for each value of t. Then by virtue of Definition 2.5 the homogeneous linear differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam stability. □

Now, we are going prove the Mittag-Leffler-Hyers-Ulam stability of the non-homogeneous linear differential equation (1.2) using Fourier transform method.

Theorem 3.2. *The differential equation* (1.2) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let l be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ be a continuously differentiable function satisfies the inequality

$$|x'(t) + l\ x(t) - r(t)| \le \epsilon E_\alpha(t^\alpha) \tag{3.4}$$

for all $t > 0$. We have to show that there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the non-homogeneous differential equation $y'(t) + l\ y(t) = r(t)$ such that $|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha)$, for any $t > 0$.

Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x'(t) + l\ x(t) - r(t)$ for each $t > 0$. In view of (3.4), we have $|p(t)| \le \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned}\mathcal{F}\{p(t)\} &= \mathcal{F}\{x'(t) + l\ x(t) - r(t)\}\\ P(\xi) &= \mathcal{F}\{x'(t)\} + l\ \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\}\\ &= -i\xi X(\xi) + l\ X(\xi) - R(\xi) = (l - i\xi)X(\xi) - R(\xi)\\ X(\xi) &= \frac{P(\xi) + R(\xi)}{(l - i\xi)}.\end{aligned}$$

Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{\{P(\xi) + R(\xi)\}\ (l + i\xi)}{l^2 - \xi^2}. \tag{3.5}$$

Let us choose $Q(\xi)$ as $\frac{1}{(l-i\xi)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l-i\xi)} \quad \Rightarrow \quad q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(l-i\xi)}\right\}.$$

Now, we set $y(t) = e^{-lt} + (r(t) * q(t))$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt}\ e^{ist}\ dt + \frac{R(\xi)}{(l-i\xi)} = \frac{R(\xi)}{(l-i\xi)} \tag{3.6}$$

Now, $\mathcal{F}\{y'(t) + l\ y(t)\} = -i\xi Y(\xi) + l\ Y(\xi) = R(\xi)$. Then by using (3.6), we have

$$\mathcal{F}\{y'(t) + l\ y(t)\} = F\{r(t)\},$$

since $\mathcal{F}$ is one-to-one operator, thus $y'(t) + l\ y(t) = r(t)$, Hence $y(t)$ is a solution of the differential equation (1.2). Then by using (3.5) and (3.6) we have

$$\begin{aligned}\mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} = X(\xi) - Y(\xi) &= \frac{\{P(\xi) + R(\xi)\}\ (l+i\xi)}{l^2 - \xi^2} - \frac{R(\xi)}{(l-i\xi)} \\ &= P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\} \\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\}\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \le |p(t)|\ \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right| \le K\epsilon E_\alpha(t^\alpha).$$

Where $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$, the integral exists for each value of t. Hence, by the virtue of Definition 2.6 the non-homogeneous differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam stability. □

Now, we prove the Mittag-Leffler-Hyers-Ulam stability of the homogeneous and non-homogeneous second order linear differential equations (1.3) and (1.4).

Theorem 3.3. *The differential equation* (1.3) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu + \nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ be a twice continuously differentiable function satisfying the inequality

$$|x''(t) + l\ x'(t) + m\ x(t)| \le \epsilon E_\alpha(t^\alpha) \tag{3.7}$$

for all $t > 0$. We will show that there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the homogeneous differential equation $y''(t) + l\ y'(t) + m\ y(t) = 0$ such that

$$|x(t) - y(t)| \le K\epsilon E_\alpha(t^\alpha),$$

for any $t > 0$. Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x''(t) + l\ x'(t) + m\ x(t)$ for each $t > 0$. In view of (3.7), we have $|p(t)| \leq \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned}
\mathcal{F}\{p(t)\} &= \mathcal{F}\{x''(t) + l\ x'(t) + m\ x(t)\}\\
P(\xi) &= \mathcal{F}\{x''(t)\} + l\ \mathcal{F}\{x'(t)\} + m\ \mathcal{F}\{x(t)\} = (\xi^2 - i\xi l + m)\ X(\xi)\\
X(\xi) &= \frac{P(\xi)}{\xi^2 - i\xi l + m}.
\end{aligned}$$

Since l, m are constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)\ (i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{3.8}$$

Let $Q(\xi) = \dfrac{1}{(i\xi - \mu)\ (i\xi - \nu)}$, then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(i\xi - \mu)\ (i\xi - \nu)} \quad \Rightarrow \quad q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(i\xi - \mu)\ (i\xi - \nu)}\right\}.$$

Now, setting $y(t)$ as $\dfrac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}$ and taking Fourier transform, we obtain

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int\limits_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}\ e^{ist}\ dt = 0. \tag{3.9}$$

Now,

$$\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = (\xi^2 - i\xi l + m)\ Y(\xi).$$

Then by using (3.9), we have $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = 0$. Since $\mathcal{F}$ is one-to-one operator, then $y''(t) + l\ y'(t) + m\ y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.3). Then by using (3.8) and (3.9) we can obtain

$$\begin{aligned}
\mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} = X(\xi) - Y(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m} = P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\}\\
\Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}
\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int\limits_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \leq |p(t)|\ \left|\int\limits_{-\infty}^{\infty} q(t-s)\ ds\right| \leq K\epsilon E_\alpha(t^\alpha).$$

Where $K = \left|\int\limits_{-\infty}^{\infty} q(t-s)\ ds\right|$, the integral exists for each value of t. Then by virtue of Definition 2.7 the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam stability. □

Theorem 3.4. *The differential equation* (1.4) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Let l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu+\nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0,\infty) \to \mathbb{F}$ is a twice continuously differentiable function satisfying the inequality

$$|x''(t) + l\ x'(t) + m\ x(t) - r(t)| \leq \epsilon E_\alpha(t^\alpha) \tag{3.10}$$

for all $t > 0$. We have to prove that there exists a solution $y : (0,\infty) \to \mathbb{F}$ satisfying the non-homogeneous differential equation $y''(t) + l\ y'(t) + m\ y(t) = r(t)$ such that

$$|x(t) - y(t)| \leq K\epsilon E_\alpha(t^\alpha),$$

for any $t > 0$. Assume that $x(t)$ is a continuously differentiable function satisfying the inequality (3.10). Let us define a function $p : (0,\infty) \to \mathbb{F}$ such that $p(t) =: x''(t) + l\ x'(t) + m\ x(t) - r(t)$ for each $t > 0$. In view of (3.10), we have $|p(t)| \leq \epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\begin{aligned}
\mathcal{F}\{p(t)\} &= \mathcal{F}\{x''(t) + l\ x'(t) + m\ x(t) - r(t)\}\\
P(\xi) &= \mathcal{F}\{x''(t)\} + l\ \mathcal{F}\{x'(t)\} + m\ \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\} = (\xi^2 - i\xi l + m)\ X(\xi) - R(\xi)\\
X(\xi) &= \frac{P(\xi) + R(\xi)}{\xi^2 - i\xi l + m}.
\end{aligned}$$

Since l, m are constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)\ (i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{3.11}$$

Taking

$$Q(\xi) = \mathcal{F}\{q(t)\} = \frac{1}{(i\xi - \mu)\ (i\xi - \nu)},$$

setting

$$y(t) = \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} + (r(t) * q(t))$$

and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int\limits_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}\ e^{ist}\ dt + \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} = \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{3.12}$$

Now,

$$\begin{aligned}
\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} &= \mathcal{F}\{y''(t)\} + l\ \mathcal{F}\{y'(t)\} + m\ \mathcal{F}\{y(t)\}\\
&= (\xi^2 - i\xi l + m)\ Y(\xi) = R(\xi).
\end{aligned}$$

Then by using (3.12), we have $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = \mathcal{F}\{r(t)\}$, since $\mathcal{F}$ is one-to-one operator, thus $y''(t) + l\ y'(t) + m\ y(t) = r(t)$, Hence $y(t)$ is a solution of the differential equation (1.4). Then by using (3.11) and (3.12) we can obtain

$$\begin{aligned}\mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} = X(\xi) - Y(\xi) &= \frac{P(\xi) + R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} - \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}\\ &= P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\}\\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\}\end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \leq |p(t)| \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right| \leq K\epsilon E_\alpha(t^\alpha).$$

Where $K = \left|\int\limits_{-\infty}^{\infty} q(t-s)\ ds\right|$, the integral exists for each value of t. Then by virtue of Definition 2.8 the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam stability. ☐

4. Mittag-Leffler-Hyers-Ulam-Rassias Stability

In the following theorems, we are going to investigate the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equations (1.1), (1.2), (1.3) and (1.4).

Theorem 4.1. *The differential equation* (1.1) *has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Let l be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ be a continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ be an integrable function satisfies

$$|x'(t) + l\ x(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.1}$$

for all $t > 0$. We will prove that, there exists a solution $y : (0, \infty) \to \mathbb{F}$ which satisfies the differential equation $y'(t) + l\ y(t) = 0$ such that

$$|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$$

for any $t > 0$. Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x'(t) + l\ x(t)$ for each $t > 0$. In view of (4.1), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)\ (l + i\xi)}{l^2 - \xi^2}. \tag{4.2}$$

Choosing $Q(\xi) = \dfrac{1}{(l - i\xi)}$, then we have $q(t) = \mathcal{F}^{-1}\left\{\dfrac{1}{(l - i\xi)}\right\}$. Now, we set $y(t) = e^{-lt}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt}\ e^{ist}\ dt = 0. \tag{4.3}$$

Hence

$$\mathcal{F}\{y'(t) + l\ y(t)\} = -i\xi Y(\xi) + l\ Y(\xi) = (l - i\xi)Y(\xi)$$

Then by using (4.3), we have $\mathcal{F}\{y'(t) + l\ y(t)\} = 0$, since $\mathcal{F}$ is one-to-one operator, thus $y'(t) + l\ y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.1). Then by using (4.2) and (4.3) we can obtain

$$\mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left| \int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds \right| \le |p(t)| \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right| \le K\phi(t)\epsilon E_\alpha(t^\alpha).$$

Where $K = \left| \int\limits_{-\infty}^{\infty} q(t-s)\ ds \right|$, the integral exists for each value of t and $\phi(t)$ is an integrable function. Then by virtue of Definition 2.9 the differential equation (1.1) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. □

Now, we prove the Mittag-Leffler-Hyers-Ulam-Rassias stability of the non-homogeneous linear differential equation (1.2) with the help of Fourier Transforms.

Theorem 4.2. *The differential equation* (1.2) *has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

PROOF. Let l be a constant in $\mathbb{F}$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ is a continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ an integrable function satisfying

$$|x'(t) + l\ x(t) - r(t)| \le \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.4}$$

for all $t > 0$. We will now prove that, there exist a solution $y : (0, \infty) \to \mathbb{F}$, which satisfies the differential equation $y'(t) + l\ y(t) = r(t)$ such that

$$|x(t) - y(t)| \le K\phi(t)\epsilon E_\alpha(t^\alpha),$$

for any $t > 0$. Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x'(t) + l\ x(t) - r(t)$ for each $t > 0$. In view of (4.4), we have $|p(t)| \le \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{\{P(\xi) + R(\xi)\}\ (l + i\xi)}{l^2 - \xi^2}. \tag{4.5}$$

Now, let us take $Q(\xi)$ as $\dfrac{1}{(l - i\xi)}$; then we have

$$\mathcal{F}\{q(t)\} = \frac{1}{(l - i\xi)} \quad \Rightarrow \quad q(t) = \mathcal{F}^{-1}\left\{\frac{1}{(l - i\xi)}\right\}.$$

We set $y(t) = e^{-lt} + (r(t) * q(t))$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} e^{-lt}\ e^{ist}\ dt + \frac{R(\xi)}{(l - i\xi)} = \frac{R(\xi)}{(l - i\xi)} \tag{4.6}$$

Now,

$$\mathcal{F}\{y'(t) + l\ y(t)\} = \mathcal{F}\{y'(t)\} + l\ \mathcal{F}\{y(t)\} = -i\xi Y(\xi) + l\ Y(\xi) = R(\xi)$$

Then by using (4.6), we have $\mathcal{F}\{y'(t) + l\ y(t)\} = F\{r(t)\}$, since $\mathcal{F}$ is one-to-one operator, thus $y'(t) + l\ y(t) = r(t)$. Hence $y(t)$ is a solution of the differential equation (1.2). Then by using (4.5) and (4.6) we can obtain

$$\mathcal{F}\{x(t) - y(t)\} = \mathcal{F}\{p(t) * q(t)\}.$$

Since the operator $\mathcal{F}$ is one-to-one and linear, it gives $x(t)-y(t) = p(t)*q(t)$. Taking modulus on both sides, we have

$$|x(t) - y(t)| = |p(t) * q(t)| = \left|\int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds\right| \leq |p(t)|\ \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right| \leq K\ \phi(t)\epsilon E_\alpha(t^\alpha).$$

If $K = \left|\int_{-\infty}^{\infty} q(t-s)\ ds\right|$ the integral exists for each value of t and $\phi(t)$ is an integrable function. Hence by the virtue of Definition 2.10 the differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. □

Now, we are going to establish the Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order homogeneous differential equation (1.3).

Theorem 4.3. *The second order linear differential equation* (1.3) *has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

PROOF. Let l, m are constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu+\nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ is a twice continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ an integrable function satisfying the inequality

$$|x''(t) + l\ x'(t) + m\ x(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.7}$$

for all $t > 0$. We will now prove that there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the homogeneous differential equation (1.3) such that

$$|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha),$$

for any $t > 0$. Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x''(t)+l\ x'(t)+m\ x(t)$ for each $t > 0$. In view of (4.7), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking Fourier transform to $p(t)$, we have

$$P(\xi) = \mathcal{F}\{x''(t)\} + l\ \mathcal{F}\{x'(t)\} + m\ \mathcal{F}\{x(t)\} = (\xi^2 - i\xi l + m)\ X(\xi)$$

$$X(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m}.$$

Since l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)\ (i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{4.8}$$

Choosing $Q(\xi)$ as $\dfrac{1}{(i\xi - \mu)\ (i\xi - \nu)}$, then we have $\mathcal{F}\{q(t)\} = \dfrac{1}{(i\xi - \mu)\ (i\xi - \nu)}$ and we define a function $y(t) = \dfrac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}$ and taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}\ e^{ist}\ dt = 0. \tag{4.9}$$

Now, $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = (\xi^2 - i\xi l + m)\ Y(\xi)$. Then by using (4.9), we have $\mathcal{F}\{y''(t) + l\,y'(t) + m\,y(t)\} = 0$, since $\mathcal{F}$ is one-to-one operator, thus $y''(t) + l\,y'(t) + m\,y(t) = 0$, Hence $y(t)$ is a solution of the differential equation (1.3). Then by using (4.8) and (4.9) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= X(\xi) - Y(\xi) = \frac{P(\xi)}{\xi^2 - i\xi l + m} \\ &= P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\} \\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$\begin{aligned} |x(t) - y(t)| = |p(t) * q(t)| &= \left| \int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds \right| \\ &\leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right| \leq K\phi(t)\epsilon E_\alpha(t^\alpha). \end{aligned}$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right|$ exists for each value of t and $\phi(t)$ is an integrable function. Then by the virtue of Definition 2.11 the homogeneous linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. □

Finally, we are going to investigate the Mittag-Leffler-Hyers-Ulam-Rassias stability of the second order non-homogeneous differential equation (1.4).

Theorem 4.4. *The second order linear differential equation* (1.4) *has the Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Let l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu\nu = m$, $\mu + \nu = -l$ and $\mu \neq \nu$. For every $\epsilon > 0$, there exists a positive constant K such that $x : (0, \infty) \to \mathbb{F}$ is a twice continuously differentiable function and $\phi : (0, \infty) \to (0, \infty)$ an integrable function satisfying the inequality

$$|x''(t) + l\ x'(t) + m\ x(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha) \tag{4.10}$$

for all $t > 0$. We have to prove that there exists a solution $y : (0, \infty) \to \mathbb{F}$ satisfying the non-homogeneous differential equation (1.4) such that $|x(t) - y(t)| \leq K\phi(t)\epsilon E_\alpha(t^\alpha)$, for any $t > 0$.

Let us define a function $p : (0, \infty) \to \mathbb{F}$ such that $p(t) =: x''(t) + l\ x'(t) + m\ x(t) - r(t)$ for each $t > 0$. In view of (4.10), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t^\alpha)$. Now, taking the Fourier transform to $p(t)$, we have

$$\begin{aligned} P(\xi) &= \mathcal{F}\{x''(t)\} + l\ \mathcal{F}\{x'(t)\} + m\ \mathcal{F}\{x(t)\} - \mathcal{F}\{r(t)\} \\ &= (\xi^2 - i\xi l + m)\ X(\xi) - R(\xi) \\ X(\xi) &= \frac{P(\xi) + R(\xi)}{\xi^2 - i\xi l + m}. \end{aligned}$$

Since l, m be constants in $\mathbb{F}$ such that there exist $\mu, \nu \in \mathbb{F}$ with $\mu + \nu = -l$, $\mu\nu = m$ and $\mu \neq \nu$, we have $(\xi^2 - i\xi l + m) = (i\xi - \mu)\ (i\xi - \nu)$. Thus

$$\mathcal{F}\{x(t)\} = X(\xi) = \frac{P(\xi) + R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{4.11}$$

Assuming $Q(\xi) = \mathcal{F}\{q(t)\} = \dfrac{1}{(i\xi - \mu)\ (i\xi - \nu)}$ and defining a function

$$y(t) = \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu} + (r(t) * q(t))$$

and also taking Fourier transform on both sides, we get

$$\mathcal{F}\{y(t)\} = Y(\xi) = \int_{-\infty}^{\infty} \frac{\mu e^{-\mu t} - \nu e^{-\nu t}}{\mu - \nu}\ e^{ist}\ dt + \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} = \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)}. \tag{4.12}$$

Now, $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = (\xi^2 - i\xi l + m)\ Y(\xi) = R(\xi)$. Then by using (4.12), we have $\mathcal{F}\{y''(t) + l\ y'(t) + m\ y(t)\} = \mathcal{F}\{r(t)\}$, since $\mathcal{F}$ is one-to-one operator; thus

$$y''(t) + l\ y'(t) + m\ y(t) = r(t).$$

Hence $y(t)$ is a solution of the differential equation (1.4). Then by using (4.11) and (4.12) we can obtain

$$\begin{aligned} \mathcal{F}\{x(t)\} - \mathcal{F}\{y(t)\} &= \frac{P(\xi) + R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} - \frac{R(\xi)}{(i\xi - \mu)\ (i\xi - \nu)} \\ &= P(\xi)\ Q(\xi) = \mathcal{F}\{p(t)\}\ \mathcal{F}\{q(t)\} \\ \Rightarrow \qquad \mathcal{F}\{x(t) - y(t)\} &= \mathcal{F}\{p(t) * q(t)\} \end{aligned}$$

Since the operator $\mathcal{F}$ is one-to-one and linear, which gives $x(t) - y(t) = p(t) * q(t)$. Taking modulus on both sides, we have

$$\begin{aligned} |x(t) - y(t)| &= |p(t) * q(t)| \\ &= \left| \int_{-\infty}^{\infty} p(t)\ q(t-s)\ ds \right| \\ &\leq |p(t)| \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right| \leq K\phi(t)\epsilon E_\alpha(t^\alpha). \end{aligned}$$

Where $K = \left| \int_{-\infty}^{\infty} q(t-s)\ ds \right|$, the integral exists for each value of t. Then by the virtue of Definition 2.12 the non-homogeneous linear differential equation (1.4) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. ☐

Conclusion: We have proved the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equations of first order and second order with constant co-efficients using the Fourier Transforms method. That is, we established the sufficient criteria for Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation of first order and second order with constant co-efficients using Fourier Transforms method. Additionally, this paper also provides another method to study the Mittag-Leffler-Hyers-Ulam stability of differential equations. Also, this paper shows that the Fourier Transform method is more convenient to study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation with constant co-efficients.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On Some Systems of Three Nonlinear Difference Equations

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Abstract

We consider in this paper, the solution of the following systems of difference equation:

$$x_{n+1}=\frac{x_{n-2}}{\pm 1+x_{n-2}y_{n-1}z_{n}},\quad y_{n+1}=\frac{y_{n-2}}{\pm 1+y_{n-2}z_{n-1}x_{n}},\ z_{n+1}=\frac{z_{n-2}}{\pm 1+z_{n-2}x_{n-1}y_{n}}$$

where the initial conditions $x_{-2},\ x_{-1},\ x_{0},\ y_{-2},\ y_{-1},\ y_{0}, z_{-2},\ z_{-1},\ z_{0}$ are arbitrary non zero real numbers.

Keywords: difference equations, recursive sequences, periodic solutions, system of difference equations, stability.
Mathematics Subject Classification: 39A10.

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1 Introduction

Difference equations related to differential equations as discrete mathematics related to continuous mathematics. Most of these models are described by nonlinear delay difference equations; see, for example, [9], [10]. The subject of the qualitative study of the nonlinear delay population models is very extensive, and the current research work tends to center around the relevant global dynamics of the considered systems of difference equations such as oscillation, boundedness of solutions, persistence, global stability of positive steady sates, permanence, and global existence of periodic solutions. See [13], [17], [19]-[22], [26], [28], [29] and the references therein. In particular, Agarwal and Elsayed [1] deal with the global stability, periodicity character and gave the solution form of some special cases of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}.$$

Camouzis et al. [5] studied the global character of solutions of the difference equation

$$x_{n+1} = \frac{\delta x_{n-2} + x_{n-3}}{A + x_{n-3}}.$$

Clark and Kulenovic [7] investigated the global asymptotic stability of the system

$$x_{n+1} = \frac{x_n}{a + cy_n}, \qquad y_{n+1} = \frac{y_n}{b + dx_n}.$$

In [9], Din studied the boundedness character, steady-states, local asymptotic stability of equilibrium points, and global behavior of the unique positive equilibrium point of a discrete predator-prey model given by

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \qquad y_{n+1} = \frac{\delta x_n y_n}{x_n + \eta y_n}.$$

Elsayed et al. [23] discussed the global convergence and periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}}.$$

Elsayed and El-Metwally [24] discussed the periodic nature and the form of the solutions of the nonlinear difference equations systems

$$x_{n+1} = \frac{x_n y_{n-2}}{y_{n-1}\left(\pm 1 \pm x_n y_{n-2}\right)}, \qquad y_{n+1} = \frac{y_n x_{n-2}}{x_{n-1}\left(\pm 1 \pm y_n x_{n-2}\right)}.$$

Gelisken and Kara [25] studied some behavior of solutions of some systems of rational difference equations of higher order and they showed that every solution is periodic with a period depends on the order.

In [27] Kurbanli discussed a three-dimensional system of rational difference equations

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} y_n - 1}, \qquad y_{n+1} = \frac{y_{n-1}}{y_{n-1} x_n - 1}, \qquad z_{n+1} = \frac{x_n}{z_{n-1} y_n}.$$

Touafek et al. [33] studied the sufficient conditions for the global asymptotic stability of the following systems of rational difference equations:

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-3} y_{n-1}}, \qquad y_{n+1} = \frac{y_{n-3}}{\pm 1 \pm y_{n-3} x_{n-1}}.$$

with a real number's initial conditions.

Our goal in this paper is to investigate the form of the solutions of the system of three difference equations

$$x_{n+1}=\frac{x_{n-2}}{\pm 1+x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{\pm 1+y_{n-2}z_{n-1}x_n},\quad z_{n+1}=\frac{z_{n-2}}{\pm 1+z_{n-2}x_{n-1}y_n},\quad (1)$$

where the initial conditions $x_{-2},\ x_{-1},\ x_0,\ y_{-2},\ y_{-1},\ y_0,\ z_{-2},\ z_{-1},\ z_0$ are arbitrary real numbers. Moreover, we obtain some numerical simulation to the equation are given to illustrate our results.

2 The System

$$x_{n+1}=\frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n}$$

In this section, we study the solution of the following system of difference equations.

$$x_{n+1}=\frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n},\quad z_{n+1}=\frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n},\qquad (2)$$

where $n\in N_0$ and the initial conditions are arbitrary real numbers.

The following theorem is devoted to the form of the solutions of system (1).

Theorem 1. Suppose that $\{x_n,y_n,z_n\}$ are solutions of the system (1). Then for $n=0,1,2,...$, we have the following formulas

$$\begin{aligned}
x_{3n-2} &= x_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)},\quad x_{3n-1}=x_{-1}\prod_{i=0}^{n-1}\frac{(1+(3i+1)x_{-1}y_0z_{-2})}{(1+(3i+2)x_{-1}y_0z_{-2})},\\
x_{3n} &= x_0\prod_{i=0}^{n-1}\frac{(1+(3i+2)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})},
\end{aligned}$$

$$\begin{aligned}
y_{3n-2} &= y_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})},\quad y_{3n-1}=y_{-1}\prod_{i=0}^{n-1}\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)},\\
y_{3n} &= y_0\prod_{i=0}^{n-1}\frac{(1+(3i+2)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2})},
\end{aligned}$$

$$\begin{aligned}
z_{3n-2} &= z_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})},\quad z_{3n-1}=z_{-1}\prod_{i=0}^{n-1}\frac{(1+(3i+1)x_0y_{-2}z_{-1})}{(1+(3i+2)x_0y_{-2}z_{-1})},\\
z_{3n} &= z_0\prod_{i=0}^{n-1}\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)},
\end{aligned}$$

Proof. For $n = 0$ the result holds. Suppose that the result holds for $n - 1$.

$$\begin{aligned}
x_{3n-5} &= x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}, \quad x_{3n-4} = x_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_{-1}y_0z_{-2})}{(1+(3i+2)x_{-1}y_0z_{-2})},\\
x_{3n-3} &= x_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})},
\end{aligned}$$

$$\begin{aligned}
y_{3n-5} &= y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}, \quad y_{3n-4} = y_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)},\\
y_{3n-3} &= y_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2})},
\end{aligned}$$

$$\begin{aligned}
z_{3n-5} &= z_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}, \quad z_{3n-4} = z_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_0y_{-2}z_{-1})}{(1+(3i+2)x_0y_{-2}z_{-1})},\\
z_{3n-3} &= z_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)}.
\end{aligned}$$

It follows from Eq.(1) that

$$\begin{aligned}
x_{3n-2} &= \frac{x_{3n-5}}{1+x_{3n-5}y_{3n-4}z_{3n-3}}\\
&= \frac{x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1+(x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)})(y_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)})(z_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)})}\\
&= \frac{x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1+x_{-2}y_{-1}z_0\prod_{i=0}^{n-2}((\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)})(\frac{(1+(3i+1)x_{-2}y_{-1}z_0)}{(1+(3i+2)x_{-2}y_{-1}z_0)})(\frac{(1+(3i+2)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)}))}\\
&= \frac{x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}}{1+x_{-2}y_{-1}z_0\prod_{i=0}^{n-2}(\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+3)x_{-2}y_{-1}z_0)})}\\
&= x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}\frac{1}{1+(\frac{x_{-2}y_{-1}z_0}{(1+(3n-3)x_{-2}y_{-1}z_0)})}\\
&= x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}(\frac{(1+(3n-3)x_{-2}y_{-1}z_0)}{(1+(3n-3)x_{-2}y_{-1}z_0)+x_{-2}y_{-1}z_0})\\
&= x_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}(\frac{(1+(3n-3)x_{-2}y_{-1}z_0)}{(1+(3n-2)x_{-2}y_{-1}z_0)}).
\end{aligned}$$

Then, we see that

$$x_{3n-2} = x_{-2} \prod_{i=0}^{n-1} \frac{(1+(3i)x_{-2}y_{-1}z_0)}{(1+(3i+1)x_{-2}y_{-1}z_0)}.$$

Also, we see from Eq.(1) that

$$\begin{aligned}
y_{3n-2} &= \frac{y_{3n-5}}{1+y_{3n-5}z_{3n-4}x_{3n-3}} \\
&= \frac{y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}}{1+(y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})})(z_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i+1)x_0y_{-2}z_{-1})}{(1+(3i+2)x_0y_{-2}z_{-1})})(x_0\prod_{i=0}^{n-2}\frac{(1+(3i+2)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})})} \\
&= \frac{y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}}{1+x_0y_{-2}z_{-1}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+3)x_0y_{-2}z_{-1})}} \\
&= y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}(\frac{1}{1+\frac{x_0y_{-2}z_{-1}}{1+(3n-3)x_0y_{-2}z_{-1}-}}) \\
&= y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}(\frac{1+(3n-3)x_0y_{-2}z_{-1}}{1+(3n-3)x_0y_{-2}z_{-1}+x_0y_{-2}z_{-1}}) \\
&= y_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}(\frac{1+(3n-3)x_0y_{-2}z_{-1}}{1+(3n-2)x_0y_{-2}z_{-1}}).
\end{aligned}$$

Then, we see that

$$y_{3n-2} = y_{-2} \prod_{i=0}^{n-1} \frac{(1+(3i)x_0y_{-2}z_{-1})}{(1+(3i+1)x_0y_{-2}z_{-1})}$$

Finally, we see that

$$\begin{aligned}
z_{3n-2} &= \frac{z_{3n-5}}{1+z_{3n-5}x_{3n-4}y_{3n-3}} \\
&= \frac{z_{-2}\prod\limits_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}}{1+(z_{-2}\prod\limits_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})})(x_{-1}\prod\limits_{i=0}^{n-2}\frac{(1+(3i+1)x_{-1}y_0z_{-2})}{(1+(3i+2)x_{-1}y_0z_{-2})})(y_0\prod\limits_{i=0}^{n-2}\frac{(1+(3i+2)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2})})} \\
&= \frac{z_{-2}\prod\limits_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}}{1+x_{-1}y_0z_{-2}\prod\limits_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+3)x_{-1}y_0z_{-2}}} \\
&= z_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}(\frac{1}{1+\frac{x_{-1}y_0z_{-2}}{1+(3n-3)x_{-1}y_0z_{-2}}}) \\
&= z_{-2}\prod_{i=0}^{n-2}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}(\frac{1+(3n-3)x_{-1}y_0z_{-2}}{1+(3n-2)x_{-1}y_0z_{-2}}).
\end{aligned}$$

Then

$$z_{3n-2} = z_{-2}\prod_{i=0}^{n-1}\frac{(1+(3i)x_{-1}y_0z_{-2})}{(1+(3i+1)x_{-1}y_0z_{-2})}.$$

This completes the proof.

3 The System $x_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n}, y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$

In this section, we obtain the form of the solutions of the system of three difference equations

$$x_{n+1} = \frac{x_{n-2}}{1+x_{n-2}y_{n-1}z_n}, \quad y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n}, \; z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}, \quad (3)$$

where $n \in N_0$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 2. Suppose that $\{x_n, y_n, z_n\}$ are solutions of the system (2). Then for $n = 0, 1, 2, ...$, we have the following formulas

$$x_{3n-2} = \frac{x_{-2}}{1+nx_{-2}y_{-1}z_0}, \quad x_{3n-1} = \frac{x_{-1}(x_{-1}y_0z_{-2}-1)}{(n+1)x_{-1}y_0z_{-2}-1}, \quad x_{3n} = \frac{x_0}{1+nx_0y_{-2}z_{-1}},$$

$$\begin{aligned}
y_{3n-2} &= \frac{(-1)^{n+1}y_{-2}(1+(n-1)x_0y_{-2}z_{-1})}{x_0y_{-2}z_{-1}-1}, \quad y_{3n-1} = (-1)^n y_{-1}(1+nx_{-2}y_{-1}z_0), \\
y_{3n} &= \frac{(-1)^n y_0((n+1)x_{-1}y_0z_{-2}-1)}{x_{-1}y_0z_{-2}-1},
\end{aligned}$$

$$z_{3n-2}=\frac{(-1)^{n+1}z_{-2}}{nx_{-1}y_0z_{-2}-1},\ z_{3n-1}=\frac{(-1)^{n+1}z_{-1}(x_0y_{-2}z_{-1}-1)}{(n-1)x_0y_{-2}z_{-1}+1},\ z_{3n}=\frac{(-1)^nz_0}{1+nx_{-2}y_{-1}z_0}.$$

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$.

$$x_{3n-5}=\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0},\ x_{3n-4}=\frac{x_{-1}(x_{-1}y_0z_{-2}-1)}{nx_{-1}y_0z_{-2}-1},\ x_{3n-3}=\frac{x_0}{1+(n-1)x_0y_{-2}z_{-1}},$$

$$\begin{aligned} y_{3n-5} &= \frac{(-1)^ny_{-2}(1+(n-2)x_0y_{-2}z_{-1})}{x_0y_{-2}z_{-1}-1},\ y_{3n-4}=(-1)^{n-1}y_{-1}(1+(n-1)x_{-2}y_{-1}z_0),\\ y_{3n-3} &= \frac{(-1)^{n-1}y_0(nx_{-1}y_0z_{-2}-1)}{x_{-1}y_0z_{-2}-1}, \end{aligned}$$

$$z_{3n-5}=\frac{(-1)^nz_{-2}}{(n-1)x_{-1}y_0z_{-2}-1},\ z_{3n-4}=\frac{(-1)^nz_{-1}(x_0y_{-2}z_{-1}-1)}{(n-2)x_0y_{-2}z_{-1}+1},\ z_{3n-3}=\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0},$$

from system (2) we can prove as follow

$$\begin{aligned} x_{3n-2} &= \frac{x_{3n-5}}{1+x_{3n-5}y_{3n-4}z_{3n-3}}\\ &= \frac{\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0}}{1+(\frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0})((-1)^{n-1}y_{-1}(1+(n-1)x_{-2}y_{-1}z_0))(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0})}\\ &= \frac{x_{-2}}{1+(n-1)x_{-2}y_{-1}z_0+x_{-2}y_{-1}z_0}=\frac{x_{-2}}{1+nx_{-2}y_{-1}z_0} \end{aligned}$$

Also, we get

$$\begin{aligned} y_{3n-1} &= \frac{y_{3n-4}}{-1+y_{3n-4}z_{3n-3}x_{3n-2}}\\ &= \frac{(-1)^{n-1}y_{-1}(1+(n-1)x_{-2}y_{-1}z_0)}{-1+((-1)^{n-1}y_{-1}(1+(n-1)x_{-2}y_{-1}z_0))(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0})(\frac{x_{-2}}{1+nx_{-2}y_{-1}z_0})}\\ &= \frac{(-1)^ny_{-1}(1+(n-1)x_{-2}y_{-1}z_0)(1+nx_{-2}y_{-1}z_0)}{1+(n-1)x_{-2}y_{-1}z_0}=(-1)^ny_{-1}(1+nx_{-2}y_{-1}z_0) \end{aligned}$$

$$\begin{aligned} z_{3n} &= \frac{z_{3n-3}}{-1+z_{3n-3}x_{3n-2}y_{3n-1}}\\ &= \frac{\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0}}{-1+(\frac{(-1)^{n-1}z_0}{1+(n-1)x_{-2}y_{-1}z_0})(\frac{x_{-2}}{1+nx_{-2}y_{-1}z_0})((-1)^ny_{-1}(1+nx_{-2}y_{-1}z_0))}\\ &= \frac{(-1)^nz_0}{1+(n-1)x_{-2}y_{-1}z_0+x_{-2}y_{-1}z_0}=\frac{(-1)^nz_0}{1+nx_{-2}y_{-1}z_0}. \end{aligned}$$

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4 The System $x_{n+1}=\frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$

In this section, we study the solution of the following system of difference equations

$$x_{n+1}=\frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n},\quad (4)$$

where $n\in N_0$ and the initial conditions are arbitrary nonzero real numbers.

Theorem 3. Suppose that $\{x_n,y_n,z_n\}$ are solutions of the system (3). Then for $n=0,1,2,...$, we have the following formulas

$$x_{3n-2}=\frac{x_{-2}}{nx_{-2}y_{-1}z_0-1},\quad x_{3n-1}=\frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-1)x_{-1}y_0z_{-2}+1},\quad x_{3n}=\frac{(-1)^n x_0}{1+nx_0y_{-2}z_{-1}},$$

$$y_{3n-2}=\frac{y_{-2}}{nx_0y_{-2}z_{-1}+1},\quad y_{3n-1}=\frac{y_{-1}(x_{-2}y_{-1}z_0-1)}{(n+1)x_{-2}y_{-1}z_0-1},\quad y_{3n}=\frac{y_0}{nx_{-1}y_0z_{-2}+1},$$

$$z_{3n-2}=\frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1},\quad z_{3n-1}=(-1)^n z_{-1}(nx_0y_{-2}z_{-1}+1),$$

$$z_{3n}=\frac{(-1)^n z_0((n+1)x_{-2}y_{-1}z_0-1)}{x_{-2}y_{-1}z_0-1}.$$

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$

$$x_{3n-5}=\frac{x_{-2}}{(n-1)x_{-2}y_{-1}z_0-1},\quad x_{3n-4}=\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1},\quad x_{3n-3}=\frac{(-1)^{n-1}x_0}{1+(n-1)x_0y_{-2}z_{-1}},$$

$$y_{3n-5}=\frac{y_{-2}}{(n-1)x_0y_{-2}z_{-1}+1},\quad y_{3n-4}=\frac{y_{-1}(x_{-2}y_{-1}z_0-1)}{nx_{-2}y_{-1}z_0-1},\quad y_{3n-3}=\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1},$$

$$z_{3n-5}=\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1},\quad z_{3n-4}=(-1)^{n-1}z_{-1}((n-1)x_0y_{-2}z_{-1}+1),$$

$$z_{3n-3}=\frac{(-1)^{n-1}z_0(nx_{-2}y_{-1}z_0-1)}{x_{-2}y_{-1}z_0-1},$$

from system (3) we can prove as follow

$$\begin{aligned}
x_{3n-1} &= \frac{x_{3n-4}}{-1+x_{3n-4}y_{3n-3}z_{3n-2}}\\
&= \frac{\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1}}{-1+(\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1})(\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1})(\frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1})}\\
&= \frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{-((n-2)x_{-1}y_0z_{-2}+1)+((-1)^n x_{-1})((-1)^{n+1}y_0z_{-2})}\\
&= \frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-1)x_{-1}y_0z_{-2}+1}.
\end{aligned}$$

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Also, we get

$$\begin{aligned}
y_{3n} &= \frac{y_{3n-3}}{1+y_{3n-3}z_{3n-2}x_{3n-1}} \\
&= \frac{\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1}}{1+(\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1})(\frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1})(\frac{(-1)^{n+1}x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-1)x_{-1}y_0z_{-2}+1})} \\
&= \frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1+y_0((-1)^{n+1}z_{-2})((-1)^{n+1}x_{-1})} \\
&= \frac{y_0}{nx_{-1}y_0z_{-2}+1}
\end{aligned}$$

$$\begin{aligned}
z_{3n-2} &= \frac{z_{3n-5}}{-1+z_{3n-5}x_{3n-4}y_{3n-3}} \\
&= \frac{\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}}{-1+(\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1})(\frac{(-1)^n x_{-1}(x_{-1}y_0z_{-2}-1)}{(n-2)x_{-1}y_0z_{-2}+1})(\frac{y_0}{(n-1)x_{-1}y_0z_{-2}+1})} \\
&= \frac{\frac{(-1)^n z_{-2}((n-2)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}}{\frac{-((n-2)x_{-1}y_0z_{-2}+1)}{(n-1)x_{-1}y_0z_{-2}+1}} \\
&= \frac{(-1)^{n+1}z_{-2}((n-1)x_{-1}y_0z_{-2}+1)}{x_{-1}y_0z_{-2}-1}.
\end{aligned}$$

This completes the proof.

5 The System $x_{n+1}=\frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n}$

In this section, we investigate the solution of the following system of difference equations

$$x_{n+1}=\frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1+z_{n-2}x_{n-1}y_n},\quad (5)$$

where the initial conditions $n\in N_0$ are arbitrary non zero real numbers. The following theorem is devoted to the form of the solutions of system (4).

Theorem 4. Suppose that $\{x_n,y_n,z_n\}$ are solutions of the system (4). Then for $n=0,1,2,...$, we have the following formulas

$$\begin{aligned}
x_{3n-2} &= \frac{(-1)^{n+1}x_{-2}((n-1)x_{-2}y_{-1}z_0+1)}{x_{-2}y_{-1}z_0-1},\ x_{3n-1}=(-1)^n x_{-1}(nx_{-1}y_0z_{-2}+1), \\
x_{3n} &= \frac{(-1)^n x_0((n+1)x_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1},
\end{aligned}$$

$$y_{3n-2}=\frac{(-1)^{n+1}y_{-2}}{nx_0y_{-2}z_{-1}-1},\ y_{3n-1}=\frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-1)x_{-2}y_{-1}z_0+1},\ y_{3n}=\frac{(-1)^n y_0}{nx_{-1}y_0z_{-2}+1},$$

$$z_{3n-2}=\frac{z_{-2}}{nx_{-1}y_0z_{-2}+1},\quad z_{3n-1}=\frac{z_{-1}(x_0y_{-2}z_{-1}-1)}{(n+1)x_0y_{-2}z_{-1}-1},\quad z_{3n}=\frac{z_0}{nx_{-2}y_{-1}z_0+1}.$$

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$

$$\begin{aligned}
x_{3n-5} &= \frac{(-1)^n x_{-2}((n-2)x_{-2}y_{-1}z_0+1)}{x_{-2}y_{-1}z_0-1},\ x_{3n-4}=(-1)^{n-1}x_{-1}((n-1)x_{-1}y_0z_{-2}+1),\\
x_{3n-3} &= \frac{(-1)^{n-1}x_0(x_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1},
\end{aligned}$$

$$y_{3n-5}=\frac{(-1)^n y_{-2}}{(n-1)x_0y_{-2}z_{-1}-1},\ y_{3n-4}=\frac{(-1)^n y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1},\ y_{3n-3}=\frac{(-1)^{n-1}y_0}{(n-1)x_{-1}y_0z_{-2}+1},$$

$$z_{3n-5}=\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1},\ z_{3n-4}=\frac{z_{-1}(x_0y_{-2}z_{-1}-1)}{nx_0y_{-2}z_{-1}-1},\ z_{3n-3}=\frac{z_0}{(n-1)x_{-2}y_{-1}z_0+1},$$

from system (4) we can prove as follow

$$\begin{aligned}
x_{3n} &= \frac{x_{3n-3}}{-1+x_{3n-3}y_{3n-2}z_{3n-1}}\\
&= \frac{\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}}{-1+(\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1})(\frac{(-1)^{n+1}y_{-2}}{nx_0y_{-2}z_{-1}-1})(\frac{z_{-1}(x_0y_{-2}z_{-1}-1)}{(n+1)x_0y_{-2}z_{-1}-1})}\\
&= \frac{\frac{(-1)^{n-1}x_0(nx_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}}{\frac{x_0y_{-2}z_{-1}-((n+1)x_0y_{-2}z_{-1}-1)}{((n+1)x_0y_{-2}z_{-1}-1)}}=\frac{(-1)^n x_0((n+1)x_0y_{-2}z_{-1}-1)}{x_0y_{-2}z_{-1}-1}
\end{aligned}$$

Also, we get

$$\begin{aligned}
y_{3n-1} &= \frac{y_{3n-4}}{-1+y_{3n-4}z_{3n-3}x_{3n-2}}\\
&= \frac{\frac{(-1)^n y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1}}{-1+(\frac{(-1)^n y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1})(\frac{z_0}{(n-1)x_{-2}y_{-1}z_0+1})(\frac{(-1)^{n+1}x_{-2}((n-1)x_{-2}y_{-1}z_0+1)}{x_{-2}y_{-1}z_0-1})}\\
&= \frac{\frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-2)x_{-2}y_{-1}z_0+1}}{\frac{(n-2)x_{-2}y_{-1}z_0+1+x_{-2}y_{-1}z_0}{(n-2)x_{-2}y_{-1}z_0+1}}=\frac{(-1)^{n+1}y_{-1}(x_{-2}y_{-1}z_0-1)}{(n-1)x_{-2}y_{-1}z_0+1}
\end{aligned}$$

$$\begin{aligned}
z_{3n-2} &= \frac{z_{3n-5}}{1+z_{3n-5}x_{3n-4}y_{3n-3}}\\
&= \frac{\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}}{1+(\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1})((-1)^{n-1}x_{-1}((n-1)x_{-1}y_0z_{-2}+1))(\frac{(-1)^{n-1}y_0}{(n-1)x_{-1}y_0z_{-2}+1})}\\
&= \frac{\frac{z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}}{\frac{(n-1)x_{-1}y_0z_{-2}+1+x_{-1}y_0z_{-2}}{(n-1)x_{-1}y_0z_{-2}+1}}=\frac{z_{-2}}{nx_{-1}y_0z_{-2}+1}
\end{aligned}$$

This completes the proof.

The following cases can be proved similarly.

6 On The System

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\ y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n},\ z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n}$$

In this section we study the solution of the following system of difference equations

$$x_{n+1} = \frac{x_{n-2}}{-1+x_{n-2}y_{n-1}z_n},\quad y_{n+1} = \frac{y_{n-2}}{-1+y_{n-2}z_{n-1}x_n},\ z_{n+1} = \frac{z_{n-2}}{-1+z_{n-2}x_{n-1}y_n},\ (6)$$

where the initial conditions $n \in N_0$ are arbitrary non zero real numbers.

Theorem 5. Let $\{x_n, y_n, z_n\}_{n=-2}^{+\infty}$ be solutions of system (5). Then

1- $\{x_n\}_{n=-2}^{+\infty}$, $\{y_n\}_{n=-2}^{+\infty}$ and $\{z_n\}_{n=-2}^{+\infty}$ and are periodic with period six i.e.,

$$x_{n+6} = x_n, \quad y_{n+6} = y_n, \quad z_{n+6} = z_n.$$

2- We have the following form

$$x_{6n-2} = x_{-2}, \quad x_{6n-1} = x_{-1}, \quad x_{6n} = x_0,$$

$$x_{6n+1} = \frac{x_{-2}}{x_{-2}y_{-1}z_0 - 1}, \quad x_{6n+2} = x_{-1}(x_{-1}y_0z_{-2} - 1), \quad x_{6n+3} = \frac{x_0}{x_0y_{-2}z_{-1} - 1},$$

$$y_{6n-2} = y_{-2}, \quad y_{6n-1} = y_{-1}, \quad y_{6n} = y_0,$$

$$y_{6n+1} = \frac{y_{-2}}{x_0y_{-2}z_{-1} - 1}, \quad y_{6n+2} = y_{-1}(x_{-2}y_{-1}z_0 - 1), \quad y_{6n+3} = \frac{y_0}{x_{-1}y_0z_{-2} - 1},$$

$$z_{6n-2} = z_{-2}, \quad z_{6n-1} = z_{-1}, \quad z_{6n} = z_0,$$

$$z_{6n+1} = \frac{z_{-2}}{x_{-1}y_0z_{-2} - 1}, \quad z_{6n+2} = z_{-1}(x_0y_{-2}z_{-1} - 1), \quad z_{6n+3} = \frac{z_0}{x_{-2}y_{-1}z_0 - 1},$$

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty} = \left\{x_{-2}, x_{-1}, x_0, \frac{x_{-2}}{x_{-2}y_{-1}z_0 - 1}, x_{-1}(x_{-1}y_0z_{-2} - 1), \frac{x_0}{x_0y_{-2}z_{-1} - 1}\right\},$$

$$\{y_n\}_{n=-2}^{+\infty} = \left\{y_{-2}, y_{-1}, y_0, \frac{y_{-2}}{x_0y_{-2}z_{-1} - 1}, y_{-1}(x_{-2}y_{-1}z_0 - 1), \frac{y_0}{x_{-1}y_0z_{-2} - 1}\right\}.$$

$$\{z_n\}_{n=-2}^{+\infty} = \left\{z_{-2}, z_{-1}, z_0, \frac{z_{-2}}{x_{-1}y_0z_{-2} - 1}, z_{-1}(x_0y_{-2}z_{-1} - 1), \frac{z_0}{x_{-2}y_{-1}z_0 - 1}\right\}.$$

7 On The System

$$x_{n+1}=\frac{x_{n-2}}{-1-x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{-1-y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{-1-z_{n-2}x_{n-1}y_n}$$

In this section we study the solution of the following system of difference equations

$$x_{n+1}=\frac{x_{n-2}}{-1-x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{-1-y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{-1-z_{n-2}x_{n-1}y_n},\tag{7}$$

where the initial conditions $n\in N_0$ are arbitrary non zero real numbers.

Theorem 6. Let $\{x_n,y_n,z_n\}_{n=-2}^{+\infty}$ be solutions of system (6). Then

1- $\{x_n\}_{n=-2}^{+\infty}$, $\{y_n\}_{n=-2}^{+\infty}$ and $\{z_n\}_{n=-2}^{+\infty}$ and are periodic with period six i.e.,

$$x_{n+6}=x_n,\quad y_{n+6}=y_n,\quad z_{n+6}=z_n.$$

2- We have the following form

$$x_{6n-2}=x_{-2},\quad x_{6n-1}=x_{-1},\quad x_{6n}=x_0,$$

$$x_{6n+1}=-\frac{x_{-2}}{x_{-2}y_{-1}z_0+1},\quad x_{6n+2}=-x_{-1}(x_{-1}y_0z_{-2}+1),\quad x_{6n+3}=-\frac{x_0}{x_0y_{-2}z_{-1}+1},$$

$$y_{6n-2}=y_{-2},\quad y_{6n-1}=y_{-1},\quad y_{6n}=y_0,$$

$$y_{6n+1}=-\frac{y_{-2}}{x_0y_{-2}z_{-1}+1},\quad y_{6n+2}=-y_{-1}(x_{-2}y_{-1}z_0+1),\quad y_{6n+3}=-\frac{y_0}{x_{-1}y_0z_{-2}+1},$$

$$z_{6n-2}=z_{-2},\quad z_{6n-1}=z_{-1},\quad z_{6n}=z_0,$$

$$z_{6n+1}=-\frac{z_{-2}}{x_{-1}y_0z_{-2}+1},\quad z_{6n+2}=-z_{-1}(x_0y_{-2}z_{-1}+1),\quad z_{6n+3}=-\frac{z_0}{x_{-2}y_{-1}z_0+1},$$

Or equivalently

$$\{x_n\}_{n=-2}^{+\infty}=\left\{x_{-2},x_{-1},x_0,-\frac{x_{-2}}{x_{-2}y_{-1}z_0+1},-x_{-1}(x_{-1}y_0z_{-2}+1),-\frac{x_0}{x_0y_{-2}z_{-1}+1}\right\},$$

$$\{y_n\}_{n=-2}^{+\infty}=\left\{y_{-2},y_{-1},y_0,-\frac{y_{-2}}{x_0y_{-2}z_{-1}+1},-y_{-1}(x_{-2}y_{-1}z_0+1),-\frac{y_0}{x_{-1}y_0z_{-2}+1}\right\}.$$

$$\{z_n\}_{n=-2}^{+\infty}=\left\{z_{-2},z_{-1},z_0,-\frac{z_{-2}}{x_{-1}y_0z_{-2}+1},-z_{-1}(x_0y_{-2}z_{-1}+1),-\frac{z_0}{x_{-2}y_{-1}z_0+1}\right\}.$$

8 The System $x_{n+1}=\frac{x_{n-2}}{1-x_{n-2}y_{n-1}z_n},\ y_{n+1}=\frac{y_{n-2}}{1-y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1-z_{n-2}x_{n-1}y_n}$

In this section, we study the solution of the following system of difference equations.

$$x_{n+1}=\frac{x_{n-2}}{1-x_{n-2}y_{n-1}z_n},\quad y_{n+1}=\frac{y_{n-2}}{1-y_{n-2}z_{n-1}x_n},\ z_{n+1}=\frac{z_{n-2}}{1-z_{n-2}x_{n-1}y_n}\tag{8}$$

where $n\in N_0$ and the initial conditions are arbitrary nonzero real numbers.

The following theorem is devoted to the form of the solutions of system (7).

Theorem 7. Suppose that $\{x_n,y_n,z_n\}$ are solutions of the system (7). Then for $n=0,1,2,...$, we have the following formulas

$$\begin{aligned} x_{3n-2} &= -x_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)},\quad x_{3n-1}=x_{-1}\prod_{i=0}^{n-1}\frac{(-1+(3i+1)x_{-1}y_0z_{-2})}{(-1+(3i+2)x_{-1}y_0z_{-2})},\\ x_{3n} &= x_0\prod_{i=0}^{n-1}\frac{(-1+(3i+2)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})},\end{aligned}$$

$$\begin{aligned} y_{3n-2} &= -y_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})},\quad y_{3n-1}=y_{-1}\prod_{i=0}^{n-1}\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)},\\ y_{3n} &= y_0\prod_{i=0}^{n-1}\frac{(-1+(3i+2)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2})},\end{aligned}$$

$$\begin{aligned} z_{3n-2} &= -z_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})},\quad z_{3n-1}=z_{-1}\prod_{i=0}^{n-1}\frac{(-1+(3i+1)x_0y_{-2}z_{-1})}{(-1+(3i+2)x_0y_{-2}z_{-1})},\\ z_{3n} &= z_0\prod_{i=0}^{n-1}\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)},\end{aligned}$$

Proof. For $n=0$ the result holds. Suppose that the result holds for $n-1$.

$$\begin{aligned} x_{3n-5} &= -x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)},\quad x_{3n-4}=x_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_{-1}y_0z_{-2})}{(-1+(3i+2)x_{-1}y_0z_{-2})},\\ x_{3n-3} &= x_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})},\end{aligned}$$

$$\begin{aligned} y_{3n-5} &= -y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})},\quad y_{3n-4}=y_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)},\\ y_{3n-3} &= y_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2})},\end{aligned}$$

$$\begin{aligned} z_{3n-5} &= -z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}, \quad z_{3n-4}=z_{-1}\prod_{i=0}^{n-2}\frac{(-1+(3i+1)x_0y_{-2}z_{-1})}{(-1+(3i+2)x_0y_{-2}z_{-1})}, \\ z_{3n-3} &= z_0\prod_{i=0}^{n-2}\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)}, \end{aligned}$$

It follows from Eq.(7) that

$$\begin{aligned} x_{3n-2} &= \frac{x_{3n-5}}{1+x_{3n-5}y_{3n-4}z_{3n-3}} \\ &= \frac{-x_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1+(-x_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)})(y_{-1}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)})(z_0\prod\limits_{i=0}^{n-2}\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)})} \\ &= \frac{-x_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1-x_{-2}y_{-1}z_0\prod\limits_{i=0}^{n-2}((\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)})(\frac{(-1+(3i+1)x_{-2}y_{-1}z_0)}{(-1+(3i+2)x_{-2}y_{-1}z_0)})(\frac{(-1+(3i+2)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)}))} \\ &= \frac{-x_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}}{1-x_{-2}y_{-1}z_0\prod\limits_{i=0}^{n-2}(\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+3)x_{-2}y_{-1}z_0)})} \\ &= -x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}\frac{1}{1+(\frac{x_{-2}y_{-1}z_0}{(-1+(3n-3)x_{-2}y_{-1}z_0)})} \\ &= -x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}(\frac{(-1+(3n-3)x_{-2}y_{-1}z_0)}{(-1+(3n-3)x_{-2}y_{-1}z_0)+x_{-2}y_{-1}z_0}) \\ &= -x_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}(\frac{(-1+(3n-3)x_{-2}y_{-1}z_0)}{(-1+(3n-2)x_{-2}y_{-1}z_0)}) \end{aligned}$$

Then, we see that

$$x_{3n-2}=-x_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_{-2}y_{-1}z_0)}{(-1+(3i+1)x_{-2}y_{-1}z_0)}$$

Also, we see from Eq.(1) that

$$\begin{aligned}
y_{3n-2} &= \frac{y_{3n-5}}{1+y_{3n-5}z_{3n-4}x_{3n-3}} \\
&= \frac{-y_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}}{1+(-y_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})})(z_{-1}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i+1)x_0y_{-2}z_{-1})}{(-1+(3i+2)x_0y_{-2}z_{-1})})(x_0\prod\limits_{i=0}^{n-2}\frac{(-1+(3i+2)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})})} \\
&= \frac{-y_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}}{1-x_0y_{-2}z_{-1}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+3)x_0y_{-2}z_{-1})}} \\
&= -y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}(\frac{1}{1+\frac{x_0y_{-2}z_{-1}}{-1+(3n-3)x_0y_{-2}z_{-1}-}}) \\
&= -y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}(\frac{-1+(3n-3)x_0y_{-2}z_{-1}}{-1+(3n-3)x_0y_{-2}z_{-1}+x_0y_{-2}z_{-1}}) \\
&= -y_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}(\frac{-1+(3n-3)x_0y_{-2}z_{-1}}{-1+(3n-2)x_0y_{-2}z_{-1}})
\end{aligned}$$

Then, we see that

$$y_{3n-2} = -y_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_0y_{-2}z_{-1})}{(-1+(3i+1)x_0y_{-2}z_{-1})}$$

Finally, we see that

$$\begin{aligned}
z_{3n-2} &= \frac{z_{3n-5}}{1+z_{3n-5}x_{3n-4}y_{3n-3}} \\
&= \frac{-z_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}}{1+(-z_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})})(x_{-1}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i+1)x_{-1}y_0z_{-2})}{(-1+(3i+2)x_{-1}y_0z_{-2})})(y_0\prod\limits_{i=0}^{n-2}\frac{(-1+(3i+2)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2})})} \\
&= \frac{-z_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}}{1-x_{-1}y_0z_{-2}\prod\limits_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+3)x_{-1}y_0z_{-2}}} \\
&= -z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}(\frac{1}{1+\frac{x_{-1}y_0z_{-2}}{-1+(3n-3)x_{-1}y_0z_{-2}-}}) \\
&= -z_{-2}\prod_{i=0}^{n-2}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}(\frac{-1+(3n-3)x_{-1}y_0z_{-2}}{-1+(3n-2)x_{-1}y_0z_{-2}})
\end{aligned}$$

Then,

$$z_{3n-2} = -z_{-2}\prod_{i=0}^{n-1}\frac{(-1+(3i)x_{-1}y_0z_{-2})}{(-1+(3i+1)x_{-1}y_0z_{-2})}$$

This completes the proof

8.1 Numerical Examples

For confirming the results of this section, we consider the following numerical example which represent solutions to the previous systems.

Example 1. We consider interesting numerical example for the difference equations system (1) with the initial conditions $x_{-2} = 13,\ x_{-1} = 0.4,\ x_0 = 3,\ y_{-2} = 0.5,\ y_{-1} = 7,\ y_0 = 3.7,\ z_{-2} = 0.9,\ z_{-1} = 17\ and\ z_0 = 0.72$. (See Fig. 1).

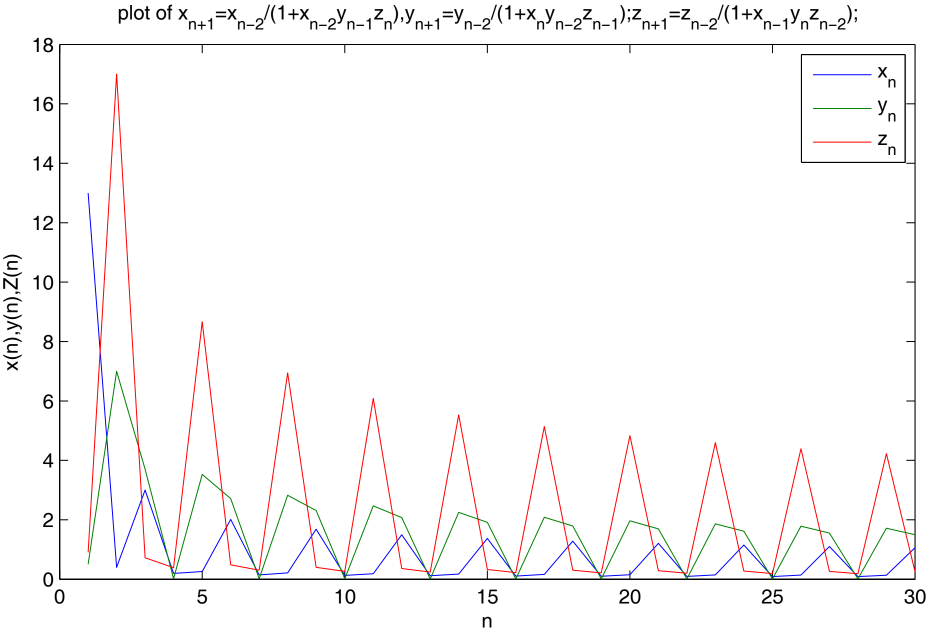

Figure 1.

Example 2. We put the initial conditions for system (2) as follows: $x_{-2} = 1.3,\ x_{-1} = -0.4,\ x_0 = 0.3,\ y_{-2} = 0.5,\ y_{-1} = 0.1,\ y_0 = -0.7,\ z_{-2} = -0.9,\ z_{-1} = 0.7\ and\ z_0 = 0.2$. (See Fig. 2).

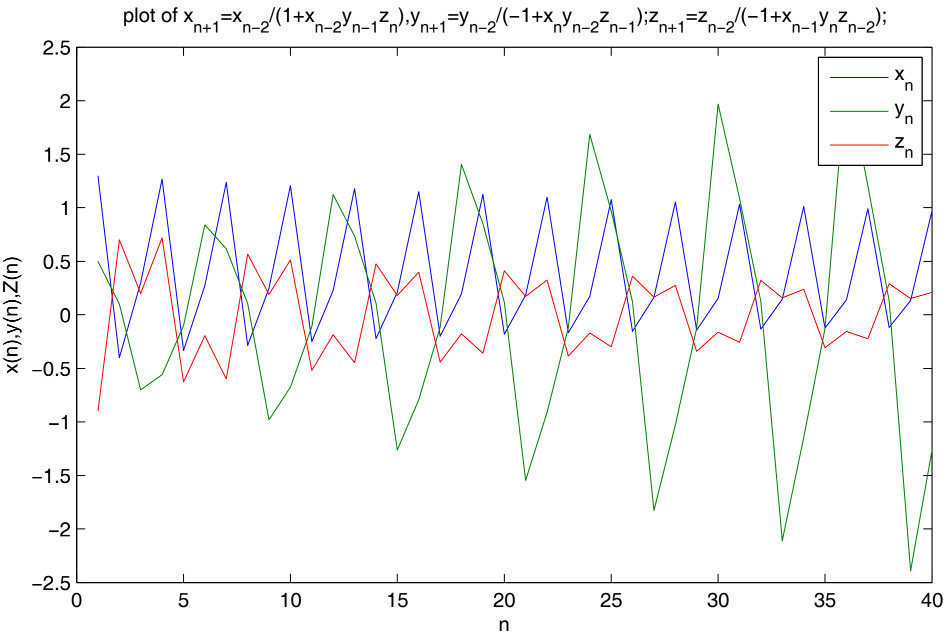

Figure 2.

Example 3. For the difference equations system (3) where the initial conditions $x_{-2} = 1.3,\ x_{-1} = 0.4,\ x_0 = 0.3,\ y_{-2} = 0.25,\ y_{-1} = 0.1,\ y_0 = 0.7,\ z_{-2} = 0.9,\ z_{-1} = 0.7$ *and* $z_0 = 0.2$. (See Fig. 3).

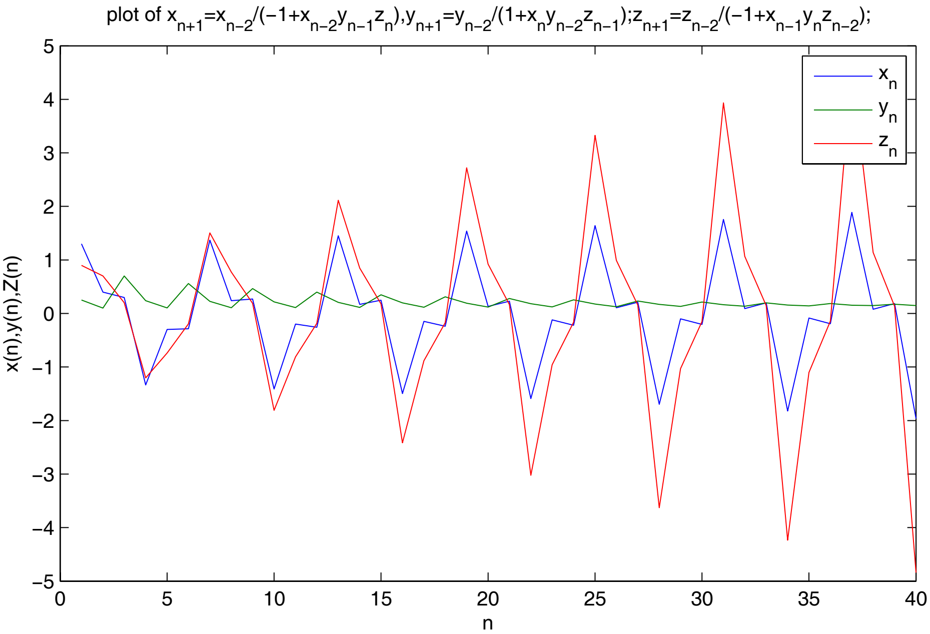

Figure 3.

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Example 4. We assume $x_{-2} = 1.3,\ x_{-1} = 0.4,\ x_0 = 0.3,\ y_{-2} = 0.25,\ y_{-1} = 0.1,$ $y_0 = 0.7,\ z_{-2} = 0.9,\ z_{-1} = 0.7$ *and* $z_0 = 0.2$ for system (4) see Fig. 4.

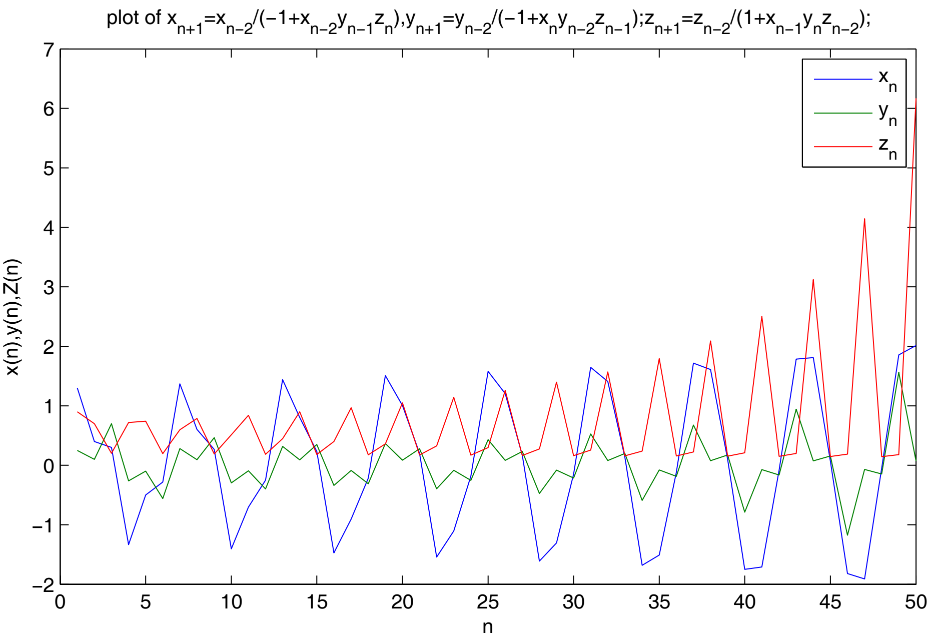

Figure 4.

Example 5. See Fig. 5, if we take system (5) with $x_{-2} = 3,\ x_{-1} = -0.4,\ x_0 = 2,\ y_{-2} = -0.5,\ y_{-1} = 0.9,\ y_0 = 0.7,\ z_{-2} = 0.19,\ z_{-1} = -0.4$ *and* $z_0 = 0.1.$

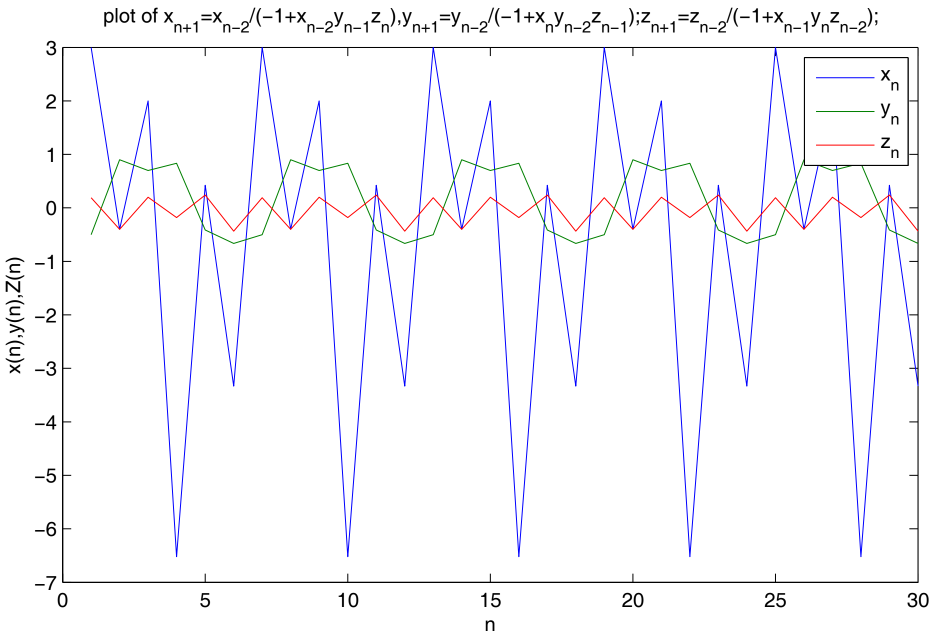

Figure 5.

Example 6. See Fig. 6, if we consider system (6) with $x_{-2} = -9,\ x_{-1} = 0.4,\ x_0 = -2,\ y_{-2} = 0.2,\ y_{-1} = 0.7,\ y_0 = 1.8,\ z_{-2} = 9,\ z_{-1} = -0.4$ *and* $z_0 = -2$.

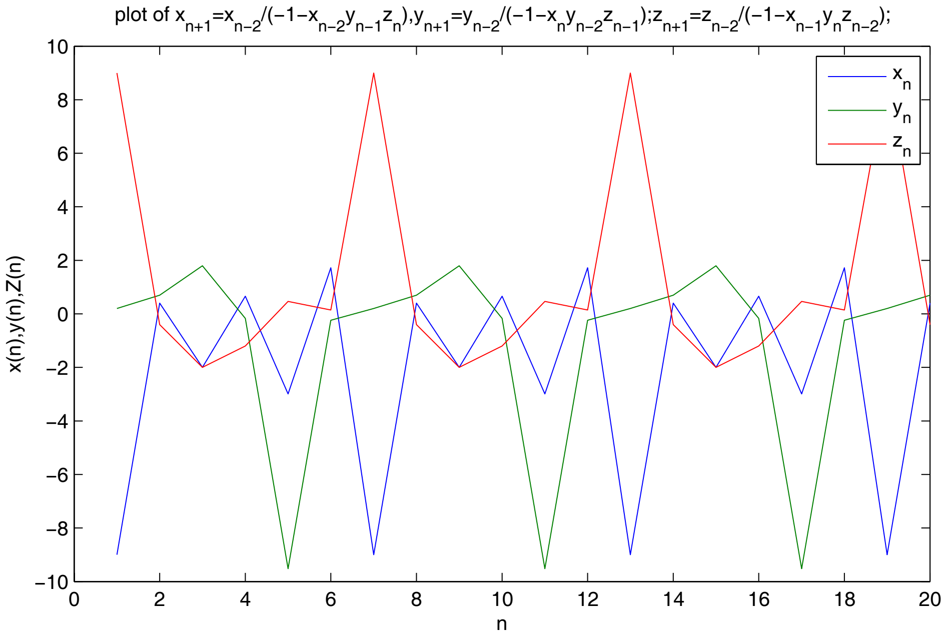

Figure 6.

Example 7. We take the difference equations system (7) with the initial conditions $x_{-2} = 9,\ x_{-1} = 4,\ x_0 = 2,\ y_{-2} = 3,\ y_{-1} = 7,\ y_0 = 18,\ z_{-2} = 11,\ z_{-1} = -4$ *and* $z_0 = 5$. (See Fig. 7).

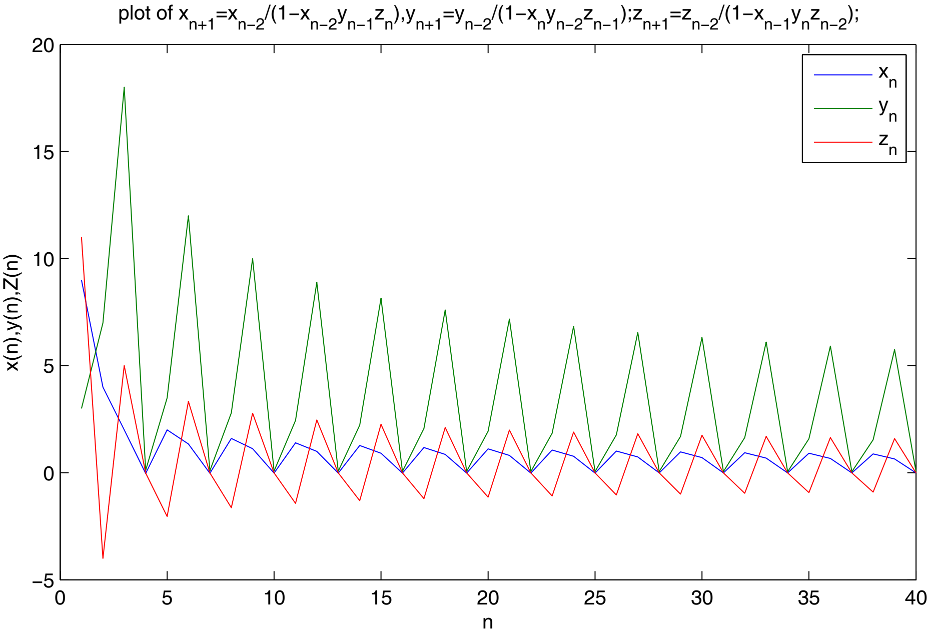

Figure 7.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

APPROXIMATION OF SOLUTIONS OF THE INHOMOGENEOUS GAUSS DIFFERENTIAL EQUATIONS BY HYPERGEOMETRIC FUNCTION

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Abstract. In this paper, we solve the inhomogeneous Gauss differential equation and apply this result to estimate the error bound occurring when an analytic function is approximated by an appropriate hypergeometric function.

1. Introduction

More than a half century ago, Ulam [22] posed the famous Ulam stability problem which was partially solved by Hyers [7] in the framework of Banach spaces. The Hyers' theorem was generalized by Aoki [4] for additive mappings. In 1978, Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy difference inequality

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p), \qquad (\varepsilon \geq 0, p \in [0,1)).$$

Since then, the stability problems of various functional equations have been studied by many authors (see [1, 6, 8, 9, 13, 15, 17, 18, 19, 20]).

Alsina and Ger [3] were the first authors who investigated the Hyers-Ulam stability of differential equations. They proved that if a differentiable function $f : I \to \mathbb{R}$ is a solution of the differential inequality $|y'(t) - y(t)| \leq \epsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a solution $f_0 : I \to \mathbb{R}$ of the differential equation $y'(t) = y(t)$ such that $|f(t)-f_0(t)| \leq 3\epsilon$ for any $t \in I$. From then on, many research papers about the Hyers-Ulam stability of differential equations have appeared in the literature, see [2, 5, 10, 11, 12, 21, 23] for instance.

The form of the homogeneous Gauss differential equation has the form

$$x(1-x)y'' + [r-(1+s+t)x]y' - sty = 0. \tag{1.1}$$

It is easy to see that

$$y_1 = 1 + \frac{st}{1!r}x + \frac{(st)(s+1)(t+1)}{2!r(r+1)}x^2 + \frac{(st)(s+1)(s+2)(t+1)(t+2)}{3!r(r+1)(r+2)}x^5 + \cdots$$

2010 *Mathematics Subject Classification.* Primary 39B82, 35B35.

Key words and phrases. Gauss differential equation; analytic function; hypergeometric function; approximation.

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and

$$\begin{aligned} y_2 =& x^{1-r}[1 + \frac{(s-r+1)(t-r+1)}{1!(2-r)}x \\ &+ \frac{(s-r+1)(s-r+2)(t-r+1)(t-r+2)}{2!(2-r)(3-r)}x^2 \\ &+ \frac{(s-r+1)(s-r+2)(s-r+3)(t-r+1)(t-r+2)(t-r+3)}{3!(2-r)(3-r)(4-r)}x^5 + ...] \end{aligned}$$

are a fundamental set of solutions of equation (1.1) (if $r \neq 1$). The series y_1 known the hypergeometric function is convergent for $|x| < 1$ and is represented by

$$y_1 = F(s, t, r, x).$$

Note that

$$y_2 = x^{1-r}F(s-r+1, t-r+1, 2-r, x)$$

is of the same type. Thus the general solution is

$$y_c = c_1y_1 + c_2y_2 = c_1F(s, t, r, x) + c_2x^{1-r}F(s-r+1, t-r+1, 2-r, x).$$

2. INHOMOGENEOUS GAUSS DIFFERENTIAL EQUATION

In this section, we consider the solution of inhomogeneous Gauss differential equation of the form

$$x(1-x)y'' + [r - (1+s+t)x]y' - sty = \sum_{m=0}^{+\infty} a_m x^m, \tag{2.1}$$

where the coefficients a_n's of the power series are given such that the radius of convergence is positive.

Theorem 2.1. *Assume that the radius of convergence of the power series* $\sum_{m=0}^{+\infty} a_m x^m$ *is* $R_0 > 0$ *and*

$$R_1 = \lim_{k\to\infty} |\frac{c_k}{c_{k+1}}| > 0. \tag{2.2}$$

Let ρ *be a positive number defined by* $\rho = min\{1, R_0, R_1\}$*. Then every solution* $y : (-\rho, \rho) \to \mathbb{C}$ *of differential equation* (2.1) *can be expressed by*

$$y(x) = y_c(x) + \sum_{m=1}^{+\infty} c_m x^m, \tag{2.3}$$

where $c_1 = \frac{1}{r}a_0$ *and*

$$\begin{aligned} c_m =& \frac{a_{m-1}}{m(m-1+r)} \\ &+ \frac{1}{m!}\sum_{i=1}^{m-1} a_{m-i-1} \prod_{j=1}^{i+1} \frac{1}{m-j+r} \prod_{j=1}^{i} (m+s-j)(m+t-j) \end{aligned} \tag{2.4}$$

for any $m \in \{2, 3, \cdots\}$.

Proof. We will show that each function $y : (-\rho, \rho) \to \mathbb{C}$ defined by (2.3) is a solution of the inhomogeneous Gauss differential equation (2.1), where y_c is a solution of homogeneous Gauss differential equation (1.1). For this purpose, it is only necessary to show that $y_p(x) = \sum_{m=1}^{\infty} c_m x^m$ satisfies differential equation (2.1). Therefore, letting $y_p(x) = \sum_{m=1}^{\infty} c_m x^m$ in differential equation (2.1), we obtain

$$\begin{aligned}\sum_{m=1}^{+\infty} m(m+1)c_{m+1}x^m + r\sum_{m=0}^{+\infty}(m+1)c_{m+1}x^m - \sum_{m=2}^{+\infty} m(m-1)c_m x^m \\ - (1+s+t)\sum_{m=1}^{+\infty} mc_m x^m - st\sum_{m=1}^{+\infty} c_m x^m = \sum_{m=0}^{+\infty} a_m x^m.\end{aligned}$$

Hence

$$rc_1 + \sum_{m=1}^{+\infty}[(m+1)(m+r)c_{m+1} - (m+s)(m+t)c_m]x^m = \sum_{m=0}^{+\infty} a_m x^m.$$

Therefore, we get $c_1 = \frac{1}{r}a_0$ and

$$c_{m+1} = \frac{1}{(m+1)(m+r)}a_m + \frac{(m+s)(m+t)}{(m+1)(m+r)}c_m, \quad (m = 1, 2, ...).$$

By some manipulations, we obtain

$$\begin{aligned}c_m =& \frac{a_{m-1}}{m(m-1+r)} \\ &+ \frac{1}{m!}\sum_{i=1}^{m-1} a_{m-i-1}\prod_{j=1}^{i+1}\frac{1}{m-j+r}\prod_{j=1}^{i}(m+s-j)(m+t-j)\end{aligned} \tag{2.5}$$

for any $m \in \{2, 3, ...\}$. The condition (2.2) implies that the radius of convergence of $y_p(x) = \sum_{m=1}^{+\infty} c_m x^m$ is R_1. By using the ratio test, we can easily show that the radius of convergence of y_c is 1. Thus y is certainly defined on $(-\rho, \rho)$. □

Corollary 2.2. *Assume that the assumptions of Theorem* 2.1 *hold. Then there exists* $C > 0$ *such that*

$$\begin{aligned}\sum_{m=1}^{+\infty} c_m x^m \leq& \sum_{m=1}^{+\infty}\frac{a_{m-1}}{m(m-1+r)}x^m \\ &+ \sum_{i=1}^{+\infty}\sum_{m=2}^{+\infty}\frac{Ca_{m-2}}{(m+i-1)^2}\prod_{j=0}^{i}(1 - \frac{-st}{(m+i-j-1)(m+i-j+s+t-1)})x^{m+i-1}.\end{aligned}$$

Proof. Since there exists a constant $C > 0$ with

$$\frac{1}{m!}\prod_{j=1}^{i+1}\frac{1}{m-j+r} \leq \frac{C}{m^2}\prod_{j=0}^{i}\frac{1}{(m-j)(m-j+s+t)}$$

for any $m = 2, 3, ...$ and for any $i = 1, 2, ...$, it follows from (2.5) that

$$\begin{aligned}
&\sum_{m=1}^{+\infty} c_m x^m = c_1 x + \sum_{m=2}^{+\infty} c_m x^m = \frac{1}{r} a_0 x + \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m \\
&\quad + \sum_{m=2}^{+\infty} \frac{1}{m!} \sum_{i=1}^{m-1} a_{m-i-1} \prod_{j=1}^{i+1} \frac{1}{m-j+r} \prod_{j=0}^{i} (m+s-j)(m+t-j) x^m \\
&\le \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{m=2}^{+\infty} \sum_{i=1}^{m-1} \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^{i} \frac{(m+s-j)(m+t-j)}{(m-j)(m-j+s+t)} x^m \\
&= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{m=2}^{+\infty} \sum_{i=1}^{m-1} \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^{i} (1 - \frac{-st}{(m-j)(m-j+s+t)}) x^m \\
&= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{i=1}^{+\infty} \sum_{m=i+1}^{+\infty} A_{mi} x^m \\
&= \sum_{m=1}^{+\infty} \frac{a_{m-1}}{m(m-1+r)} x^m + \sum_{i=1}^{+\infty} \sum_{m=2}^{+\infty} A_{m+i-1i} x^{m+i-1},
\end{aligned}$$

where we define

$$A_{mi} := \frac{C a_{m-i-1}}{m^2} \prod_{j=0}^{i} (1 - \frac{-st}{(m-j)(m-j+s+t)})$$

for all $i = 1, 2, \cdots$ and $m = 2, 3, \cdots$. ☐

3. Approximation property of hypergeometric function

In this section, we investigate an approximation property of hypergeometric functions. More precisely, we will prove that if an analytic function satisfies the condition (2.2), then it can be approximated by a hypergeometric function. Suppose that y is a given function expressed as a power series of the form

$$y(x) = \sum_{m=0}^{\infty} b_m x^m, \tag{3.1}$$

whose radius of convergence is $R_0 > 0$. Then we obtain

$$\begin{aligned}
x(1-x)y'' + &[r - (1+s+t)x]y' - sty \\
&= \sum_{m=0}^{\infty} [(m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m] x^m \\
&= \sum_{m=0}^{\infty} a_m x^m,
\end{aligned} \tag{3.2}$$

where we define

$$a_m := (m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m \tag{3.3}$$

for all $m \in \{0, 1, 2, 3, \cdots\}$.

Lemma 3.1. *If the a_m's, the b_m's and the c_m's are as defined in* (3.3), (3.1) *and* (2.4), *then*

$$c_m = b_m - \frac{b_0}{m!}\prod_{j=1}^{m}\frac{1}{m-j+r}\prod_{j=1}^{m-1}(m+s-j)(m+t-j) \tag{3.4}$$

for all $m \in \{0, 1, 2, 3, ...\}$.

Proof. The proof is clear by induction on m. For $m = 1$ and by (3.3) we have

$$c_1 = \frac{a_0}{r} = \frac{1}{r}(rb_1 - stb_0) = b_1 - \frac{st}{r}b_0. \tag{3.5}$$

Assume now that formula (3.3) is true for some m. It follows from (2.4), (3.3) and (3.4) that

$$\begin{aligned} c_{m+1} &= \frac{a_m}{(m+1)(m+r)} + \frac{(m+s)(m+t)}{(m+1)(m+r)}c_m \\ &= \frac{1}{(m+1)(m+r)}\Big((m+1)(m+r)b_{m+1} - (m+s)(m+t)b_m\Big) \\ &+ \frac{(m+s)(m+t)}{(m+1)(m+r)}\Big(b_m - \frac{b_0}{m!}\prod_{j=1}^{m}\frac{1}{m-j+r}\prod_{j=1}^{m-1}(m+s-j)(m+t-j)\Big) \\ &= b_{m+1} - \frac{b_0}{(m+1)!}\prod_{j=1}^{m+1}\frac{1}{m+1-j+r}\prod_{j=1}^{m}(m+1+s-j)(m+1+t-j), \end{aligned}$$

as desired. ☐

Theorem 3.2. *Let R and R_0 be positive constants with $R < R_0$. Assume that $y : (-R, R) \to \mathbb{C}$ is a function of the form* (3.1) *whose radius of convergence is R_1. Also, b_m's and c_m's are given by* (3.3) *and* (3.4), *respectively. If $R < min\{1, R_0, R_1\}$, then there exist a hypergeometric function $y_h : (-R, R) \to \mathbb{C}$ and a constant $d > 0$ such that $|y(x) - y_h(x)| \leq d\frac{x}{1-x}$ for all $x \in (-R, R)$.*

Proof. We assume that y can be represented by a power series (3.1) whose radius of convergence is $R < R_0$. So

$$x(1-x)\sum_{m=2}^{+\infty} m(m-1)b_m x^{m-2} + [r - (1+s+t)x]\sum_{m=1}^{+\infty} mb_m x^{m-1} - st\sum_{m=0}^{+\infty} mb_m x^m$$

is also a power series whose radius of convergence is R_0, more precisely, in view of (3.2) and (3.3), we have

$$\begin{aligned} x(1-x)\sum_{m=0}^{+\infty} m(m-1)b_m x^{m-2} + [r - (1+s+t)x]\sum_{m=0}^{+\infty} mb_m x^{m-1} \\ - st\sum_{m=0}^{+\infty} mb_m x^m = \sum_{m=0}^{+\infty} a_m x^m \end{aligned}$$

for all $x \in (-R, R)$. Since the power series $\sum_{m=0}^{+\infty} a_m x^m$ is absolutely convergent on its interval of convergence, which includes the interval $[-R, R]$ and the power series $\sum_{m=0}^{+\infty} |a_m x^m|$ is continuous on $[-R, R]$. So there exists a constant $d_1 > 0$ with

$$\sum_{m=0}^{n} |a_m x^m| \leq d_1$$

for all integers $n \geq 0$ and for any $x \in (-R, R)$.

On the other hand, since

$$\sum_{k=1}^{+\infty} |\frac{-st}{(m-k-1)(m-k-1+t+s)}| \leq \frac{st\pi^2}{6} =: d_2, \quad (m = 2, 3, ...),$$

we have

$$|\prod_{k=1}^{+\infty} (1 - \frac{-st}{(m-k-1)(m-k-1+t+s)}| \leq d_2, \quad (m = 2, 3, \cdots)$$

(see [16, Theorem 6.6.2]). Hence, substituting $i - j$ for k in the above infinite product, there exists a constant d_3 with

$$|\prod_{j=0}^{i} (1 - \frac{-st}{(m-i-j-1)(m-i-j-1+t+s)}| \leq d_3$$

for all $i = 1, 2, \cdots$ and $m = 2, 3, \cdots$. Therefore, it follows Lemma 2.2 that

$$|\sum_{m=0}^{\infty} c_m x^m| \leq d_1 d_3 \frac{x}{1-x} \tag{3.6}$$

for all $x \in (-R_0, R_0)$. This completes the proof of our theorem. □

Corollary 3.3. *Assume that R and R_0 are positive constants with $R < R_0$. Let $y : (R, R_0) \to C$ be a function which can be represented by a power series of the form (3.1) whose radius of convergence is R_0. Moreover, assume that there exists a positive number R_1 satisfying the condition (2.2) with b_m's and c_m's given in (3.1) and Lemma 3.1. If $R < min\{1, R_0, R_1\}$ then there exists a hypergeometric function $y_h : (-R, R) \to \mathbb{C}$ such that $|y(x) - y_h(x)| = O(x)$ as $x \to 0$.*

Example 3.4. *Now, we will introduce an example concerning the hypergeometric function for differential equation (2.1) with $st = \frac{1}{16}$. Given a constant R with $0 < R < 1$ and assume that a function $y : (-R, R) \longrightarrow \mathbb{C}$ can be expressed as a power series of the form (3.1), where*

$$b_m = \{ {}^{0, \quad m=0}_{\frac{1}{4^m}, \quad m \geq 1.}$$

It is easy to see that the radius of convergence of the above power series is $R_1 = 4$. Since $b_0 = 0$ it follows from Lemma 3.1 that $c_m = b_m$ for each $m \in \{0, 1, 2, 3, ...\}$. Moreover, there exists a positive constant R_1 such that the condition (2.2) is satisfied

$$R_1 = \lim_{k \to +\infty} |\frac{c_k}{c_{k+1}}| = \lim_{k \to +\infty} |\frac{b_k}{b_{k+1}}| = 4.$$

Now we assume $r = s = t = \frac{1}{4}$. *Then we get*

$$\begin{aligned}\sum_{m=0}^{+\infty} |a_m x^m| &\leq \frac{1}{16} + \frac{15}{64}|x| + \sum_{m=2}^{+\infty} \frac{4(m+\frac{1}{4})^2 - (m+1)(m+\frac{1}{4})}{4^{m+2}}|x|^m \\ &\leq \frac{1}{16} + \frac{15}{64} + \sum_{m=2}^{+\infty} \frac{3m(m+\frac{1}{4})}{4^{m+2}} \\ &\leq \frac{1}{16} + \frac{15}{64} + \sum_{m=2}^{+\infty} \frac{1}{2^{m+2}} \leq \frac{1}{16} + \frac{15}{64} + \frac{1}{8} = \frac{27}{64}\end{aligned}$$

for all $x \in (-R, R)$. *Since* $R < min\{1, R_0, R_1\} = 1$, *we can conclude from* (3.6) *that there exists a solution function* $y_h : (-R, R) \to \mathbb{C}$ *of the Gauss differential equation* (2.1) *with* $r = s = t = \frac{1}{4}$ *satisfying* $|y(x) - y_h(x)| \leq \frac{27}{64}\frac{x}{1-x}$ *for all* $x \in (-R, R)$.

ACKNOWLEDGMENTS

This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

ON TOPOLOGICAL ROUGH GROUPS

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Abstract. In this paper, we give an introduction for rough groups and rough homomorphisms. Then we present some properties related to topological rough subgroups and rough subsets. Finally we construct the product of topological rough groups and give an illustrated example.

1. Introduction

In [2], Bagirmaz *et al.* introduced the concept of topological rough groups. They extended the notion of a topological group to include algebraic structures of rough groups. In addition, they presented some examples and properties.

The main purpose of this paper is to introduce some basic definitions and results about topological rough groups and topological rough subgroups. We also introduce the Cartesian product of topological rough groups.

This paper is as follows: Section 2 gives basic results and definitions on rough groups and rough homomorphisms. In Section 3, following results and definitions of [2], we give some more interesting and nice results about topological rough groups. Finally, in Section 4 we prove that the product of topological rough groups is a topological rough group. Further, an example is provided.

This paper has been produced from the PhD thesis of Alharbi registered in King Abdulaziz University.

2. Rough groups and rough homomorphisms

First, we give the definition of rough groups introduced by Biswas and Nanda in 1994 [3].

2010 *Mathematics Subject Classification.* Primary: 22A05, 54A05. Secondary: 03E25.

Key words and phrases. rough group; topological rough group; topological rough subgroup; product of topological rough groups.

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Let (U, R) be an approximation space. For a subset $X \subseteq U$,

$$\overline{X} = \{[x]_R : [x]_R \cap X \neq \emptyset\}$$

and

$$\underline{X} = \{[x]_R : [x]_R \subseteq X\}.$$

Suppose that $*$ is a binary operation defined on U. We will use xy instead of $x * y$ for each composition of elements $x, y \in U$ as well as for composition of subsets XY, where $X, Y \subseteq U$.

Definition 2.1. [2] *Let $G = (\underline{G}, \overline{G})$ be a rough set in the approximation space (U, R). Then $G = (\underline{G}, \overline{G})$ is called a rough group if the following conditions are satisfied:*

1. *for all $x, y \in G$, $xy \in \overline{G}$ (closed);*
2. *for all $x, y, z \in \overline{G}$, $(xy)z = x(yz)$ (associative law);*
3. *for all $x \in G$, there exists $e \in \overline{G}$ such that $xe = ex = x$ (e is the rough identity element);*
4. *for all $x \in G$, there exists $y \in G$ such that $xy = yx = e$ (y is the rough inverse element of x. It is denoted as x^{-1}).*

Definition 2.2. [2] *A nonempty rough subset $H = (\underline{H}, \overline{H})$ of a rough group $G = (\underline{G}, \overline{G})$ is called a rough subgroup if it is a rough group itself.*

A rough set $G = (\underline{G}, \overline{G})$ is a trivial rough subgroup of itself. Also the rough set $e = (\underline{e}, \overline{e})$ is a trivial rough subgroup of the rough group G if $e \in G$.

Theorem 2.1. [2] *A rough subset H is a rough subgroup of the rough group G if the two conditions are satisfied:*

1. *for all $x, y \in H, xy \in \overline{H}$;*
2. *for all $y \in H, y^{-1} \in H$.*

Also, a rough normal subgroup can be defined. Let N be a rough subgroup of the rough group G. Then N is called a rough normal subgroup of G if for all $x \in G, xN = Nx$.

Definition 2.3. [4] *Let (U_1, R_1) and (U_2, R_2) be two approximation spaces and $*, *'$ be two binary operations on U_1 and U_2, respectively. Suppose that $G_1 \subseteq U_1$, $G_2 \subseteq U_2$ are rough groups. If the mapping $\varphi : \overline{G_1} \to \overline{G_2}$ satisfies $\varphi(x * y) = \varphi(x) *' \varphi(y)$ for all $x, y \in \overline{G_1}$, then φ is called* **a rough homomorphism**.

Definition 2.4. [4] *A rough homomorphism φ from a rough group G_1 to a rough group G_2 is called:*

1. *a rough epimorphism (or surjective) if $\varphi : \overline{G_1} \to \overline{G_2}$ is onto.*
2. *a rough embedding (or monomorphism) if $\varphi : \overline{G_1} \to \overline{G_2}$ is one-to -one.*
3. *a rough isomorphism if $\varphi : \overline{G_1} \to \overline{G_2}$ is both onto and one-to-one.*

3. TOPOLOGICAL ROUGH GROUPS

We study a topological rough group, which has an ordinary topology on a rough group, i.e., a topology τ on $\overline{G}$ induced a subspace topology τ_G on G. Suppose that (U, R) is an approximation space with a binary operation $*$ on U. Let G be a rough group in U.

Definition 3.1. [2] *A topological rough group is a rough group G with a topology τ_G on $\overline{G}$ satisfying the following conditions:*

(1) *the product mapping $f : G \times G \to \overline{G}$ defined by $f(x, y) = xy$ is continuous with respect to a product topology on $G \times G$ and the topology τ on G induced by τ_G;*

(2) *the inverse mapping $\iota : G \to G$ defined by $\iota(x) = x^{-1}$ is continuous with respect to the topology τ on G induced by τ_G.*

Elements in the topological rough group G are elements in the original rough set G with ignoring elements in approximations.

Example 3.1. *Let $U = \{\overline{0}, \overline{1}, \overline{2}\}$ be any group with 3 elements. Let $U/\mathcal{R} = \{\{\overline{0}, \overline{2}\}, \{\overline{1}\}\}$ be a classification of equivalent relation. Let $G = \{\overline{1}, \overline{2}\}$. Then $\underline{G} = \{\overline{1}\}$ and $\overline{G} = \{\overline{0}, \overline{1}, \overline{2}\} = U$. A topology on $\overline{G}$ is $\tau_G = \{\varphi, \overline{G}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1}, \overline{2}\}\}$ and the relative topology is $\tau = \{\varphi, G, \{\overline{1}\}, \{\overline{2}\}\}$. The conditions in Definition 3.1 are satisfied and hence G is a topological rough group.*

Example 3.2. *Let $U = \mathbb{R}$ and $U/\mathcal{R} = \{\{x : x \geqslant 0\}, \{x : x < 0\}\}$ be a partition of $\mathbb{R}$. Consider $G = \mathbb{R}^* = \mathbb{R} - 0$. Then G is a rough group with addition. It is also a topological rough group with the standard topology on $\mathbb{R}$.*

Example 3.3. *Consider $U = S_4$ the set of all permutations of four objects. Let $(*)$ be the multiplication operation of permutations. Let*

$$U/\mathcal{R} = \{E_1, E_2, E_3, E_4\}$$

be a classification of U, where

$$E_1 = \{1, (12), (13), (14), (23), (24), (34)\},$$

$$E_2 = \{(123), (132), (142), (124), (134), (143), (234), (243)\},$$

$$E_3 = \{(1234), (1243), (1342), (1324), (1423), (1432)\},$$

$$E_4 = \{(12)(34), (13)(24), (14)(23)\}.$$

Let $G = \{(12), (123), (132)\}$. Then $\overline{G} = E_1 \cup E_2$. Clearly, G is a rough group. Consider a topology on $\overline{G}$ as $\tau_G = \{\varphi, \overline{G}, \{(12)\}, \{1, (123), (132)\}, \{1, (12), (123), (132)\}\}$. Then the relative topology on G is $\tau = \{\varphi, G, \{(12)\}, \{(123), (132)\}\}$. The conditions in Definition 3.1 are satisfied and hence G is a topological rough group.

Proposition 3.1. [2] *Let G be a topological rough group and fix $a \in G$. Then*

(1) *the mapping* $L_a : G \to \overline{G}$ *defined by* $L_a(x) = ax$, *is one-to-one and continuous for all* $x \in G$.
(2) *the mapping* $R_a : G \to \overline{G}$ *defined by* $R_a(x) = xa$, *is one-to-one and continuous for all* $x \in G$.
(3) *the inverse mapping* $\iota : G \to G$ *is a homeomorphism for all* $x \in G$.

Proposition 3.2. [2] *Let* G *be a topological rough group. Then* $G = G^{-1}$.

Proposition 3.3. [2] *Let* G *be a topological rough group and* $V \subseteq G$. *Then* V *is open (resp. closed) if and only if* V^{-1} *is open (resp. closed).*

Proposition 3.4. [2] *Let* G *be a topological rough group and* W *be an open set in* $\overline{G}$ *with* $e \in W$. *Then there exists an open set* V *with* $e \in V$ *such that* $V = V^{-1}$ *and* $VV \subseteq W$.

Proposition 3.5. [2] *Let* G *be a rough group. If* $G = \overline{G}$, *then* G *is a topological group.*

Definition 3.2. *Let* G *be a topological rough group. Then a subset* U *of* G *is called rough symmetric if* $U = U^{-1}$.

From the definition of rough subgroups, we obtain the following result.

Corollary 3.1. *Every rough subgroup of a topological rough group is rough symmetric.*

Theorem 3.1. *Let* G *be a topological rough group. Then the closure of any rough symmetric subset* A *of* G *is again rough symmetric.*

Proof. Since the inverse mapping $\iota : G \to G$ is a homeomorphism, $cl(A) = (cl(A))^{-1}$. □

Theorem 3.2. *Let* G *be a topological rough group and* H *be a rough subgroup. Then* $cl(H)$ *is a rough group in* $\overline{G}$.

Proof. (1) Identity element: $H \subseteq cl(H)$ implies that $\overline{H} \subseteq \overline{cl(H)}$ and so $e \in \overline{cl(H)}$. Since $cl(H) \subseteq G$, we have $ex = xe = x$ for all $x \in cl(H)$.
(2) Inverse element: $cl(H)^{-1} \subseteq cl(H^{-1}) = cl(H)$.
(3) Closed under product: Let $x, y \in cl(H)$. Then $xy \in \overline{G}$, which implies that there exists an open set $U \in \overline{G}$ such that $xy \in U$. We will prove that $U \wedge H \neq \varphi$. Consider the multiplication mapping $\mu : G \times G \to \overline{G}$. This implies that there exist open sets W, V of G such that $x \in W, y \in V$, $W \wedge H \neq \varphi, V \wedge H \neq \varphi$. Since the topology on G is a relative topology on $\overline{G}$, there exist open sets W', V' of $\overline{G}$ such that $W \subseteq W', V \subseteq V'$. Hence $W' \wedge H \neq \varphi, V' \wedge H \neq \varphi$. Then $\mu(W \times V) \wedge H \neq \varphi$, but we have $\mu(W \times V) \subseteq U$, which implies $H \wedge U \neq \varphi$. So $xy \in cl(H) \subseteq \overline{cl(H)}$. This implies that $cl(H)$ is a rough group of $\overline{G}$.

Thus $cl(H)$ is a rough group in $\overline{G}$. □

Definition 3.3. *Let (X, τ) be a topological rough space of approximation space $(U, \mathcal{R})$, and let $\mathcal{B} \subseteq \tau$ be a base for τ. For $x \in X$, the family*

$$\mathcal{B}_x = \{O \in \mathcal{B} : x \in O\} \subseteq \mathcal{B}$$

is called a base at x.

Theorem 3.3. *Let G be a topological rough space with $\overline{G}$ group. For $g \in \overline{G}$, the base at g is equal to*

$$\mathcal{B}_g = \{gO : O \in \mathcal{B}_e\},$$

where e is the identity element of a rough group G.

4. Product of topological rough groups

Let $(U, \mathcal{R}_1)$ and $(V, \mathcal{R}_2)$ be approximation spaces with binary operations $*_1$ and $*_2$, respectively. Consider the Cartesian product of U and V: let $x, x^{'} \in U$ and $y, y^{'} \in V$. Then $(x, y), (x^{'}, y^{'}) \in U \times V$. Define $*$ as $(x, y) * (x^{'}, y^{'}) = (x *_1 x^{'}, y *_2 y^{'})$. Then $*$ is a binary operation on $U \times V$. In [1], Alharbi *et al.* proved that the product of equivalence relations is also an equivalence relation on $U \times V$.

Theorem 4.1. [1] *Let $G_1 \subseteq U$ and $G_2 \subseteq V$ be two rough groups. Then the Cartesian product $G_1 \times G_2$ is a rough group.*

The following conditions are satisfied:

1. For all $(x, y), (x^{'}, y^{'}) \in G_1 \times G_2$, $(x_1, y_1^{'}) * (x_2, y_2^{'}) = (x_1 *_1 x_2, y_1^{'} *_2 y_2^{'}) \in \overline{G_1} \times \overline{G_2}$.
2. Associative law is satisfied over all elements in $\overline{G_1} \times \overline{G_2}$.
3. There exists an identity element $(e, e^{'}) \in \overline{G_1} \times \overline{G_2}$ such that $\forall (x, x^{'}) \in G_1 \times G_2, (x, x^{'}) \times (e, e^{'}) = (e, e^{'}) \times (x, x^{'}) = (ex, e^{'}x^{'}) = (x.x')$.
4. For all $(x, x^{'}) \in G_1 \times G_2$, there exists an element $(y, y^{'}) \in G_1 \times G_2$ such that $(x, x^{'}) * (y, y^{'}) = (y, y^{'}) * (x, x^{'}) = (e, e^{'})$.

Example 4.1. *Consider Example 3.1 where $U = \{\overline{0}, \overline{1}, \overline{2}\}$ and $U/R = \{\{\overline{0}, \overline{2}\}, \{\overline{1}\}\}$. Then the Cartesian product $U \times U$ is as follows:*

$$U \times U = \{(\overline{0}, \overline{0}), (\overline{0}, \overline{2}), (\overline{0}, \overline{1}), (\overline{2}, \overline{0}), (\overline{2}, \overline{2}), (\overline{2}, \overline{1}), (\overline{1}, \overline{0}), (\overline{1}, \overline{2}), (\overline{1}, \overline{1})\}.$$

Then the new classification is

$$\{\{(\overline{0}, \overline{0}), (\overline{0}, \overline{2}), (\overline{2}, \overline{0}), (\overline{2}, \overline{2})\}, \{(\overline{0}, \overline{1}), (\overline{2}, \overline{1})\}, \{(\overline{1}, \overline{0}), (\overline{1}, \overline{2})\}, \{(\overline{1}, \overline{1})\}\}.$$

Consider the rough group $G = \{\overline{1}, \overline{2}\}$. Then the Cartesian product $G \times G$ is

$$G \times G = \{(\overline{2}, \overline{2}), (\overline{2}, \overline{1}), (\overline{1}, \overline{2}), (\overline{1}, \overline{1})\},$$

where $\overline{G \times G} = \overline{G} \times \overline{G} = U \times U$. From the definition of a rough group, we have that

(1) *the multiplication of elements in* $G \times G$ *is closed under* $\overline{G} \times \overline{G}$, *i.e.* $(\overline{2}, \overline{2})(\overline{2}, \overline{2}) = (\overline{1}, \overline{1}), (\overline{2}, \overline{2})(\overline{2}, \overline{1}) = (\overline{1}, \overline{0}), (\overline{2}, \overline{2})(\overline{1}, \overline{1}) = (\overline{0}, \overline{0}), (\overline{2}, \overline{2})(\overline{1}, \overline{2}) = (\overline{0}, \overline{1}), (\overline{2}, \overline{1})(\overline{2}, \overline{1}) = (\overline{1}, \overline{2})$,
$(\overline{2}, \overline{1})(\overline{1}, \overline{1}) = (\overline{0}, \overline{2}), (\overline{2}, \overline{1})(\overline{1}, \overline{2}) = (\overline{0}, \overline{0}), (\overline{1}, \overline{1})(\overline{1}, \overline{1}) = (\overline{2}, \overline{2}), (\overline{1}, \overline{1})(\overline{1}, \overline{2}) = (\overline{2}, \overline{0})$;
(2) *there exists* $(\overline{0}, \overline{0}) \in \overline{G} \times \overline{G}$ *such that for every* $(g, g') \in G \times G$, *we have* $(\overline{0}, \overline{0})(g, g') = (g, g')$;
(3) *for every element of* $G \times G$, *there exists an inverse element in* $G \times G$, *where* $(\overline{1}, \overline{1})^{-1} = (\overline{2}, \overline{2}) \in G \times G$, $(\overline{2}, \overline{1})^{-1} = (\overline{1}, \overline{2}) \in G \times G$;
(4) *the associative law is satisfied.*

Hence $G \times G$ *is a rough group.*

From Example 3.1, we have $\tau_G = \{\varphi, \overline{G}, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1}, \overline{2}\}\}$ *as a topology on* $\overline{G}$. *Then* $\tau_G \times \tau_G$ *is the product topology of* $\overline{G} \times \overline{G}$. *Also we have* $\tau = \{\varphi, G, \{\overline{1}\}, \{\overline{2}\}\}$ *as a relative topology on* G. *So* $\tau \times \tau$ *is a topology on* $G \times G$ *induced by* $\tau_G \times \tau_G$.

Consider the multiplication mapping $\mu : (G \times G) \times (G \times G) \to \overline{G} \times \overline{G}$. *This mapping is continuous with respect to topology* $\tau \times \tau$ *and the product topology on* $(G \times G) \times (G \times G)$. *Also, we can show that the inverse mapping* $\iota : G \times G \to G \times G$ *is continuous. Hence* $G \times G$ *is a topological rough group.*

ACKNOWLEDGEMENT

The authors wish to thank the Deanship for Scientific Research (DSR) at King Abdulaziz University for financially funding this project under grant no. KEP-PhD-2-130-39. C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

ON THE FARTHEST POINT PROBLEM IN BANACH SPACES

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Abstract. A long standing conjecture in theory of Banach spaces is:" Every uniquely remotal set in a Banach is a singleton". This is known as the farthest point Conjecture. In an attempt to solve this problem, we give our contribution toward solving it, in the positive direction, by proving that every such subset E in the sequence space ℓ^1 is a singleton.

1. Introduction

Let X be a normed space, and E be a closed and bounded subset of X. We define the real valued function $D(.,E) : X \to \mathbb{R}$ by

$$D(x,E) = \sup\{\|x-e\| : e \in E\},$$

the farthest distance function. We say that E is remotal if for every $x \in X$, there exists $e \in E$ such that $D(x,E) = \|x-e\|$. In this case, we denote the set $\{e \in E : D(x,E) = \|x-e\|\}$ by $F(x,E)$. It is clear that $F(.,E) : X \to E$ is a multi-valued function. However, if $F(.,E) : X \to E$ is a single-valued function, then E is called uniquely remotal. In such case, we denote $F(x,E)$ by $F(x)$, if no confusion arises.

The study of remotal and uniquely remotal sets has attracted many mathematicians in the last decades, due to its connection with the geometry of Banach spaces. We refer the reader to [1], [3], [5], [6] and [8] for samples of these studies. However, uniquely remotal sets are of special interest. In fact, one of the most interesting and hitherto unsolved problems in the theory of farthest points, known as the the farthest point problem, which is stated as: If every point of a normed space X admits a unique farthest point in a given bounded subset E, then must E be a singleton ?.

This problem gained its importance when Klee [4] proved that: singletoness of uniquely remotal sets is equivalent to convexity of Chybechev sets in Hilbert spaces (which is an open problem too, in the theory of nearest points).

Since then, a considerable work has been done to answer this question, and many partial results have been obtained toward solving this problem. We refer the reader to [1], [3], [6] and [8] for some related work on uniquely remotal sets.

1991 *Mathematics Subject Classification.* Primary 46B20; Secondary 41A50; 41A65.

Key words and phrases. Uniquely remotal, Singleton, Banach space, farthest point problem.

Centers of sets have played a major role in the study of uniquely remotal sets, see [1], [2] and [3]. Recall that a center c of a subset E of a normed space X is an element $c \in X$ such that

$$D(c, E) = \inf_{x \in X} D(x, E).$$

Whether a set has a center or not is another question. However, in inner product spaces, any closed bounded set does have a center [1].

In [7] it was proved that if E is a uniquely remotal subset of a normed space, admitting a center c, and if F, restricted to the line segment $[c, F(c)]$ is continuous at c, then E is a singleton. Then recently, a generalization has been obtained in [9], where the authors proved the singletoness of uniquely remotal sets if the farthest point mapping F restricted to $[c, F(c)]$ is partially continuous at c. Furthermore, a generalization of Klee's result in [4], "If a compact subset E, with a center c, is uniquely remotal in a normed space X, then E must be a singleton", was also obtained in [9].

In this article, we prove that every uniquely remotal subset of the sequence space $\ell^1(\mathbb{R})$ is a singleton. Recall that $\ell^1(\mathbb{R}) = \{x = (x_n) : x_n \in \mathbb{R} \; and \; \sum_{n=1}^{\infty} |x_n| < \infty\}$.

2. Preliminaries

In this section, we prove the following propositions that play a key role in the proof of the main result.

Throughout the rest of the paper, F will denote the farthest distance single-valued function associated with a uniquely remotal set E.

Proposition 2.1. *Let E be a uniquely rematal subset of a Banach space X. Let (x_n) be a sequence in X such that (x_n) converges to $x \in X$. If $F(x_n) = y$ for all n, where $y \in E$, then $F(x) = y$.*

Proof. Suppose that $F(x) \neq y$. Since E is uniquely remotal, then there exists $w \in E$ such that $F(x) = w$. Further, there exists $\epsilon > 0$ such that $||x - w|| > ||x - y|| + \epsilon$. Also, there exists $n_0 \in \mathbb{N}$ such that $||x_n - x|| < \frac{\epsilon}{2}$ for all $n \geq n_0$. Therefore, for $m \geq n_0$

$$\begin{aligned} ||x_m - w|| &\geq ||x - w|| - ||x_m - x|| \\ &> ||x - y|| + \epsilon - \frac{\epsilon}{2} \\ &> ||x_m - y|| + \frac{\epsilon}{2} - ||x_m - x|| \\ &> ||x_m - y||. \end{aligned}$$

This contradicts that $y = F(x_m)$. Hence, we must have $F(x) = y$.

□

Proposition 2.2. *Let K be a compact subset of a Banach space X and E be uniquely remotal in X. Then there exist $x \in K$ and $e \in E$ such that*

$$D(E,K) = \sup\{||y-\theta|| : y \in K,\ \theta \in E\} = ||e-x||.$$

Proof. From the definition of $D(E,K)$, there exist two sequences (e_n) and (x_n) in E and K respectively such that

$$D(E,K) = \lim_{n\to\infty} ||e_n - x_n||.$$

Since K is compact, then there exists a subsequence (x_{n_k}) of (x_n) such that (x_{n_k}) converges to x in K. So,

$$D(E,K) = \lim_{k\to\infty} ||e_{n_k} - x_{n_k}||.$$

The definition of $D(E,K)$ implies that $D(E,K) \geq ||e' - x'||$ for all $e' \in E$ and $x' \in K$. Therefore,

$$\lim_{k\to\infty} ||e_{n_k} - x_{n_k}|| \geq ||x - F(x)||.$$

But

$$||e_{n_k} - x_{n_k}|| \leq ||e_{n_k} - x|| + ||x - x_{n_k}|| \leq ||x_{n_k} - x|| + ||x - F(x)||.$$

Thus

$$\lim_{k\to\infty} ||x_{n_k} - y_{n_k}|| \leq ||x - F(x)||.$$

Since $x \in K$ and $F(x) \in E$, it follows that $D(E,K) = ||x - F(x)||$, which ends the proof.

□

3. Main Results

Let E be a uniquely remotal subset of a Banach space X. Let x_0 be an element in X and $e_0 \in E$ be the unique farthest point from x_0, i.e $F(x_0) = e_0$. Consider the closed ball

$$B[x_0, ||x_0 - e_0||] = B[x_0, D(x_0, E)].$$

Then clearly e_0 lies on the boundary of $B[x_0, D(x_0,E)]$.

Let $J = \{B[y, ||y - e_0|| : F(y) = e_0\}$, and define the relation " $\leq$ " on J as follows:

$$B_1 \leq B_2 \ if\ B_2 \subseteq B_1.$$

It is easy to see that the relation " $\leq$ " is a partial order.

Now, we claim the following.

Theorem 3.1. *J has a maximal element.*

Proof. Let T be any chain in J. Consider the net $\{||y_\alpha - e_0|| : \alpha \in I\}$. Notice that if $B_{\alpha_1} \leq B_{\alpha_2}$ then $||y_{\alpha_2} - e_0|| \leq ||y_{\alpha_1} - e_0||$. Let $r = \inf_{\alpha\in I} ||y_\alpha - e_0||$. Then it is easy to see that if the infimum is attained at some α_0, then $B_{\alpha_0}[y_{\alpha_0}, ||y_{\alpha_0} - e_0||]$ is an upper bound for T. If the infimum is not attained then there exists a sequence

(B_n) in T such that $\lim\limits_{n\to\infty} ||y_n - e_0|| = \inf\limits_{\alpha\in I} ||y_\alpha - e_0|| = r$.

We claim that (y_n) has a convergent subsequence. If not, then there exists $\epsilon > 0$ such that $||y_n - y_m|| > \epsilon$ for all $n,\ m$. Clearly we can assume that $\epsilon < r$.

Since $\lim\limits_{n\to\infty} ||y_n - e_0|| = r$, then there exists $n_0 \in \mathbb{N}$ such that $||y_n - e_0|| < r + \frac{\epsilon}{2}$ for all $n \geq n_0$. But $||y_{n_0} - y_{n_0+1}|| > \epsilon$, so $B_{n_0} \subseteq B_{n_0+1}$. Farther, $r \leq ||y_{n_0} - e_0||$ and $||y_{n_0+1} - e_0|| < r + \frac{\epsilon}{2}$. Without loss of generality, we can assume, for simplicity, that $y_{n_0} = 0$. Then the element $v = (1 + \frac{r}{||y_{n_0+1}||} y_{n_0+1}) \in B_{n_0+1}$.

Now, $||v - 0|| = ||v|| = ||y_{n_0+1}|| + r > r + \epsilon$. Thus, $v \notin B_{n_0}$ which contradicts the fact that $B_{n_0+1} \subseteq B_{n_0}$. Hence, there is a subsequence (y_{n_k}) that converges to some element, say y. By assumption $F(y_{n_k}) = e_0$ for all n_k, which implies by Proposition 2.1 that $F(y) = e_0$. Thus, $B[y, ||y - e_0||] \in J$.

It suffices now to show that $B[y, ||y - e_0||] \subseteq B_\alpha$ for all $\alpha \in I$. If this is not true then there exists $w \in B[y, ||y - e_0||]$ such that $w \notin B_{m_1}$ for some m_1. Since (B_n) is a chain, then $w \notin B_{n_k}$ for all $n_k > m_1$. Furthermore, $||w - y_{n_k}|| > r + \epsilon'$ for some $\epsilon' > 0$ and all $n_k > m_1$.

But $||w - y_{n_k}|| \leq ||y_{n_k} - y|| + ||y - w||$, where $||y_{n_k} - y|| \to 0$ and $||y - w|| < ||y - F(y)|| = ||y - r||$. It follows that $\liminf\limits_{n_k} ||w - y_{n_k}|| \leq r$, which contradicts the fact that $||w - y_{n_k}|| > r + \epsilon'$. This means that $B[y, ||y - e_0||]$ is an upper bound for the chain T. Hence, By Zorn's lemma J has a maximal element.

□

Now we are ready to prove the main result of this paper.

Theorem 3.2. *Every uniquely remotal set in $\ell^1(\mathbb{R})$ is a singleton.*

Proof. Let E be a uniquely remotal set in ℓ^1, and let $\hat{e}$ be the unique farthest point in E from 0, i.e. $F(0) = \hat{e}$. By Theorem 3.1, $J = \{B[y, ||y - \hat{e}||] : F(y) = \hat{e}\}$ has a maximal element say $B[\hat{v}, ||\hat{v} - \hat{e}||]$.

Without loss of generality, we may assume that $\hat{v} = 0$ and $||\hat{e}|| = 1$ so that the maximal element is the unit ball of ℓ^1. Let $\hat{e} = (b_1, b_2, b_3, \dots)$. Since $||\hat{e}|| = 1$ then with no loss of generality we can assume that $b_1 \neq 0$. Further, assume $b_1 > 0$. So, $b_1 > \frac{1}{m_0}$ for some $m_0 \in \mathbb{N}$.

Let $\delta_1 = (1, 0, 0, \dots)$ and consider the sequence $(\frac{\delta_1}{n})$ in ℓ^1, where $n > m_0$. Then $F(\frac{\delta_1}{n}) \neq \hat{e}$ for all $n > m_0$, since if $F(\frac{\delta_1}{n}) = \hat{e}$ for some $n > m_0$, then for $w \in B[\frac{\delta_1}{n}, ||\frac{\delta_1}{n} - \hat{e}||]$, we have $||w|| - ||\frac{\delta_1}{n}|| \leq ||w - \frac{\delta_1}{n}|| \leq ||\frac{\delta_1}{n} - \hat{e}||$. But $b_1 > \frac{1}{n}$, so $||\frac{\delta_1}{n} - \hat{e}|| = ||\hat{e}|| - \frac{1}{n} = ||\hat{e}|| - ||\frac{\delta_1}{n}||$. Thus, $||w|| \leq ||\hat{e}|| = 1$ and accordingly $w \in B[0, 1]$, which contradicts the maximality of $B[0, 1]$. Hence, $F(\frac{\delta_1}{n}) \neq \hat{e}$ for all $n > m_0$.

Let $F(\frac{\delta_1}{n}) = z_n = (c_1^n, c_2^n, c_3^n, \dots)$. Then we must have $c_1^n < \frac{1}{n}$ for all $n > m_0$. Otherwise, we obtain that $||z_n - \frac{\delta_1}{n}|| = ||z_n|| - ||\frac{\delta_1}{n}|| \leq 1 - \frac{1}{n} = ||\hat{e} - \frac{\delta_1}{n}||$, which contradicts the fact that $F(\frac{\delta_1}{n}) = z_n$.

Now, since $\frac{\delta_1}{n} \to 0$, then $||z_n|| \to 1$. Further, the sequence c_1^n converges to λ, where $\lambda \leq 0$.

Consider the set $P = \{b_1\delta_1\}$. Then, clearly $D(\hat{e}, P) = \sum_{j=2}^{\infty} |b_j| < 1$. Also, $D(z_n, P) = ||z_n - b_1\delta_1|| = |c_1^n - b_1| + \sum_{j=2}^{\infty} |c_j^n|$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} D(z_n, P) &= (b_1 + |\lambda|) + \lim_{n\to\infty} \sum_{j=2}^{\infty} |c_j^n| \\ &= b_1 + |\lambda| + (1 - |\lambda|) \\ &= 1 + b_1 \end{aligned}$$

Since $D(P, E) \geq D(P, z_n)$ for all n, we get that $D(P, E) \geq 1 + b_1$. On the other hand, $D(P, E) = \sup\limits_{e\in E} ||b_1\delta_1 - e|| \leq b_1 + 1$, since $||e|| \leq 1$ for every $e \in E$. Thus,

$$D(P, E) = 1 + b_1.$$

By Proposition 2.2, $D(P, E) = ||b_1\delta_1 - e'||$ for some $e' \in E$. So,

$$1 + b_1 \leq b_1 + ||e'|| \leq 1 + b_1,$$

which implies that $||e'|| = 1$. Therefore, e' is another farthest point in E from 0, i.e. $F(0) = \{e', \hat{e}\}$, which contradicts the unique remotality of E. Hence, E must be a singleton. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On the stability of 3-Lie homomorphisms and 3-Lie derivations

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Abstract. In this paper, we prove the Hyers-Ulam stability of 3-Lie homomorphisms in 3-Lie algebras for Cauchy-Jensen functional equation. We also prove the Hyers-Ulam stability of 3-Lie derivations on 3-Lie algebras for Cauchy-Jensen functional equation.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations had been first raised by Ulam [21]. In 1941, Hyers [10] gave a first affirmative answer to the question of Ulam for Banach spaces. The generalizations of this result have been published by Aoki [2] for $(0 < p < 1)$, Rassias [19] for $(p < 0)$ and Gajda [8] for $(p > 1)$ for additive mappings and linear mappings by a general control function $\theta(\|x\|^p + \|y\|^p)$, respectively. In 1994, Găvruta [9] generalized these theorems for approximate additive mappings controlled by the unbounded Cauchy difference with regular conditions, i.e., who replaced $\theta(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Several stability problems for various functional equations have been investigated in [1, 4, 6, 7, 12, 14, 15, 16, 17, 18, 20].

A Lie algebra is a Banach algebra endowed with the Lie product

$$[x, y] := \frac{(xy - yx)}{2}.$$

Similarly, a 3-Lie algebra is a Banach algebra endowed with the product

$$\Big[[x, y], z\Big] := \frac{[x, y]z - z[x, y]}{2}.$$

Let A and B be two 3-Lie algebras. A $\mathbb{C}$-linear mapping $H : A \to B$ is called a 3-Lie homomorphism if

$$H([[x, y], z]) = [[H(x), H(y)], H(z)]$$

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[0]Keywords: Jensen functional equation, 3-Lie algebra, 3-Lie homomorphisms, 3-Lie derivation, Hyers-Ulam stability.

[0]2010 Mathematics Subject Classification. Primary 39B52; 39B82; 22D25; 17A40.

for all $x, y, z \in A$. A $\mathbb{C}$-linear mapping $D : A \to A$ is called a 3-Lie derivation if

$$D\Big([[x, y], z]\Big) = [[D(x), y], z] + [[x, D(y)], z] + [[x, y,], D(z)]$$

for all $x, y, z \in A$ (see [22]).

Throughout this paper, we suppose that A and B are two 3-Lie algebras. For convenience, we use the following abbreviation for a given mapping $f : A \to B$

$$\begin{aligned} &D_\mu f(x, y, z) := f(\frac{\mu x + \mu y}{2} + \mu z) + f(\frac{\mu x + \mu z}{2} + \mu y) + f(\frac{\mu y + \mu z}{2} + \mu x) \\ &- 2\mu f(x) - 2\mu f(y) - 2\mu f(z) \end{aligned}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and all $x, y, z \in A$.

Throughout this paper, assume that A is a 3-Lie algebra with norm $\|\cdot\|$ and that B is a 3-Lie algebra with norm $\|\cdot\|$.

2. Stability of 3-Lie homomorphisms in 3-Lie algebras

We need the following lemmas which have been given in for proving the main results.

Lemma 2.1. ([11]) *Let X be a uniquely 2-divisible abelian group and Y be linear space. A mapping $f : X \to Y$ satisfies*

$$f(\frac{x + y}{2} + z) + f(\frac{x + z}{2} + y) + f(\frac{y + z}{2} + x) = 2[f(x) + f(y) + f(z)] \tag{2.1}$$

for all $x, y, z \in X$ if and only if $f : X \to Y$ is additive.

Lemma 2.2. *Let X and Y be linear spaces and let $f : X \to Y$ be a mapping such that*

$$D_\mu f(x, y, z) = 0 \tag{2.2}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then the mapping $f : X \to Y$ is $\mathbb{C}$-linear.

Proof. By Lemma 2.2, f is additive. Letting $y = z = 0$ in (2.1), we get $2f\left(\mu\frac{x}{2}\right) = \mu f(x)$ and so $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1$. By the same reasoning as in the proof of [13, Theorem 2.1], the mapping $f : X \to Y$ is $\mathbb{C}$-linear. ☐

In the following, we investigate the Hyers-Ulam stability of (2.1).

Theorem 2.3. *Let $\varphi : A^3 \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x, y, z) := \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) < \infty \tag{2.3}$$

for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping satisfying

$$\|D_\mu f(x, y, z)\| \le \varphi(x, y, z), \tag{2.4}$$

$$\|f([[x, y], z]) - [[f(x), f(y)], f(z)]\| \le \varphi(x, y, z) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{1}{6}\widetilde{\varphi}(x, x, x) \tag{2.6}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x = y = z$ in (2.4), we get

$$\|3f(2x) - 6f(x)\| \le \varphi(x, x, x) \tag{2.7}$$

for all $x \in A$. If we replace x by $2^n x$ in (2.7) and divide both sides by $3 \cdot 2^{n+1}$. then we get

$$\|\frac{1}{2^{n+1}} f(2^{n+1}x) - \frac{1}{2^n} f(2^n x)\| \le \frac{1}{3 \cdot 2^{n+1}} \varphi(2^n x, 2^n x, 2^n x)$$

for all $x \in A$ and all nonnegative integers n. Hence

$$\begin{aligned} \|\frac{1}{2^{n+1}} f(2^{n+1}x) - \frac{1}{2^m} f(2^m x)\| &= \|\sum_{k=m}^{n} [\frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^k} f(2^k x)]\| \\ &\le \sum_{k=m}^{n} \|\frac{1}{2^{k+1}} f(2^{k+1}x) - \frac{1}{2^k} f(2^k x)\| \\ &\le \frac{1}{6} \sum_{k=m}^{n} \frac{1}{2^k} \varphi(2^k x, 2^k x, 2^k x) \end{aligned} \tag{2.8}$$

for all $x \in A$ and all nonnegative integers $n \ge m \ge 0$. It follows from (2.3) and (2.8) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence in B for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges for all $x \in A$. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.8), we get (2.6). It follows from (2.3) that

$$\begin{aligned} \|D_\mu H(x, y, z)\| &= \lim_{n\to\infty} \frac{1}{2^n} \|D_\mu f(2^n x, 2^n y, 2^n z)\| \\ &\le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0 \end{aligned}$$

for all $x, y, z \in A$ and all $\mu \in \mathbb{T}^1$. So $D_\mu H(x, y, z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By Lemma 2.2, the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.5) that

$$\begin{aligned} &\|H([[x, y], z]) - [[H(x), H(y)], H(z)]\| \\ &= \lim_{n\to\infty} \frac{1}{8^n} \|f([[2^n x, 2^n y], 2^n z]) - [[f(2^n x), f(2^n y)], f(2^n z)]\| \\ &\le \lim_{n\to\infty} \frac{1}{8^n} \varphi(2^n x, 2^n y, 2^n z) \le \lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0 \end{aligned}$$

for all $x, y, z \in A$. Thus

$$H([[x, y], z]) = [[H(x), H(y)], H(z)]$$

for all $x, y, z \in A$.

Therefore, the mapping $H : A \to B$ is a 3-Lie homomorphism. ☐

Corollary 2.4. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be positive real numbers such that $p_1, p_2, p_3 < 1$ and $q_1, q_2, q_3 < 3$. Suppose that $f : A \to B$ is a mapping such that*

$$\|D_\mu f(x, y, z)\| \le \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3}), \tag{2.9}$$

$$\|f([[x, y], z]) - [[f(y), f(z)], f(x)]\| \le \varepsilon(\|x\|^{q_1} + \|y\|^{q_2} + \|z\|^{q_3}) \tag{2.10}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{\theta}{3}\{\frac{1}{2 - 2^{p_1}}\|x\|^{p_1} + \frac{1}{2 - 2^{p_2}}\|x\|^{p_2} + \frac{1}{2 - 2^{p_3}}\|x\|^{p_3}\}$$

for all $x \in A$.

Theorem 2.5. *Let $\Phi : A^3 \to [0, \infty)$ be a function such that*

$$\sum_{n=1}^{\infty} 8^n \psi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}) < \infty \tag{2.11}$$

for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping such that

$$\|D_\mu f(x, y, z)\|_B \le \psi(x, y, z),$$

$$\|f([[x, y], z]) - [[f(x), f(y)], f(z)]\| \le \psi(x, y, z)$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\| \le \frac{1}{6}\widetilde{\psi}(x, x, x) \tag{2.12}$$

for all $x \in A$, where $\widetilde{\psi}(x, y, z) := \sum_{n=1}^{\infty} 2^n \psi(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n})$ for all $x, y, z \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.3, there exists a unique 3-Lie homomorphism $H : A \to B$ satisfying (2.12). The mapping $H : A \times A \to B$ is given by

$$H(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

The rest of the proof is similar to the proof of Theorem 2.3 ☐

Corollary 2.6. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and q_3 be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : A \to B$ is a mapping satisfying (2.9) and (2.10). Then there exists a unique 3-Lie homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\| \le \frac{\theta}{3}\{\frac{1}{2^{p_1} - 2}\|x\|^{p_1} + \frac{1}{2^{p_2} - 2}\|x\|^{p_2} + \frac{1}{2^{p_3} - 2}\|x\|^{p_3}\}$$

for all $x \in A$.

3. Stability of 3-Lie derivations on 3-Lie algebras

In this section, we prove the Hyers-Ulam stability of 3-Lie derivations on 3-Lie algebras for the functional equation $D_\mu f(x,y,z)=0$.

Theorem 3.1. *Let $\varphi : A^3 \to [0,\infty)$ be a function satisfying (2.3). Suppose that $f : A \to A$ is a mapping satisfying*

$$\|D_\mu f(x,y,z)\| \le \varphi(x,y,z),$$

$$\|f([[x,y],z]) - [[f(x),y],z] - [[x,f(y)],z] - [[x,y],f(z)]\| \le \varphi(x,y,z) \tag{3.1}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique 3-Lie derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \frac{1}{6}\widetilde{\varphi}(x,x,x) \tag{3.2}$$

for all $x \in A$, where $\widetilde{\varphi}$ is given in Theorem 2.3.

Proof. By the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (3.2) and

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. It follows from (3.1) that

$$\begin{aligned}
&\|D([[x,y],z]) - [[D(x),y],z] - [[x,D(y)],z] - [[x,y],D(z)]\| \\
&= \lim_{n\to\infty} \frac{1}{8^n}\|f([[2^n x,2^n y],2^n z]) - [[f(2^n x),2^n y],2^n z] - [[2^n x,f(2^n y)],2^n z] - [[2^n x,2^n x],f(2^n z)]\| \\
&\le \lim_{n\to\infty} \frac{1}{8^n}\varphi(2^n x,2^n y,2^n z) = 0
\end{aligned}$$

for all $x,y,z \in A$. So

$$D([[x,y],z]) = [[D(x),y],z] + [[x,G(y)],z] + [[x,y],D(z)]$$

for all $x,y,z \in A$. Therefore, the mapping $D : A \to A$ is a 3-Lie derivation. □

Corollary 3.2. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be positive real numbers such that $p_1, p_2, p_3 < 1$ and $q_1, q_2, q_3 < 3$. Suppose that $f : A \to A$ is a mapping such that*

$$\|D_\mu f(x,y,z)\| \le \theta(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3}), \tag{3.3}$$

$$\|f([[x,y],z]) - [[f(x),y],z] - [[x,f(y)],z] - [[x,y],f(z)]\| \le \varepsilon(\|x\|^{q_1} + \|y\|^{q_2} + \|z\|^{q_3}) \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique 3-Lie derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \frac{\theta}{3}\{\frac{1}{2-2^{p_1}}\|x\|^{p_1} + \frac{1}{2-2^{p_2}}\|x\|^{p_2} + \frac{1}{2-2^{p_3}}\|x\|^{p_3}\}$$

for all $x \in A$.

Theorem 3.3. *Let $\psi : A^3 \to [0,\infty)$ be a function satisfying* (2.11)*. Suppose that $f : A \to A$ is a mapping satisfying*

$$\|D_\mu f(x,y,z)\| \le \psi(x,y,z),$$

$$\|f([[x,y],z]) - [[f(x),y],z] - [[x,f(y)],z] - [[x,y],f(z)]\| \le \psi(x,y,z)$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique 3-Lie derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \frac{1}{6}\widetilde{\psi}(x,x,x) \tag{3.5}$$

for all $x \in A$, where $\widetilde{\psi}$ is given in Theorem 2.5.

Proof. By the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (3.5) and

$$D(x) := \lim_{n\to\infty} 2^n f(\frac{x}{2^n})$$

for all $x \in A$.

The rest of proof is similar to the proof Theorem 3.1. ☐

Corollary 3.4. *Let $\varepsilon, \theta, p_1, p_2, p_3, q_1, q_2$ and q_3 be non-negative real numbers such that $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : A \to B$ is a mapping satisfying (3.3) and (3.4). Then there exists a unique 3-Lie derivation $D : A \to A$ such that*

$$\|f(x) - H(x)\| \le \frac{\theta}{3}\{\frac{1}{2^{p_1}-2}\|x\|^{p_1} + \frac{1}{2^{p_2}-2}\|x\|^{p_2} + \frac{1}{2^{p_3}-2}\|x\|^{p_3}\}$$

for all $x \in A$.

Acknowledgments

This work was supported by Daejin University.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

NEUTROSOPHIC EXTENDED TRIPLET GROUPS AND HOMOMORPHISMS IN C^*-ALGEBRAS

JUNG RYE LEE, CHOONKIL PARK*, AND XIAOHONG ZHANG

Abstract. Çelik, Shalla and Olgun [2] defined neutro-homomorphisms in neutrosophic extended triplet groups and Zhang et al. [8] investigated neutro-homomorphisms in neutrosophic extended triplet groups.

In this note, we apply the results on neutro-homomorphisms in neutrosophic extended triplet groups to investigate C^*-algebra homomorphisms in unital C^*-algebras.

1. Introduction and preliminaries

As an extension of fuzzy sets and intuitionistic fuzzy sets, Smarandache [4] proposed the new concept of neutrosophic sets.

Definition 1.1. ([5, 6]) Let N be a nonempty set together with a binary operation $*$. Then N is called a neutrosophic extended triplet set if, for any $a \in N$, there exist a neutral of a (denoted by neut(a)) and an opposite of a (denoted by anti(a)) such that neut(a) $\in N$, anti(a) $\in N$ and

$$\begin{aligned} a * \mathrm{neut}(a) &= \mathrm{neut}(a) * a = a, \\ a * \mathrm{anti}(a) &= \mathrm{anti}(a) * a = \mathrm{neut}(a). \end{aligned}$$

The triplet (a, neut(a), anti(a)) is called a neutrosophic extended triplet.

Note that, for a neutrosophic triplet set $(N, *)$ and $a \in N$, neut(a) and anti(a) may not be unique.

Definition 1.2. ([5, 6]) Let $(N, *)$ be a neutrosophic extended triplet set.

Then N is called a neutrosophic extended triplet group (NETG) if the following conditions hold:

(1) $(N, *)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$;

(2) $(N, *)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

N is called a commutative neutrosophic extended triplet group if, for all $a, b \in N$, $a * b = b * a$.

Let A be a unital C^*-algebra with multiplication operation $\bullet$, unit e and unitary group $U(A) := \{u \in A \mid u^* \bullet u = u \bullet u^* = e\}$. Then $u \bullet v \in U(A)$ and $(u \bullet v) \bullet w = u \bullet (v \bullet w)$ for all $u, v, w \in U(A)$ (see [3]). So $(U(A), \bullet)$ is an NETG.

2010 *Mathematics Subject Classification.* Primary 46L05, 03E72, 94D05.

Key words and phrases. neutro-homomorphism; neutrosophic extended triplet group; homomorphism in unital C^*-algebra; perfect neutrosophic extended triplet group.

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Proposition 1.3. ([7]) *Let* $(N, *)$ *be an NETG. Then*
(1) *neut(a) is unique for each* $a \in N$*;*
(2) *neut(a)* $*$ *neut(a)= neut(a) for each* $a \in N$*.*

Note that $u \bullet e = e \bullet u = u$ for any $u \in (U(A), \bullet)$. By Proposition 1.3, $\text{neut}(u) = e$ for each $u \in U(A)$.

Definition 1.4. ([7]) Let $(N, *)$ be an NETG. Then N is called a weak commutative neutrosophic extended triplet group (briefly, WCNETG) if $a * \text{neut}(b) = \text{neut}(b) * a$ for all $a, b \in N$.

Since $\text{neut}(v) = e$ for all $v \in U(A)$, $u \bullet \text{neut}(v) = \text{neut}(v) \bullet u$ for all $u, v \in U(A)$. So $(U(A), \bullet)$ is a WCNETG.

2. Neutrosophic extended triplet groups and C^*-algebra homomorphisms in unital C^*-algebras

Definition 2.1. ([8]) Let $(N, *)$ be a WCNETG. Then N is called a perfect NETG if anti(neut(a))= neut(a) for all $a \in N$.

Since $\text{anti}(e) = e$ and $\text{neut}(u) = e$ for all $u \in U(A)$, $\text{anti}(\text{neut}(u)) = \text{neut}(u) = e$ for all $u \in U(A)$. Thus $(U(A), \bullet)$ is a perfect NETG.

Definition 2.2. ([1, 2]) Let $(N_1, *)$ and $(N_2, *)$ be neutrosophic extended triplet groups. A mapping $f : N_1 \to N_2$ is called a neutro-homomorphism if

$$f(x * y) = f(x) * f(y)$$

for all $x, y \in N_1$.

From now on, assume that A is a unital C^*-algebra with multiplication operation $\bullet$, unit e and unitary group $U(A)$ and that B is a unital C^*-algebra with multiplication operation $\bullet$ and unitary group $U(B)$.

Definition 2.3. Let $(U(A), \bullet)$ and $(U(B), \bullet)$ be unitary groups of unital C^*-algebras A and B, respectively. A mapping $h : U(A) \to U(B)$ is called a neutro-$*$-homomorphism if

$$\begin{aligned} h(u \bullet v) &= h(u) \bullet h(v), \\ h(u^*) &= h(u)^* \end{aligned}$$

for all $u, v \in U(A)$.

Theorem 2.4. *Let* A *and* B *be unital* C^**-algebras. Let* $H : A \to B$ *be a* $\mathbb{C}$*-linear mapping and let* $h : (U(A), \bullet) \to (U(B), \bullet)$ *be a neutro-*$*$*-homomorphism. If* $H|_{U(A)} = h$*, then* $H : A \to B$ *is a* C^**-algebra homomorphism.*

Proof. Since $H : A \to B$ is $\mathbb{C}$-linear and $x, y \in A$ are finite linear combinations of unitary elements (see [3]), i.e., $x = \sum_{j=1}^m \lambda_j u_j,\ y = \sum_{i=1}^n \mu_i v_i$ $(\lambda_j, \mu_i \in \mathbb{C},\ u_j, v_i \in U(A))$,

$$\begin{aligned} H(x \bullet y) &= H((\sum_{j=1}^m \lambda_j u_j) \bullet (\sum_{i=1}^n \mu_i v_i)) = H(\sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i (u_j \bullet v_i)) \\ &= \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i H(u_j \bullet v_i) = \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i h(u_j \bullet v_i) \\ &= \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i h(u_j) \bullet h(v_i) = \sum_{j=1}^m \sum_{i=1}^n \lambda_j \mu_i H(u_j) \bullet H(v_i) \\ &= H(\sum_{j=1}^m \lambda_j u_j) \bullet H(\sum_{i=1}^n \mu_i v_i) = H(x) \bullet H(y) \end{aligned}$$

for all $x, y \in A$.

Since $H : A \to B$ is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [3]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ $(\lambda_j \in \mathbb{C},\ u_j \in U(A))$,

$$\begin{aligned} H(x^*) &= H((\sum_{j=1}^m \lambda_j u_j)^*) = H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} h(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} h(u_j)^* \\ &= \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* = H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in A$. Thus the $\mathbb{C}$-linear mapping $H : A \to B$ is a C^*-algebra homomorphism. □

3. Conclusions

In this note, we have studied unitary groups of unital C^*-algbras as neutrosophic extended triplet groups and have extended neutro-homomorphisms in neutrosophic extended triplet groups to neutro-$*$-homomorphisms in unitary groups of unital C^*-algebras. We have obtained C^*-algebra homomorphisms in unital C^*-algebras by using neutro-$*$-homomorphisms in unitary groups of unital C^*-algebras.

Acknowledgments

This work was supported by Daejin University.

Competing interests

The authors declare that they have no competing interests.

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Orthogonal stability of a quadratic functional inequality: a fixed point approach

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Abstract. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping from an orthogonality space $(\mathcal{X}, \perp)$ into a real Banach space $(\mathcal{Y}, \|\cdot\|)$. Using fixed point method, we prove the Hyers-Ulam stability of the orthogonally quadratic functional inequality

$$\begin{aligned}\Big\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) \\ + f\Big(\frac{z-x-y}{2}\Big) - f(x) - f(y) \Big\| \quad \leq \quad \Big\| f(z) \Big\| \qquad (0.1)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y$, $x \perp z$ and $y \perp z$.

Keywords: Hyers-Ulam stability; quadratic functional equation; fixed point method; quadratic functional inequality; orthogonality space.

1. Introduction and preliminaries

Studying functional equations by focusing on their approximate and exact solutions conduces to one of the most substantial significant study brunches in functional equations, what we call *"the theory of stability of functional equations"*. This theory specifically analyzes relationships between approximate and exact solutions of functional equations. Actually a functional equation is considered to be *stable* if one can find an exact solution for any approximate solution of that certain functional equation. Another related and close term in this area is *superstability*, which has a similar nature and concept to the stability problem. As a matter of fact, superstability for a given functional equation occurs when any approximate solution is an exact solution too. In such this situation the functional equation is called *superstable*.

In 1940, the most preliminary form of stability problems was proposed by Ulam [58]. He gave a talk and asked the following: "when and under what conditions does an exact solution of a functional equation near an approximately solution of that exist?"

In 1941, this question that today is considered as the source of the stability theory, was formulated and solved by Hyers [26] for the Cauchy's functional equation in Banach spaces. Then the result of Hyers was generalized by Aoki [1] for additive mappings and by Rassias [47] for linear mappings by considering the unbounded Cauchy difference $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p), (\varepsilon >$

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[0]2010 Mathematics Subject Classification: 39B55, 39B52, 47H10.
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$0, p \in [0,1)$). In 1994, Găvruţa [23] provided a further generalization of Rassias' theorem in which he replaced the unbounded Cauchy difference by the general control function $\varphi(x,y)$ for the existence of a unique linear mapping. The first author treating the stability of the quadratic functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ was Skof [55] by proving that if f is a mapping from a normed space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ satisfying $\big\|f(x+y)+f(x-y)-2f(x)-2f(y)\big\| \le \varepsilon$, for some $\varepsilon > 0$, then there is a unique quadratic mapping $g : \mathcal{X} \to \mathcal{Y}$ such that $\big\|f(x)-g(x)\big\| \le \frac{\varepsilon}{2}$. Cholewa [13] extended the Skof's theorem by replacing $\mathcal{X}$ by an abelian group $\mathcal{G}$. The Skof's result was later generalized by Czerwik [14] in the spirit of Ulam-Hyers-Rassias. For more epochal information and various aspects about the stability of functional equations theory, we refer the reader to the monographs ([6, 11, 12, 15, 16, 20, 27, 30, 41, 42, 43, 46], [48]–[51], [54]), which also include many interesting results concerning the stability of different functional equations in many various spaces.

Assume that $(\mathcal{X}, \langle\cdot,\cdot\rangle)$ is a real inner product space with the usual Hilbert norm $\|\cdot\| = \langle\cdot,\cdot\rangle^{\frac{1}{2}}$. Moreover, consider the orthogonal Cauchy functional equation

$$f(x+y)=f(x)+f(y), \qquad x\perp y$$

in which $\perp$ is an abstract orthogonality relation. By the Pythagorean theorem, $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x)=\|x\|^2=\langle x,x\rangle$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus orthogonally Cauchy functional equation is not equivalent to the classic Cauchy equation on the whole inner product space $(\mathcal{X}, \langle\cdot,\cdot\rangle)$.

Pinsker [44] characterized orthogonally additive functionals on an inner product space when the orthogonality is the ordinary one in such spaces. Sundaresan [56] generalized this result to arbitrary Banach spaces equipped with the Birkhoff-James orthogonality. The orthogonal Cauchy functional equation was first investigated by Gudder and Strawther [25]. They defined $\perp$ by a system consisting of five axioms and described the general semi-continuous real-valued solution of conditional Cauchy functional equation. In 1985, Rätz [52] introduced his new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Furthermore, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [53] investigated the problem in a rather more general framework.

We now recall the concept of orthogonality space in the sense of Rätz [52], and then proceed it to prove our results for the orthogonally functional inequality (0.1).

Definition 1.1. Suppose $\mathcal{X}$ is a real vector space with dim $\mathcal{X} \ge 2$ and $\perp$ is a binary relation on $\mathcal{X}$ with the following properties:

($\mathcal{O}_1$) totality of $\perp$ for zero: $x\perp 0$, $0\perp x$ for all $x \in \mathcal{X}$;

($\mathcal{O}_2$) independence: if $x, y \in \mathcal{X} - 0$, $x\perp y$, then x, y are linearly independent;

($\mathcal{O}_3$) homogeneity: if $x, y \in \mathcal{X}$, $x\perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

($\mathcal{O}_4$) the Thalesian property: if $\mathcal{P}$ is a 2-dimensional subspace of $\mathcal{X}$, $x \in \mathcal{P}$ and $\lambda \in \mathbb{R}_+$, which is the set of nonnegative real numbers, then there exists $y_0 \in \mathcal{P}$ such that $x \perp y_0$ and $x + y_0 \perp \lambda x - y_0$.

The pair $(\mathcal{X}, \perp)$ is called an orthogonality space and it becomes an orthogonality normed space when the orthogonality space equipped with a normed structure.

Some interesting examples are

(i) The trivial orthogonality on a vector space $\mathcal{X}$ defined by $(\mathcal{O}_1)$, and for non-zero elements $x, y \in \mathcal{X}$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ given by $x \perp y$ if and only if $\langle x, y \rangle = 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(\mathcal{X}, \| \cdot \|)$ defined by $x \perp y$ if and only if $\big\| x + \lambda y \big\| \geq \big\| x \big\|$ for all $\lambda \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in \mathcal{X}$. Clearly examples (i) and (ii) are symmetric but example (iii) is not. It is remarkable to note, however, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric. There are several orthogonality notions on a real normed space such as Birkhoff-James, Boussouis, Singer, Carlsson, unitary-Boussouis, Roberts, Phythagorean, isosceles and Diminnie (see [3]–[5], [10, 18, 29]).

Ger and Sikorska [24] investigated the orthogonal stability of the Cauchy functional equation $f(x + y) = f(x) + f(y)$, namely, they showed that if f is a mapping from an orthogonality space $\mathcal{X}$ into a real Banach space $\mathcal{Y}$ and $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$, for all $x, y \in \mathcal{X}$ with $x \perp y$ and some $\varepsilon > 0$, then there exists exactly one orthogonally additive mapping $g : \mathcal{X} \to \mathcal{Y}$ such that $\|f(x) - g(x)\| \leq \frac{16}{3}\varepsilon$, for all $x \in \mathcal{X}$.

Consider the classic quadratic functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ on the real inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$. Then the important parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + \|y\|^2$$

which holds entirely in a square norm on an inner product space, shows that $f : \mathcal{X} \to \mathbb{R}$ defined by $f(x) = \|x\|^2 = \langle x, x \rangle$, is a solution for the quadratic functional equation on the whole inner product space $\mathcal{X}$, (particularly in where $x \perp y$).

The orthogonally quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \quad x \perp y$$

was first investigated by Vajzović [59] when $\mathcal{X}$ is a Hilbert space, $\mathcal{Y}$ is the scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, Drljević [19], Fochi [22], Moslehian [34, 35] and Szabó [57] generalized the Vajzović's results. See also [36, 37, 40].

The following quadratic 3-variables functional equation

$$\begin{aligned} f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) + f\Big(\frac{z-x-y}{2}\Big) \\ = f(x) + f(y) + f(z) \end{aligned} \tag{1.1}$$

has been introduced and solved by S. Farhadabadi, J. Lee and C. Park on vector spaces in [21]. It has been also shown that the functional equation (1.1) is equivalent to the classic quadratic functional equation in vector spaces. In any inner product space $(\mathcal{X}, \langle\cdot,\cdot\rangle)$, it is easy to verify that

$$\left\langle \frac{x+y+z}{2}, \frac{x+y+z}{2} \right\rangle + \left\langle \frac{x-y-z}{2}, \frac{x-y-z}{2} \right\rangle + \left\langle \frac{y-x-z}{2}, \frac{y-x-z}{2} \right\rangle + \left\langle \frac{z-x-y}{2}, \frac{z-x-y}{2} \right\rangle = \langle x,x\rangle + \langle y,y\rangle + \langle z,z\rangle$$

for all $x,y,z \in \mathcal{X}$. For this obvious reason, similar to the classic quadratic functional equation, the mapping $f(x) = \langle x,x\rangle$ can also be a solution for the 3-variables equation (1.1) on the whole inner product space $\mathcal{X}$ (particularly, for the case $x \perp y, y \perp z$ and $x \perp z$).

Fixed point theory has a basic role in applications of many considerable branches in mathematics specially in stability problems. In 1996, Isac and Rassias [28] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. In view of the fact that, we will use methods related to fixed point theory, we give briefly some useful information, a definition and a fundamental result in fixed point theory.

Definition 1.2. Let $\mathcal{X}$ be a set. A function $d : \mathcal{X} \times \mathcal{X} \to [0,\infty]$ is called a *generalized metric* on $\mathcal{X}$ if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y \in \mathcal{X}$;

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in \mathcal{X}$.

Theorem 1.3. ([7, 17]) *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty, \quad \forall n \geq n_0$;

(2) the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of $\mathcal{J}$;

(3) y^* is the unique fixed point of $\mathcal{J}$ in the set $\mathcal{Y} = \{y \in \mathcal{X} \mid d(\mathcal{J}^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, \mathcal{J}y)$ *for all* $y \in \mathcal{Y}$.

In 2003, Cădariu and Radu [7, 8, 45] exerted the above definition and fixed point theorem to prove some stability problems for the Jensen and Cauchy functional equations. During the last decade, by applying fixed point methods, stability problems of several functional equations have been extensively investigated by a number of authors (see [2, 8, 9, 31, 33, 38, 39, 45]).

Throughout this paper, $(\mathcal{X}, \perp)$ is an orthogonality space and $(\mathcal{Y}, \|\cdot\|)$ is a real Banach space.

2. Solution and Hyers-Ulam stability of the functional inequality (0.1)

In this section, we first solve the orthogonally quadratic functional inequality (0.1) by proving an orthogonal superstability proposition, and then we prove its Hyers-Ulam stability in orthogonality spaces.

Definition 2.1. A mapping $f : \mathcal{X} \to \mathcal{Y}$ is called an (exact) *orthogonally quadratic mapping* if

$$f(x+y) + f(x-y) = f(x) + f(y) \tag{2.1}$$

for all $x, y \in \mathcal{X}$, with $x \perp y$. And it is called an *approximate orthogonally quadratic mapping* if

$$\begin{aligned}\Big\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) \\ + f\Big(\frac{z-x-y}{2}\Big) - f(x) - f(y) \Big\| \quad &\le \quad \Big\| f(z) \Big\|\end{aligned} \tag{2.2}$$

for all $x, y, z \in \mathcal{X}$ with $x \perp y, y \perp z$ and $x \perp z$.

Proposition 2.2. *Each approximate orthogonally quadratic mapping in the form of* (2.2) *is also an* (*exact*) *orthogonally quadratic mapping satisfying* (2.1).

Proof. Assume that $f : \mathcal{X} \to \mathcal{Y}$ is an approximate orthogonally quadratic mapping satisfying (2.2).

Since $0 \perp 0$, letting $x = y = z = 0$ in (2.2), we have

$$\|2f(0)\| \le \|f(0)\| = 0$$

and so $f(0) = 0$.

Since $(x+y) \perp 0$ for all $x, y \in \mathcal{X}$, replacing x, y and z by $x+y, 0$ and 0 in (2.2), respectively, we conclude that

$$\Big\| 2f\Big(\frac{x+y}{2}\Big) + 2f\Big(\frac{-x-y}{2}\Big) - f(x+y) \Big\| \le \|f(0)\| = 0,$$

which implies

$$f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{-x-y}{2}\Big) = \frac{1}{2} f(x+y) \tag{2.3}$$

for all $x, y \in \mathcal{X}$ (particularly, with $x \perp y$).

Replacing y by $-y$ in the above equality, we get

$$f\Big(\frac{x-y}{2}\Big) + f\Big(\frac{y-x}{2}\Big) = \frac{1}{2} f(x-y) \tag{2.4}$$

for all $x, y \in \mathcal{X}$ (particularly, with $x \perp y$).

Since $x \perp 0$ for all $x \in \mathcal{X}$, letting $z = 0$ in (2.2), we obtain

$$\begin{aligned}\Big\| f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{x-y}{2}\Big) + f\Big(\frac{y-x}{2}\Big) + f\Big(\frac{-x-y}{2}\Big) \\ - f(x) - f(y) \Big\| \quad &\le \quad \|f(0)\| = 0\end{aligned}$$

and so

$$f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{-x-y}{2}\Big) + f\Big(\frac{x-y}{2}\Big) + f\Big(\frac{y-x}{2}\Big) = f(x) + f(y) \tag{2.5}$$

for all $x, y \in \mathcal{X}$ with $x \perp y$.

It follows from (2.3), (2.4) and (2.5) that

$$\frac{1}{2} f(x+y) + \frac{1}{2} f(x-y) = f(x) + f(y)$$

for all $x, y \in \mathcal{X}$ with $x \perp y$, which is the equation (2.1). Hence $f : \mathcal{X} \to \mathcal{Y}$ is an (exact) orthogonally quadratic mapping. □

Theorem 2.3. *Let $\varphi : \mathcal{X}^3 \to [0,\infty)$ be a function such that $\varphi(0,0,0) = 0$ and there exists an $\alpha < 1$ with*

$$\varphi(x,y,z) \le 4\alpha\varphi\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right) \tag{2.6}$$

for all $x,y,z \in \mathcal{X}$, with $x\perp y$, $y\perp z$ and $x\perp z$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying

$$\begin{aligned}\Bigg\| f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big)\\ + f\Big(\frac{z-x-y}{2}\Big) - f(x) - f(y)\Bigg\| \le \Big\|f(z)\Big\| + \varphi(x,y,z)\end{aligned} \tag{2.7}$$

for all $x,y,z \in \mathcal{X}$, with $x\perp y$, $y\perp z$ and $x\perp z$. Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\| \le \frac{\alpha}{1-\alpha}\varphi(x,0,0) \tag{2.8}$$

for all $x \in \mathcal{X}$.

Proof. Consider the set $\mathcal{S} := \big\{h : \mathcal{X} \to \mathcal{Y}\big\}$ and introduce the generalized metric on $\mathcal{S}$:

$$d(g,h) = \inf\Big\{\mu \in \mathbb{R}_+ : \big\|g(x) - h(x)\big\| \le \mu\varphi(x,0,0), \quad \forall x \in \mathcal{X}\Big\},$$

where, as usual, $\inf\emptyset = +\infty$. It is easy to show that $(\mathcal{S}, d)$ is complete (see [32]).

Now we consider the linear mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ such that

$$\mathcal{J}g(x) := \frac{1}{4}g(2x)$$

for all $g \in \mathcal{S}$ and all $x \in \mathcal{X}$.

Since $0\perp 0$, letting $x = y = z = 0$ in (2.7), we have

$$2\big\|f(0)\big\| \le \big\|f(0)\big\| + \varphi(0,0,0).$$

So $f(0) = 0$.

Since $x\perp 0$ for all $x \in \mathcal{X}$, letting $y = z = 0$ in (2.7), we get $\Big\|4f\left(\frac{x}{2}\right) - f(x)\Big\| \le \varphi(x,0,0)$ for all $x \in \mathcal{X}$. Dividing both sides by 4, putting $2x$ instead of x and then using (2.6), we obtain

$$\left\|\frac{1}{4}f(2x) - f(x)\right\| \le \frac{1}{4}\varphi(2x,0,0) \le \alpha\varphi(x,0,0)$$

for all $x \in \mathcal{X}$, which clearly yields

$$d(\mathcal{J}f, f) \le \alpha. \tag{2.9}$$

Let $g,h \in \mathcal{S}$ be given such that $d(g,h) = \varepsilon$. Then $\big\|g(x) - h(x)\big\| \le \varepsilon\varphi(x,0,0)$ for all $x \in \mathcal{X}$. Hence the definition of $\mathcal{J}g$ and (2.6), result that

$$\big\|\mathcal{J}g(x) - \mathcal{J}h(x)\big\| = \left\|\frac{1}{4}g(2x) - \frac{1}{4}h(2x)\right\| \le \frac{1}{4}\varepsilon\varphi(2x,0,0) \le \alpha\varepsilon\varphi(x,0,0)$$

for all $x \in \mathcal{X}$, which implies that $d(\mathcal{J}g, \mathcal{J}h) \le \alpha\varepsilon = \alpha d(g,h)$ for all $g,h \in \mathcal{S}$.

Thus $\mathcal{J}$ is a strictly contractive mapping with Lipschitz constant $\alpha < 1$.

According to Theorem 1.3, there exists a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

(1) $\mathcal{Q}$ is a fixed point of $\mathcal{J}$, i.e., $\mathcal{J}\mathcal{Q} = \mathcal{Q}$, and so

$$\frac{1}{4}\mathcal{Q}(2x) = \mathcal{Q}(x) \tag{2.10}$$

for all $x \in \mathcal{X}$. The mapping $\mathcal{Q}$ is a unique fixed point of $\mathcal{J}$ in the set

$$\mathcal{M} = \{g \in \mathcal{S} : d(g, f) < \infty\}.$$

This signifies that $\mathcal{Q}$ is a unique mapping satisfying (2.10) such that there exists a $\mu \in (0, \infty)$ satisfying

$$\|f(x) - \mathcal{Q}(x)\| \le \mu\varphi(x, 0, 0)$$

for all $x \in \mathcal{X}$;

(2) $d(\mathcal{J}^n f, \mathcal{Q}) \to 0$ as $n \to \infty$. So, we conclude that

$$\lim_{n\to\infty} \frac{1}{4^n} f(2^n x) = \mathcal{Q}(x) \tag{2.11}$$

for all $x \in \mathcal{X}$;

(3) $d(f, \mathcal{Q}) \le \frac{1}{1-\alpha} d(f, \mathcal{J}f)$, which gives by (2.9) the inequality

$$d(f, \mathcal{Q}) \le \frac{\alpha}{1-\alpha}.$$

This proves that the inequality (2.8) holds.

To end the proof we show that $\mathcal{Q}$ is an orthogonally quadratic mapping.

By (2.11), (2.7), (2.6) and the fact that $\alpha < 1$,

$$\begin{aligned}
&\Big\|\mathcal{Q}\Big(\frac{x+y+z}{2}\Big) + \mathcal{Q}\Big(\frac{x-y-z}{2}\Big) + \mathcal{Q}\Big(\frac{y-x-z}{2}\Big) + \mathcal{Q}\Big(\frac{z-x-y}{2}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \mathcal{Q}(x) - \mathcal{Q}(y)\Big\| \\
&= \lim_{n\to\infty} \frac{1}{4^n}\Big\| f\big(2^{n-1}(x+y+z)\big) + f\big(2^{n-1}(x-y-z)\big) \\
&\qquad + f\big(2^{n-1}(y-z-x)\big) + f\big(2^{n-1}(z-x-y)\big) - f\big(2^n x\big) - f\big(2^n y\big)\Big\| \\
&\le \Big\| \lim_{n\to\infty} \frac{1}{4^n} f\big(2^n z\big)\Big\| + \lim_{n\to\infty} \frac{1}{4^n}\varphi\big(2^n x, 2^n y, 2^n z\big) \\
&\le \|\mathcal{Q}(z)\| + \lim_{n\to\infty} \alpha^n \varphi(x, y, z) \\
&= \|\mathcal{Q}(z)\|
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $x \perp z$ and $y \perp z$. And, now applying Proposition 2.2, we obatin that $\mathcal{Q}$ is an orthogonally quadratic mapping and the proof is complete. □

Theorem 2.4. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function such that $\varphi(0, 0, 0) = 0$ and there exists an $\alpha < 1$ with*

$$\varphi(x, y, z) \le \frac{\alpha}{4}\varphi\big(2x, 2y, 2z\big)$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y, y \perp z$ and $x \perp z$. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying (2.7). Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - \mathcal{Q}(x)\| \le \frac{1}{1-\alpha}\varphi(x, 0, 0) \tag{2.12}$$

for all $x \in \mathcal{X}$.

Proof. Let $(\mathcal{S}, d)$ be the generalized metric space defined in the proof of Theorem 2.3.

Now we consider the linear mapping $\mathcal{J} : \mathcal{S} \to \mathcal{S}$ such that

$$\mathcal{J}g(x) := 4g\Big(\frac{x}{2}\Big)$$

for all $x \in \mathcal{X}$.

Similar to the proof of Theorem 2.3, from (2.7) one can get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi(x, 0, 0)$$

for all $x \in \mathcal{X}$, which means $d(f, \mathcal{J}f) \leq 1$.

We can also show that $\mathcal{J}$ is a strictly contractive mapping with Lipschitz constant $\alpha < 1$. So by applying Theorem 1.3 again, we have

$$d(f, \mathcal{Q}) \leq \frac{1}{1-\alpha} d(f, \mathcal{J}f) \leq \frac{1}{1-\alpha}$$

which implies that the inequality (2.12) holds.

The rest of the proof is similar to the proof of the previous theorem. □

Corollary 2.5. *Let $\mathcal{X}$ be a normed orthogonality space. Let δ be a nonnegative real number and $p \neq 2$ be a positive real number. Let $f : \mathcal{X} \to \mathcal{Y}$ be an even mapping satisfying*

$$\begin{aligned}\Big\|f\Big(\frac{x+y+z}{2}\Big) + f\Big(\frac{x-y-z}{2}\Big) + f\Big(\frac{y-x-z}{2}\Big) + f\Big(\frac{z-x-y}{2}\Big) - f(x) - f(y)\Big\| \\ \leq \Big\|f(z)\Big\| + \delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$, with $x \perp y$, $y \perp z$ and $x \perp z$. Then there exists a unique orthogonally quadratic mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - \mathcal{Q}(x)\big\| \leq \frac{2^p}{|2^p - 4|}\delta\|x\|^p$$

for all $x \in \mathcal{X}$.

Proof. Define $\varphi(x, y, z) := \delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$ for all $x, y, z \in \mathcal{X}$.

First assume that $0 < p < 2$.

Take $\alpha := 2^{p-2}$. Since $p < 2$, obviously $\alpha < 1$. Hence there exists an $\alpha < 1$ such that

$$\begin{aligned}\varphi(x, y, z) &= \delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big) \\ &= 4\alpha 2^{-p}\delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big) \\ &= 4\alpha\delta\Big(\Big\|\frac{x}{2}\Big\|^p + \Big\|\frac{y}{2}\Big\|^p + \Big\|\frac{z}{2}\Big\|^p\Big) \\ &= 4\alpha\varphi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ (particularly, with $x \perp y, y \perp z$ and $x \perp z$). The recent term allows to use Theorem 2.3. So by applying Theorem 2.3, it follows from (2.8) that

$$\big\|f(x) - \mathcal{Q}(x)\big\| \leq \frac{2^p}{4 - 2^p}\delta\|x\|^p$$

for all $x \in \mathcal{X}$.

For the case $p > 2$, taking $\alpha := 2^{2-p}$, and then applying Theorem 2.4, we similarly obtain the desired result. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

INTEGRAL INEQUALITIES FOR ASYMMETRIZED SYNCHRONOUS FUNCTIONS

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Abstract. In this paper we establish some integral inequalities for the product of asymmetrized synchronous/asynchronous functions. Some examples for integrals of monotonic functions, including power, logarithmic and sin functions are also provided.

1. Introduction

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{1.1}$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],t\in[a,b].$$

It is obvious that for any function f we have $\breve{f}+\tilde{f}=f$.

If f is convex on $[a,b]$, then for any $t_1,\ t_2\in[a,b]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\breve{f}(\alpha t_1+\beta t_2)&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(a+b-\alpha t_1-\beta t_2)\right]\\&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(\alpha(a+b-t_1)+\beta(a+b-t_2))\right]\\&\leq\frac{1}{2}\left[\alpha f(t_1)+\beta f(t_2)+\alpha f(a+b-t_1)+\beta f(a+b-t_2)\right]\\&=\frac{1}{2}\alpha\left[f(t_1)+f(a+b-t_1)\right]+\frac{1}{2}\beta\left[f(t_2)+f(a+b-t_2)\right]\\&=\alpha\breve{f}(t_1)+\beta\breve{f}(t_2),\end{aligned}$$

which shows that $\breve{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=t^3$. We have [6]

$$\breve{f}_0(t):=\frac{1}{2}\left[t^3+(a+b-t)^3\right]=\frac{3}{2}(a+b)t^2-\frac{3}{2}(a+b)^2t+\frac{1}{2}(a+b)^3$$

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Monotonic functions, Synchronous functions, Čebyšev's inequality, Integral inequalities.

for any $t \in \mathbb{R}$.

Since the second derivative $\left(\breve{f}_0\right)''(t) = 3(a+b)$, $t \in \mathbb{R}$, then $\breve{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2} > 0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2} < 0$. Therefore if $a < 0 < b$ with $\frac{a+b}{2} > 0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\breve{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity [6], see also [9] for an equivalent definition.

Definition 1. *We say that the function* $f : [a,b] \to \mathbb{R}$ *is symmetrized convex (concave) on the interval* $[a,b]$ *if the symmetrical transform* $\breve{f}$ *is convex (concave) on* $[a,b]$.

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the closed convex cone of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a,b] \subsetneq SCon[a,b]. \tag{1.2}$$

Also, if $[c,d] \subset [a,b]$ and $f \in SCon[a,b]$, then this does not imply in general that $f \in SCon[c,d]$.

We have the following result [6], [9] :

Theorem 1. *Assume that* $f : [a,b] \to \mathbb{R}$ *is symmetrized convex and integrable on the interval* $[a,b]$. *Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

We also have [6]:

Theorem 2. *Assume that* $f : [a,b] \to \mathbb{R}$ *is symmetrized convex on the interval* $[a,b]$. *Then for any* $x \in [a,b]$ *we have the bounds*

$$f\left(\frac{a+b}{2}\right) \le \breve{f}(x) \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

For a monograph on Hermite-Hadamard type inequalities see [8].

In a similar way, we can introduce the following concept as well:

Definition 2. *We say that the function* $f : [a,b] \to \mathbb{R}$ *is asymmetrized monotonic nondecreasing (nonincreasing) on the interval* $[a,b]$ *if the anti-symmetrical transform* $\tilde{f}$ *is monotonic nondecreasing (nonincreasing) on the interval* $[a,b]$.

If f is monotonic nondecreasing on $[a,b]$, then for any $t_1, t_2 \in [a,b]$ we have

$$\begin{aligned}\tilde{f}(t_2) - \tilde{f}(t_1) &= \frac{1}{2}[f(t_2) - f(a+b-t_2)] - \frac{1}{2}[f(t_1) - f(a+b-t_1)] \\ &= \frac{1}{2}[f(t_2) - f(t_1)] + \frac{1}{2}[f(a+b-t_1) - f(a+b-t_2)] \\ &\ge 0,\end{aligned}$$

which shows that $f : [a,b] \to \mathbb{R}$ is asymmetrized monotonic nondecreasing on the interval $[a,b]$.

Consider the real numbers $a < b$ and define the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = t^2$. We have

$$\tilde{f}_0(t) := \frac{1}{2}\left[t^2 - (a+b-t)^2\right] = (a+b)t - \frac{1}{2}(a+b)^2$$

and $\left(\tilde{f_0}\right)'(t) = a+b$, therefore $f : [a,b] \to \mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a,b]$ provided $\frac{a+b}{2} > 0\,(<0)$. So, if we take $a < 0 < b$ with $\frac{a+b}{2} > 0$, then f is asymmetrized monotonic nondecreasing on $[a,b]$ but not monotonic nondecreasing on $[a,b]$.

If we denote by $\mathcal{M}^{\nearrow}[a,b]$ the closed convex cone of monotonic nondecreasing functions defined on $[a,b]$ and by $\mathcal{AM}^{\nearrow}[a,b]$ the closed convex cone of asymmetrized monotonic nondecreasing functions, then from the above remarks we can conclude that

$$\mathcal{M}^{\nearrow}[a,b] \subsetneq \mathcal{AM}^{\nearrow}[a,b]. \tag{1.5}$$

Also, if $[c,d] \subset [a,b]$ and $f \in \mathcal{AM}^{\nearrow}[a,b]$, then this does not imply in general that $f \in \mathcal{AM}^{\nearrow}[c,d]$.

We recall that the pair of functions (f,g) defined on $[a,b]$ are called *synchronous (asynchronous)* on $[a,b]$ if

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0 \tag{1.6}$$

for any $t,\ s \in [a,b]$. It is clear that if both functions (f,g) are monotonic nondecreasing (nonincreasing) on $[a,b]$ then they are synchronous on $[a,b]$. There are also functions that change monotonicity on $[a,b]$, but as a pair they are still synchronous. For instance if $a < 0 < b$ and $f,g : [a,b] \to \mathbb{R}$, $f(t) = t^2$ and $g(t) = t^4$, then

$$(f(t) - f(s))(g(t) - g(s)) = \left(t^2 - s^2\right)\left(t^4 - s^4\right) = \left(t^2 - s^2\right)^2\left(t^2 + s^2\right) \geq 0$$

for any $t,\ s \in [a,b]$, which show that (f,g) is synchronous.

Definition 3. *We say that the pair of functions (f,g) defined on $[a,b]$ is called asymmetrized synchronous (asynchronous) on $[a,b]$ if the pair of transforms $\left(\tilde{f},\tilde{g}\right)$ is synchronous (asynchronous) on $[a,b]$, namely*

$$\left(\tilde{f}(t) - \tilde{f}(s)\right)(\tilde{g}(t) - \tilde{g}(s)) \geq (\leq)\, 0 \tag{1.7}$$

for any $t,s \in [a,b]$.

It is clear that if f,g are asymmetrized monotonic nondecreasing (nonincreasing) on $[a,b]$ then they are asymmetrized synchronous on $[a,b]$.

One of the most important results for synchronous (asynchronous) and integrable functions f, g on $[a,b]$ is the well-known *Čebyšev's inequality*:

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \geq (\leq)\, \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1.8}$$

For integral inequalities of Čebyšev's type, see [1]-[5], [7], [10]-[18] and the references therein.

Motivated by the above results, we establish in this paper some inequalities for asymmetrized synchronous (asynchronous) functions on $[a,b]$. Some examples for power, logarithm and sin functions are provided as well.

2. Main Results

We have the following result:

Theorem 3. *Assume that f, g are asymmetrized synchronous (asynchronous) and integrable functions on $[a,b]$. Then*

$$\int_a^b \tilde{f}(t)\, g(t)\, dt \geq (\leq)\, 0. \tag{2.1}$$

Proof. We consider only the case of symmetrized synchronous and integrable functions.

1. By the Čebyšev's inequality (1.8) for $\left(\tilde{f}, \tilde{g}\right)$ we get

$$\frac{1}{b-a}\int_a^b \tilde{f}(t)\,\tilde{g}(t)\, dt \geq \frac{1}{b-a}\int_a^b \tilde{f}(t)\, dt \frac{1}{b-a}\int_a^b \tilde{g}(t)\, dt. \tag{2.2}$$

We have

$$\int_a^b \tilde{f}(t)\, dt = \frac{1}{2}\left[\int_a^b f(t)\, dt - \int_a^b f(a+b-t)\, dt\right] = 0$$

since, by the change of variable $s = a+b-t$, $t \in [a,b]$,

$$\int_a^b f(a+b-t)\, dt = \int_a^b f(s)\, ds.$$

Also,

$$\begin{aligned}\int_a^b \tilde{f}(t)\,\tilde{g}(t) &= \frac{1}{4}\int_a^b [f(t) - f(a+b-t)][g(t) - g(a+b-t)]\, dt \\ &= \frac{1}{4}\int_a^b [f(t)\, g(t) + f(a+b-t)\, g(a+b-t)]\, dt \\ &\quad - \frac{1}{4}\int_a^b [f(t)\, g(a+b-t) + f(a+b-t)\, g(t)]\, dt \\ &= \frac{1}{4}\left[\int_a^b f(t)\, g(t)\, dt + \int_a^b f(a+b-t)\, g(a+b-t)\, dt\right] \\ &\quad - \frac{1}{4}\left[\int_a^b f(t)\, g(a+b-t)\, dt + \int_a^b f(a+b-t)\, g(t)\, dt\right] \\ &= \frac{1}{2}\left(\int_a^b f(t)\, g(t)\, dt - \int_a^b f(a+b-t)\, g(t)\, dt\right) \\ &= \int_a^b \tilde{f}(t)\, g(t)\, dt\end{aligned} \tag{2.3}$$

since, by the change of variable $s = a+b-t$, $t \in [a,b]$, we have

$$\int_a^b f(a+b-t)\, g(a+b-t)\, dt = \int_a^b f(t)\, g(t)\, dt$$

and

$$\int_a^b f(t)\, g(a+b-t)\, dt = \int_a^b f(a+b-t)\, g(t)\, dt.$$

By (2.2) we then get the desired result (2.1).

2. An alternative proof is as follows. Since $\left(\tilde{f},\tilde{g}\right)$ are synchronous, then

$$\left[\tilde{f}(t)-\tilde{f}\left(\frac{a+b}{2}\right)\right]\left[\tilde{g}(t)-\tilde{g}\left(\frac{a+b}{2}\right)\right]\geq 0$$

for any $t\in[a,b]$, which is equivalent to

$$\tilde{f}(t)\,\tilde{g}(t)\geq 0 \text{ for any } t\in[a,b], \tag{2.4}$$

or to

$$[f(t)-f(a+b-t)][g(t)-g(a+b-t)]\geq 0 \text{ for any } t\in[a,b].$$

This is a property of interest for asymmetrized synchronous functions.

If we integrate the inequality (2.4) and use the identity (2.3) we get the desired result (2.1). □

Remark 1. *The inequality (2.1) can be written in an equivalent form as*

$$\int_a^b f(t)\,g(t)\,dt\geq\int_a^b f(a+b-t)\,g(t)\,dt,$$

or as

$$\int_a^b f(t)\,g(t)\,dt\geq\int_a^b \breve{f}(t)\,g(t)\,dt.$$

Theorem 4. *If both f, g are asymmetrized monotonic nondecreasing (nonincreasing) and integrable functions on $[a,b]$, then*

$$\frac{1}{4}|f(b)-f(a)|\,|g(b)-g(a)|\geq\frac{1}{b-a}\int_a^b\tilde{f}(t)\,g(t)\,dt\geq 0, \tag{2.5}$$

and

$$\frac{1}{2}\min\left\{|f(b)-f(a)|\frac{1}{b-a}\int_a^b|g(t)|\,dt,|g(b)-g(a)|\frac{1}{b-a}\int_a^b|f(t)|\,dt\right\}\geq\frac{1}{b-a}\int_a^b\tilde{f}(t)\,g(t)\,dt\geq 0. \tag{2.6}$$

Proof. Assume that both f, g are asymmetrized monotonic nondecreasing and integrable functions on $[a,b]$, then they are asymmetrized synchronous and by (2.1) we get the second inequality in (2.5).

We also have

$$\tilde{f}(a)\leq\tilde{f}(t)\leq\tilde{f}(b)$$

for any $t\in[a,b]$, namely

$$-\frac{1}{2}[f(b)-f(a)]\leq\frac{1}{2}[f(t)-f(a+b-t)]\leq\frac{1}{2}[f(b)-f(a)],$$

for any $t\in[a,b]$, which implies that $\frac{1}{2}[f(b)-f(a)]\geq 0$ and

$$\frac{1}{2}|f(t)-f(a+b-t)|\leq\frac{1}{2}[f(b)-f(a)] \tag{2.7}$$

for any $t\in[a,b]$.

Similarly, we have $\frac{1}{2}[g(b)-g(a)]\geq 0$ and

$$\frac{1}{2}|g(t)-g(a+b-t)|\leq\frac{1}{2}[g(b)-g(a)] \tag{2.8}$$

for any $t \in [a,b]$.

If we multiply (2.7) and (2.8), then we get

$$\begin{aligned}\frac{1}{4}&[f(t)-f(a+b-t)][g(t)-g(a+b-t)] \\ &= \frac{1}{4}|[f(t)-f(a+b-t)][g(t)-g(a+b-t)]| \\ &\le \frac{1}{4}[f(b)-f(a)][g(b)-g(a)]\end{aligned} \tag{2.9}$$

for any $t \in [a,b]$.

Since

$$\begin{aligned}0 &\le \int_a^b \tilde{f}(t)g(t)\,dt = \frac{1}{4}\int_a^b [f(t)-f(a+b-t)][g(t)-g(a+b-t)]\,dt \\ &\le \frac{1}{4}[f(b)-f(a)][g(b)-g(a)](b-a),\end{aligned}$$

where for the last inequality we used (2.9), hence we get the first inequality in (2.5).

Also, we observe that

$$0 \le \int_a^b \tilde{f}(t)g(t)\,dt = \int_a^b \left|\tilde{f}(t)g(t)\right|dt \le \frac{1}{2}[f(b)-f(a)]\int_a^b |g(t)|\,dt$$

and since

$$\int_a^b \tilde{f}(t)g(t)\,dt = \int_a^b f(t)\tilde{g}(t)\,dt,$$

then also

$$\int_a^b f(t)\tilde{g}(t)\,dt \le \frac{1}{2}[g(b)-g(a)]\int_a^b |f(t)|\,dt$$

and the inequality (2.6) is also proved. □

Remark 2. *If the functions f, $g : [a,b] \to \mathbb{R}$ are either both of them nonincreasing or nondecreasing on $[a,b]$, then they are integrable and we have the inequalities (2.5) and (2.6).*

We have the following refinement of the inequality in (2.1).

Theorem 5. *Assume that f, g are asymmetrized synchronous and integrable functions on $[a,b]$. Then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \tilde{f}(t)g(t)\,dt \\ &\ge \left|\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right||\tilde{g}(t)|\,dt - \frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right|dt\frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt\right| \ge 0.\end{aligned} \tag{2.10}$$

Proof. By the continuity property of modulus, we have

$$\begin{aligned}\left[\tilde{f}(t)-\tilde{f}(s)\right][\tilde{g}(t)-\tilde{g}(s)] &= \left|\left[\tilde{f}(t)-\tilde{f}(s)\right][\tilde{g}(t)-\tilde{g}(s)]\right| \\ &= \left|\tilde{f}(t)-\tilde{f}(s)\right||\tilde{g}(t)-\tilde{g}(s)| \\ &\ge \left|\left|\tilde{f}(t)\right|-\left|\tilde{f}(s)\right|\right||\tilde{g}(t)-\tilde{g}(s)| \\ &= \left|\left(\left|\tilde{f}(t)\right|-\left|\tilde{f}(s)\right|\right)(\tilde{g}(t)-\tilde{g}(s))\right|\end{aligned}$$

for any $t, s \in [a, b]$.

Taking the double integral mean on $[a, b]^2$ and using the properties of the integral versus the modulus, we have

$$\begin{aligned} (2.11) \quad & \frac{1}{(b-a)^2}\int_a^b\int_a^b \left[\tilde{f}(t)-\tilde{f}(s)\right][\tilde{g}(t)-\tilde{g}(s)]\,dtds \\ & \geq \left|\frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\left|\tilde{f}(t)\right|-\left|\tilde{f}(s)\right|\right)(|\tilde{g}(t)|-|\tilde{g}(s)|)\,dtds\right|. \end{aligned}$$

Since, by Korkine's identity we have

$$\begin{aligned} & \frac{1}{(b-a)^2}\int_a^b\int_a^b \left[\tilde{f}(t)-\tilde{f}(s)\right][\tilde{g}(t)-\tilde{g}(s)]\,dtds \\ & = 2\left[\frac{1}{b-a}\int_a^b \tilde{f}(t)\,\tilde{g}(t)\,dt-\frac{1}{b-a}\int_a^b \tilde{f}(t)\,dt\frac{1}{b-a}\int_a^b \tilde{g}(t)\,dt\right] \\ & = \frac{2}{b-a}\int_a^b \tilde{f}(t)\,\tilde{g}(t)\,dt \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(b-a)^2}\int_a^b\int_a^b \left(\left|\tilde{f}(t)\right|-\left|\tilde{f}(s)\right|\right)(|\tilde{g}(t)|-|\tilde{g}(s)|)\,dtds \\ & = 2\left[\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right||\tilde{g}(t)|\,dt-\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right|dt\frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt\right], \end{aligned}$$

hence by (2.11) we have

$$\begin{aligned} & \frac{1}{b-a}\int_a^b \tilde{f}(t)\,\tilde{g}(t)\,dt \\ & \geq \left|\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right||\tilde{g}(t)|\,dt-\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right|dt\frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt\right|. \end{aligned}$$

By using the identity (2.3) we get the desired result (2.10). □

Remark 3. *We remark that, if $\left(\tilde{f}, g\right)$ are synchronous, then by a similar argument to the one above for $g \leftrightarrow \tilde{g}$ we have*

$$\begin{aligned} (2.12) \quad & \frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt \\ & \geq \left|\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right||g(t)|\,dt-\frac{1}{b-a}\int_a^b \left|\tilde{f}(t)\right|dt\frac{1}{b-a}\int_a^b |g(t)|\,dt\right| \geq 0. \end{aligned}$$

Also, since

$$\frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt = \frac{1}{b-a}\int_a^b f(t)\,\tilde{g}(t)\,dt,$$

then if we assume that $(f,\tilde{g})$ *are synchronous we also have*

$$\begin{aligned}(2.13)\quad &\frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt\\ &\geq \left|\frac{1}{b-a}\int_a^b |f(t)|\,|\tilde{g}(t)|\,dt-\frac{1}{b-a}\int_a^b |f(t)|\,dt\frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt\right|\geq 0.\end{aligned}$$

Now, if f *and* g *have the same monotonicity, then* $\left(\tilde{f},\tilde{g}\right)$, $\left(\tilde{f},g\right)$, $(f,\tilde{g})$ *are synchronous and we have*

$$(2.14)\quad \frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt\geq \max\left\{\left|C\left(\tilde{f},\tilde{g}\right)\right|,\left|C\left(\tilde{f},g\right)\right|,|C(f,\tilde{g})|\right\}\geq 0,$$

where

$$C(h,\ell):=\frac{1}{b-a}\int_a^b |h(t)\,\ell(t)|\,dt-\frac{1}{b-a}\int_a^b |h(t)|\,dt\frac{1}{b-a}\int_a^b |\ell(t)|\,dt$$

provided h *and* ℓ *are integrable on* $[a,b]$.

We say that the function $h:[a,b]\to\mathbb{R}$ is *H-r-Hölder continuous* with the constant $H>0$ and power $r\in(0,1]$ if

$$(2.15)\quad |h(t)-h(s)|\leq H\,|t-s|^r$$

for any t, $s\in[a,b]$. If $r=1$ we call that h is *L-Lipschitzian* when $H=L>0$.

Theorem 6. *Assume that* f, g *are asymmetrized synchronous with* f *is* H_1-r_1-*Hölder continuous and* g *is* H_2-r_2-*Hölder continuous on* $[a,b]$. *Then*

$$(2.16)\quad \frac{1}{4(r_1+r_2+1)}H_1H_2(b-a)^{r_1+r_2}\geq\frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt\geq 0.$$

If particular, if f *is* L_1-*Lipschitzian and* g *is* L_2-*Lipschitzian, then*

$$(2.17)\quad \frac{1}{12}L_1L_2(b-a)^2\geq\frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt\geq 0.$$

Proof. From (2.3) we have

$$\begin{aligned}0&\leq\int_a^b \tilde{f}(t)\,g(t)\,dt=\frac{1}{4}\int_a^b [f(t)-f(a+b-t)]\,[g(t)-g(a+b-t)]\,dt\\ &=\frac{1}{4}\int_a^b |[f(t)-f(a+b-t)]\,[g(t)-g(a+b-t)]|\,dt\\ &\leq\frac{1}{4}H_1H_2\int_a^b |2t-a-b|^{r_1+r_2}\,dt=\frac{2^{r_1+r_2}}{4}H_1H_2\int_a^b \left|t-\frac{a+b}{2}\right|^{r_1+r_2}dt\\ &=\frac{2}{2^{2-r_1-r_2}}H_1H_2\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)^{r_1+r_2}dt=\frac{2}{2^{2-r_1-r_2}}H_1H_2\frac{\left(\frac{b-a}{2}\right)^{r_1+r_2+1}}{r_1+r_2+1}\\ &=\frac{1}{4(r_1+r_2+1)}H_1H_2(b-a)^{r_1+r_2+1},\end{aligned}$$

which is equivalent to the desired result (2.16). □

3. Some Examples

Consider the identity function $\ell : [a,b] \to \mathbb{R}$ defined by $\ell(t) = t$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we have

$$\begin{aligned}\frac{1}{4}(b-a)[g(b)-g(a)] &\geq \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) g(t)\,dt \\ &\geq \max\{|C_{1,\ell}(g)|, |C_{2,\ell}(g)|, |C_{3,\ell}(g)|\} \geq 0,\end{aligned} \tag{3.1}$$

where

$$C_{1,\ell}(g) := \frac{1}{b-a}\int_a^b \left|\left(t-\frac{a+b}{2}\right)\tilde{g}(t)\right| dt - \frac{1}{4}\int_a^b |\tilde{g}(t)|\,dt,$$

$$C_{2,\ell}(g) := \frac{1}{b-a}\int_a^b \left|\left(t-\frac{a+b}{2}\right) g(t)\right| dt - \frac{1}{4}\int_a^b |g(t)|\,dt$$

and

$$C_{3,\ell}(g) := \frac{1}{b-a}\int_a^b |t\tilde{g}(t)|\,dt - \frac{1}{b-a}\int_a^b |t|\,dt \frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt.$$

If g is monotonic nondecreasing and L-*Lipschitzian* on $[a,b]$, then by (2.17) we get

$$\frac{1}{12}L(b-a)^2 \geq \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) g(t)\,dt \ \ (\geq 0). \tag{3.2}$$

Consider the power function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f(t) = t^p$ with $p > 0$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we get

$$\begin{aligned}\frac{1}{4}(b^p-a^p)[g(b)-g(a)] &\geq \frac{1}{b-a}\int_a^b \left[\frac{t^p-(a+b-t)^p}{2}\right] g(t)\,dt \\ &\geq \max\{|C_{1,p}(g)|, |C_{2,p}(g)|, |C_{3,p}(g)|\} \geq 0,\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}C_{1,p}(g) &:= \frac{1}{b-a}\int_a^b \left|\frac{t^p-(a+b-t)^p}{2}\right| |\tilde{g}(t)|\,dt \\ &\quad - \frac{1}{b-a}\int_a^b \left|\frac{t^p-(a+b-t)^p}{2}\right| dt \frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt,\end{aligned}$$

$$\begin{aligned}C_{2,p}(g) &:= \frac{1}{b-a}\int_a^b \left|\frac{t^p-(a+b-t)^p}{2}\right| |g(t)|\,dt \\ &\quad - \frac{1}{b-a}\int_a^b \left|\frac{t^p-(a+b-t)^p}{2}\right| dt \frac{1}{b-a}\int_a^b |g(t)|\,dt\end{aligned}$$

and

$$C_{3,p}(g) := \int_a^b t^p |\tilde{g}(t)|\,dt - \frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\frac{1}{b-a}\int_a^b |\tilde{g}(t)|\,dt.$$

If g is monotonic nondecreasing and L-*Lipschitzian* on $[a,b]$, then by (2.17) we get

$$\begin{aligned}&\frac{p}{12}L(b-a)^2 \begin{cases} b^{p-1} \text{ if } p \geq 1 \\ \\ a^{p-1} \text{ if } p \in (0,1)\end{cases} \\ &\geq \frac{1}{b-a}\int_a^b \left[\frac{t^p-(a+b-t)^p}{2}\right] g(t)\,dt \ \ (\geq 0).\end{aligned} \tag{3.4}$$

Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R},\ f=\ln$. If g is monotonic nondecreasing, then by (2.5) and (2.14) we have

$$\frac{1}{4}\ln\left(\frac{b}{a}\right)[g(b)-g(a)]\geq\frac{1}{2(b-a)}\int_a^b\ln\left(\frac{t}{a+b-t}\right)g(t)\,dt \geq\max\left\{\left|C_{1,\ln}(g)\right|,\left|C_{2,\ln}(g)\right|,\left|C_{3,\ln}(g)\right|\right\}\geq 0, \tag{3.5}$$

where

$$C_{1,\ln}(g):=\frac{1}{b-a}\int_a^b\left|\ln\left(\frac{t}{a+b-t}\right)\right|^{1/2}\left|\tilde{g}(t)\right|dt -\frac{1}{b-a}\int_a^b\left|\ln\left(\frac{t}{a+b-t}\right)\right|^{1/2}dt\frac{1}{b-a}\int_a^b\left|\tilde{g}(t)\right|dt,$$

$$C_{2,\ln}(g):=\frac{1}{b-a}\int_a^b\left|\ln t\right|\left|\tilde{g}(t)\right|dt-\frac{1}{b-a}\int_a^b\left|\ln t\right|dt\frac{1}{b-a}\int_a^b\left|\tilde{g}(t)\right|dt$$

and

$$C_{1,\ln}(g):=\frac{1}{b-a}\int_a^b\left|\ln\left(\frac{t}{a+b-t}\right)\right|^{1/2}\left|g(t)\right|dt -\frac{1}{b-a}\int_a^b\left|\ln\left(\frac{t}{a+b-t}\right)\right|^{1/2}dt\frac{1}{b-a}\int_a^b\left|g(t)\right|dt.$$

If g is monotonic nondecreasing and L-*Lipschitzian* on $[a,b]$, then by (2.17) we get

$$\frac{1}{6a}L(b-a)^2\geq\frac{1}{b-a}\int_a^b\ln\left(\frac{t}{a+b-t}\right)g(t)\,dt\ \ (\geq 0). \tag{3.6}$$

Consider the function $f:[a,b]\subset\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to\mathbb{R},\ f=\sin$. If g is monotonic nondecreasing, then by (2.5) we have

$$\frac{1}{2}\sin\left(\frac{b-a}{2}\right)[g(b)-g(a)]\geq\frac{1}{b-a}\int_a^b\sin\left(t-\frac{a+b}{2}\right)g(t)\,dt\geq 0. \tag{3.7}$$

If g is monotonic nondecreasing and L-*Lipschitzian* on $[a,b]$, then by (2.17) we get

$$\frac{1}{12}L(b-a)^2\times\begin{cases}\cos b\text{ if }-\frac{\pi}{2}\leq a<b\leq 0,\\ \max\{\cos a,\cos b\}\ \text{ if }-\frac{\pi}{2}\leq a<0<b\leq\frac{\pi}{2},\\ \cos a\text{ if }0\leq a<b\leq\frac{\pi}{2}\end{cases} \geq\frac{1}{b-a}\cos\left(\frac{a+b}{2}\right)\int_a^b\sin\left(t-\frac{a+b}{2}\right)g(t)\,dt\ \ (\geq 0). \tag{3.8}$$

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FURTHER INEQUALITIES FOR HEINZ OPERATOR MEAN

S. S. DRAGOMIR[1,2]

Abstract. In this paper we obtain some new inequalities for Heinz operator mean.

1. Introduction

Throughout this paper A, B are positive invertible operators on a complex Hilbert space $(H, \langle \cdot, \cdot \rangle)$. We use the following notations for operators and $\nu \in [0,1]$

$$A\nabla_{\nu}B := (1-\nu)A + \nu B,$$

the *weighted operator arithmetic mean,* and

$$A\sharp_{\nu}B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2},$$

the *weighted operator geometric mean* [13]. When $\nu = \frac{1}{2}$ we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

Define the *Heinz operator mean* by

$$H_{\nu}(A,B) := \frac{1}{2}\left(A\sharp_{\nu}B + A\sharp_{1-\nu}B\right).$$

The following interpolatory inequality is obvious

$$A\sharp B \le H_{\nu}(A,B) \le A\nabla B \tag{1.1}$$

for any $\nu \in [0,1]$.

The famous *Young inequality* for scalars says that if a, $b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \le (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{1.3}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

In the recent paper [1] we have obtained the following additive and multiplicative reverse of Young's inequality

$$0 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{1.4}$$

1991 *Mathematics Subject Classification.* 26D15; 26D10, 47A63, 47A30, 15A60.

Key words and phrases. Young's Inequality, Convex functions, Arithmetic mean-Geometric mean inequality, Heinz means.

and

$$1 \leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.5}$$

for any $a,b>0$ and $\nu \in [0,1]$, where K is Kantorovich's constant.

The operator version of (1.4) is as follows [1]:

Theorem 1. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0,1]$.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$0 \leq A\nabla_{\nu}B - A\sharp_{\nu}B \leq \nu(1-\nu)(h-1)\ln hA \tag{1.6}$$

and, in particular

$$0 \leq A\nabla B - A\sharp B \leq \frac{1}{4}(h-1)\ln hA. \tag{1.7}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$0 \leq A\nabla_{\nu}B - A\sharp_{\nu}B \leq \nu(1-\nu)\frac{h-1}{h}\ln hA \tag{1.8}$$

and, in particular

$$0 \leq A\nabla B - A\sharp B \leq \frac{1}{4}\frac{h-1}{h}\ln hA. \tag{1.9}$$

The operator version of (1.5) is [1]:

Theorem 2. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions*

(i) $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$;

(ii) $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$;

we have

$$A\nabla_{\nu}B \leq \exp[4\nu(1-\nu)(K(h)-1)]A\sharp_{\nu}B \tag{1.10}$$

and, in particular

$$A\nabla B \leq \exp[K(h)-1]A\sharp B. \tag{1.11}$$

For other recent results on geometric operator mean inequalities, see [2]-[12], [14] and [16]-[17].

We recall that *Specht's ratio* is defined by [15]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.12}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In the recent paper [6] we obtained amongst other the following result for the Heinz operator mean of A, B that are positive invertible operators that satisfy the condition $mA \leq B \leq MA$ for some constants $M > m > 0$,

$$\omega(m,M)A\sharp B \leq H_{\nu}(A,B) \leq \Omega(m,M)A\sharp B, \tag{1.13}$$

where

$$\Omega(m,M):=\begin{cases} S\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \\ \max\left\{S\left(m^{|2\nu-1|}\right), S\left(M^{|2\nu-1|}\right)\right\} \text{ if } m\le 1\le M, \\ \\ S\left(M^{|2\nu-1|}\right) \text{ if } 1<m, \end{cases}$$

and

$$\omega(m,M):=\begin{cases} S\left(M^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } M<1, \\ \\ 1 \text{ if } m\le 1\le M, \\ \\ S\left(m^{\left|\nu-\frac{1}{2}\right|}\right) \text{ if } 1<m. \end{cases}$$

Motivated by the above results we establish in this paper some new additive and multiplicative reverse inequalities for the Heinz operator mean.

2. Additive Reverse Inequalities for Heinz Mean

We have the following generalization of Theorem 1:

Theorem 3. *Assume that A, B are positive invertible operators and the constants $M>m>0$ are such that*

$$mA\le B\le MA. \tag{2.1}$$

Then for any $\nu\in[0,1]$ we have

$$(0\le)\, A\nabla_\nu B - A\sharp_\nu B \le \nu(1-\nu)\,\Omega(m,M)\,A \tag{2.2}$$

where

$$\Omega(m,M):=\begin{cases} (m-1)\ln m \textit{ if } M<1, \\ \\ \max\{(m-1)\ln m, (M-1)\ln M\} \textit{ if } m\le 1\le M, \\ \\ (M-1)\ln M \textit{ if } 1<m. \end{cases} \tag{2.3}$$

In particular, we have

$$(0\le)\, A\nabla B - A\sharp B \le \frac{1}{4}\Omega(m,M)\,A. \tag{2.4}$$

Proof. We consider the function $D:(0,\infty)\to[0,\infty)$ defined by $D(x)=(x-1)\ln x$. We have that $D'(x)=\ln x+1-\frac{1}{x}$ and $D''(x)=\frac{x+1}{x^2}$ for $x\in(0,\infty)$. This shows that the function is convex on $(0,\infty)$, monotonic decreasing on $(0,1)$ and monotonic increasing on $[1,\infty)$ with the minimum 0 realized in $x=1$.

From the inequality (1.4) we have

$$(0\le)\,(1-\nu)+\nu x-x^\nu\le\nu(1-\nu)\,D(x)$$

for any $x>0$, $\nu\in[0,1]$ and hence

$$(0\le)\,(1-\nu)I+\nu X-X^\nu\le\nu(1-\nu)\max_{m\le x\le M}D(x) \tag{2.5}$$

for the positive operator X that satisfies the condition $0<mI\le X\le MI$ for $0<m<M$ and $\nu\in[0,1]$.

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (2.5) we get

$$(1-\nu) I + \nu A^{-1/2}BA^{-1/2} - \left(A^{-1/2}BA^{-1/2}\right)^{\nu} \leq \nu(1-\nu) \max_{m\leq x\leq M} D(x) \tag{2.6}$$

Now, if we multiply (2.6) in both sides with $A^{1/2}$ we get

$$\begin{aligned}(0\leq)\,&(1-\nu) A + \nu B - A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu} A^{1/2} \\ &\leq \nu(1-\nu) \max_{m\leq x\leq M} D(x) A\end{aligned} \tag{2.7}$$

for any $\nu \in [0,1]$.

Finally, since

$$\max_{m\leq x\leq M} D(x) = \begin{cases} (m-1)\ln m \text{ if } M<1, \\ \\ \max\{(m-1)\ln m, (M-1)\ln M\} \text{ if } m\leq 1\leq M, \\ \\ (M-1)\ln M \text{ if } 1<m, \end{cases}$$

then by (2.7) we get the desired result (2.2). □

Corollary 1. *With the assumptions of Theorem 3 we have*

$$(0\leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)\,\Omega(m,M) A. \tag{2.8}$$

Proof. From (2.2) we have by replacing ν with $1-\nu$ that

$$(0\leq)\, A\nabla_{1-\nu} B - A\sharp_{1-\nu} B \leq \nu(1-\nu)\,\Omega(m,M) A. \tag{2.9}$$

Adding (2.2) with (2.9) and dividing by 2 we get (2.8). □

Corollary 2. *Let A, B be two positive operators. For positive real numbers m, m', M, M', put $h := \frac{M}{m}$, $h' := \frac{M'}{m'}$ and let $\nu \in [0,1]$.*

(i) If $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, then

$$(0\leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)(h-1)\ln hA. \tag{2.10}$$

(ii) If $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$(0\leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)\left(\frac{h-1}{h}\right)\ln hA. \tag{2.11}$$

Proof. If the condition (i) is valid, then we have

$$I < \frac{M'}{m'}I = h'I \leq X \leq hI = \frac{M}{m}I,$$

which, by (2.8) gives the desired result (2.10).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h}I \leq X \leq \frac{1}{h'}I < I,$$

which, by (2.8) gives

$$(0\leq)\, A\nabla B - H_{\nu}(A,B) \leq \nu(1-\nu)\left(\frac{1}{h}-1\right)\ln\frac{1}{h}$$

that is equivalent to (2.11). □

Theorem 4. *With the assumptions of Theorem 3 we have*

$$(0\leq)\,H_{\nu}(A,B)-A\sharp B\leq\frac{1}{4m^{1-\nu}}\max_{x\in[m,M]}D\left(x^{2\nu-1}\right)A,\tag{2.12}$$

where the function $D:(0,\infty)\to[0,\infty)$ *is defined by* $D(x)=(x-1)\ln x$ *(see the proof of Theorem 3).*

Proof. From the inequality (1.4) we have for $\nu=\frac{1}{2}$

$$(0\leq)\,\frac{c+d}{2}-\sqrt{cd}\leq\frac{1}{4}(c-d)(\ln c-\ln d)\tag{2.13}$$

for any $c,\ d>0$.

If we take in (2.13) $c=a^{1-\nu}b^{\nu}$ and $d=a^{\nu}b^{1-\nu}$ then we get

$$\frac{a^{1-\nu}b^{\nu}+a^{\nu}b^{1-\nu}}{2}-\sqrt{ab}\leq\frac{1}{4}\left(a^{1-\nu}b^{\nu}-a^{\nu}b^{1-\nu}\right)\left(\ln a^{1-\nu}b^{\nu}-\ln a^{\nu}b^{1-\nu}\right)\tag{2.14}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

This inequality is of interest in itself.

Now, if we take in (2.14) $a=1$ and $b=x$, then we get

$$\begin{aligned}0&\leq\frac{x^{\nu}+x^{1-\nu}}{2}-\sqrt{x}\leq\frac{1}{4}\left(x^{\nu}-x^{1-\nu}\right)\left(\ln x^{\nu}-\ln x^{1-\nu}\right)\\&=\frac{2\nu-1}{4}\left(x^{\nu}-x^{1-\nu}\right)\ln x=\frac{1}{4x^{1-\nu}}\left(x^{2\nu-1}-1\right)\ln x^{2\nu-1}\\&=\frac{1}{4x^{1-\nu}}D\left(x^{2\nu-1}\right)\end{aligned}\tag{2.15}$$

for any $x>0$ and $\nu\in[0,1]$.

Now, if $x\in[m,M]\subset(0,\infty)$, then by (2.15) we get the upper bound

$$(0\leq)\,\frac{x^{\nu}+x^{1-\nu}}{2}-\sqrt{x}\leq\frac{1}{4m^{1-\nu}}\max_{x\in[m,M]}D\left(x^{2\nu-1}\right).$$

Using the continuous functional calculus, we then have

$$(0\leq)\,\frac{X^{\nu}+X^{1-\nu}}{2}-X^{1/2}\leq\frac{1}{4m^{1-\nu}}\max_{x\in[m,M]}D\left(x^{2\nu-1}\right)\tag{2.16}$$

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI\leq A^{-1/2}BA^{-1/2}\leq MI$ and by taking $X=A^{-1/2}BA^{-1/2}$ in (2.16) we get

$$\begin{aligned}0&\leq\frac{\left(A^{-1/2}BA^{-1/2}\right)^{\nu}+\left(A^{-1/2}BA^{-1/2}\right)^{1-\nu}}{2}-\left(A^{-1/2}BA^{-1/2}\right)^{1/2}\\&\leq\frac{1}{4m^{1-\nu}}\max_{x\in[m,M]}D\left(x^{2\nu-1}\right)\end{aligned}\tag{2.17}$$

for any $\nu\in[0,1]$.

Now, if we multiply (2.17) in both sides with $A^{1/2}$ we get the desired result (2.12). □

Corollary 3. *Let A, B be as in Corollary 2.*

(i) If $0 < mI \le A \le m'I < M'I \le B \le MI$, *then*

$$(0 \le) H_\nu(A,B) - A\sharp B \le \frac{1}{4(h')^{1-\nu}} \begin{cases} \left(h^{2\nu-1}-1\right)\ln h^{2\nu-1} & \text{if } \nu \in \left[\frac{1}{2},1\right], \\ \left((h')^{2\nu-1}-1\right)\ln (h')^{2\nu-1} & \text{if } \nu \in \left[0,\frac{1}{2}\right). \end{cases} \tag{2.18}$$

(ii) If $0 < mI \le B \le m'I < M'I \le A \le MI$, *then*

$$(0 \le) H_\nu(A,B) - A\sharp B \le \frac{1}{4}h^{1-\nu} \begin{cases} \left(h^{-2\nu+1}-1\right)\ln h^{-2\nu+1} & \text{if } \nu \in \left[\frac{1}{2},1\right], \\ \left((h')^{-2\nu+1}-1\right)\ln (h')^{-2\nu+1} & \text{if } \nu \in \left[0,\frac{1}{2}\right). \end{cases} \tag{2.19}$$

Proof. If the condition (i) is valid, then we have

$$I < \frac{M'}{m'}I = h'I \le X \le hI = \frac{M}{m}I,$$

which, by (2.12) gives

$$0 \le H_\nu(A,B) - A\sharp B \le \frac{1}{4(h')^{1-\nu}} \max_{x\in[h',h]} D\left(x^{2\nu-1}\right)A. \tag{2.20}$$

Observe that, if $\nu \in \left[\frac{1}{2},1\right]$, then

$$\max_{x\in[h',h]} D\left(x^{2\nu-1}\right) = D\left(h^{2\nu-1}\right) = \left(h^{2\nu-1}-1\right)\ln h^{2\nu-1}.$$

If $\nu \in \left[0,\frac{1}{2}\right]$, then

$$\max_{x\in[h',h]} D\left(x^{2\nu-1}\right) = D\left((h')^{2\nu-1}\right) = \left((h')^{2\nu-1}-1\right)\ln (h')^{2\nu-1}.$$

By (2.20) we get the desired result (2.18).

If the condition (ii) is valid, then we have

$$0 < \frac{1}{h}I \le X \le \frac{1}{h'}I < I,$$

which, by (2.12) gives

$$0 \le H_\nu(A,B) - A\sharp B \le \frac{1}{4\left(\frac{1}{h}\right)^{1-\nu}} \max_{x\in\left[\frac{1}{h},\frac{1}{h'}\right]} D\left(x^{2\nu-1}\right)A. \tag{2.21}$$

If $\nu \in \left[\frac{1}{2},1\right]$, then

$$\max_{x\in\left[\frac{1}{h},\frac{1}{h'}\right]} D\left(x^{2\nu-1}\right) = D\left(\left(\frac{1}{h}\right)^{2\nu-1}\right) = D\left((h)^{-2\nu+1}\right).$$

If $\nu \in \left[0,\frac{1}{2}\right]$, then

$$\max_{x\in\left[\frac{1}{h},\frac{1}{h'}\right]} D\left(x^{2\nu-1}\right) = D\left(\left(\frac{1}{h'}\right)^{2\nu-1}\right) = D\left((h')^{-2\nu+1}\right).$$

By (2.21) we get the desired result (2.19). □

3. Multiplicative Reverse Inequalities for Heinz Mean

We have the following generalization of Theorem 2:

Theorem 5. *Assume that A, B are positive invertible operators and the constants $M > m > 0$ are such that the condition (2.1) is valid. Then for any $\nu \in [0,1]$ we have*

$$A\nabla_{\nu}B \leq A\sharp_{\nu}B \exp\left[4\nu\left(1-\nu\right)\left(\mathcal{F}\left(m,M\right)-1\right)\right] \tag{3.1}$$

where

$$\mathcal{F}\left(m,M\right) := \begin{cases} K\left(m\right) & \text{if } M < 1, \\ \\ \max\left\{K\left(m\right), K\left(M\right)\right\} & \text{if } m \leq 1 \leq M, \\ \\ K\left(M\right) & \text{if } 1 < m, \end{cases}$$

In particular, we have

$$A\nabla B \leq A\sharp B \exp\left[\mathcal{F}\left(m,M\right)-1\right]. \tag{3.2}$$

Proof. From the inequality (1.5) we have for $a = 1$ and $b = x$ that

$$\begin{aligned} (1-\nu)+\nu x &\leq x^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(K\left(\frac{1}{x}\right)-1\right)\right] \\ &= x^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right] \end{aligned} \tag{3.3}$$

for any $x > 0$ and hence

$$\begin{aligned} \left(1-\nu\right)I+\nu X &\leq X^{\nu}\max_{m\leq x\leq M}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right] \\ &= X^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(\max_{m\leq x\leq M}K\left(x\right)-1\right)\right] \end{aligned} \tag{3.4}$$

for any operator X with the property that $0 < mI \leq X \leq MI$ and for any $\nu \in [0,1]$.

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \leq A^{-1/2}BA^{-1/2} \leq MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (3.4) we get

$$\begin{aligned} &\left(1-\nu\right)I+\nu A^{-1/2}BA^{-1/2} \\ &\leq \left(A^{-1/2}BA^{-1/2}\right)^{\nu}\max_{m\leq x\leq M}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right] \\ &= \left(A^{-1/2}BA^{-1/2}\right)^{\nu}\exp\left[4\nu\left(1-\nu\right)\left(\max_{m\leq x\leq M}K\left(x\right)-1\right)\right] \end{aligned} \tag{3.5}$$

for any $\nu \in [0,1]$.

Now, if we multiply (3.5) in both sides with $A^{1/2}$ we get

$$\begin{aligned} &\left(1-\nu\right)A+\nu BA \\ &\leq A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}\max_{m\leq x\leq M}\exp\left[4\nu\left(1-\nu\right)\left(K\left(x\right)-1\right)\right] \\ &= A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{\nu}A^{1/2}\exp\left[4\nu\left(1-\nu\right)\left(\max_{m\leq x\leq M}K\left(x\right)-1\right)\right] \end{aligned} \tag{3.6}$$

for any $\nu \in [0,1]$.

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Since

$$\max_{m\leq x\leq M} K(x)=\begin{cases} K(m) \text{ if } M<1, \\ \\ \max\{K(m), K(M)\} \text{ if } m\leq 1\leq M, \\ \\ K(M) \text{ if } 1<m, \end{cases}$$

then by (3.6) we get the desired result (3.1). □

Corollary 4. *With the assumptions of Theorem 5 we have*

$$A\nabla B\leq \exp\left[4\nu(1-\nu)(\digamma(m,M)-1)\right]H_\nu(A,B). \tag{3.7}$$

Corollary 5. *For two positive operators* A, B *and positive real numbers* m, m', M, M' *satisfying either of the following conditions:*

(i) $0<mI\leq A\leq m'I<M'I\leq B\leq MI$;

(ii) $0<mI\leq B\leq m'I<M'I\leq A\leq MI$;

we have

$$A\nabla B\leq \exp\left[4\nu(1-\nu)(K(h)-1)\right]H_\nu(A,B). \tag{3.8}$$

We also have:

Theorem 6. *Assume that* A, B *are positive invertible operators and the constants* $M>m>0$ *are such that the condition (2.1) is valid. Then for any* $\nu\in[0,1]$ *we have*

$$H_\nu(A,B)\leq \exp\left[\Theta_\nu(m,M)-1\right]A\sharp B \tag{3.9}$$

where

$$\Theta_\nu(m,M):=\begin{cases} K\left(m^{|2\nu-1|}\right) \text{ if } M<1, \\ \\ \max\left\{K\left(m^{|2\nu-1|}\right), K\left(M^{|2\nu-1|}\right)\right\} \text{ if } m\leq 1\leq M, \\ \\ K\left(M^{|2\nu-1|}\right) \text{ if } 1<m. \end{cases} \tag{3.10}$$

Proof. From the inequality (1.5) we have for $\nu=\frac{1}{2}$

$$\frac{\frac{c+d}{2}}{\sqrt{cd}}\leq \exp\left(K\left(\frac{c}{d}\right)-1\right) \tag{3.11}$$

for any $c,d>0$.

If we take in (3.11) $c=a^{1-\nu}b^{\nu}$ and $d=a^{\nu}b^{1-\nu}$ then we get

$$\frac{a^{1-\nu}b^{\nu}+a^{\nu}b^{1-\nu}}{2}\leq \exp\left(K\left(\left(\frac{a}{b}\right)^{1-2\nu}\right)-1\right)\sqrt{ab} \tag{3.12}$$

for any a, $b>0$ for any $\nu\in[0,1]$.

This is an inequality of interest in itself.

If we take in (2.19) $a=x$ and $b=1$, then we get

$$\frac{x^{1-\nu}+x^{\nu}}{2}\leq \exp\left(K\left(x^{1-2\nu}\right)-1\right)\sqrt{x}, \tag{3.13}$$

for any $x>0$.

Now, if $x\in[m,M]\subset(0,\infty)$ then by (2.20) we have

$$\frac{x^{1-\nu}+x^{\nu}}{2}\leq \sqrt{x}\exp\left(\max_{x\in[m,M]}K\left(x^{1-2\nu}\right)-1\right) \tag{3.14}$$

for any $x \in [m, M]$.

If $\nu \in \left(0, \frac{1}{2}\right)$, then

$$\max_{x\in[m,M]} K\left(x^{1-2\nu}\right) = \begin{cases} K\left(m^{1-2\nu}\right) \text{ if } M < 1, \\ \\ \max\left\{K\left(m^{1-2\nu}\right), K\left(M^{1-2\nu}\right)\right\} \text{ if } m \le 1 \le M, \\ \\ K\left(M^{1-2\nu}\right) \text{ if } 1 < m. \end{cases}$$

If $\nu \in \left(\frac{1}{2}, 1\right)$, then

$$\begin{aligned} \max_{x\in[m,M]} K\left(x^{1-2\nu}\right) &= \max_{x\in[m,M]} K\left(x^{2\nu-1}\right) \\ &= \begin{cases} K\left(m^{2\nu-1}\right) \text{ if } M < 1, \\ \\ \max\left\{K\left(m^{2\nu-1}\right), K\left(M^{2\nu-1}\right)\right\} \text{ if } m \le 1 \le M, \\ \\ K\left(M^{2\nu-1}\right) \text{ if } 1 < m. \end{cases} \end{aligned}$$

Therefore, by (3.14) we have

$$\frac{x^{1-\nu} + x^{\nu}}{2} \le \exp\left[\Theta(m, M) - 1\right]\sqrt{x} \tag{3.15}$$

for any $x \in [m, M] \subset (0, \infty)$ and for any $\nu \in [0, 1]$.

If X is an operator with $mI \le X \le MI$, then by (3.15) we have

$$\frac{X^{1-\nu} + X^{\nu}}{2} \le \exp\left[\Theta(m, M) - 1\right] X^{1/2}.$$

If the condition (2.1) holds true, then by multiplying in both sides with $A^{-1/2}$ we get $mI \le A^{-1/2}BA^{-1/2} \le MI$ and by taking $X = A^{-1/2}BA^{-1/2}$ in (3.15) we get

$$\begin{aligned} &\frac{1}{2}\left[\left(A^{-1/2}BA^{-1/2}\right)^{1-\nu} + \left(A^{-1/2}BA^{-1/2}\right)^{\nu}\right] \\ &\le \exp\left[\Theta(m, M) - 1\right]\left(A^{-1/2}BA^{-1/2}\right)^{1/2}. \end{aligned} \tag{3.16}$$

Now, if we multiply (3.16) in both sides with $A^{1/2}$ we get the desired result (3.9). □

Finally, we have

Corollary 6. *For two positive operators A, B and positive real numbers m, m', M, M' satisfying either of the following conditions:*

(i) $0 < mI \le A \le m'I < M'I \le B \le MI$,

(ii) $0 < mI \le B \le m'I < M'I \le A \le MI$,

we have for $h = \frac{M}{m}$ and $h' = \frac{M'}{m'}$ that

$$H_{\nu}(A, B) \le \exp\left[K\left(h^{|2\nu-1|}\right) - 1\right] A\sharp B, \tag{3.17}$$

where $\nu \in [0, 1]$.

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Global Dynamics of Monotone Second Order Difference Equation

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Abstract. We investigate the global character of the difference equation of the form

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots$$

with several period-two solutions, where f is decreasing in the first variable and is increasing in the second variable. We show that the boundaries of the basins of attractions of different locally asymptotically stable equilibrium solutions or period-two solutions are in fact the global stable manifolds of neighboring saddle or non-hyperbolic equilibrium solutions or period-two solutions. We illustrate our results with the complete study of global dynamics of a certain rational difference equation with quadratic terms.

Keywords. asymptotic stability, attractivity, bifurcation, difference equation, global, local stability, period two;

AMS 2000 Mathematics Subject Classification: 37B25, 37D10, 39A20, 39A30.

1 Introduction and Preliminaries

Let I be some interval of real numbers and let $f \in C^1[I \times I, I]$ be such that $f(I \times I) \subseteq \mathcal{K}$ where $\mathcal{K} \subseteq I$ is a compact set. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, \dots \tag{1}$$

where f is a continuous and decreasing in the first variable and increasing in the second variable. The following result gives a general information about global behavior of solutions of Equation (1).

Theorem 1 *([4])*
Let $I \subseteq R$ and let $f \in C[I \times I, I]$ be a function which is non-decreasing in first and non-increasing in second argument. Then for every solution of Equation (1) the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of (1) converges to either an equilibrium or period-two solution or to the singular point on the boundary. Consequently, most important question becomes determining the basins of attraction of these solutions as well as the unbounded solutions. The answer to this question follows from an application of the theory of monotone maps in the plane which will be presented in Preliminaries.

In [1, 2, 3] authors consider difference equation (1) with several equilibrium solutions as well as the period-two solutions and determine the basins of attraction of different equilibrium solutions and the period-two solutions. In this paper we consider Equation (1) which has up to two equilibrium solutions and up to two minimal period-two solutions which are in South-East ordering. More precisely, we will give sufficient conditions for the precise description of the basins of attraction of different equilibrium solutions and period-two solutions. The results can be immediately extended to the case of any number of the equilibrium solutions and the period-two solutions by replicating our main results.

This paper is organized as follows. In the rest of this section we will recall several basic results on competitive systems in the plane from [7, 15, 16, 17] which are included for completeness of presentation. Our main results about some global dynamics scenarios for monotone systems in the plane and their application to global dynamics of

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Equation (1) are given in section 2. As an application of the results from section 2 in section 3 the global dynamics of difference equation

$$x_{n+1} = \frac{\gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \ldots \tag{2}$$

with all non-negative parameters and initial conditions is presented. All global dynamic scenarios for Equation (1) will be illustrated in the case of Equation (2), which global dynamics can be shortly described as the sequence of exchange of stability bifurcations between an equilibrium and one or two period-two solutions.

We now give some basic notions about monotone maps in the plane.

Definition 2 *Let R be a subset of $\mathbb{R}^2$ with nonempty interior, and let $T : R \to R$ be a map (i.e., a continuous function). Set $T(x, y) = (f(x, y), g(x, y))$. The map T is* competitive *if $f(x, y)$ is non-decreasing in x and non-increasing in y, and $g(x, y)$ is non-increasing in x and non-decreasing in y. If both f and g are nondecreasing in x and y, we say that T is* cooperative. *If T is competitive (cooperative), the associated system of difference equations*

$$\left\{ \begin{array}{lcl} x_{n+1} & = & f(x_n, y_n) \\ y_{n+1} & = & g(x_n, y_n) \end{array} \right. , \quad n = 0, 1, \ldots, \quad (x_{-1}, x_0) \in R \tag{3}$$

is said to be competitive (cooperative). The map T and associated difference equations system are said to be strongly competitive *(*strongly cooperative*) if the adjectives non-decreasing and non-increasing are replaced by increasing and decreasing.*

Consider a partial ordering $\preceq$ on $\mathbb{R}^2$. Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $y \preceq x$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on $\mathcal{R}$ if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the South-East (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$. Now we can show that a map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is *cooperative* and a map monotone with respect to the South-East ordering is *competitive.*

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \ldots, 4$ to be the usual four quadrants based at $x = (x_1, x_2)$ and numbered in a counterclockwise direction, for example, $Q_1(x) = \{\mathbf{y} = (y_1, y_2) \in R^2 : \ x_1 \leq y_1, \ x_2 \leq y_2\}$. Basin of attraction of a fixed point $(\bar{x}, \bar{y})$ of a map T, denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to $(\bar{x}, \bar{y})$. Similarly, we define a basin of attraction of a periodic point of period p. The fixed point $A(x, y)$ of the map T is said to be *non-hyperbolic point of stable type* if one of the roots of characteristic equation evaluated in A is 1 or -1 and the second root is in $(-1, 1)$.

The next four results, from [16, 17], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [7, 19] and in [18].

Theorem 3 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\overline{\mathrm{x}} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap int\,(Q_1(\overline{\mathrm{x}}) \cup Q_3(\overline{\mathrm{x}}))$ is nonempty (i.e., $\overline{\mathrm{x}}$ is not the NW or SE vertex of $\mathcal{R}$), and T is strongly competitive on Δ. Suppose that the following statements are true.*

a. The map T has a C^1 extension to a neighborhood of $\overline{\mathrm{x}}$.

b. The Jacobian $J_T(\overline{\mathrm{x}})$ of T at $\overline{\mathrm{x}}$ has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through $\overline{\mathrm{x}}$ that is invariant and a subset of the basin of attraction of $\overline{\mathrm{x}}$, such that $\mathcal{C}$ is tangential to the eigenspace E^λ at $\overline{\mathrm{x}}$, and $\mathcal{C}$ is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of $\mathcal{C}$ in the interior of $\mathcal{R}$ are either fixed points or minimal period-two points. In the latter case, the set of endpoints of $\mathcal{C}$ is a minimal period-two orbit of T.

Theorem 4 *For the curve $\mathcal{C}$ of Theorem 3 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

i. The map T has no fixed points nor periodic points of minimal period two in Δ.

ii. The map T has no fixed points in Δ, $\det J_T(\overline{\mathrm{x}}) > 0$*, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.*

iii. The map T has no points of minimal period-two in Δ, $\det J_T(\overline{\mathrm{x}}) < 0$*, and $T(x) = \overline{\mathrm{x}}$ has no solutions $x \in \Delta$.*

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 3 reduces just to $|\lambda| < 1$. This follows from a change of variables [19] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 5 *Assume the hypotheses of Theorem 3, and let $\mathcal{C}$ be the curve whose existence is guaranteed by Theorem 3. If the endpoints of $\mathcal{C}$ belong to $\partial\mathcal{R}$, then $\mathcal{C}$ separates $\mathcal{R}$ into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{se} y\}, \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{se} x\}, \tag{4}$$

such that the following statements are true.

(i) $\mathcal{W}_-$ is invariant, and $\operatorname{dist}(T^n(x), Q_2(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_-$.

(ii) $\mathcal{W}_+$ is invariant, and $\operatorname{dist}(T^n(x), Q_4(\overline{x})) \to 0$ as $n \to \infty$ for every $x \in \mathcal{W}_+$.

(B) If, in addition to the hypotheses of part (A), $\overline{x}$ is an interior point of $\mathcal{R}$ and T is C^2 and strongly competitive in a neighborhood of $\overline{x}$, then T has no periodic points in the boundary of $Q_1(\overline{x}) \cup Q_3(\overline{x})$ except for $\overline{x}$, and the following statements are true.

(iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_2(\overline{x})$ for $n \geq n_0$.

(iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \operatorname{int} Q_4(\overline{x})$ for $n \geq n_0$.

If T is a map on a set $\mathcal{R}$ and if $\overline{x}$ is a fixed point of T, the stable set $\mathcal{W}^s(\overline{x})$ of $\overline{x}$ is the set $\{x \in \mathcal{R} : T^n(x) \to \overline{x}\}$ and unstable set $\mathcal{W}^u(\overline{x})$ of $\overline{x}$ is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^{0} \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1},\ x_0 = x, \text{ and } \lim_{n \to -\infty} x_n = \overline{x} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\overline{x})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\overline{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on $\mathcal{R}$, the sets $\mathcal{W}^s(\overline{x})$ and $\mathcal{W}^u(\overline{x})$ are the stable and unstable manifolds of $\overline{x}$.

Theorem 6 *In addition to the hypotheses of part (B) of Theorem 5, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve $\mathcal{C}$ of Theorem 3 has endpoints in $\partial\mathcal{R}$, then $\mathcal{C}$ is the stable set $\mathcal{W}^s(\overline{x})$ of $\overline{x}$, and the unstable set $\mathcal{W}^u(\overline{x})$ of $\overline{x}$ is a curve in $\mathcal{R}$ that is tangential to E^μ at $\overline{x}$ and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\overline{x})$ in $\mathcal{R}$ are fixed points of T.*

Remark 7 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of Equation (1) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to Equation (1) is a strictly competitive map on $I \times I$, see [16].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Equation (1) to obtain the equivalent system

$$\begin{array}{l} u_{n+1} = v_n \\ v_{n+1} = f(v_n, u_n) \end{array}, \quad n = 0, 1, \ldots .$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u),\ f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [16].

Remark 8 The characteristic equation of Equation (1) at an equilibrium solution $(\bar{x}, \bar{x})$:

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \tag{5}$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 3-6 depends on the nonexistence of minimal period-two solution.

2 Main Results

In this section we present some global dynamics scenarios which feasibility will be illustrated in Section 3.

Theorem 9 *Consider the competitive map T generated by the system (3) on a rectangular region $\mathcal{R}$ with nonempty interior. Suppose T has no minimal period-two solutions in $\mathcal{R}$, is strongly competitive on int $\mathcal{R}$, is C^2 in a neighborhood of any fixed point and b. of Theorem 3 holds.*

(a) *Assume that T has a saddle fixed points E_1, E_3 and locally asymptotically stable fixed point E_2, such that $E_1 \preceq_{se} E_2 \preceq_{se} E_3$, and E_0, which is South-west corner of the region $\mathcal{R}$ is either repeller or singular point. Furthermore assume that $E_1 \preceq_{se} E_0 \preceq_{se} E_3$ and that the ray through E_0 and E_1 (resp. E_0 and E_2) is stable manifold of E_1 (resp. E_2). If T has no period-two solutions then every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ converges to E_2.*

(b) *Assume that T has locally asymptotically stable fixed points E_1, E_3 and a saddle fixed point E_2, such that $E_1 \preceq_{se} E_2 \preceq_{se} E_3$, and E_0, which is South-west corner of the region $\mathcal{R}$ is either repeller or singular point. Furthermore assume that $E_1 \preceq_{se} E_0 \preceq_{se} E_3$ and that the ray through E_0 and E_1 (resp. E_0 and E_3) is attracted to E_1 (resp. E_3). If T has no period-two solutions then every solution which starts below (resp. above) the stable manifold $\mathcal{W}^s(E_2)$ converges to E_1 (resp. E_3).*

(c) *Assume that T has exactly five fixed points $E_1, \ldots, E_5$, $E_1 \preceq_{se} E_2 \preceq_{se} E_3 \preceq_{se} E_4 \preceq_{se} E_5$ where E_1, E_3, E_5 are saddle points, and E_2, E_4 are locally asymptotically stable points. Assume that E_0, which is South-west corner of the region $\mathcal{R}$, is either repeller or singular point such that $E_1 \preceq_{se} E_0 \preceq_{se} E_5$ and that the ray through E_0 and E_1 (resp. E_0 and E_5) is part of the basin of attraction of E_1 (resp. E_5). If T has no period-two solutions then every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_1)$ and $\mathcal{W}^s(E_3)$ converges to E_2 while every solution which starts in the interior of the region bounded by the global stable manifolds $\mathcal{W}^s(E_3)$ and $\mathcal{W}^s(E_5)$ converges to E_4.*

(d) *Assume that T has exactly five fixed points $E_1, \ldots, E_5$, $E_1 \preceq_{se} E_2 \preceq_{se} E_3 \preceq_{se} E_4 \preceq_{se} E_5$ where E_1, E_3, E_5 are locally asymptotically stable points, and E_2, E_4 are saddle points. Assume that E_0, which is South-west corner of the region $\mathcal{R}$, is either repeller or singular point such that $E_1 \preceq_{se} E_0 \preceq_{se} E_5$ and that the ray through E_0 and E_1 (resp. E_0 and E_5) is part of the basin of attraction of E_1 (resp. E_5). If T has no period-two solutions then every solution which starts below (resp. above) the stable manifold $\mathcal{W}^s(E_4)$ (resp. $\mathcal{W}^s(E_2)$) converges to E_5 (resp. E_1). Every solution which starts between the stable manifolds $\mathcal{W}^s(E_2)$ and $\mathcal{W}^s(E_4)$ converges to E_3.*

Proof.

(a) The existence of the global stable and unstable manifolds of the saddle point equilibria E_1 and E_3 is guaranteed by Theorems 3 - 6. In view of uniqueness of these manifolds we have that $\mathcal{W}^s(E_1)$ has end points in E_0 and $(0, \infty)$ while $\mathcal{W}^s(E_3)$ has end points in E_0 and $(\infty, 0)$. Furthermore $\mathcal{W}^u(E_1)$ and $\mathcal{W}^u(E_3)$ have end points in E_2. Now, by Corollary 2 in [16] every solution which starts in the interior of the ordered interval $[[E_1, E_2]]$ is attracted to E_2 and similarly every solution which starts in the interior of the ordered interval $[[E_2, E_3]]$ is attracted to E_2. Furthermore, for every $(x_0, y_0) \in [[E_1, E_3]] \setminus ([[E_1, E_2]] \cup [[E_2, E_3]] \cup \{E_0\})$ one can find the points $(x_l, y_l) \in [[E_1, E_2]]$ and $(x_u, y_u) \in [[E_1, E_2]]$ such that $(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$ and so $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u)), n \geq 1$, which implies that $T^n((x_0, y_0)) \to E_2$. Finally, for every $(x_0, y_0) \in \mathcal{R} \setminus ([[E_1, E_3]] \cup \{E_0\})$ one can find the points $(x_L, y_L) \in \mathcal{W}^u(E_1), (x_U, y_U) \in \mathcal{W}^u(E_3)$ such that $(x_L, y_L) \preceq_{se} (x_0, y_0) \preceq_{se} (x_U, y_U)$ which implies that $T^n((x_0, y_0))$ will eventually enter $[[E_1, E_3]]$ and so it will converge to E_2.

(b) The existence of the stable and unstable manifolds of the saddle point equilibrium E_2 is guaranteed by Theorems 3-6. The endpoints of the unstable manifold are E_1 and E_3. First one can assume that the initial point $(x_0, y_0) \in [[E_1, E_2]] \setminus \{E_0\}$. In view of Corollary 2 in [16] the interior of $[[E_1, E_2]]$ is subset of the basin of attraction of E_1. If the initial point $(x_0, y_0) \notin [[E_1, E_2]]$ but it is between$\mathcal{W}^s(E_1)$ and the ray through E_0 and E_1 then one can find te points (x_l, y_l) the ray through E_0 and E_1 and $(x_u, y_u) \in \mathcal{W}^s(E_1)$ such that $(x_l, y_l) \preceq_{se} (x_0, y_0) \preceq_{se} (x_u, y_u)$ and so $T^n((x_l, y_l)) \preceq_{se} T^n((x_0, y_0)) \preceq_{se} T^n((x_u, y_u)), n \geq 1$, which means $T^n((x_0, y_0))$ will eventually enter $[[E_1, E_2]]$ and so $T^n((x_0, y_0)) \to E_2$.

The proof when the initial point (x_0, y_0) is below $\mathcal{W}^s(E_2)$ is similar.

(c) The proof is similar to the one in case (a) and will be ommitted. This dynamic scenario is a replication of dynamic scenario in (a).

(d) The proof is similar to the one in case (b) and will be ommitted. This dynamic scenario is exactly replication of dynamic scenario in (b).

■

In the case of Equation (1) we have the following results which are direct application of Theorem 9.

Theorem 10 *Consider Equation* (1) *and assume that f is decreasing in first and increasing in the second variable on the set $(a,b)^2$, where a is either the repeller or a singular point of f, such that f is C^2 in a neighborhood of any fixed point.*

(a) Assume that Equation (1) *has locally asymptotically stable equilibrium solutions $\bar{x} > a$ and the unique saddle point minimal period-two solution $\{P_1, Q_1\}$, $P_1 \preceq_{se} (a,a) \preceq_{se} Q_1$. Assume that the stable manifold of P_1 (resp. Q_1) is the line through (a,a) and P_1 (resp. the line through (a,a) and Q_1). Then the equilibrium $\bar{x}$ is globally asymptotically stable for all $x_{-1}, x_0 > a$.*

(b) Assume that Equation (1) *has the saddle equilibrium solution $\bar{x} > a$ and the unique locally asymptotically stable minimal period-two solution $\{P_1, Q_1\}$, $P_1 \preceq_{se} (a,a) \preceq_{se} Q_1$. Assume that the stable manifold of P_1 (resp. Q_1) is the line through (a,a) and P_1 (resp. the line through (a,a) and Q_1). Then the period-two solution $\{P_1, Q_1\}$ attracts all initial points off the global stable manifold $\mathcal{W}^s(E(\bar{x},\bar{x}))$.*

(c) Assume that Equation (1) *has a saddle equilibrium solution $\bar{x} > a$. Assume that Equation* (1) *has two minimal period-two solutions $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ such that $P_1 \preceq_{se} P_2 \preceq_{se} E(\bar{x},\bar{x}) \preceq_{se} Q_2 \preceq_{se} Q_1$, where $\{P_2, Q_2\}$ is locally asymptotically stable and $\{P_1, Q_1\}$ is a saddle point and assume that the global stable manifold of P_1 (resp. Q_1) is the line through (a,a) and P_1 (resp. the line through (a,a) and Q_1). Then every solution which starts off the union of global stable manifolds $\mathcal{W}^s(E(\bar{x},\bar{x})) \cup \mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1)$ converges to the period-two solution $\{P_2, Q_2\}$.*

(d) Assume that Equation (1) *has locally asymptotically stable equilibrium solution $\bar{x} > a$. Asume that Equation* (1) *has two minimal period-two solutions $\{P_1, Q_1\}$ and $\{P_2, Q_2\}$ such that $P_1 \preceq_{se} P_2 \preceq_{se} E(\bar{x},\bar{x}) \preceq_{se} Q_2 \preceq_{se} Q_1$, where $\{P_1, Q_1\}$ is locally asymptotically stable and $\{P_2, Q_2\}$ is a saddle point. If the line through (a,a) and P_1 (resp. the line through (a,a) and Q_1) is a part of the basin of attraction of $\{P_1, Q_1\}$ then every solution which starts between the stable manifolds $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ converges to $\bar{x}$ while every solution which starts below $\mathcal{W}^s(Q_2)$ (resp. above $\mathcal{W}^s(P_2)$) converges to the period-two solution $\{P_1, Q_1\}$.*

Proof.

(a) In view of Remark 7 the second iterate T^2 of the map T associated with Equation (1) is strictly competitive. Applying Theorem 9 part (a) to T^2, where we set $E_1 = P_1, E_2 = (\bar{x},\bar{x}), E_3 = Q_1$ we complete the proof.

(b) The proof follows from Theorem 9 part (b) applied to T^2, where we set $E_1 = P_1, E_2 = (\bar{x},\bar{x}), E_3 = Q_1$ and observation that locally asymptotically stable fixed point (resp. saddle point) for T has the same character for T^2.

(c) The proof is similar to the proof in case (a) and will be ommitted.

(d) The proof follows from Theorem 9 part (d) applied to T^2, where we set $E_1 = P_1, E_2 = P_2, E_3 = (\bar{x},\bar{x}), E_4 = Q_2, E_5 = Q_1$ and the observation that locally asymptotically stable fixed point (resp. saddle point) for T has the same character for T^2.

■

Remark 11 The term "saddle point" in formulation of statements of Theorems 9 and 10 can be replaced by the term "non-hyperbolic point of stable type". Results related to Theorem 9 were obtained in [1, 2] and the results related to Theorem 10 were obtained in [6, 9, 10]. Furthermore Cases (b) and (c) of Theorem 9 can be extended to the case when we have any odd number of the equilibrium points which alternate its stability between two types: locally asymptotically stable and saddle points or non-hyperbolic equilibrium points of the stable type. The transition from Case (a) to Case (b) and from Case (c) to Case (d) in Theorem 9 is an exchange of stability bifurcation, while in the case of Theorem 10 these two bifurcations are two global period doubling bifurcations.

3 Case study: Equation $x_{n+1} = \frac{\gamma x_{n-1}}{Ax_n^2+Bx_nx_{n-1}+Cx_{n-1}}$

We investigate global behavior of Equation (2), where the parameters γ, A, B, C are positive numbers and the initial conditions x_{-1}, x_0 are arbitrary nonnegative numbers such that $x_{-1} + x_0 > 0$. Equation (2) is a special case of equations

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, \dots \tag{6}$$

and

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, \dots \tag{7}$$

The comprehensive linearized stability analysis of Equation (6) was given in [9] and some special cases were considered in [10]. Some special cases of Equation (7) have been considered in the series of papers [5, 6, 11, 12, 19]. Describing the global dynamics of Equation (7) is a formidable task as this equation contains as a special cases many equations with complicated dynamics, such as the linear fractional difference equation

$$x_{n+1} = \frac{Dx_n + Ex_{n-1} + F}{dx_n + ex_{n-1} + f}, \quad n = 0, 1, ... \tag{8}$$

Equation (2) has 0 as a singular point and the first quadrant as the region $\mathcal{R}$.

3.1 Local stability analysis

By using the substitution $y_n = \frac{C}{\gamma} x_n$ Equation (2) is reduced to the equation

$$x_{n+1} = \frac{x_{n-1}}{A'x_n^2 + B'x_n x_{n-1} + x_{n-1}}, n = 0, 1, ... \tag{9}$$

where $A' = \frac{\gamma^2}{C^2} A$ and $B' = \frac{\gamma^2}{C^2} B$. In the sequel we consider Equation (9) where A' and B' will be replaced with A and B respectively.

First, we notice that under the conditions on parameters all solutions of Equation (9) are in interval $(0, 1]$ and that 0 is a singular point.

Equation (9) has the unique positive equilibrium $\bar{x}$ given by

$$\bar{x} = \frac{-1+\sqrt{1+4(A+B)}}{2(A+B)}. \tag{10}$$

The partial derivatives associated to Equation (9) at the equilibrium $\bar{x}$ are

$$f'_x = \left.\frac{-y(2Ax+By)}{(Ax^2+Bxy+y)^2}\right|_{\bar{x}} = -\frac{4(2A+B)}{\left(1+\sqrt{1+4A+4B}\right)^2}, \quad f'_y = \left.\frac{Ax^2}{(Ax^2+Bxy+y)^2}\right|_{\bar{x}} = \frac{4A}{\left(1+\sqrt{1+4A+4B}\right)^2}.$$

Characteristic equation associated to Equation (9) at the equilibrium is

$$\lambda^2 + \frac{4(2A+B)}{\left(1+\sqrt{1+4A+4B}\right)^2}\lambda - \frac{4A}{\left(1+\sqrt{1+4A+4B}\right)^2} = 0.$$

By applying the linearized stability Theorem, see [13], we obtain the following result.

Theorem 12 *The unique positive equilibrium solution $\bar{x}$ of Equation (9) is:*

- *i) locally asymptotically stable when $B + 3A > 4A^2$;*
- *ii) a saddle point when $B + 3A < 4A^2$;*
- *iii) a non-hyperbolic point of stable type (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \frac{1}{4A} < 1$) when $B + 3A = 4A^2$.*

In the next lemma we prove the existence of period two solutions of Equation (9).

Lemma 13 *Equation (9) has the minimal period-two solution $\{(0, 1), (1, 0)\}$ and the minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$, where*

$$\phi = \frac{A-\sqrt{(A-B)(A(-3+4A)-B}-B}{2A(A-B)} \text{ and } \psi = \frac{A+\sqrt{(A-B)(A(-3+4A)-B}-B}{2A(A-B)} \tag{11}$$

if and only if

$$\frac{3}{4} < A < 1 \text{ and } B + 3A < 4A^2 \quad \text{or} \quad A > 1 \text{ and } B + 3A > 4A^2.$$

Proof. A minimal period-two solution is a positive solution of the following system

$$\left\{\begin{array}{r} x+(B-A)y-1=0 \\ -Axy+y=0. \end{array}\right. \tag{12}$$

where $\phi+\psi=x$ and $\phi\psi=y$. The system (12) has two solutions $x=1,\ y=0$ and

$$x=\frac{1}{A},\quad y=\frac{A-1}{A(B-A)}.$$

For second solution we have that $x, y, x^2-4y>0$ if and only if

$$\frac{3}{4}<A<1 \text{ and } B+3A<4A^2 \quad \text{or} \quad A>1 \text{ and } B+3A>4A^2.$$

Now, ϕ and ψ are solution of equation

$$t^2-\frac{1}{A}t-\frac{A-1}{A(B-A)}=0,$$

and the proof is complete. ■

The following theorem describes the local stability nature of the period-two solutions.

Theorem 14 *Consider Equation* (9).

i) The minimal period two solution $\{(0,1),(1,0)\}$ *is:*

- *a) locally asymptotically stable when* $A>1$*;*
- *b) a saddle point when* $A<1$*;*
- *c) a non-hyperbolic point of the stable type when* $A=1$.

ii) The minimal period two solution $\{P(\phi,\psi), Q(\psi,\phi)\}$*, given by* (11) *is:*

- *a) locally asymptotically stable when* $\frac{3}{4}<A<1$ *and* $B+3A<4A^2$*;*
- *b) a saddle point when* $A>1$ *and* $B+3A>4A^2$.

iii) If $A=B=1$ *the minimal period two solution* $\{\phi, 1-\phi\}$ *(*$0<\phi<1$*) is non-hyperbolic.*

Proof. In order to prove this theorem, we associate the second iterate map to Equation (9). We have

$$T^2\begin{pmatrix} u \\ v \end{pmatrix}=\begin{pmatrix} g(u,v) \\ h(u,v) \end{pmatrix}$$

where

$$g(u,v)=\frac{u}{Av^2+Buv+u},\quad h(u,v)=\frac{v}{v+\frac{Au^2}{\left(Av^2+Buv+u\right)^2}+\frac{Buv}{Av^2+Buv+u}}.$$

The Jacobian of the map T^2 has the following form

$$J_{T^2}\begin{pmatrix} \phi \\ \psi \end{pmatrix}=\begin{pmatrix} g'_u(\phi,\psi) & g'_v(\phi,\psi) \\ h'_u(\phi,\psi) & h'_v(\phi,\psi) \end{pmatrix}$$

where

$$g'_u=\frac{Av^2}{\left(Av^2+Buv+u\right)^2},\quad g'_v=-\frac{u(Bu+2Av)}{\left(Av^2+Buv+u\right)^2},$$

$$h'_u=-\frac{Av^3(u+Buv+Av^2)(Buv(1+Bv)+A(2u+Bv^3))}{(A^2v^5+u^2v(1+Bv)(1+B+Bv)+Au(u+v^3(2+B+2Bv))^2},$$

$$h'_v=\frac{u(u+Buv+Av^2)(B^2u^2v^2(1+Bv)+A^2v^2(5u+2Bv^3)+Au(u+3Buv+Bv^3(2+3Bv^3)))}{(A^2v^5+u^2v(1+Bv)(1+B+Bv)+Au(u+v^3(2+B+2Bv))^2}.$$

Set

$$\mathcal{S}=g'_u(\phi,\psi)+h'_v(\phi,\psi)$$

and

$$\mathcal{D}=g'_u(\phi,\psi)h'_v(\phi,\psi)-g'_v(\phi,\psi)h'_u(\phi,\psi).$$

After some lengthy calculation one can see that:

i) for the minimal period-two solution $\{(0,1),(1,0)\}$ we have

$$\mathcal{S}=\frac{1}{A} \text{ and } \mathcal{D}=0$$

and applying the linearized stability Theorem [13] we obtain that the minimal period-two solution $\{(0,1),(1,0)\}$ of Equation (9) is:

   a) locally asymptotically stable when $A>1$;
   b) a saddle point when $A<1$;
   c) a non-hyperbolic point of the stable type when $A=1$.

ii) For the positive minimal period two solution $\{P(\phi,\psi),Q(\psi,\phi)\}$ we have

$$\mathcal{S}=\frac{6A^4+A(B-2)B-B^2-3A^3(3+2B)+A^2(4+B(6+B))}{A^2(A-B)^2}, \quad \mathcal{D}=\frac{(A-1)^2}{(A-B)^2}.$$

Applying the linearized stability Theorem [13] we obtain that the minimal period-two solution $\{P(\phi,\psi),Q(\psi,\phi)\}$ of Equation (9) is:

   a) locally asymptotically stable when $\frac{3}{4}<A<1$ and $B+3A<4A^2$;
   b) a saddle point when $A>1$ and $B+3A>4A^2$.

iii) If $A=B=1$ then

$$\mathcal{S}=1+\phi^2(1-\phi)^2, \quad \mathcal{D}=\phi^2(1-\phi)^2$$

from which the proof follows.

■

3.2 Global results and basins of attraction

In this section we present global dynamics results for Equation (9).

Theorem 15 *If $B+3A>4A^2$ and $0<A<1$ then Equation (9) has a unique equilibrium solution $E(\overline{x},\overline{x})$ given by* (10) *which is locally asymptotically stable and the minimal period-two solution $\{P(0,1),Q(1,0)\}$ which is a saddle point. Furthermore, the global stable manifold of the period-two solution $\{P,Q\}$ is given by $\mathcal{W}^s(\{P,Q\})=\mathcal{W}^s(P)\cup\mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are the coordinate axes. The basin of attraction $\mathcal{B}(E)=\{(x,y): x\geq 0, y\geq 0\}$. More precisely*

i) If $(u_0,v_0)\in\mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P, and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to Q.

ii) If $(u_0,v_0)\in\mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to Q, and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to P.

iii) If $(u_0,v_0)\in\mathcal{R}\setminus(\mathcal{W}^s(P)\cup\mathcal{W}^s(Q))$ (the region between $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$) then the sequence $\{(u_n,v_n)\}$ is attracted to $E(\overline{x},\overline{x})$.

See Figure 1 for visual illustration.

Proof. The proof is direct application of Theorem 10 part (a). ■

Theorem 16 *If $B+3A>4A^2$ and $A=1$ then Equation (9) has a unique equilibrium solution $E(\overline{x},\overline{x})$ which is locally asymptotically stable and the minimal period-two solution, $\{P(0,1),Q(1,0)\}$ which is a non-hyperbolic point of stable type. Furthermore, the global stable manifold of the period-two solution $\{P,Q\}$ is given by $\mathcal{W}^s(\{P,Q\})=\mathcal{W}^s(P)\cup\mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are the coordinate axes. The global dynamics is given in Theorem 15.*

Proof. In view of Remark 11 the proof is direct application of Theorem 10 part (a).
■

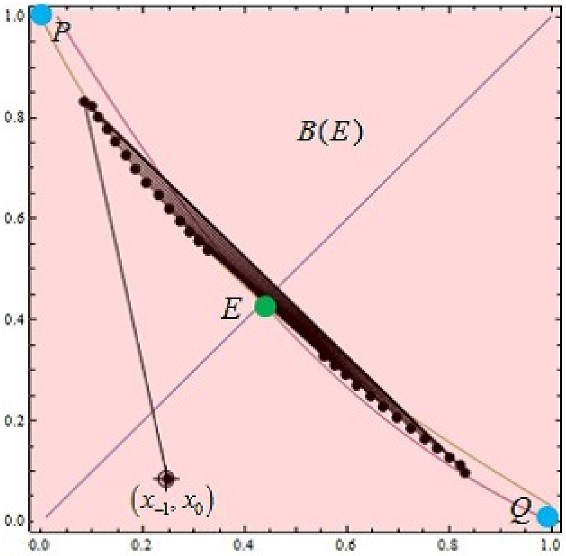

Figure 1: Visual illustration of Theorem 15.

Theorem 17 *If $B+3A>4A^2$ and $A>1$ then Equation (9) has a unique equilibrium solution $E(\overline{x},\overline{x})$ which is locally asymptotically stable and two minimal period-two solutions $\{P_1(0,1),Q_1(1,0)\}$ which is locally asymptotically stable and $\{P_2(\phi,\psi),Q_2(\psi,\phi)\}$ given by* (11)*, which is a saddle point. Furthermore, the global stable manifold of the period-two solution $\{P_2,Q_2\}$ is given by $\mathcal{W}^s(\{P_2,Q_2\})=\mathcal{W}^s(P_2)\cup\mathcal{W}^s(Q_2)$ where $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ are continuous increasing curves, that divide the first quadrant into two connected components, namely*

$\mathcal{W}_1^+:=\{x\in\mathcal{R}\setminus\mathcal{W}^s(P_2):\exists y\in\mathcal{W}^s(P_2) \text{ with } y\preceq_{se}x\}$ *and* $\mathcal{W}_1^-:=\{x\in\mathcal{R}\setminus\mathcal{W}^s(P_2):\exists y\in\mathcal{W}^s(P_2) \text{ with } x\preceq_{se}y\}$

$\mathcal{W}_2^+:=\{x\in\mathcal{R}\setminus\mathcal{W}^s(Q_2):\exists y\in\mathcal{W}^s(Q_2) \text{ with } y\preceq_{se}x\}$ *and* $\mathcal{W}_2^-=\{x\in\mathcal{R}\setminus\mathcal{W}^s(Q_2):\exists y\in\mathcal{W}^s(Q_2) \text{ with } x\preceq_{se}y\}$

respectively such that the following statements are true.

i) If $(u_0,v_0)\in\mathcal{W}^s(P_2)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P_2 and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to Q_2.

ii) If $(u_0,v_0)\in\mathcal{W}^s(Q_2)$ then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to Q_2 and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ is attracted to P_2.

iii) If $(u_0,v_0)\in\mathcal{W}_1^-$ (the region above $\mathcal{W}^s(P_2)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ is attracted to P_1 and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to Q_1.

iv) If $(u_0,v_0)\in\mathcal{W}_2^+$ (the region below $\mathcal{W}^s(Q_2)$) then the subsequence of even-indexed terms $\{(u_{2n},v_{2n})\}$ tends to Q_1 and the subsequence of odd-indexed terms $\{(u_{2n+1},v_{2n+1})\}$ tends to P_1.

v) If $(u_0,v_0)\in\mathcal{W}_1^+\cap\mathcal{W}_2^-$ (the region between $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$) then the sequence $\{(u_n,v_n)\}$ is attracted to $E(\overline{x},\overline{x})$.

Shortly the basin of attraction of E is the region between $\mathcal{W}^s(P_2)$ and $\mathcal{W}^s(Q_2)$ while the rest of the first quadrant without $\mathcal{W}^s(P_2)\cup\mathcal{W}^s(Q_2)\cup(0,0)$ is the basin of attraction of $\{P_1,Q_1\}$.

See Figure 2 for visual illustration.

Proof. The proof is direct application of Theorem 10 part (d).

■

Theorem 18 *If $B+3A<4A^2$ and $\frac{3}{4}<A<1$ then Equation (9) has a unique equilibrium solution $E(\overline{x},\overline{x})$ which is a saddle point and minimal period-two solution $\{P_1(0,1),Q_1(1,0)\}$ which is a saddle point and $\{P_2(\phi,\psi),Q_2(\psi,\phi)\}$, given by* (11) *which is locally asymptotically stable. Furthermore, there exists a set $\mathcal{C}_E$ which is an invariant subset of the basin of attraction of E. The set $\mathcal{C}_E$ is a graph of a strictly increasing continues function of the first variable*

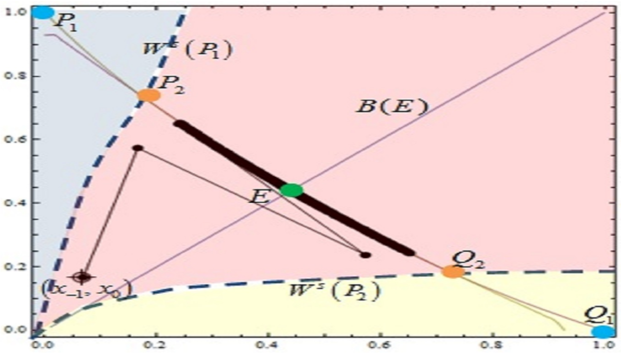

Figure 2: Visual illustration of Theorem 17.

on $(0,\infty)$ *interval and separates* $\mathcal{R}\setminus(0,0)$*, where* $\mathcal{R}=(0,\infty)\times(0,\infty)$ *into two connected and invariant components* $\mathcal{W}^-(\overline{x},\overline{x})$ *and* $\mathcal{W}^+(\overline{x},\overline{x})$*. The global stable manifold of the period-two solution* $\{P_1,Q_1\}$ *is given by* $\mathcal{W}^s(\{P_1,Q_1\})=\mathcal{W}^s(P_1)\cup\mathcal{W}^s(Q_1)$ *where* $\mathcal{W}^s(P_1)$ *and* $\mathcal{W}^s(Q_1)$ *are continuous nondecreasing curves which represent the coordinate axes. The basin of attraction of* $\{P_2,Q_2\}$ *is the first quadrant without* $\mathcal{W}^s(P_1)\cup\mathcal{W}^s(Q_1)\cup(0,0)\cup\mathcal{C}_E$*. More precisely*

- *i) Every initial point* (u_0,v_0) *in* $\mathcal{C}_E$ *is attracted to* E*.*
- *ii) If* $(u_0,v_0)\in\mathcal{W}^s(P_1)$ *then the subsequence of even-indexed terms* $\{(u_{2n},v_{2n})\}$ *is attracted to* P_1 *and the subsequence of odd-indexed terms* $\{(u_{2n+1},v_{2n+1})\}$ *is attracted to* Q_1*.*
- *iii) If* $(u_0,v_0)\in\mathcal{W}^s(Q_1)$ *then the subsequence of even-indexed terms* $\{(u_{2n},v_{2n})\}$ *is attracted to* Q_1 *and the subsequence of odd-indexed terms* $\{(u_{2n+1},v_{2n+1})\}$ *is attracted to* P_1*.*
- *iv) If* $(u_0,v_0)\in\mathcal{W}^-(\overline{x},\overline{x})$ *(the region between* $\mathcal{C}_E$ *and* $\mathcal{W}^s(P_1)$*) then the subsequence of even-indexed terms* $\{(u_{2n},v_{2n})\}$ *is attracted to* P_2 *and the subsequence of odd-indexed terms* $\{(u_{2n+1},v_{2n+1})\}$ *tends to* Q_2*.*
- *v) If* $(u_0,v_0)\in\mathcal{W}^+(\overline{x},\overline{x})$ *(the region between* $\mathcal{C}_E$ *and* $\mathcal{W}^s(Q_1)$*) then the subsequence of even-indexed terms* $\{(u_{2n},v_{2n})\}$ *tends to* Q_2 *and the subsequence of odd-indexed terms* $\{(u_{2n+1},v_{2n+1})\}$ *tends to* P_2*.*

See Figure 3 for visual illustration.

Proof. Theorem 12 implies that there exists a unique equilibrium solution $E(\overline{x},\overline{x})$ which is a saddle point and Theorem 14 implies that minimal period-two solution $\{P_1(0,1),Q_1(1,0)\}$ is a saddle point and $\{P_2(\phi,\psi),Q_2(\psi,\phi)\}$ is locally asymptotically stable. Now the proof is direct application of Theorem 10 part (c). ■

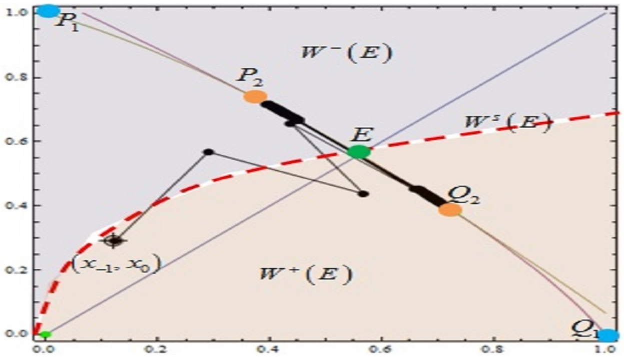

Figure 3: Visual illustration of Theorem 18.

Theorem 19 *If* $B+3A<4A^2$ *and* $A=1$ *then Equation (9) has a unique equilibrium solution* $E(\overline{x},\overline{x})$*, which is a saddle point and the minimal period-two solution* $\{P_1(0,1),Q_1(1,0)\}$ *which is a non-hyperbolic point of stable type.*

Furthermore, the global stable manifold $\mathcal{W}^s(E)$ is continuous increasing curve which divides first quadrant and the global stable manifold of the period-two solution $\{P_1, Q_1\}$ is given by $\mathcal{W}^s(\{P_1, Q_1\}) = \mathcal{W}^s(P_1) \cup \mathcal{W}^s(Q_1)$ where $\mathcal{W}^s(P_1)$ and $\mathcal{W}^s(Q_1)$ are the coordinate axes. The basin of attraction $\mathcal{B}(\{P_1, Q_1\}) = \{(x, y) : x \geq 0, y \geq 0\} \setminus (\mathcal{W}^s(E) \cup (0, 0))\}$. More precisely

- *i) Every initial point (u_0, v_0) in $\mathcal{W}^s(E)$ is attracted to E.*
- *i) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P_1.*
- *iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P_1 and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q_1.*

See Figure 4 for visual illustration.

Proof. From Theorem 12 Equation (9) has a unique equilibrium point $E(\overline{x}, \overline{x})$ which is a saddle point. Theorem 14 implies that the period-two solution $\{P, Q\}$ is a non-hyperbolic point. In view of Remark 11 the proof is direct application of Theorem 10 part (b). ■

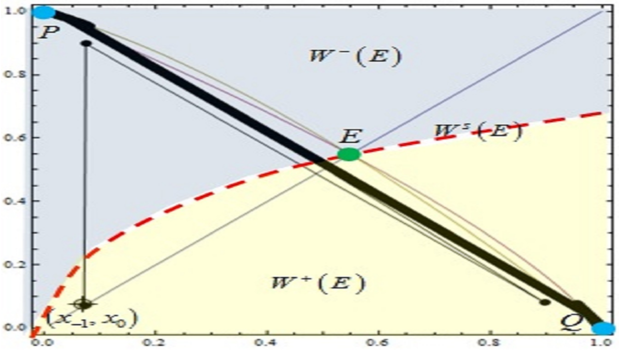

Figure 4: Visual illustration of Theorem 19.

Theorem 20 *If $B + 3A < 4A^2$ and $A > 1$ then Equation (9) has a unique equilibrium solution $E(\overline{x}, \overline{x})$ which is a saddle point and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is locally asymptotically stable. The global behavior is the same as in Theorem 19.*

Proof. The proof is direct application of Theorem 10 part (b). ■

Theorem 21 *Assume that $B + 3A = 4A^2$.*

- *a) If $\frac{3}{4} < A < 1$ then Equation (9) has a unique equilibrium point $E(\overline{x}, \overline{x})$ which is a non-hyperbolic point of stable type and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is a saddle point. Then every initial point (u_0, v_0) in $\mathcal{R}$ is attracted to E.*
- *b) If $A > 1$ then Equation (9) has a unique equilibrium solution $E(\overline{x}, \overline{x})$ which is a non-hyperbolic point of the stable type and the minimal period-two solution $\{P(0, 1), Q(1, 0)\}$ which is locally asymptotically stable. The global behavior is the same as in Theorem 19.*
- *c) If $A = 1$ then Equation (9) has a unique equilibrium solution $E(\overline{x}, \overline{x})$ and infinitely many minimal period-two solution $\{P(\phi, 1 - \phi), Q(1 - \phi, \phi)\}$ $(0 < \phi < 1)$ which are a non-hyperbolic points of stable type.*
  - *i) There exists a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E*
  - *ii) For every minimal period-two solution $\{P(\phi, 1 - \phi), Q(1 - \phi, \phi)\}$ $(0 < \phi < 1)$ there exists the global stable manifold given by $\mathcal{W}^s(\{P, Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are continuous increasing curves. If $(u_0, v_0) \in \mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to P and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to Q. If $(u_0, v_0) \in \mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to Q and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to P. The union of these stable manifolds and $\mathcal{C}_E$ foliates the first quadrant without the singular point $(0, 0)$.*

See Figure 5 for visual illustration.

Proof.

a) From Theorem 12 Equation (9) has a unique equilibrium point $E(\overline{x}, \overline{x}) = (\frac{1}{2A}, \frac{1}{2A})$ which is non-hyperbolic of stable type. From Theorem 14 Equation (9) has a unique minimal period-two solution $\{P_1(0,1), Q_1(1,0)\}$ which is a saddle point. In view of Remark 11 the proof is direct application of Theorem 10 part (a).

b) From Theorem 12 Equation (9) has a unique equilibrium point $E(\overline{x}, \overline{x}) = (\frac{1}{2A}, \frac{1}{2A})$, which is non-hyperbolic of stable type. From Theorem 14 Equation (9) has a unique minimal period-two solution $\{P_1(0,1), Q_1(1,0)\}$ which is locally asymptotically stable point. In view of Remark 11 the proof is direct application of Theorem 10 part (b).

c) From Theorem 12 Equation (9) has a unique equilibrium point $E(\overline{x}, \overline{x}) = (\frac{1}{2A}, \frac{1}{2A})$ which is non-hyperbolic. All conditions of Theorem 5 are satisfied, which yields the existence a continuous increasing curve $\mathcal{C}_E$ which is a subset of the basin of attraction of E. The proof of the statement ii) follows from Theorems 3, 5, 14 and Theorem 5 in [8].

■

Remark 22 The global dynamics of Equation (9) can be described in the language of bifurcation theory as follows: when $B + 3A \neq 4A^2$, then the period-doubling bifurcation happens when A is passing through the value 1 in such a way that for $A > 1$ new interior period-two solution emerges and exchange stability with already existing period-two solution on the boundary. Another bifurcation happens when $B + 3A < 4A^2$ in which case the stability of the unique equilibrium changes from local attractor to the saddle point. Finally, there is a bifurcation at another critical value $B + 3A = 4A^2$ when A is passing through the critical value 1, which is one of exchange stability between the unique equilibrium and unique period-two solution, with specific dynamics at $A = 1$, when there is an infinite number of period-two solutions which basins of attraction filled up the first quadrant without the origin. See [16] for similar results.

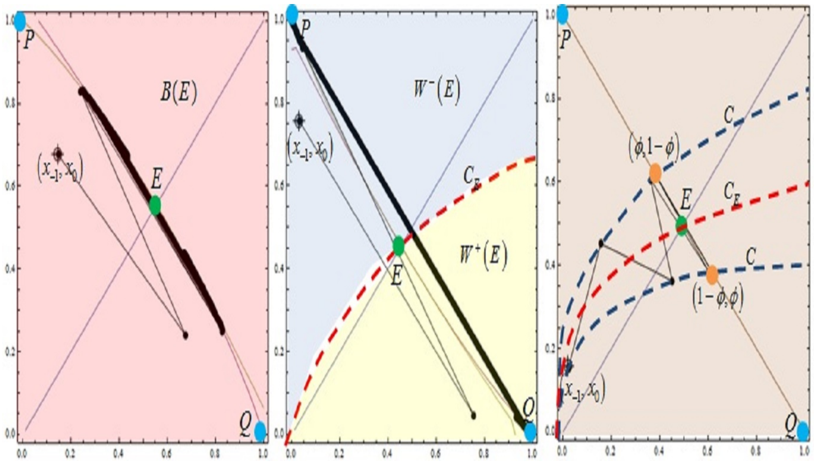

Figure 5: Visual illustration of Theorem 21.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.1, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Global Dynamics of Generalized Second-Order Beverton–Holt Equations of Linear and Quadratic Type

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Abstract. We investigate second-order generalized Beverton–Holt difference equations of the form

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \ldots,$$

where f is a function nondecreasing in both arguments, the parameter a is a positive constant, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers in the domain of f. We will discuss several interesting examples of such equations and present some general theory. In particular, we will investigate the local and global dynamics in the event f is a certain type of linear or quadratic polynomial, and we explore the existence problem of period-two solutions.

Keywords. attractivity, difference equation, invariant sets, periodic solutions, stable set .

AMS 2010 Mathematics Subject Classification: 39A20, 39A28, 39A30, 92D25

1 Introduction and Preliminaries

Consider the following second-order difference equation:

$$x_{n+1} = \frac{af(x_n, x_{n-1})}{1 + f(x_n, x_{n-1})}, \quad n = 0, 1, \ldots. \tag{1}$$

Here f is a continuous function nondecreasing in both arguments, the parameter a is a positive real number, and the initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers in the domain of f. Equation (1) is a generalization of the first-order Beverton–Holt equation

$$x_{n+1} = \frac{ax_n}{1 + x_n}, \quad n = 0, 1, \ldots, \tag{2}$$

where $a > 0$ and $x_0 \geq 0$. The global dynamics of Equation (2) may be summarized as follows, see [9, 15]:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a \leq 1 \\ a-1 & \text{if } a > 1 \text{ and } x_0 > 0. \end{cases} \tag{3}$$

Many variations of Equation (2) have been studied. German biochemist Leonor Michaelis and Canadian physician Maud Menten used the model in their study of enzyme kinetics in 1913; see [20]. Additionally, Jacques Monod, a French biochemist, happened upon the model empirically in his study of microorganism growth around 1942; see [20]. It was not until 1957 that fisheries scientists Ray Beverton and Sidney Holt used the model in their study of population dynamics, see [1, 9]. The so-called Monod differential equation [20] is given by

$$\frac{1}{N} \cdot \frac{dN}{dt} = \frac{rS}{a + S}, \tag{4}$$

where $N(t)$ is the concentration of bacteria at time t, $\frac{dN}{dt}$ is the growth rate of the bacteria, $S(t)$ is the concentration of the nutrient, r is the maximum growth rate of the bacteria, and a is a half-saturation

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constant (when $S = a$, the right-hand side of Equation (4) equals $r/2$). Based on experimental data, the following system of two differential equations for the nutrient S and bacteria N, as presented in [20], is given by

$$\frac{dS}{dt} = -\frac{1}{\gamma}N\frac{rS}{a+S}, \quad \frac{dN}{dt} = N\frac{rS}{a+S}, \tag{5}$$

where the constant γ is called the growth yield. Both Equation (4) and System (5) contain the function $f(x) = rx/(a + x)$ known as the Monod function, Michaelis-Menten function, Beverton–Holt function, or Holling function of the first kind; see [1, 5, 9, 11].

One possible two-generation population model based on Equation (2),

$$x_{n+1} = \frac{a_1 x_n}{1+x_n} + \frac{a_2 x_{n-1}}{1+x_{n-1}}, \quad n = 0, 1, \ldots, \tag{6}$$

where $a_i > 0$ for $i = 1, 2$ and $x_{-1}, x_0 \geq 0$, was considered in [18]. The global dynamics of Equation (6) may be summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a_1 + a_2 \leq 1 \\ a_1 + a_2 - 1 & \text{if } a_1 + a_2 > 1 \text{ and } x_0 + x_{-1} > 0. \end{cases}$$

This result was extended in [5] to the case of a k-generation population model based on Equation (2) of the form

$$x_{n+1} = \sum_{i=0}^{k-1} \frac{a_i x_{n-i}}{1+x_{n-i}}, \quad n = 0, 1, \ldots, \tag{7}$$

where $a_i \geq 0$ for $i = 0, 1, \ldots, k-1$, $\sum\limits_{i=0}^{k-1} a_i > 0$, and $x_{1-k}, \ldots, x_0 \geq 0$. It was shown that the global dynamics of Equation (7) may be given precisely by (3), where $a = \sum\limits_{i=0}^{k-1} a_i$ and we consider all initial conditions positive.

The simplest model of Beverton–Holt type which exhibits two coexisting attractors and the Allee effect is the sigmoid Beverton–Holt (or second-type Holling) difference equation

$$x_{n+1} = \frac{a x_n^2}{1+x_n^2}, \quad n = 0, 1, \ldots, \tag{8}$$

where $a > 0$ and $x_0 \geq 0$. The dynamics of Equation (8) may be concisely summarized as follows:

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } a < 2 \text{ or } (a \geq 2 \text{ and } x_0 < \overline{x}_-) \\ \overline{x}_- & \text{if } a \geq 2 \text{ and } x_0 = \overline{x}_- \\ \overline{x}_+ & \text{if } a \geq 2 \text{ and } x_0 > \overline{x}_-, \end{cases} \tag{9}$$

where $\overline{x}_-$ and $\overline{x}_+$ are the two positive equilibria when $a \geq 2$; see [1, 5]. One possible two-generation population model based on Equation (8),

$$x_{n+1} = \frac{a_1 x_n^2}{1+x_n^2} + \frac{a_2 x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{10}$$

where $a_i > 0$ for $i = 1, 2$ and $x_{-1}, x_0 \geq 0$, was considered in [4]. However, the summary of the global dynamics of Equation (10) is not an immediate extension of the global dynamics of Equation (8) as given in

(9); see [4]. Equation (10) can have up to three equilibrium solutions and up to three period-two solutions. In the case when Equation (10) has three equilibrium solutions and three period-two solutions, the zero equilibrium, the larger positive equilibrium, and one period-two solution are attractors with substantial basins of attraction, which together with the remaining equilibrium and the global stable manifolds of the saddle-point period-two solutions exhaust the first quadrant of initial conditions. This behavior happens when the coefficient a_2 is in some sense dominant to a_1; see [4]. Such behavior is typical for other models in population dynamics such as

$$x_{n+1} = \frac{a_1 x_n}{1+x_n} + \frac{a_2 x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \ldots$$

and

$$x_{n+1} = a_1 x_n + \frac{a_2 x_{n-1}^2}{1+x_{n-1}^2}, \quad n = 0, 1, \ldots,$$

which were also investigated in [4]. In the case of a k-generation population model based on the sigmoid Beverton–Holt difference equation with $k > 2$, one can expect to have attractive period-k solutions as well as chaos.

The first model of the form given in Equation (1), where f is a linear function in both variables (that is, $f(u,v) = cu + dv$ for $c, d, u, v \geq 0$) was considered in [19] to describe the global dynamics in part of the parametric space. Here we will extend the results from [19] to the whole parametric space. In this paper we will then restrict ourselves to the case when $f(u,v)$ is a quadratic polynomial, which will give similar global dynamics to that presented for Equation (10). The corresponding dynamic scenarios will be essentially the same for any polynomial function of the type $f(u,v) = cu^k + du^m$ where $c, d \geq 0$ and m, k are positive integers. Higher values of m and k may only create additional equilibria and period-two solutions but should replicate the global dynamics seen in the quadratic case presented in this paper. The global dynamics of some higher-order transcendental-type generalized Beverton-Holt equation was considered in [3].

Let the function $F : [0,\infty)^2 \to [0,a)$ be defined as follows:

$$F(u,v) = \frac{af(u,v)}{1+f(u,v)}. \tag{11}$$

Then Equation (1) becomes $x_{n+1} = F(x_n, x_{n-1})$ for all $n = 0, 1, \ldots$, where $F(u,v)$ is nondecreasing in both of its arguments.

The following theorem from [2] immediately applies to Equation (1).

Theorem 1 *Let I be a set of real numbers and $F : I \times I \to I$ be a function which is nondecreasing in the first variable and nondecreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = F(x_n, x_{n-1}), \quad x_{-1}, x_0 \in I, \quad n = 0, 1, \ldots, \tag{12}$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution are eventually monotonic.

The consequence of Theorem 1 is that every bounded solution of Equation (12) converges to either an equilibrium, a period-two solution, or to a singular point on the boundary. It should be noticed that Theorem 1 is specific for second-order difference equations and does not extend to difference equations of order higher than two. Furthermore, the powerful theory of monotone maps in the plane [16, 17] can be applied to Equation (1) to determine the boundaries of the basins of attraction of the equilibrium solutions and period-two solutions. Finally, when $f(u,v)$ is a polynomial function, all computation needed to determine the local stability of all equilibrium solutions and period-two solutions is reduced to the theory of counting the number of zeros of polynomials in a given interval, as given in [12]. This theory will give more precise results than the global attractivity and global asymptotic stability results in [7, 8]. However, in the case of difference equations of the form

$$x_{n+1} = \frac{ag(x_n, x_{n-1}, \ldots, x_{n+1-k})}{1 + g(x_n, x_{n-1}, \ldots, x_{n+1-k})}, \quad n = 0, 1, \ldots, \quad k \geq 1,$$

where $a > 0$ and g is nondecreasing in all its arguments, Theorem 1 does not apply for $k > 2$, but the results from [7, 8, 13] can give global dynamics in some regions of the parametric space.

The following theorem from [10] is often useful in determining the global attractivity of a unique positive equilibrium.

Theorem 2 *Let $I \subseteq [0, \infty)$ be some open interval and assume that $F \in C[I \times I, (0, \infty)]$ satisfies the following conditions:*

(i) $F(x, y)$ is nondecreasing in each of its arguments;

(ii) Equation (12) has a unique positive equilibrium point $\overline{x} \in I$ and the function $F(x, x)$ satisfies the ***negative feedback condition****:*

$$(x - \overline{x})(F(x, x) - x) < 0 \text{ for every } x \in I \backslash \{\overline{x}\}.$$

Then every positive solution of Equation (12) with initial conditions in I converges to $\overline{x}$.

The following result from [4] will be used to describe the global dynamics of Equation (1).

Theorem 3 *Assume that difference equation (12) has three equilibrium points $U_1 \leq \bar{x}_0 < \bar{x}_{SW} < \bar{x}_{NE}$ where the equilibrium points $\bar{x}_0$ and $\bar{x}_{NE}$ are locally asymptotically stable. Further, assume that there exists a minimal period-two solution $\{\Phi_1, \Psi_1\}$ which is a saddle point such that $(\Phi_1, \Psi_1) \in int(Q_2(E_{SW}))$. In this case there exist four continuous curves $\mathcal{W}^s(\Phi_1, \Psi_1)$, $\mathcal{W}^s(\Psi_1, \Phi_1), \mathcal{W}^u(\Phi_1, \Psi_1), \mathcal{W}^u(\Psi_1, \Phi_1)$, where $\mathcal{W}^s(\Phi_1, \Psi_1), \mathcal{W}^s(\Psi_1, \Phi_1)$ are passing through the point E_{SW}, and are graphs of decreasing functions. The curves $\mathcal{W}^u(\Phi_1, \Psi_1), \mathcal{W}^u(\Psi_1, \Phi_1)$ are the graphs of increasing functions and are starting at E_0. Every solution which starts below $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$ in the North-east ordering converges to E_0 and every solution which starts above $\mathcal{W}^s(\Phi_1, \Psi_1) \cup \mathcal{W}^s(\Psi_1, \Phi_1)$ in the North-east ordering converges to E_{NE}, i.e. $\mathcal{W}^s(\Phi_1, \Psi_1) = \mathcal{C}_1^+ = \mathcal{C}_2^+$ and $\mathcal{W}^s(\Psi_1, \Phi_1) = \mathcal{C}_1^- = \mathcal{C}_2^-$.*

This paper is organized as follows. The next section deals with the local stability of equilibrium solutions and period-two solutions of the general second-order difference equation (12), where $F(u, v)$ is nondecreasing in both of its arguments. In view of the results for monotone maps in [16, 17] and their applications to second-order difference equations in [4, 5], the local dynamics of the equilibrium solutions and period-two solutions will determine the global dynamics in hyperbolic cases and some nonhyperbolic cases as well. The third section will provide some examples of global dynamic scenarios of Equation (1) when the function $f(u, v)$ is either linear in both variables or linear in one variable and quadratic in the other variable. The obtained results will be interesting from a modeling point of view as they show that the appearance of period-two solutions with substantial basins of attraction (sets which contain open subsets) is controlled by the coefficient of the x_{n-1} term that is affected by the size of the grandparents' population. The same phenomenon was observed in the case of Equation (10).

2 Local Stability

In this section we provide general conditions to determine the local stability of equilibrium solutions and period-two solutions.

It is clear that $x_n \leq a$ for all $n \geq 1$. In light of Theorem 1, since all solutions are bounded, if there are no singular points on the boundary of the domain of F, it immediately follows that all solutions to Equation (1) converge to an equilibrium or a period-two solution.

An equilibrium $\overline{x}$ of Equation (1) satisfies

$$\overline{x}(1+f(\overline{x},\overline{x})) = af(\overline{x},\overline{x}). \tag{13}$$

Clearly $\overline{x}_0 = 0$ is an equilibrium point if and only if $(0,0)$ is in the domain of f and $f(0,0)=0$.

The linearized equation of Equation (1) about an equilibrium $\overline{x}$ is

$$z_{n+1} = F_u(\overline{x},\overline{x})z_n + F_v(\overline{x},\overline{x})z_{n-1}, \quad n = 0,1,\ldots.$$

Since f is a nondecreasing function, it follows that $F_u(\overline{x},\overline{x}) \geq 0, F_v(\overline{x},\overline{x}) \geq 0$. Therefore, if

$$\lambda(\overline{x}) = F_u(\overline{x},\overline{x}) + F_v(\overline{x},\overline{x}) = \frac{a(f_u(\overline{x},\overline{x}) + f_v(\overline{x},\overline{x}))}{(1+f(\overline{x},\overline{x}))^2}, \tag{14}$$

then in view of Corollary 2 of [13] we may conclude that

$$\overline{x} \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } \lambda(\overline{x}) < 1 \\ \text{nonhyperbolic} & \text{if } \lambda(\overline{x}) = 1 \\ \text{unstable} & \text{if } \lambda(\overline{x}) > 1. \end{cases}$$

Further, Theorem 2.13 of [15] implies that if $\overline{x}$ is unstable, then

$$\overline{x} \text{ is} \begin{cases} \text{a repeller} & \text{if } \delta(\overline{x}) > 1 \\ \text{nonhyperbolic} & \text{if } \delta(\overline{x}) = 1 \\ \text{a saddle point} & \text{if } \delta(\overline{x}) < 1, \end{cases}$$

where

$$\delta(\overline{x}) = F_v(\overline{x},\overline{x}) - F_u(\overline{x},\overline{x}) = \frac{a(f_v(\overline{x},\overline{x}) - f_u(\overline{x},\overline{x}))}{(1+f(\overline{x},\overline{x}))^2}. \tag{15}$$

Let (ϕ,ψ) be a period-two solution of Equation (1). The Jacobian matrix of the corresponding map $T = G^2$, where $G(u,v) = (v, F(v,u))$ and F is given by Equation (11), is given in Theorem 12 of [6]. The linearized equation evaluated at (ϕ,ψ) is

$$\lambda^2 - TrJ_T(\phi,\psi)\lambda + DetJ_T(\phi,\psi) = 0,$$

where

$$TrJ_T(\phi,\psi) = D_2F(\psi,\phi) + D_1F(F(\psi,\phi),\psi) \cdot D_1F(\psi,\phi) + D_2F(F(\psi,\phi),\psi)$$

and

$$DetJ_T(\phi,\psi) = D_2F(F(\psi,\phi),\psi) \cdot D_2F(\psi,\phi).$$

3 Examples

In this section we present four examples of different forms of Equation (1) where the transition function $f(u,v)$ is linear or quadratic polynomial in its variables which effects the global dynamics.

3.1 Linear-Linear: $f(u,v) = cu + dv$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1})}{1 + cx_n + dx_{n-1}}, \quad n = 0, 1, \ldots, \tag{16}$$

where $c \geq 0$ and $d > 0$. If $d = 0$, then Equation (16) becomes Equation (2) after a reduction of parameters. By Equation (13) we know that $\overline{x}_0 = 0$ is always a fixed point and $\overline{x}_+ = \frac{a(c+d)-1}{c+d}$ is a unique positive fixed point for $a(c+d) > 1$.

Since $\lambda(\overline{x}_0) = a(c+d)$, we have that

$$\overline{x}_0 \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } a(c+d) < 1 \\ \text{nonhyperbolic} & \text{if } a(c+d) = 1 \\ \text{unstable} & \text{if } a(c+d) > 1. \end{cases}$$

Further, notice that

$$\lambda(\overline{x}_+) = \frac{a(c+d)}{\left(1 + \left(\frac{a(c+d)-1}{c+d}\right) \cdot (c+d)\right)^2} = \frac{1}{a(c+d)} < 1$$

for all values of parameters for which $\overline{x}_+$ exists. Therefore

$$\overline{x}_+ = \frac{a(c+d)-1}{c+d} \text{ is always locally asymptotically stable.}$$

Note that there is an exchange in stability from $\overline{x}_0$ to $\overline{x}_+$ as the parametric value $a(c+d)$ passes through 1.

We next search for period-two solutions. Suppose there exists such a solution $\{\psi, \phi, \psi, \phi, \ldots\}$ with $\phi \neq \psi$. Then $\{\psi, \phi\}$ satisfies the following system:

$$\begin{cases} \psi & = \dfrac{af(\phi,\psi)}{1 + f(\phi,\psi)} & = \dfrac{a(c\phi + d\psi)}{1 + c\phi + d\psi} \\ \phi & = \dfrac{af(\psi,\phi)}{1 + f(\psi,\phi)} & = \dfrac{a(c\psi + d\phi)}{1 + c\psi + d\phi} \end{cases}. \tag{17}$$

Notice that

$$\psi - \phi = \frac{a(d-c)(\psi - \phi)}{(1 + c\phi + d\psi)(1 + c\psi + d\phi)},$$

whence we deduce that $d > c$ and $(1 + c\phi + d\psi)(1 + d\psi + d\phi) = a(d-c)$. Now

$$\psi + \phi = \frac{a\left((c+d)(\psi + \phi) + 2(c\phi + d\psi)(c\psi + d\phi)\right)}{a(d-c)},$$

or equivalently,

$$2c(\psi + \phi) + 2(c\phi + d\psi)(c\psi + d\phi) = 0.$$

Since $\psi + \phi > 0$, it must be the case that $c = 0$, and then $2d^2\psi\phi = 0$ so that one of either ϕ or ψ equals zero. Without loss of generality assume $\phi = 0$. But then $\psi = \frac{ad\psi}{1+d\psi}$, and hence $\psi = \frac{ad-1}{d} = \overline{x}_+$. Thus the only non-equilibrium solution of System (17) is the period-two solution $\{\overline{x}_+, 0, \overline{x}_+, 0, \ldots\}$, which exists for $ad > 1$ and $c = 0$. Now we formulate our main result about the global dynamics of Equation (16).

Theorem 4 *Consider Equation* (16).

(a) If $a(c+d) \leq 1$, then $\overline{x}_0 = 0$ is a global attractor of all solutions.

(b) If $c = 0$ and $ad > 1$, then there exists a period-two solution $\left\{\frac{ad-1}{d}, 0, \frac{ad-1}{d}, 0, \ldots\right\}$. $\overline{x}_+$ is a global attractor of all solutions with positive initial conditions. Any solution with exactly one initial condition equal to zero will converge to the period-two solution.

(c) If $c > 0$ and $a(c+d) > 1$, $\overline{x}_+$ is a global attractor of all nonzero solutions.

Proof.

(a) If $a(c+d) \leq 1$, then $\overline{x}_0 = 0$ is the only equilibrium, and no period-two solutions exist. By Theorem 1 all solutions must converge to zero.

(b) Suppose $c = 0$ and $ad > 1$, and consider $I = (0, \infty)$. Notice that

$$F(x,x) = \frac{adx}{1+dx} \gtreqless x \iff \overline{x}_+ \gtreqless x,$$

and therefore by Theorem 2 we have that all solutions with initial conditions in I converge to $\overline{x}_+$. Now suppose one initial condition is zero, so without loss of generality assume $x_{-1} = 0$ and $x_0 > 0$. Then $x_1 = 0$ and

$$x_2 = \frac{adx_0}{1+dx_0} \gtreqless x_0 \iff \frac{ad-1}{d} = \overline{x}_+ \gtreqless x_0.$$

Further, one can show $x_2 \lesseqgtr \overline{x}_+ \iff x_0 \lesseqgtr \overline{x}_+$. By induction, $\lim_{k\to\infty} x_{2k} = \overline{x}_+$ and $x_{2k-1} = 0$ for all $k = 0, 1, \ldots$. Thus all solutions with exactly one initial condition equal to zero will converge to the period-two solution $\{\overline{x}_+, 0, \overline{x}_+, 0, \ldots\}$.

(c) When $c > 0$ and $a(c+d) > 1$, $\overline{x}_+$ is locally asymptotically stable while $\overline{x}_0$ is unstable. As in the proof of (b) we can employ Theorem 2 to show that all solutions with positive initial conditions must converge to $\overline{x}_+$. Since $c > 0$ and $d > 0$, if $x_0 + x_{-1} > 0$, then $x_1 = F(x_0, x_{-1}) > 0$ (and also $x_2 > 0$), so the solution eventually has consecutive positive terms and must converge to $\overline{x}_+$.

□

3.2 Translated Linear-Linear: $f(u,v) = cu + dv + k$

We briefly consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1} + k)}{1 + cx_n + dx_{n-1} + k}, \quad n = 0, 1, \ldots, \tag{18}$$

where $c \geq 0$, $d \geq 0$, $c+d>0$, and $k>0$. We notice in this example $f(0,0)=k>0$, so the origin cannot be an equilibrium. More specifically, an equilibrium of Equation (18) must satisfy

$$(c+d)\overline{x}^2 + (k+1-a(c+d))\,\overline{x} - ak = 0$$

Since $c+d>0$ and $ak>0$ by Descartes' Rule of Signs it must be the case that there exists a unique positive equilibrium $\overline{x}_+$.

Theorem 5 *Consider Equation* (18) *such that $c+d>0$ and $k>0$. The unique positive equilibrium $\overline{x}_+$ is a global attractor.*

Proof. The result follows from a straightforward application of Theorem 1.4.8 of [14]. □

3.3 Quadratic-Linear: $f(u,v) = cu^2 + dv$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n^2 + dx_{n-1})}{1 + cx_n^2 + dx_{n-1}}, \quad n = 0, 1, \ldots. \tag{19}$$

Remark 1 For the analysis that follows, we will consider Equation (19) with $c>0$ and $d>0$. Notice that when $c=0$ Equation (19) is a special case of Equation (16), and the global dynamics for this case is discussed in Theorem 4. When $d=0$ Equation (19) is essentially Equation (8), the dynamics of which may be seen in (9).

An equilibrium solution of Equation (19) satisfies

$$c\overline{x}^3 + d\overline{x}^2 + \overline{x} = ac\overline{x}^2 + ad\overline{x}$$

so that all nonzero equilibria satisfy

$$c\overline{x}^2 + (d-ac)\overline{x} + (1-ad) = 0, \tag{20}$$

whence we easily deduce the possible solutions

$$\overline{x}_\pm = \frac{ac - d \pm \sqrt{(d-ac)^2 + 4c(ad-1)}}{2c},$$

which are real if and only if $R = (d-ac)^2 + 4c(ad-1) \geq 0$.
Notice that

$$R \geq 0 \iff d^2 - 2acd + a^2c^2 + 4acd - 4c \geq 0 \iff (ac+d)^2 \geq 4c. \tag{21}$$

Here we have that

$$\lambda(\overline{x}) = \frac{a(2c\overline{x} + d)}{(1 + c\overline{x}^2 + d\overline{x})^2}.$$

Theorem 6 *Equation* (19) *always has the zero equilibrium $\overline{x}_0 = 0$, and*

$$\overline{x}_0 \textit{ is} \begin{cases} \textit{locally asymptotically stable} & \textit{if } ad < 1 \\ \textit{nonhyperbolic} & \textit{if } ad = 1 \\ \textit{a repeller} & \textit{if } ad > 1. \end{cases}$$

Proof. The proof follows from the fact that $\lambda(\overline{x}_0) = \delta(\overline{x}_0) = ad$. □

The next result gives the local stability of positive equilibrium solutions.

Theorem 7 *Assume $c > 0$ and $d > 0$.*

(1) Suppose either
(a) $d \geq ac$ and $1 \geq ad$, or
(b) $d < ac$, $1 > ad$, and $R < 0$.
Then Equation (19) has no positive equilibria.

(2) Suppose either
(a) $1 < ad$, or
(b) $d < ac$ and $1 = ad$.
Then Equation (19) has the positive equilibrium solution $\overline{x}_+$, and it is locally asymptotically stable.

(3) Suppose $d < ac$, $1 > ad$, and $R = 0$. Then Equation (19) has the positive equilibrium solution $\overline{x}_\pm$, and it is nonhyperbolic of stable type (that is one characteristic value is $\lambda_1 = \pm 1$ and the other $|\lambda_1| < 1$).

(4) Suppose $d < ac$, $1 > ad$, and $R > 0$. Then Equation (19) has two positive equilibria, $\overline{x}_+$ and $\overline{x}_-$; $\overline{x}_+$ is locally asymptotically stable, and $\overline{x}_-$ is a saddle point.

Proof. The existence of positive equilibria follows from Descartes' Rule of Signs. Using Equation (14), notice that

$$\lambda(\overline{x}) = \frac{a(2c\overline{x}+d)}{(1+c\overline{x}^2+d\overline{x})^2} = \frac{a(2c\overline{x}+d)}{(a(c\overline{x}+d))^2} = \frac{2c\overline{x}+d}{a(c\overline{x}+d)^2} = \frac{1}{a(c\overline{x}+d)} + \frac{c\overline{x}}{a(c\overline{x}+d)^2}.$$

Further, for the parametric values for which $\overline{x}_+$ exists,

$$\begin{aligned} \lambda(\overline{x}_+) \leq 1 &\iff \frac{c\overline{x}_+}{a(c\overline{x}_+ + d)^2} \leq \frac{a(c\overline{x}_+ + d) - 1}{a(c\overline{x}_+ + d)} \\ &\iff c\overline{x}_+ \leq (c\overline{x}_+ + d)\left(a(c\overline{x}_+ + d) - 1\right) = (c\overline{x}_+ + d)(c\overline{x}_+^2 + d\overline{x}_+) \\ &\iff c \leq (c\overline{x}_+ + d)^2 \\ &\iff 4c \leq (2c\overline{x}_+ + 2d)^2 = (ac + d + \sqrt{R})^2, \end{aligned}$$

which is true by (21). Thus if $R > 0$, $\overline{x}_+$ is locally asymptotically stable, and if $R = 0$, $\overline{x}_\pm$ is nonhyperbolic. In the latter case the characteristic equation of the linearization of Equation (19) about $\overline{x}_\pm$, $y^2 = F_u(\overline{x}_\pm, \overline{x}_\pm)y + F_v(\overline{x}_\pm, \overline{x}_\pm)$, reduces to $acy^2 - (ac-d)y - d = 0$, which has characteristic values $y_1 = 1$ and $y_2 = -\frac{d}{ac}$, where $-1 < y_2 < 0$ since $ac > d$. Thus in this case $\overline{x}_\pm$ is nonhyperbolic of stable type. When $\overline{x}_-$ exists, then

$$\begin{aligned} \lambda(\overline{x}_-) > 1 &\iff 4c > (ac + d - \sqrt{R})^2 \\ &\iff 4c + (ac+d)\sqrt{R} > (ac+d)^2 \\ &\iff (ac+d)\sqrt{R} > (ac+d)^2 - 4c = R \\ &\iff (ac+d)^2 > R = (ac+d)^2 - 4c, \end{aligned}$$

which is true since $c > 0$. To show more specifically that $\overline{x}_-$ is a saddle point when $R > 0$, we must show that $\delta(\overline{x}_-) < 1$, where δ is defined by Equation (15). Notice

$$\delta(\overline{x}_-) = \frac{a(d-2c\overline{x}_-)}{(1+c\overline{x}_-^2+d\overline{x}_-)^2} = \frac{a(d-2c\overline{x}_-)}{(a(c\overline{x}_-+d))^2} = \frac{4(d-2c\overline{x}_-)}{a(2c\overline{x}_-+2d)^2} = \frac{4\left(2d - ac + \sqrt{R}\right)}{a(ac+d-\sqrt{R})^2},$$

and so we have that

$$\begin{aligned}\delta(\overline{x}_-) < 1 &\iff 4\left(2d - ac + \sqrt{R}\right) < a\left(ac + d - \sqrt{R}\right)^2\\ &\iff (2 + a(ac+d))\sqrt{R} < a(ac+d)^2 - 4d.\end{aligned}$$

The right-hand side of the latter inequality is positive since $a(ac+d)^2 - 4d > 4ac - 4d = 4(ac-d) > 0$ by assumption. But then

$$\begin{aligned}\delta(\overline{x}_-) < 1 &\iff (2 + a(ac+d))^2\left((ac+d)^2 - 4c\right) < \left(a(ac+d)^2 - 4d\right)^2\\ &\iff 3a^3c^2d + 6a^2cd^2 + 3ad^3 - 3a^2c^2 - 2acd - 3d^2 - 4c < 0\\ &\iff (ad-1)\left(3d^2 + 3a^2c^2 + 2c(3ad+2)\right) < 0,\end{aligned}$$

which is automatically true since the latter factor is strictly positive and $ad < 1$. Thus indeed $\overline{x}_-$ is a saddle point when it exists for $R > 0$. □

Theorem 8 *There exist no minimal period-two solutions to Equation* (19) *if* $c, d > 0$.

Proof. Suppose there exist $\phi, \psi > 0$ with $\phi \neq \psi$ such that

$$\begin{cases} \psi &= \dfrac{af(\phi,\psi)}{1+f(\phi,\psi)} &= \dfrac{a(c\phi^2 + d\psi)}{1 + c\phi^2 + d\psi} \\[2ex] \phi &= \dfrac{af(\psi,\phi)}{1+f(\psi,\phi)} &= \dfrac{a(c\psi^2 + d\phi)}{1 + c\psi^2 + d\phi}\end{cases}. \tag{22}$$

From System (22) we notice that

$$\psi - \phi = \frac{a(\psi-\phi)(d - c(\psi+\phi))}{(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi)},$$

whence it immediately follows that $(1 + c\phi^2 + d\psi)(1 + c\psi^2 + d\phi) = a(d - c(\psi+\phi))$. But then

$$\psi + \phi = \frac{2(c\phi^2 + d\psi)(c\psi^2 + d\phi) + c(\psi^2 + \phi^2) + d(\psi+\phi)}{d - c(\psi+\phi)}.$$

Thus we have that necessarily

$$2\phi\psi = \frac{2a^2(c\phi^2 + d\psi)(c\psi^2 + d\phi)}{a(d - c(\psi+\phi))} = a\left((\psi+\phi) - \frac{c(\psi^2+\phi^2) + d(\psi+\phi)}{d - c(\psi+\phi)}\right) > 0$$

since both $\psi, \phi > 0$. But this implies that

$$\begin{aligned}&(\psi+\phi)(d - c(\psi+\phi)) > c(\psi^2+\phi^2) + d(\psi+\phi)\\ \iff\ & d(\psi+\phi) - c(\psi+\phi)^2 > c(\psi^2+\phi^2) + d(\psi+\phi)\\ \iff\ & 0 > c(\psi^2+\phi^2) + c(\psi+\phi)^2,\end{aligned}$$

a clear contradiction since $c > 0$.

Now suppose there exists a period-two solution $\{\phi, \psi, \phi, \psi, \ldots\}$ with $\phi \neq \psi$ but $\phi\psi = 0$. Suppose without loss of generality that $\phi = 0$. Now

$$\begin{cases} \psi & = \dfrac{af(0,\psi)}{1+f(0,\psi)} & = \dfrac{ad\psi}{1+d\psi} \\[2ex] 0 & = \dfrac{af(\psi,0)}{1+f(\psi,0)} & = \dfrac{ac\psi^2}{1+c\psi^2} \end{cases},$$

which immediately leads to the contradiction $\psi = \phi = 0$ for $c > 0$. Thus Equation (19) has no minimal period-two solutions. □

The next result describes the global dynamics of Equation (19).

Theorem 9 *Consider Equation* (19) *under the condition* $c > 0$ *and* $d > 0$.

(1) *Suppose either*
   (a) $d \geq ac$ *and* $1 \geq ad$, *or*
   (b) $d < ac$, $1 > ad$, *and* $R < 0$.
   Then $\overline{x}_0$ *is a global attractor of all solutions.*

(2) *Suppose either*
   (a) $1 < ad$, *or*
   (b) $d < ac$ *and* $1 = ad$.
   Then $\overline{x}_+$ *is a global attractor of all nonzero solutions.*

(3) *Suppose* $d < ac$, $1 > ad$, *and* $R = 0$. *Then Equation* (19) *has the equilibria* $\overline{x}_0 = 0$, *which is locally asymptotically stable, and* $\overline{x}_\pm$, *which is nonhyperbolic of stable type. There exists a continuous curve* $\mathcal{C}$ *passing through* $E = (\overline{x}_\pm, \overline{x}_\pm)$ *such that* $\mathcal{C}$ *is the graph of a decreasing function. The set of initial conditions* $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ *is the union of two disjoint basins of attraction, namely* $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E)$, *where* $E_0 = (\overline{x}_0, \overline{x}_0)$,

$$\mathcal{B}(E_0) = \{(x_{-1}, x_0) : (x_{-1}, x_0) \prec_{ne} (x, y) \text{ for some } (x, y) \in \mathcal{C}\}, \text{ and}$$
$$\mathcal{B}(E) = \{(x_{-1}, x_0) : (x, y) \prec_{ne} (x_{-1}, x_0) \text{ for some } (x, y) \in \mathcal{C}\} \cup \mathcal{C}.$$

(4) *Suppose* $d < ac$, $1 > ad$, *and* $R > 0$. *Then Equation* (19) *has the equilibria* $\overline{x}_0 = 0$, *which is locally asymptotically stable,* $\overline{x}_-$, *which is a saddle point, and* $\overline{x}_+$, *which is locally asymptotically stable. There exist two continuous curves* $\mathcal{W}^s(E_-)$ *and* $\mathcal{W}^u(E_-)$, *both passing through* $E_- = (\overline{x}_-, \overline{x}_-)$, *such that* $\mathcal{W}^s(E_-)$ *is the graph of a decreasing function and* $\mathcal{W}^u(E_-)$ *is the graph of an increasing function. The set of initial conditions* $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ *is the union of three disjoint basins of attraction, namely* $Q_1 = \mathcal{B}(E_0) \cup \mathcal{B}(E_-) \cup \mathcal{B}(E_+)$, *where* $E_0 = (\overline{x}_0, \overline{x}_0)$, $E_+ = (\overline{x}_+, \overline{x}_+)$, $\mathcal{B}(E_-) = \mathcal{W}^s(E_-)$,

$$\mathcal{B}(E_0) = \{(x_{-1}, x_0) : (x_{-1}, x_0) \prec_{ne} (x, y) \text{ for some } (x, y) \in \mathcal{W}^s(E_-)\}, \text{ and}$$
$$\mathcal{B}(E_+) = \{(x_{-1}, x_0) : (x, y) \prec_{ne} (x_{-1}, x_0) \text{ for some } (x, y) \in \mathcal{W}^s(E_-)\}$$

Proof. (1) The proof in this case follows from Theorems 1, 7, and 8 along with the fact that $\overline{x}_0 = 0$ is the sole equilibrium of Equation (19).

(2) The proof used to show that all solutions with positive initial conditions converge to $\overline{x}_+$ follows from an application of Theorem 2 (as used above in the proof of Theorem 4). Notice that $x_1 = F(x_0, x_{-1}) > 0$ if either $x_0 > 0$ or $x_{-1} > 0$ (and similar for x_2), so $I = (0, \infty)$ is an attracting and invariant interval. Thus all nonzero solutions must converge to $\overline{x}_+$.

(3) The proof follows from an application of Theorems 1-4 of [17] applied to the *cooperative* second iterate of the map corresponding to Equation (19). The proof is completely analogous to the proof of Theorem 5 in [4], so we omit the details.

(4) The proof follows from an immediate application of Theorem 5 in [4]. □

3.4 Linear-Quadratic: $f(u, v) = cu + dv^2$

We consider the difference equation

$$x_{n+1} = \frac{a(cx_n + dx_{n-1}^2)}{1 + cx_n + dx_{n-1}^2}, \quad n = 0, 1, \ldots. \tag{23}$$

Remark 2 For the analysis that follows, we will consider Equation (23) with $c > 0$ and $d > 0$. Notice that when $d = 0$ Equation (23) reduces to Equation (2), a special case of Equation (16). When $c = 0$ Equation (23) is essentially Equation (8) with delay.

An equilibrium of (23) satisfies

$$d\overline{x}^3 + c\overline{x}^2 + \overline{x} = ac\overline{x} + ad\overline{x}^2$$

so that all nonzero equilibria satisfy

$$d\overline{x}^2 + (c - ad)\overline{x} + (1 - ac) = 0, \tag{24}$$

whence we easily deduce the possible solutions

$$\overline{x}_\pm = \frac{ad - c \pm \sqrt{(c - ad)^2 + 4d(ac - 1)}}{2d},$$

which are real if and only if $R = (c - ad)^2 + 4d(ac - 1) \geq 0$.
Notice that

$$R \geq 0 \iff c^2 - 2acd + a^2d^2 + 4acd - 4d \geq 0 \iff (ad + c)^2 \geq 4d. \tag{25}$$

Here we have that

$$\lambda(\overline{x}) = \frac{a(c + 2d\overline{x})}{(1 + c\overline{x} + d\overline{x}^2)^2}.$$

Theorem 10 *Equation* (23) *always has the zero equilibrium* $\overline{x}_0 = 0$, *and*

$$\overline{x}_0 \text{ is} \begin{cases} \text{locally asymptotically stable} & \text{if } ac < 1 \\ \text{nonhyperbolic} & \text{if } ac = 1 \\ \text{unstable} & \text{if } ac > 1. \end{cases}$$

Proof. The proof follows from the fact that $\lambda(\overline{x}_0) = ac$. □

Theorem 11 *Consider Equation* (23) *and assume* $c > 0$ *and* $d > 0$.

(1) Suppose either
(a) $c \geq ad$ *and* $1 \geq ac$, *or*
(b) $c < ad$, $1 > ac$, *and* $R < 0$.
Then Equation (23) *has no positive equilibria.*

(2) Suppose either
(a) $1 < ac$, *or*
(b) $c < ad$ *and* $1 = ac$.
Then Equation (23) *has the positive equilibrium solution* $\overline{x}_+$, *and it is locally asymptotically stable.*

(3) Suppose $c < ad$, $1 > ac$, *and* $R = 0$. *Then Equation* (23) *has the positive equilibrium solution* $\overline{x}_\pm$, *and it is nonhyperbolic of stable type.*

(4) Suppose $c < ad$, $1 > ac$, *and* $R > 0$. *Then Equation* (23) *has two positive equilibria,* $\overline{x}_+$ *and* $\overline{x}_-$; $\overline{x}_+$ *is locally asymptotically stable, and* $\overline{x}_-$ *is unstable.*
Let $K = a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d$.
(i) If $K < 0$, *then* $\overline{x}_-$ *is a saddle point.*
(ii) If $K > 0$, *then* $\overline{x}_-$ *is a repeller.*
(iii) If $K = 0$, *then* $\overline{x}_-$ *is nonhyperbolic of unstable type (that is one characteristic value is* $\lambda_1 = \pm 1$ *and the other* $|\lambda_1| > 1$*).*

Proof. Much of the analysis is similar to the considerations in the proof of Theorem 7. Notice that

$$\lambda(\overline{x}) = \frac{a(c+2d\overline{x})}{(1+c\overline{x}+d\overline{x}^2)^2} = \frac{a(c+2d\overline{x})}{(a(c+d\overline{x}))^2} = \frac{c+2d\overline{x}}{a(c+d\overline{x})^2} = \frac{1}{a(c+d\overline{x})} + \frac{d\overline{x}}{a(c+d\overline{x})^2}.$$

For the parametric values for which $\overline{x}_+$ exists,

$$\begin{aligned} \lambda(\overline{x}_+) \leq 1 &\iff \frac{d\overline{x}_+}{a(c+d\overline{x}_+)^2} \leq \frac{a(c+d\overline{x}_+)-1}{a(c+d\overline{x}_+)} \\ &\iff d\overline{x}_+ \leq (c+d\overline{x}_+)\left(a(c+d\overline{x}_+)-1\right) = (c+d\overline{x}_+)(c\overline{x}_+ + d\overline{x}_+^2) \\ &\iff d \leq (c+d\overline{x}_+)^2 \\ &\iff 4d \leq (2c+2d\overline{x}_+)^2 = (ad+c+\sqrt{R})^2, \end{aligned}$$

which is true by (25). Thus if $R > 0$, $\overline{x}_+$ is locally asymptotically stable, and if $R = 0$, $\overline{x}_\pm$ is nonhyperbolic. In the latter case the characteristic equation of the linearization of Equation (23) about $\overline{x}_\pm$, $y^2 = F_u(\overline{x}_\pm, \overline{x}_\pm)y + F_v(\overline{x}_\pm, \overline{x}_\pm)$, reduces to $ady^2 - cy + c - ad = 0$, which has characteristic values $y_1 = 1$ and $y_2 = \frac{c-ad}{ad}$, where $-1 < y_2 < 0$ since $ad > c$. Thus in this case $\overline{x}_\pm$ is nonhyperbolic of stable type.
When $\overline{x}_-$ exists,

$$\begin{aligned} \lambda(\overline{x}_-) > 1 &\iff 4d > (ad+c-\sqrt{R})^2 \\ &\iff 4d + (ad+c)\sqrt{R} > (ad+c)^2 \\ &\iff (ad+c)\sqrt{R} > (ad+c)^2 - 4d = R \\ &\iff (ad+c)^2 > R = (ad+c)^2 - 4d \end{aligned}$$

which is true since $d > 0$. To more specifically classify $\overline{x}_-$, we must calculate $\delta(\overline{x}_-)$. Notice

$$\delta(\overline{x}_-) = \frac{a(2d\overline{x}_- - c)}{(1+c\overline{x}_- + d\overline{x}_-^2)^2} = \frac{a(2d\overline{x}_- - c)}{(a(c+d\overline{x}_-))^2} = \frac{4(2d\overline{x}_- - c)}{a(2c+2d\overline{x}_-)^2} = \frac{4\left(ad-2c-\sqrt{R}\right)}{a(ad+c-\sqrt{R})^2},$$

and so we have that

$$\begin{aligned}\delta(\overline{x}_-) \gtrless 1 &\iff 4\left(ad - 2c - \sqrt{R}\right) \gtrless a\left(ad + c - \sqrt{R}\right)^2 \\ &\iff (a(ad+c)-2)\sqrt{R} \gtrless a(ad+c)^2 - 4ad + 4c = aR + 4c.\end{aligned}$$

Notice that $R > 0$ automatically implies $a(ad+c) > 2$, as

$$0 < (ad+c)^2 - 4d < a^2d^2 + 2acd + a^2d^2 - 4d = 2d\,(a(ad+c)-2)$$

since $c < ad$. Therefore we may square both sides to obtain

$$\begin{aligned}\delta(\overline{x}_-) \gtrless 1 &\iff (a(ad+c)-2)^2 R \gtrless (aR+4c)^2 \\ &\iff R\left(a^2(ad+c)^2 - 4a(ad+c) + 4\right) \gtrless a^2R^2 + 8acR + 16c^2 \\ &\iff R\left(a^2R - 4ac + 4\right) \gtrless a^2R^2 + 8acR + 16c^2 \\ &\iff R(1-3ac) - 4c^2 \gtrless 0 \\ &\iff a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d \gtrless 0.\end{aligned}$$

Thus if

$$K = a^2d^2 + 14acd - 3c^2 - 3a^3cd^2 - 6a^2c^2d - 3ac^3 - 4d, \tag{26}$$

$K < 0$ implies $\overline{x}_-$ is a saddle point and $K > 0$ implies it is a repeller. If $K = 0$, $\overline{x}_-$ is nonhyperbolic, and we expect in such case to be nonhyperbolic of unstable type. Indeed one can show that in the event $K = 0$, the characteristic equation of the linearization of Equation (23) about $\overline{x}_-$, $y^2 = F_u(\overline{x}_-, \overline{x}_-)y + F_v(\overline{x}_-, \overline{x}_-)$, has roots $y_1 = -1$ and $y_2 = F_u(\overline{x}_-, \overline{x}_-) + 1 > 1$, which immediately shows the desired result. □

The investigation of the existence of periodic solutions of Equation (23) is an interesting one that involves a thorough analysis of potential parametric cases. This analysis will reveal the potential for the existence of several nonzero periodic solutions. The juxtaposition of Equation (19) with Equation (23) illustrates an interesting phenomenon in which, loosely speaking, the dominance of the delay term x_{n-1} contributes to the possibility of periodic solutions arising.

A minimal period-two solution $\{\phi, \psi, \phi, \psi, \ldots\}$ with $\phi, \psi > 0$ and $\phi \neq \psi$ must satisfy

$$\begin{cases} \psi &= \dfrac{af(\phi,\psi)}{1+f(\phi,\psi)} = \dfrac{a(c\phi + d\psi^2)}{1 + c\phi + d\psi^2} \\[2ex] \phi &= \dfrac{af(\psi,\phi)}{1+f(\psi,\phi)} = \dfrac{a(c\psi + d\phi^2)}{1 + c\psi + d\phi^2} \end{cases}. \tag{27}$$

Eliminating either ψ or ϕ from System (27) we obtain

$$\left(d\phi^2 + (c-ad)\phi + (1-ac)\right)h(\phi) = 0, \quad \text{or} \quad \left(d\psi^2 + (c-ad)\psi + (1-ac)\right)h(\psi) = 0,$$

where

$$\begin{aligned}h(x) = {}& -d^3x^6 + d^2(c+2ad)x^5 - d(c^2 + 2d + 3acd + a^2d^2)x^4 + d(c + 3ac^2 + 2ad + 3a^2cd)x^3 \\ & - (c^2 + ac^3 + d + 2acd + 3a^2c^2d + a^3cd^2)x^2 + ac(1+ac)(2c+ad)x - a^2c^2(1+ac).\end{aligned} \tag{28}$$

Since $dx^2 + (c-ad)x + (1-ac) \neq 0$ for any x that is not a solution of the equilibrium equation (24), minimal period-two solutions must be the solutions of the equation

$$h(x) = 0. \tag{29}$$

Theorem 12 *Any real solutions of Equation* (29) *are positive numbers for* $c, d > 0$*, and there exist up to three minimal period-two solutions of Equation* (23)*. Furthermore, let* K *be as defined in Equation* (26)*, and define the following expressions:*

$$\begin{aligned}
J &= 4a^5cd^4 - 8a^4c^2d^3 + 12a^3c^3d^2 - 24a^3cd^3 - 8a^2c^4d + 28a^2c^2d^2 - a^2d^3 + 4ac^5 + 4ac^3d \\
&\quad + 32acd^2 + 4c^4 + 8c^2d + 4d^2 \\
\Delta_1 &= 6d^6 \\
\Delta_2 &= d^{10}\left(8a^2d^2 - 16acd - 7c^2 - 24d\right) \\
\Delta_3 &= -2d^{12}\left(8a^5cd^5 + 13a^4c^2d^4 + 10a^3c^3d^3 - 44a^3cd^4 + 4a^2c^4d^2 - 34a^2c^2d^3 - 4a^2d^4 - 19ac^5d\right. \\
&\quad \left. +14ac^3d^2 + 44acd^3 + 6c^6 + 7c^4d + 5c^2d^2 + 16d^3\right)
\end{aligned}$$

$$\begin{aligned}
\Delta_4 &= c^2d^{13}\left(-16a^9cd^8 - 12a^8c^2d^7 + 24a^7c^3d^6 + 152a^7cd^7 - 68a^6c^4d^5 + 80a^6c^2d^6 + 8a^6d^7 + 48a^5c^5d^4\right. \\
&\quad -164a^5c^3d^5 - 464a^5cd^6 - 60a^4c^6d^3 + 20a^4c^4d^4 - 180a^4c^2d^5 - 64a^4d^6 + 56a^3c^7d^2 - 332a^3c^5d^3 \\
&\quad +388a^3c^3d^4 + 488a^3cd^5 - 48a^2c^8d + 272a^2c^6d^2 + 255a^2c^4d^3 + 152a^2c^2d^4 + 136a^2d^5 + 24ac^9 \\
&\quad \left. +8ac^7d + 124ac^5d^2 + 180ac^3d^3 - 152acd^4 + 24c^8 + 68c^6d + 32c^4d^2 - 44c^2d^3 - 32d^4\right)
\end{aligned}$$

$$\begin{aligned}
\Delta_5 &= 2c^4d^{13}J\left(3a^8c^2d^6 + 2a^7cd^6 - 18a^6c^2d^5 - a^6d^6 + 6a^5c^5d^3 + 10a^5c^3d^4 - 8a^5cd^5 - 10a^4c^4d^3\right. \\
&\quad +44a^4c^2d^4 + 6a^4d^5 + 54a^3c^5d^2 - 25a^3c^3d^3 - 6a^3cd^4 + 3a^2c^8 - 8a^2c^6d + 35a^2c^4d^2 - 39a^2c^2d^3 \\
&\quad \left. -9a^2d^4 + 6ac^7 + 2ac^5d + 4ac^3d^2 + 14acd^3 + 3c^6 + 10c^4d + 11c^2d^2 + 4d^3\right)
\end{aligned}$$

$$\Delta_6 = a^2c^6d^{14}(ac+1)KJ^2.$$

(1) If $\Delta_i > 0$ *for all* $2 \leq i \leq 6$ *then Equation* (29) *has six real roots. Consequently, Equation* (23) *has three minimal period-two solutions.*
(2) If $\Delta_j \leq 0$ *for some* $2 \leq j \leq 5$ *and* $\Delta_i > 0$ *for* $i \neq j$*, then Equation* (29) *has two distinct real roots and two pairs of conjugate imaginary roots. Consequently, Equation* (23) *has one minimal period-two solution.*
(3) If $\Delta_i \leq 0$*,* $\Delta_{i+1} \geq 0$ *(such that at least one of these is strict) for some* $2 \leq i \leq 4$*, and if* $\Delta_6 < 0$*, then Equation* (29) *has three pairs of conjugate imaginary roots. Consequently, Equation* (23) *has no minimal period-two solutions.*

Proof. The proof of the first statement follows from Descartes' Rule of Signs. Let disc(h) denote the 12×12 discrimination matrix as defined in [12]:

$$\text{disc}(h) = \begin{bmatrix} a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6a_6 & 5a_5 & 4a_4 & 3a_3 & 2a_2 & a_1 \end{bmatrix}.$$

Here a_k equals the coefficient of the degree-k term of h as defined in Equation (28); that is, $a_6 = -d^3$, $a_5 = d^2(c + 2ad)$, $a_4 = -d(c^2 + 2d + 3acd + a^2d^2)$, $a_3 = d(c + 3ac^2 + 2ad + 3a^2cd)$, $a_2 = -(c^2 + ac^3 + d + 2acd + 3a^2c^2d + a^3cd^2)$, $a_1 = ac(1+ac)(2c+ad)$, and $a_0 = -a^2c^2(1+ac)$. Let Δ_k denote the determinant of the submatrix of disc(h) formed by its first $2k$ rows and $2k$ columns for $k = 1, 2, \ldots, 6$. Then the values of Δ_k are listed above, and the veracity of the statements above may now be verified by employing Theorem 1 of [12]. Notice that $\Delta_1 > 0$ for all $d > 0$. $\square$

Remark 3 The parametric conditions discussed above do not exhaust all of the parametric space but cover a substantial region of parameters for which Equation (23) possesses hyperbolic dynamics.

We will use the sufficient conditions provided in Theorems 10, 11, and 12 to obtain some global dynamic scenarios discussed in [4]. We will not investigate the dynamics of Equation (23) when it has one or no positive fixed point since in such cases the dynamics should be similar to the dynamics of Equation (19) discussed in Theorem 9. The following theorem relies on results from [4] and summarizes potential hyperbolic dynamic scenarios for Equation (23) in the event it possesses three fixed points and zero, one, or three pairs of hyperbolic period-two points. In particular, Theorem 3 is applicable to case (ii) of the following result. See also the statement and proof of Theorem 11 in [4].

Theorem 13 *Consider Equation* (23) *and assume* $0 < c < ad, ac < 1$ *such that* $R > 0$.

(i) If $\Delta_i > 0$ *for all* $2 \leq i \leq 6$ *then Equation* (23) *has three equilibria* $\overline{x}_0 < \overline{x}_- < \overline{x}_+$, *where* $\overline{x}_0$ *and* $\overline{x}_+$ *are locally asymptotically stable and* $\overline{x}_-$ *is a repeller, and three minimal period-two solutions* $\{\phi_1, \psi_1\}$, $\{\phi_2, \psi_2\}$, *and* $\{\phi_3, \psi_3\}$. *Here* $(\phi_1, \psi_1) \prec_{ne} (\phi_2, \psi_2) \prec_{ne} (\phi_3, \psi_3)$, $\{\phi_1, \psi_1\}$ *and* $\{\phi_3, \psi_3\}$ *are saddle points, and* $\{\phi_2, \psi_2\}$ *is locally asymptotically stable. The global behavior of Equation* (23) *is described by Theorem 8 of [4]. In this case there exist four continuous curves* $\mathcal{W}^s(\phi_1, \psi_1)$, $\mathcal{W}^s(\psi_1, \phi_1)$, $\mathcal{W}^s(\phi_3, \psi_3)$, $\mathcal{W}^s(\psi_3, \phi_3)$ *that have endpoints at* $E_- = (\overline{x}_-, \overline{x}_-)$ *and are graphs of decreasing functions. Every solution which starts below* $\mathcal{W}^s(\phi_1, \psi_1) \cup \mathcal{W}^s(\psi_1, \phi_1)$ *in the northeast ordering converges to* $E_0 = (\overline{x}_0, \overline{x}_0)$ *and every solution which starts above* $\mathcal{W}^s(\phi_3, \psi_3) \cup \mathcal{W}^s(\psi_3, \phi_3)$ *in the northeast ordering converges to* $E_+ = (\overline{x}_+, \overline{x}_+)$. *Every solution which starts above* $\mathcal{W}^s(\phi_1, \psi_1) \cup \mathcal{W}^s(\psi_1, \phi_1)$ *and below* $\mathcal{W}^s(\phi_3, \psi_3) \cup \mathcal{W}^s(\psi_3, \phi_3)$ *in the northeast ordering converges to* $\{\phi_2, \psi_2\}$. *For example, this happens for* $a = 1$, $c = \frac{389}{2176}$, *and* $d = \frac{249}{64}$.

(ii) *If* $\Delta_j \leq 0$ *for some* $2 \leq j \leq 5$ *and* $\Delta_i > 0$ *for* $i \neq j$*, then Equation* (23) *has three equilibria* $\overline{x}_0 < \overline{x}_- < \overline{x}_+$*, where* $\overline{x}_0$ *and* $\overline{x}_+$ *are locally asymptotically stable and* $\overline{x}_-$ *is a repeller, and one period-two solution* $\{\phi_1, \psi_1\}$*, which is a saddle point. The global behavior of Eq.* (23) *is described by Theorem 7 of [4]. In this case there exist four continuous curves* $\mathcal{W}^s(\phi_1,\psi_1), \mathcal{W}^s(\psi_1,\phi_1), \mathcal{W}^u(\phi_1,\psi_1), \mathcal{W}^u(\psi_1,\phi_1)$*, where* $\mathcal{W}^s(\phi_1,\psi_1), \mathcal{W}^s(\psi_1,\phi_1)$ *have endpoints at* $E_- = (\overline{x}_-, \overline{x}_-)$ *and are graphs of decreasing functions. The curves* $\mathcal{W}^u(\phi_1,\psi_1), \mathcal{W}^u(\psi_1,\phi_1)$ *are graphs of increasing functions and start at* $E_0 = (\overline{x}_0, \overline{x}_0)$*. Every solution which starts below* $\mathcal{W}^s(\phi_1,\psi_1) \cup \mathcal{W}^s(\psi_1,\phi_1)$ *in the northeast ordering converges to* E_0 *and every solution which starts above* $\mathcal{W}^s(\phi_1,\psi_1) \cup \mathcal{W}^s(\psi_1,\phi_1)$ *in the northeast ordering converges to* $E_+ = (\overline{x}_+, \overline{x}_+)$ *. For example, this happens for* $a = 1$*,* $c = \frac{1}{5}$*, and* $d = \frac{237}{64}$*.*

(iii) *If* $\Delta_i \leq 0$ *and* $\Delta_{i+1} \geq 0$ *(such that at least one of these is strict) for some* $2 \leq i \leq 4$*, and if* $\Delta_6 < 0$*, then Eq.* (23) *has three equilibria* $\overline{x}_0 < \overline{x}_- < \overline{x}_+$*, where* $\overline{x}_0$ *and* $\overline{x}_+$ *are locally asymptotically stable and* $\overline{x}_-$ *is a saddle point, and no period-two solution. The global behavior of Equation* (23) *is described by Theorem 5 of [4] or Theorem 9 case (4). For example, this happens for* $a = 1$*,* $c = \frac{493}{1024}$*, and* $d = \frac{157}{48}$*.*

Equation (23) exhibits global dynamics similar to that of Equation (10), which was investigated in [4]. Therefore, we pose the following conjecture.

Conjecture 1 *There exists a topological conjugation between the maps in Equations* (10) *and* (23)*.*

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Volume 29, Number 2 March 2021
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS, LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)
Editor in Chief: George Anastassiou
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An international publication of Eudoxus Press, LLC, of TN.

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A study of a coupled system of nonlinear second-order ordinary differential equations with nonlocal integral multi-strip boundary conditions on an arbitrary domain

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Abstract

In this paper, we study a nonlinear system of second order ordinary differential equations with nonlocal integral multi-strip coupled boundary conditions. Leray-Schauder alternative criterion, Schauder fixed point theorem and Banach contraction mapping principle are employed to obtain the desired results. Examples are constructed for the illustration of the obtained results. We emphasize that our results are new and enhance the literature on boundary value problems of coupled systems of ordinary differential equations. Several new results appear as special cases of our work.

Keywords: System of ordinary differential equations; integral boundary condition; multi-strip; existence; fixed point.
MSC 2000: 34A34, 34B10, 34B15.

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1 Introduction

This paper is concerned with the following coupled system of nonlinear second-order ordinary differential equations:

$$\begin{cases} u''(t) = f(t, u(t), v(t)), \ t \in [a, b], \\ v''(t) = g(t, u(t), v(t)), \ t \in [a, b], \end{cases} \tag{1.1}$$

supplemented with the nonlocal integral multi-strip coupled boundary conditions of the form:

$$\begin{cases} \displaystyle\int_a^b u(s)ds = \sum_{j=1}^m \gamma_j \int_{\xi_j}^{\eta_j} v(s)ds + \lambda_1, \int_a^b u'(s)ds = \sum_{j=1}^m \rho_j \int_{\xi_j}^{\eta_j} v'(s)ds + \lambda_2, \\ \displaystyle\int_a^b v(s)ds = \sum_{j=1}^m \sigma_j \int_{\xi_j}^{\eta_j} u(s)ds + \lambda_3, \int_a^b v'(s)ds = \sum_{j=1}^m \delta_j \int_{\xi_j}^{\eta_j} u'(s)ds + \lambda_4, \end{cases} \tag{1.2}$$

where $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $a < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_m < \eta_m < b$, and γ_j, ρ_j, σ_j and δ_j $\in \mathbb{R}^+$ $(j = 1, 2, \ldots, m)$, $\lambda_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$.

Mathematical modeling of several real world phenomena lead to the occurrence of nonlinear boundary value problems of differential equations. During the past few decades, the topic of boundary value problems has evolved as an important and interesting area of investigation in view of its extensive applications in diverse disciplines such as fluid mechanics, mathematical physics, etc. For application details, we refer the reader to the text [1], while some recent works on boundary value problems of ordinary differential equations can be found in the papers ([2]-[5]).

Much of the literature on boundary value problems involve classical boundary conditions. However, these conditions cannot cater the complexities of the physical and chemical processes occurring within the domain. In order to cope with this situation, the concept of nonlocal boundary conditions was introduced. Such conditions relate the boundary values of the unknown function to its values at some interior positions of the domain. For a detailed account of nonlocal nonlinear boundary value problems, for instance, see ([6]-[16]) and the references cited therein.

Computational fluid dynamics (CFD) technique are directly concerned with the boundary data [1]. However, the assumption of circular cross-section in the fluid flow problems is not justifiable in many situations. The concept of integral boundary conditions played a key role in resolving this issue as such conditions can be applied to arbitrary shaped structures. Integral boundary conditions are also found to be quite useful in the study of thermal and hydrodynamic problems. In fact, one can find numerous applications of integral boundary conditions in the fields like chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. ([17]-[20]). For some recent results on boundary value problems integral boundary conditions, we refer the reader to a series of articles ([21]-[32]) and the references cited therein.

Motivated by the importance of nonlocal and integral boundary conditions, we introduce a new kind of coupled integral boundary conditions (1.2) and solve a nonlinear coupled system of second-order ordinary differential equations (1.1) equipped with these conditions. Our main results rely on Leray-Schauder alternative and Banach contraction mapping principle.

The rest of the paper is organized as follows. In Section 2, we present an auxiliary lemma. The main results for the problem (1.1) and (1.2) are discussed in Section 3. We also construct examples illustrating the obtained results. The paper concludes with some interesting observations.

2 An auxiliary lemma

The following lemma plays a key role in defining the solution for the problem (1.1) − (1.2).

Lemma 2.1 *For* $f_1, g_1 \in C([a,b],\mathbb{R})$*, the solution of the linear system of differential equations*

$$\begin{aligned} u''(t) &= f_1(t),\ t \in [a,b], \\ v''(t) &= g_1(t),\ t \in [a,b], \end{aligned} \tag{2.1}$$

subject to the boundary conditions (1.2) *is equivalent to the system of integral equations*

$$\begin{aligned} u(t) &= \int_a^t (t-s) f_1(s) ds \\ &\quad -\frac{1}{A_3}\Big\{ \int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s) + L_1 + (b-a)A_2(t-a)\Big](b-s) f_1(s) ds \\ &\quad + \int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j) + L_2 + A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\ &\quad \times (b-s) g_1(s) ds\Big\} + \frac{1}{A_3}\Big\{\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\gamma_j A_1 (b-a)(s-p) + \rho_j L_1 \\ &\quad + \rho_j (b-a) A_2 (t-a)\Big] g_1(p) dp ds + \sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j A_1 \sum_{j=1}^m \gamma_j(\eta_j-\xi_j)(s-p) \\ &\quad + \delta_j L_2 + \delta_j A_2 (t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] f_1(p) dp ds\Big\} + \Omega_1(t), \end{aligned} \tag{2.2}$$

$$v(t) = \int_a^t (t-s) g_1(s) ds - \frac{1}{A_3}\Big\{ \int_a^b \Big[\frac{A_1\Big((b-a)^2 - A_2\Big)}{2\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)}(b-s)$$

$$
\begin{aligned}
&+L_3+A_2(t-a)\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big](b-s)f_1(s)ds\\
&+\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)+L_4+A_2(b-a)(t-a)\Big](b-s)g_1(s)ds\Big\}\\
&+\frac{1}{A_3}\Big\{\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[(s-p)\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}+\rho_jL_3 \qquad (2.3)\\
&+\delta_jA_2(t-a)\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big]g_1(p)dpds\\
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1(b-a)(s-p)+\delta_jL_4+\delta_jA_2(b-a)(t-a)\Big]f_1(p)dpds\Big\}\\
&+\Omega_2(t),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= (b-a)^2-\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)^2\Big), &(2.4)\\
A_2 &= (b-a)^2-\Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)^2\Big), \quad A_3=A_1A_2\neq 0, &(2.5)\\
L_1 &= (b-a)\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big(\sum_{j=1}^{m}\sigma_j+\sum_{j=1}^{m}\delta_j\Big)\\
&\quad -\frac{(b-a)^4}{2}-\frac{(b-a)^2}{2}\Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)^2\Big), &(2.6)\\
L_2 &= \sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big(\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)^2\Big)+(b-a)^2\Big)\\
&\quad -\frac{(b-a)^3}{2}\sum_{j=1}^{m}(\eta_j-\xi_j)(\rho_j+\gamma_j), &(2.7)\\
L_3 &= \sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big((b-a)^2(\sigma_j+\delta_j)-A_2\delta_j\Big)\\
&\quad +\frac{(b-a)^3}{2}\Big[\frac{A_2-(b-a)^2}{\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)}-\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big], &(2.8)\\
L_4 &= \frac{(b-a)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}\sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big[\sigma_j\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)^2
\end{aligned}
$$

$$+(b-a)^2 - A_2\Big] - \frac{(b-a)^4}{2} + \frac{(b-a)^2}{2\sum_{j=1}^m \gamma_j}\sum_{j=1}^m \rho_j\Big(A_2-(b-a)^2\Big), \tag{2.9}$$

$$\begin{aligned}\Omega_1(t) &= \frac{1}{A_3}\Big\{A_1(b-a)\lambda_1 + \Big[L_1 + A_2(b-a)(t-a)\Big]\lambda_2 + A_1\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)\lambda_3 \\ &\quad + \Big[L_2 + A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big]\lambda_4\Big\},\end{aligned} \tag{2.10}$$

$$\begin{aligned}\Omega_2(t) &= \frac{1}{A_3}\Big\{\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)}\lambda_1 + \Big[L_3 + A_2(t-a)\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\Big]\lambda_2 \\ &\quad + A_1(b-a)\lambda_3 + \Big[L_4 + A_2(b-a)(t-a)\Big]\lambda_4\Big\}.\end{aligned} \tag{2.11}$$

Proof. Integrating the linear system (2.1) twice from a to t, we get

$$u(t) = c_1 + c_2(t-a) + \int_a^t (t-s)f_1(s)ds, \tag{2.12}$$

$$v(t) = c_3 + c_4(t-a) + \int_a^t (t-s)g_1(s)ds, \tag{2.13}$$

where c_1, c_2, c_3 and c_4 are arbitrary real constants.
Using the boundary conditions (1.2) in (2.12) and (2.13), together with notations (2.4), we obtain

$$\begin{aligned}&(b-a)c_1 + \frac{(b-a)^2}{2}c_2 - \sum_{j=1}^m \gamma_j(\eta_j-\xi_j)c_3 - \sum_{j=1}^m \gamma_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)c_4 \\ &= -\int_a^b \frac{(b-s)^2}{2}f_1(s)ds + \sum_{j=1}^m \gamma_j\int_{\xi_j}^{\eta_j}\int_a^s (s-p)g_1(p)dpds + \lambda_1,\end{aligned} \tag{2.14}$$

$$(b-a)c_2 - \sum_{j=1}^m \rho_j(\eta_j-\xi_j)c_4 = -\int_a^b (b-s)f_1(s)ds + \sum_{j=1}^m \rho_j\int_{\xi_j}^{\eta_j}\int_a^s g_1(p)dpds + \lambda_2, \tag{2.15}$$

$$\begin{aligned}&-\sum_{j=1}^m \sigma_j(\eta_j-\xi_j)c_1 - \sum_{j=1}^m \sigma_j\Big(\frac{(\eta_j-a)^2}{2} - \frac{(\xi_j-a)^2}{2}\Big)c_2 + (b-a)c_3 + \frac{(b-a)^2}{2}c_4 \\ &= -\int_a^b \frac{(b-s)^2}{2}g_1(s)ds + \sum_{j=1}^m \sigma_j\int_{\xi_j}^{\eta_j}\int_a^s (s-p)f_1(p)dpds + \lambda_3,\end{aligned} \tag{2.16}$$

$$-\sum_{j=1}^m \delta_j(\eta_j-\xi_j)c_2 + (b-a)c_4 = -\int_a^b (b-s)g_1(s)ds + \sum_{j=1}^m \delta_j\int_{\xi_j}^{\eta_j}\int_a^s f_1(p)dpds + \lambda_4. \tag{2.17}$$

Solving the equations (2.15) and (2.17) for c_2 and c_4, we find that

$$\begin{aligned} c_2 &= \frac{1}{A_1}\Big[-\int_a^b (b-s)\Big((b-a)f_1(s)+\sum_{j=1}^m \rho_j(\eta_j-\xi_j)g_1(s)\Big)ds \\ &+\sum_{j=1}^m \rho_j\int_{\xi_j}^{\eta_j}\int_a^s \Big((b-a)g_1(p)+\sum_{j=1}^m \delta_j(\eta_j-\xi_j)f_1(p)\Big)dpds \\ &+(b-a)\lambda_2+\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\lambda_4\Big], \end{aligned} \tag{2.18}$$

$$\begin{aligned} c_4 &= \frac{1}{A_1}\Big[-\int_a^b (b-s)\Big(\sum_{j=1}^m \delta_j(\eta_j-\xi_j)f_1(s)+(b-a)g_1(s)\Big)ds \\ &+\sum_{j=1}^m \delta_j\int_{\xi_j}^{\eta_j}\int_a^s \Big(\sum_{j=1}^m \rho_j(\eta_j-\xi_j)g_1(p)+(b-a)f_1(p)\Big)dpds \\ &+\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big]. \end{aligned} \tag{2.19}$$

Using (2.18) and (2.19) in (2.14) and (2.16) and then solving the resulting equations for c_1 and c_3, we obtain

$$\begin{aligned} c_1 &= \frac{1}{A_3}\Big\{-\int_a^b \Big[\frac{1}{2}(b-a)A_1(b-s)+L_1\Big](b-s)f_1(s)ds \\ &-\int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)+L_2\Big](b-s)g_1(s)ds \\ &+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s \Big[A_1\gamma_j(b-a)(s-p)+\rho_jL_1\Big]g_1(p)dpds \\ &+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s \Big[A_1\sum_{j=1}^m \gamma_j\sigma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\Big]f_1(p)dpds \\ &+A_1(b-a)\lambda_1+L_1\lambda_2+A_1\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)\lambda_3+L_2\lambda_4\Big\}, \end{aligned}$$

$$\begin{aligned} c_3 &= \frac{1}{A_3}\Big\{-\int_a^b \Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{2\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)}(b-s)+L_3\Big](-s)f_1(s)ds \\ &-\int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s)+L_4\Big](b-s)g_1(s)ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}\sum_{j=1}^m\gamma_j(s-p)+\rho_jL_3\Big]g_1(p)dpds\\
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1(b-a)(s-p)+\delta_jL_4\Big]f_1(p)dpds\\
&+\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}\lambda_1+L_3\lambda_2+A_1(b-a)\lambda_3+L_4\lambda_4\Big\}.
\end{aligned}
$$

Inserting the values of c_1, c_2, c_3 and c_4 in (2.12) and (2.13), we get the solutions (2.2) and (2.3). The converse follows by direct computation. This completes the proof. □

3 Main results

Let us introduce the space $\mathcal{X}=\{u(t)|u(t)\in C([a,b])\}$ equipped with norm $\|u\|=\sup\{|u(t)|, t\in[a,b]\}$. Obviously $(\mathcal{X},\|\cdot\|)$ is a Banach space and consequently, the product space $(\mathcal{X}\times\mathcal{X},\|u,v\|)$ is a Banach space with norm $\|(u,v)\|=\|u\|+\|v\|$ for $(u,v)\in\mathcal{X}\times\mathcal{X}$.

By Lemma 2.1, we define an operator $\mathcal{T}:\mathcal{X}\times\mathcal{X}\to\mathcal{X}\times\mathcal{X}$ as

$$\mathcal{T}(u,v)(t):=(\mathcal{T}_1(u,v)(t),\mathcal{T}_2(u,v)(t)),$$

where

$$
\begin{aligned}
\mathcal{T}_1(u,v)(t) \;=\;& \int_a^t(t-s)f(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)\\
&+L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds\\
&-\int_a^b\Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]\\
&\times(b-s)g(s,u(s),v(s))ds+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p) \qquad (3.1)\\
&+\rho_jL_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds\\
&+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\\
&+\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\}+\Omega_1(t),
\end{aligned}
$$

$$
\mathcal{T}_2(u,v)(t) \;=\; \int_a^t(t-s)g(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{A_1\Big((b-a)^2-A_2\Big)}{2\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)}(b-s)
$$

$$
\begin{aligned}
&+L_3+A_2(t-a)\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big](b-s)f(s,u(s),v(s))ds\\
&-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)+L_4+A_2(b-a)(t-a)\Big](b-s)\\
&\times g(s,u(s),v(s))ds+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[(s-p)\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)} \qquad (3.2)\\
&+\rho_jL_3+\delta_jA_2(t-a)\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big]g(p,u(p),v(p))dpds\\
&+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1(b-a)(s-p)+\delta_jL_4+\delta_jA_2(b-a)(t-a)\Big]\\
&\times f(p,u(p),v(p))dpds\Big\}+\Omega_2(t).
\end{aligned}
$$

In order to prove our main results, we need the following assumptions.

(H_1) There exist real constants $m_i, n_i \geq 0,\ (i=1,2)$ and $m_0>0,\ n_0>0$ such that $\forall u,v\in\mathbb{R}$, we have

$$
\begin{aligned}
|f(t,u,v)| &\leq m_0+m_1|u|+m_2|v|,\\
|g(t,u,v)| &\leq n_0+n_1|u|+n_2|v|.
\end{aligned}
$$

(H_2) There exist nonnegative functions $\alpha(t),\ \beta(t)\in L(0,1)$ and $u,v\in\mathbb{R}$, such that

$$
\begin{aligned}
|f(t,u,v)| &\leq \alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2},\ \epsilon_1,\epsilon_2>0,\ 0<p_1,p_2<1,\\
|g(t,u,v)| &\leq \beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2},\ d_1,d_2>0,\ 0<l_1,l_2<1.
\end{aligned}
$$

(H_3) There exist ℓ_1 and ℓ_2 such that for all $t\in[a,b]$ and $u_i,v_i\in\mathbb{R},\ i=1,2$, we have

$$
\begin{aligned}
|f(t,u_1,v_1)-f(t,u_2,v_2)| &\leq \ell_1(|u_1-u_2|+|v_1-v_2|),\\
|g(t,u_1,v_1)-g(t,u_2,v_2)| &\leq \ell_2(|u_1-u_2|+|v_1-v_2|).
\end{aligned}
$$

For the sake of convenience in the forthcoming analysis, we set

$$
\begin{aligned}
q_1 \;=\; & \frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_1|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^4}{2}\\
&+|A_1|\Big(\sum_{j=1}^{m}\gamma_j\Big)\Big(\sum_{j=1}^{m}\sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+|A_2|(b-a)\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \qquad (3.3)\\
\bar{q}_1 \;=\; &\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\\
&+|A_1|(b-a)\sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \qquad (3.4)\\
q_2 \;=\; &\frac{1}{|A_3|}\Big\{\Big|\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)}\Big|\frac{(b-a)^3}{6}+|L_3|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^3}{2}\\
&\times\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)+|A_1|(b-a)\sum_{j=1}^{m}\sigma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\delta_j|L_4|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^{m}\delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \qquad (3.5)\\
\bar{q}_2 \;=\; &\frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_4|\frac{(b-a)^2}{2}+|A_2|\frac{(b-a)^4}{2}\\
&+\Big|\frac{A_1\Big((b-a)^2-A_2\Big)}{\sum_{j=1}^{m}(\eta_j-\xi_j)}\Big|\sum_{j=1}^{m}\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\rho_j|L_3|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)\Big(\sum_{j=1}^{m}\delta_j\Big)\Big(\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j))\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}, \qquad (3.6)
\end{aligned}
$$

$$\bar{\lambda}_1=\sup_{t\in[a,b]}|\Omega_1(t)|,\quad \bar{\lambda}_2=\sup_{t\in[a,b]}|\Omega_2(t)|. \qquad (3.7)$$

Moreover, we set

$$Q_1=q_1+q_2,\quad Q_2=\bar{q}_1+\bar{q}_2,\quad \bar{\lambda}=\bar{\lambda}_1+\bar{\lambda}_2, \qquad (3.8)$$

where q_i, $\bar{q}_i$ and $\bar{\lambda}_i$ (i=1,2) are given in the equations (3.3) – (3.7) and

$$Q_0=\min\{1-(Q_1m_1+Q_2n_1),1-(Q_1m_2+Q_2n_2)\},\ m_i,n_i\geq 0\ (i=1,2). \qquad (3.9)$$

3.1 Existence of solutions

In this subsection, we discuss the existence of solutions for the problem (1.1)-(1.2) by using standard fixed poit theorems.

Lemma 3.1 *(Leray-Schauder alternative [33]). Let $T : K \to K$ be a completely continuous operator (i.e., a map that restricted to any bounded set in K is compact). Let $\omega(T) = \{x \in K : x = \varphi T(x)$ for some $0 < \varphi < 1\}$. Then either the set $\omega(T)$ is unbounded, or T has at least one fixed point.*

Theorem 3.2 *Assume that condition (H_1) holds. In addition it is assumed that*

$$Q_1 m_1 + Q_2 n_1 < 1 \quad \text{and} \quad Q_1 m_2 + Q_2 n_2 < 1, \tag{3.10}$$

where Q_1 and Q_2 are given by (3.8). Then there exist at least one solution for problem (1.1) − (1.2) on $[a, b]$

Proof. First of all, we show that the operator $\mathcal{T} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is completely continuous. Notice that the operator $\mathcal{T}$ is continuous as the functions f and g are continuous. Let $\Upsilon \subset \mathcal{X} \times \mathcal{X}$ be bounded. Then there exist positive constants κ_f and κ_g such that $|f(t, u(t), v(t))| \le \kappa_f$, $|g(t, u(t), v(t))| \le \kappa_g$, $\forall (u, v) \in \Upsilon$. Then, for any $(u, v) \in \Upsilon$, we can obtain

$$\begin{aligned}
|\mathcal{T}_1(u,v)(t)| = \quad & \sup_{t\in[a,b]} \Big| \int_a^t (t-s) f(s,u(s),v(s))ds - \frac{1}{A_3}\Big\{ \int_a^b \Big[\frac{1}{2}A_1(b-a)(b-s) \\
& + L_1 + (b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds \\
& + \int_a^b \Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j) + L_2 + A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times (b-s)g(s,u(s),v(s))ds\Big\} + \frac{1}{A_3}\Big\{\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\gamma_j A_1(b-a)(s-p) \\
& + \rho_j L_1 + \rho_j(b-a)A_2(t-a)\Big] g(p,u(p),v(p))dpds \\
& + \sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j A_1 \sum_{j=1}^m \gamma_j(\eta_j-\xi_j)(s-p) + \delta_j L_2 \\
& + \delta_j A_2(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] f(p,u(p),v(p))dpds\Big\} + \Omega_1(t)\Big| \\
& \le \kappa_f\Big\{\frac{(b-a)^2}{2} + \frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^4}{6} + |L_1|\frac{(b-a)^2}{2} + |A_2|\frac{(b-a)^4}{2} \\
& + |A_1|\Big(\sum_{j=1}^m \gamma_j\Big)\Big(\sum_{j=1}^m \sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!} - \frac{(\xi_j-a)^3}{3!}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)\Big(\sum_{j=1}^{m}\rho_j\Big)\Big(\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big]\Big\}\\
&+\kappa_g\Big\{\frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^{m}\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2}\\
&+|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\\
&+|A_1|(b-a)\sum_{j=1}^{m}\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\}+\bar{\lambda}_1\\
&\le \kappa_f q_1+\kappa_g\bar{q}_1+\bar{\lambda}_1,
\end{aligned}
$$

which implies that

$$\|\mathcal{T}_1(u,v)\|\le \kappa_f q_1+\kappa_g\bar{q}_1+\bar{\lambda}_1.$$

Similarly, it can be found that

$$\|\mathcal{T}_2(u,v)\|\le \kappa_f q_2+\kappa_g\bar{q}_2+\bar{\lambda}_2.$$

Consequently, we get $\|\mathcal{T}(u,v)(t)\|\le \kappa_f Q_1+\kappa_g Q_2+\bar{\lambda}$ (Q_1 , Q_2 and $\bar{\lambda}$ are given by (3.8)), which implies that the operator $\mathcal{T}$ is uniformly bounded. Next, we show that $\mathcal{T}$ is equicontinuous. For $t_1,t_2\in[a,b]$ with $t_1<t_2$, we have

$$
\begin{aligned}
&|\mathcal{T}_1(u,v)(t_2)-\mathcal{T}_1(u,v)(t_1)|\\
\le\;& \kappa_f\Big|\int_a^{t_1}\Big[(t_2-s)-(t_1-s)\Big]ds+\int_{t_1}^{t_2}(t_2-s)ds\Big|\\
&+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\int_a^b(b-a)(b-s)ds+\Big(\sum_{j=1}^{m}\delta_j\Big)\int_{\xi_j}^{\eta_j}\int_a^s\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)dpds\Big]\\
&+\kappa_g\Big[\int_a^b\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)(b-s)ds+\sum_{j=1}^{m}\int_{\xi_j}^{\eta_j}\int_a^s\rho_j(b-a)dpds\Big]\\
&+(b-a)\lambda_2+\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\lambda_4\Big\}
\end{aligned}
$$

$$\begin{aligned}
&\leq \kappa_f\Big[(t_2-t_1)(t_1-a)+\frac{(t_2-t_1)^2}{2}\Big]+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\frac{(b-a)^3}{2}\\
&+\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\kappa_g\Big[\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\frac{(b-a)^2}{2}+(b-a)\sum_{j=1}^m\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+(b-a)\lambda_2+\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\lambda_4\Big\}\to 0 \text{ independent of } u \text{ and } v \text{ as } (t_2-t_1)\to 0.
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
&|\mathcal{T}_2(u,v)(t_2)-\mathcal{T}_2(u,v)(t_1)|\\
&\leq \kappa_g\Big[(t_2-t_1)(t_1-a)+\frac{(t_2-t_1)^2}{2}\Big]+\frac{(t_2-t_1)}{|A_1|}\Big\{\kappa_f\Big[\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\frac{(b-a)^3}{6}\\
&+(b-a)\sum_{j=1}^m\delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\kappa_g\Big[\frac{(b-a)^3}{2}+\Big(\sum_{j=1}^m\delta_j\Big)\Big(\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\
&+\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big\}\to 0 \text{ independent of } u \text{ and } v \text{ as } (t_2-t_1)\to 0.
\end{aligned}$$

Finally, we will verify that the set $\omega=\{(u,v)\in\mathcal{X}\times\mathcal{X}|(u,v)=\varphi\mathcal{T}(u,v),0<\varphi<1\}$ is bounded. Let $(u,v)\in\omega$. Then $(u,v)=\varphi\mathcal{T}(u,v)$ and for any $t\in[a,b]$, we have

$$u(t)=\varphi\mathcal{T}_1(u,v)(t),\quad v(t)=\varphi\mathcal{T}_2(u,v)(t).$$

Then

$$\begin{aligned}
|u(t)| &\leq q_1(m_0+m_1\|u\|+m_2\|v\|)+\bar{q}_1(n_0+n_1\|u\|+n_2\|v\|)+\bar{\lambda}_1\\
&= q_1m_0+\bar{q}_1n_0+(q_1m_1+\bar{q}_1n_1)\|u\|+(q_1m_2+\bar{q}_1n_2)\|v\|+\bar{\lambda}_1,
\end{aligned}$$

and

$$\begin{aligned}
|v(t)| &\leq q_2(m_0+m_1\|u\|+m_2\|v\|)+\bar{q}_2(n_0+n_1\|u\|+n_2\|v\|)+\bar{\lambda}_2\\
&= q_2m_0+\bar{q}_2n_0+(q_2m_1+\bar{q}_2n_1)\|u\|+(q_2m_2+\bar{q}_2n_2)\|v\|+\bar{\lambda}_2.
\end{aligned}$$

Hence, we have

$$\|u\|+\|v\| \leq (q_1+q_2)m_0+(\bar{q}_1+\bar{q}_2)n_0+[(q_1+q_2)m_1+(\bar{q}_1+\bar{q}_2)n_1]\|u\|$$

$$+[(q_1+q_2)m_2+(\bar{q}_1+\bar{q}_2)n_2]\|v\|+\bar{\lambda}_1+\bar{\lambda}_2,$$

which, in view of (3.9) and (3.10), yields

$$\|(u,v)\|\leq\frac{Q_1m_0+Q_2n_0+\bar{\lambda}}{Q_0},$$

for any $t\in[a,b]$, which proves that the set ω is bounded. Hence, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Therefore, the problem $(1.1)-(1.2)$ has at least one solution on $[a,b]$. This completes the proof. □

Next, we apply Schauder fixed point theorem to prove the existence of solutions for the problem (1.1)-(1.2) by imposing the the sub-growth condition on the nonlinear functions involved in the problem.

Theorem 3.3 *Assume that* (H_2) *holds. Then, there exist at least one solution on* $[a,b]$ *for the problem* $(1.1)-(1.2)$.

Proof. Define a set Y in the Banach space $\mathcal{X}\times\mathcal{X}$ by

$$Y=\{(u,v)\in\mathcal{X}\times\mathcal{X}:\|(u,v)\|\leq y\},$$

where

$$y\geqslant\max\{7\bar{\lambda},7Q_1\alpha(t),7Q_2\beta(t),(7Q_1\epsilon_1)^{\frac{1}{1-p_1}},(7Q_1\epsilon_2)^{\frac{1}{1-p_2}},(7Q_2d_1)^{\frac{1}{1-l_1}},(7Q_2d_2)^{\frac{1}{1-l_1}}\}.$$

In order to show that $\mathcal{T}:Y\to Y$. We have

$$\begin{aligned}|\mathcal{T}_1(u,v)(t)|=\quad&\sup_{t\in[a,b]}\Big|\int_a^t(t-s)f(s,u(s),v(s))ds-\frac{1}{A_3}\Big\{\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)\\&+L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds\\&+\int_a^b\Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]\\&\times(b-s)g(s,u(s),v(s))ds\Big\}+\frac{1}{A_3}\Big\{\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p)\\&+\rho_jL_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds\\&+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\\&+\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\}+\Omega_1(t)\Big|\end{aligned}$$

$$\leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_1+\bar{\lambda}_1,$$

which implies that

$$\|\mathcal{T}_1(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_1+\bar{\lambda}_1.$$

Analogously, we have

$$\|\mathcal{T}_2(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)q_2+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)\bar{q}_2+\bar{\lambda}_2.$$

In consequence,

$$\|\mathcal{T}(u,v)\| \leq \Big(\alpha(t)+\epsilon_1|u|^{p_1}+\epsilon_2|v|^{p_2}\Big)Q_1+\Big(\beta(t)+d_1|u|^{l_1}+d_2|v|^{l_2}\Big)Q_2+\bar{\lambda} \leq y,$$

where Q_1 , Q_2 and $\bar{\lambda}$ are given by (3.8). Therefore, we conclude that $\mathcal{T}: Y \to Y$, where $\mathcal{T}_1(u,v)(t)$ and $\mathcal{T}_2(u,v)(t)$ are continuous on $[a,b]$.

Now we prove that $\mathcal{T}$ is completely continuous operator by fixing that

$$G=\max_{t\in[a,b]}|f(t,u(t),v(t))|, \quad H=\max_{t\in[a,b]}|g(t,u(t),v(t))|.$$

Letting $t,\tau \in [a,b]$ with $a<t<\tau<b$ and $(u,v)\in Y$, we get

$$\begin{aligned}&|\mathcal{T}_1(u,v)(\tau)-\mathcal{T}_1(u,v)(t)|\\ &\leq G\Big[(\tau-t)(t-a)+\frac{(\tau-t)^2}{2}\Big]+\frac{(\tau-t)}{|A_1|}\Big\{G\Big[\frac{(b-a)^3}{2}\\ &+\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\ &+H\Big[\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\frac{(b-a)^2}{2}+(b-a)\sum_{j=1}^m\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\\ &+(b-a)\lambda_2+\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\lambda_4\Big\} \to 0 \text{ as } (\tau-t)\to 0.\end{aligned}$$

In a similar manner, one can obtain

$$\begin{aligned}&|\mathcal{T}_2(u,v)(\tau)-\mathcal{T}_2(u,v)(t)|\\ &\leq H\Big[(\tau-t)(t-a)+\frac{(\tau-t)^2}{2}\Big]+\frac{(\tau-t)}{|A_1|}\Big\{G\Big[\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\frac{(b-a)^3}{6}\\ &+(b-a)\sum_{j=1}^m\delta_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\end{aligned}$$

$$+H\Big[\frac{(b-a)^3}{2}+\Big(\sum_{j=1}^{m}\delta_j\Big)\Big(\sum_{j=1}^{m}\rho_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]$$
$$+\sum_{j=1}^{m}\delta_j(\eta_j-\xi_j)\lambda_2+(b-a)\lambda_4\Big\}\to 0 \text{ as } (\tau-t)\to 0.$$

Thus the operator $\mathcal{T}Y\subset Y$ is equicontinuous and uniformaly bounded set. Hence $\mathcal{T}$ is a completely continuous operator. So, by Schauder fixed point theorem, there exist a solution to the problem $(1.1)-(1.2)$. □

3.2 Uniqueness of solutions

Here we establish the uniqueness of solutions for the problem $(1.1)-(1.2)$ by means of Banach's contraction mapping principle.

Theorem 3.4 *Assume that* (H_3) *holds and that*

$$Q_1\ell_1+Q_2\ell_2<1, \tag{3.11}$$

where Q_1 *and* Q_2 *are given by* (3.8). *Then the problem* $(1.1)-(1.2)$ *has a unique solution on* $[a,b]$.

Proof. Define $\sup_{t\in[a,b]}|f(t,0,0)|=N_1, \sup_{t\in[a,b]}|g(t,0,0)|=N_2$ and

$$r\ge\frac{Q_1N_1+Q_2N_2+\bar{\lambda}}{1-(Q_1\ell_1+Q_2\ell_2)}.$$

Then we show that $\mathcal{T}B_r\subset B_r$, where $B_r=\{(u,v)\in\mathcal{X}\times\mathcal{X}:\|(u,v)\|\le r\}$. For any $(u,v)\in B_r, t\in[a,b]$, we find that

$$\begin{aligned}|f(s,u(s),v(s))| &= |f(s,u(s),v(s))-f(s,0,0)+f(s,0,0)|\\ &\le |f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)|\\ &\le \ell_1(\|u\|+\|v\|)+N_1\le\ell_1\|(u,v)\|+N_1\le\ell_1r+N_1,\end{aligned}$$

and

$$\begin{aligned}|g(s,u(s),v(s))| &= |g(s,u(s),v(s))-g(s,0,0)+g(s,0,0)|\\ &\le |g(s,u(s),v(s))-g(s,0,0)|+|g(s,0,0)|\\ &\le \ell_2(\|u\|+\|v\|)+N_2\le\ell_2\|(u,v)\|+N_2\le\ell_2r+N_2.\end{aligned}$$

Then, for $(u,v)\in B_r$, we obtain

$$|\mathcal{T}_1(u,v)(t)|\le \sup_{t\in[a,b]}\Big|\int_a^t(t-s)f(s,u(s),v(s))ds+\frac{1}{A_3}\Big\{-\int_a^b\Big[\frac{1}{2}A_1(b-a)(b-s)$$

$$
\begin{aligned}
&+L_1+(b-a)A_2(t-a)\Big](b-s)f(s,u(s),v(s))ds\\
&-\int_a^b\Big[\frac{1}{2}A_1(b-s)\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+L_2+A_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]\\
&\times(b-s)g(s,u(s),v(s))ds+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\gamma_jA_1(b-a)(s-p)\\
&+\rho_jL_1+\rho_j(b-a)A_2(t-a)\Big]g(p,u(p),v(p))dpds\\
&+\sum_{j=1}^m\int_{\xi_j}^{\eta_j}\int_a^s\Big[\sigma_jA_1\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)(s-p)+\delta_jL_2\\
&+\delta_jA_2(t-a)\sum_{j=1}^m\rho_j(\eta_j-\xi_j)\Big]f(p,u(p),v(p))dpds\Big\}+\Omega_1(t)\Big|\\
&\le[\ell_1r+N_1]\times\Big\{\frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^4}{6}+|L_1|\frac{(b-a)^2}{2}\\
&+|A_2|\frac{(b-a)^4}{2}+\sum_{j=1}^m\delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_1|\Big(\sum_{j=1}^m\gamma_j\Big)\Big(\sum_{j=1}^m\sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)\\
&+|A_2|(b-a)\Big(\sum_{j=1}^m\rho_j\Big)\Big(\sum_{j=1}^m\delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}\\
&+[\ell_2r+N_2]\times\Big\{\frac{1}{|A_3|}\Big\{|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^m\gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2}\\
&+|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^m\rho_j(\eta_j-\xi_j)+|A_1|(b-a)\\
&\times\sum_{j=1}^m\gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)+\sum_{j=1}^m\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\\
&+|A_2|(b-a)^2\sum_{j=1}^m\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big\}+\bar{\lambda}_1\\
\le\ &q_1(\ell_1r+N_1)+\bar{q}_1(\ell_2r+N_2)+\bar{\lambda}_1.
\end{aligned}
$$

Hence

$$\|\mathcal{T}_1(u,v)\|\le q_1(\ell_1r+N_1)+\bar{q}_1(\ell_2r+N_2)+\bar{\lambda}_1.$$

Likewise, we find that

$$\|\mathcal{T}_2(u,v)\|\le q_2(\ell_1r+N_1)+\bar{q}_2(\ell_2r+N_2)+\bar{\lambda}_2.$$

From the above estimates, it follows that that $\|\mathcal{T}(u,v)\| \le r$.

Next we show that the operator $\mathcal{T}$ is a contraction. For $(u_1,v_1),\ (u_2,v_2) \in \mathcal{X}\times\mathcal{X}$, we have

$$
\begin{aligned}
& |\mathcal{T}_1(u_1,v_1)(t)-\mathcal{T}_1(u_2,v_2)(t)| \\
\le\ & \sup_{t\in[a,b]}\Big\{\int_a^t (t-s)\Big|f(s,u_1(s),v_1(s))-f(s,u_2(s),v_2(s))\Big|ds \\
& +\frac{1}{|A_3|}\Big\{\int_a^b \Big[\frac{1}{2}|A_1|(b-a)(b-s)+L_1+(b-a)|A_2|(t-a)\Big](b-s) \\
& \times\Big|f(s,u_1(s),v_1(s))-f(s,u_2(s),v_2(s))\Big|ds \\
& +\int_a^b \Big[\frac{1}{2}|A_1|(b-s)\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)+L_2+|A_2|(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times(b-s)\Big|g(s,u_1(s),v_1(s))-g(s,u_2(s),v_2(s))\Big|ds\Big\} \\
& +\frac{1}{|A_3|}\Big\{\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\gamma_j|A_1|(b-a)(s-p)+\rho_j L_1+\rho_j(b-a)|A_2|(t-a)\Big] \\
& \times\Big|g(p,u_1(p),v_1(p))-g(p,u_2(p),v_2(p))\Big|dpds \\
& +\sum_{j=1}^m \int_{\xi_j}^{\eta_j}\int_a^s \Big[\sigma_j|A_1|\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)(s-p)+\delta_j L_2+\delta_j|A_2|(t-a)\sum_{j=1}^m \rho_j(\eta_j-\xi_j)\Big] \\
& \times\Big|f(p,u_1(p),v_1(p))-f(p,u_2(p),v_2(p))\Big|dpds\Big\} \\
\le\ & \ell_1(|u_1-u_2|+|v_1-v_2|)\times\Big\{\frac{(b-a)^2}{2}+\frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^4}{6}+|L_1|\frac{(b-a)^2}{2} \\
& +|A_2|\frac{(b-a)^4}{2}+|A_1|\Big(\sum_{j=1}^m \gamma_j\Big)\Big(\sum_{j=1}^m \sigma_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big) \\
& +\sum_{j=1}^m \delta_j|L_2|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big) \\
& +|A_2|(b-a)\Big(\sum_{j=1}^m \rho_j\Big)\Big(\sum_{j=1}^m \delta_j(\eta_j-\xi_j)\Big)\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\} \\
& +\ell_2(|u_1-u_2|+|v_1-v_2|)\times\Big\{\frac{1}{|A_3|}\Big[|A_1|\frac{(b-a)^3}{6}\sum_{j=1}^m \gamma_j(\eta_j-\xi_j)+|L_2|\frac{(b-a)^2}{2} \\
& +|A_2|\frac{(b-a)^3}{2}\sum_{j=1}^m \rho_j(\eta_j-\xi_j)+|A_1|(b-a)\sum_{j=1}^m \gamma_j\Big(\frac{(\eta_j-a)^3}{3!}-\frac{(\xi_j-a)^3}{3!}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\rho_j|L_1|\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)+|A_2|(b-a)^2\sum_{j=1}^{m}\rho_j\Big(\frac{(\eta_j-a)^2}{2}-\frac{(\xi_j-a)^2}{2}\Big)\Big]\Big\}\\
\leq\ & (\ell_1 q_1+\ell_2\bar{q}_1)(|u_1-u_2|+|v_1-v_2|),
\end{aligned}
$$

which yields

$$\|\mathcal{T}_1(u_1,v_1)-\mathcal{T}_1(u_2,v_2)\|\leq(\ell_1 q_1+\ell_2\bar{q}_1)(|u_1-u_2|+|v_1-v_2|).$$

Similarly,

$$\|\mathcal{T}_2(u_1,v_1)-\mathcal{T}_2(u_2,v_2)\|\leq(\ell_1 q_2+\ell_2\bar{q}_2)(|u_1-u_2|+|v_1-v_2|).$$

So, it follows from the above inequalities that

$$\|\mathcal{T}(u_1,v_1)-\mathcal{T}(u_2,v_2)\|\leq(Q_1\ell_1+Q_2\ell_2)(\|u_1-u_2\|+\|v_1-v_2\|),$$

where Q_1 and Q_2 are given by (3.8). By the given assumption (3.11), it follows that the operator $\mathcal{T}$ is a contraction. Thus, by Banach's contraction mapping principle, we deduce that the operator $\mathcal{T}$ has a fixed point, which corresponds to a unique solution of the problem (1.1)-(1.2) on $[a,b]$. □

Example 3.5 *Consider the following second order system of ordinary differential equations*

$$
\begin{cases}
u''(t)=\dfrac{1}{10+t^2}\Big(\dfrac{|u|}{1+|u|}+v(t)\Big)+e^{-t}, & t\in[2,3],\\
v''(t)=\dfrac{1}{3\sqrt{32+t^2}}\Big(u(t)+\tan^{-1}v(t)\Big)+\cos(t-2), & t\in[2,3],
\end{cases}
\tag{3.12}
$$

subject to the boundary conditions

$$
\begin{cases}
\displaystyle\int_2^3 u(s)ds=\sum_{j=1}^{3}\gamma_j\int_{\xi_j}^{\eta_j}v(s)ds+2, \int_2^3 u'(s)ds=\sum_{j=1}^{3}\rho_j\int_{\xi_j}^{\eta_j}v'(s)ds+1,\\
\displaystyle\int_2^3 v(s)ds=\sum_{j=1}^{3}\sigma_j\int_{\xi_j}^{\eta_j}u(s)ds+\frac{3}{2}, \int_2^3 v'(s)ds=\sum_{j=1}^{3}\delta_j\int_{\xi_j}^{\eta_j}u'(s)ds+\frac{1}{2},
\end{cases}
\tag{3.13}
$$

where $a=2, b=3, m=3, \lambda_1=2, \lambda_2=1, \lambda_3=3/2, \lambda_4=1/2, \gamma_1=2/5, \gamma_2=21/40, \gamma_3=13/20, \rho_1=1/3, \rho_2=1/2, \rho_3=2/3, \sigma_1=3/7, \sigma_2=5/7, \sigma_3=1, \delta_1=3/8, \delta_2=5/8, \delta_3=7/8, \xi_1=15/7, \eta_1=16/7, \xi_2=17/7, \eta_2=18/7, \xi_3=19/7, \eta_3=20/7.$

Using the given data, we find that $\ell_1=\frac{1}{7}, \ell_2=\frac{1}{9}$, $A_1\approx 0.827806\neq 0, A_2\approx 0.793367\neq 0, A_3\approx 0.656754, |L_1|=0.03337, |L_2|\approx 0.225389, |L_3|\approx 0.027121, |L_4|\approx 0.185097, q_1\approx 1.963984, q_2\approx 1.422591, \bar{q}_1\approx 1.290164$ and $\bar{q}_2\approx 1.851349$. Also $Q_1\ell_1+Q_2\ell_2\approx 0.832853<1$ (Q_1 and Q_2 are given by (3.8)). Thus, all the conditions of Theorem 3.4 are satisfied. Hence it follows by the conclusion of Theorem 3.4 that the problem (3.12) – (3.13) has a unique solution on $[2,3]$.

4 Conclusions

The salient features of this work includes *(i)* considering a coupled system of nonlinear ordinary differential equations on an arbitrary domain *(ii)* a new kind of integral multi-strip coupled boundary conditions. The results obtained for the given problem are new and significantly contribute to the existing literature on the topic. As a special case, our results correspond to the uncoupled integral boundary conditions of the form:

$$\int_a^b u(s)ds = \lambda_1, \int_a^b u'(s)ds = \lambda_2; \int_a^b v(s)ds = \lambda_3, \int_a^b v'(s)ds = \lambda_4,$$

if we take all $\gamma_j = 0, \rho_j = 0, \sigma_j = 0, \delta_j = 0$ $(j = 1, \ldots, m)$ in the results of this paper.

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Explicit identities involving truncated exponential polynomials and phenomenon of scattering of their zeros

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Abstract : In this paper, we study differential equations arising from the generating functions of truncated exponential polynomials. We give explicit identities for the truncated polynomials. Using numerical investigation, we observe the behavior of complex roots of the truncated polynomials $e_n(x)$. By means of numerical experiments, we demonstrate a remarkably regular structure of the complex roots of the truncated polynomials $e_n(x)$.

Key words : Differential equations, complex roots, truncated polynomials.

AMS Mathematics Subject Classification : 05A19, 11B83, 34A30, 65L99.

1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, tangent numbers and polynomials, Genocchi numbers and polynomials, Laguerre polynomials, and Hermite polynomials. These numbers and polynomials possess many interesting properties and arising in many areas of mathematics, physics, and applied engineering(see [1-14]). By using software, many mathematicians can explore concepts much more easily than in the past. The ability to create and manipulate figures on the computer screen enables mathematicians to quickly visualize and produce many problems, examine properties of the figures, look for patterns, and make conjectures. This capability is especially exciting because these steps are essential for most mathematicians to truly understand even basic concept. Numerical experiments of Euler polynomials, Bernoulli polynomials, tangent polynomials, Genocchi polynomials, Laguerre polynomials, and Hermite polynomials have been the subject of extensive study in recent year and much progress have been made both mathematically and computationally. Using computer, a realistic study for the zeros of truncated polynomials $e_n(x)$ is very interesting. The main purpose of this paper is to observe an interesting phenomenon of 'scattering' of the zeros of the truncated polynomials $e_n(x)$ in complex plane. Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ denotes the set of nonnegative integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers. We first give the definitions of the truncated exponential polynomials. It should be mentioned that the definition of truncated exponential polynomials $e_n(x)$ can be found in [1, 3]. The truncated exponential polynomials $e_n(x)$ are defined by means of the generating function:

$$\left(\frac{1}{1-t}\right) e^{xt} = \sum_{n=0}^{\infty} e_n(x) t^n, \quad |t| < 1. \tag{1.1}$$

We recall that G. Dattoli and M. Migliorati(see [3]) studied some properties of truncated exponential polynomials $e_n(x)$. The truncated exponential polynomials $e_n(x)$ satisfy the following relations

$$\frac{d}{dx} e_n(x) = e_{n-1}(x),$$

$$e_{n+1}(x) = \left(1 + \frac{x}{n+1}\left(1 - \frac{d}{dx}\right)\right) e_n(x).$$

The Miller-Lee polynomials $G_n^{(k)}(x)$(see [1]), are defined by means of the following generating function

$$\left(\frac{1}{1-t}\right)^{k+1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)t^n. \tag{1.2}$$

Differential equations arising from the generating functions of special polynomials are studied by many authors in order to give explicit identities for special polynomials. In this paper, we study linear differential equations arising from the generating functions of truncated exponential polynomials $e_n(x)$. We give explicit identities for truncated exponential polynomials $e_n(x)$.

2. Differential equations associated with truncated exponential polynomials

In this section, we study linear differential equations arising from the generating functions of truncated exponential polynomials. Let

$$F = F(t,x) = \left(\frac{1}{1-t}\right) e^{xt}. \tag{2.1}$$

Then, by (2.1), we get

$$\begin{aligned} F^{(1)} &= \frac{d}{dt}F(t,x) = \frac{d}{dt}\left(\frac{1}{1-t}\right)e^{xt} \\ &= \left(\frac{1}{1-t}\right)^2 e^{xt} + x\left(\frac{1}{1-t}\right)e^{xt} \\ &= \left(\frac{1}{1-t} + x\right)F(t,x), \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F^{(2)} &= \left(\frac{d}{dt}\right)^2 F(t,x) \\ &= \left(\frac{1}{1-t}\right)^2 e^{xt}F(t,x) + \left(\frac{1}{1-t} + x\right)F(t,x)F^{(1)} \\ &= \left(\left(\frac{2}{1-t}\right)^2 + \left(\frac{2x}{1-t} + x^2\right)\right)F(t,x). \end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t,x) \\ &= \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right)F(t,x), \quad (N = 0,1,2,\ldots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we obtain

$$\begin{aligned}
F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\
&= \left(\sum_{i=0}^{N} ia_i(N,x)(1-t)^{-i-1}\right)F(t,x) + \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right)F^{(1)}(t,x) \\
&= \left(\sum_{i=0}^{N} ia_i(N,x)(1-t)^{-i-1}\right)F(t,x) \\
&\qquad + \left(\sum_{i=0}^{N} a_i(N,x)(1-t)^{-i}\right)\left((1-t)^{-1}+x\right)F(t,x) \\
&= \left(\sum_{i=0}^{N} (i+1)a_i(N,x)(1-t)^{-i-1}\right)F(t,x) + \left(\sum_{i=0}^{N} xa_i(N,x)(1-t)^{-i}\right)F(t,x) \\
&= \left(\sum_{i=0}^{N} xa_i(N,x)(1-t)^{-i}\right)F(t,x) + \left(\sum_{i=1}^{N+1} ia_{i-1}(N,x)(1-t)^{-i}\right)F(t,x).
\end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = \left(\sum_{i=0}^{N+1} a_i(N+1,x)(1-t)^{-i}\right)F(t,x). \tag{2.6}$$

By (2.5) and (2.6), we have

$$\begin{aligned}
&\left(\sum_{i=0}^{N} xa_i(N,x)(1-t)^{-i}\right)F(t,x) + \left(\sum_{i=1}^{N+1} ia_{i-1}(N,x)(1-t)^{-i}\right)F(t,x) \\
&= \left(\sum_{i=0}^{N+1} a_i(N+1,x)(1-t)^{-i}\right)F(t,x)..
\end{aligned} \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$\begin{aligned}
a_0(N+1,x) &= xa_0(N,x), \\
a_{N+1}(N+1,x) &= (N+1)a_N(N,x),
\end{aligned} \tag{2.8}$$

and

$$a_i(N+1,x) = xa_i(N,x) + ia_{i-1}(N,x), (1 \le i \le N). \tag{2.9}$$

In addition, by (2.2) and (2.4), we get

$$F = F^{(0)} = a_0(0,x)F(t,x) = F(t,x). \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_0(0,x) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned}
&(1-t)^{-1}F(t,x) + xF(t,x) \\
&= \sum_{i=0}^{1} a_i(1,x)(1-t)^{-i}F(t,x) \\
&= a_0(1,x)F(t,x) + a_1(1,x)(1-t)^{-1}F(t,x).
\end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_0(1,x) = x, \quad a_1(1,x) = 1. \tag{2.13}$$

From (2.8), we note that

$$a_0(N+1,x)=xa_0(N,x)=x^2a_0(N-1,x)=\cdots=x^{N+1},$$

and

$$a_{N+1}(N+1,x)=(N+1)a_N(N,x)=\cdots=(N+1)!. \tag{2.14}$$

For $i=1,2,3$ in (2.9), we get

$$\begin{aligned} a_1(N+1,x)&=\sum_{k=0}^{N}x^k a_0(N-k,x),\\ a_2(N+1,x)&=2\sum_{k=0}^{N-1}x^k a_1(N-k,x), \text{ and}\\ a_3(N+1,x)&=3\sum_{k=0}^{N-2}x^k a_2(N-k,x). \end{aligned}$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$a_i(N+1,x)=i\sum_{k=0}^{N-i+1}x^k a_{i-1}(N-k,x). \tag{2.15}$$

Now, we give explicit expressions for $a_i(N+1,x)$. By (2.14) and (2.15), we get

$$a_1(N+1,x)=\sum_{k_1=0}^{N}x^{k_1}a_0(N-k_1,x)=x^N(N+1),$$

$$a_2(N+1,x)=2\sum_{k_1=0}^{N-1}x^{k_1}a_1(N-k_1,x)=2!\sum_{k_1=0}^{N-1}x^{N-1}(N-k_1),$$

and

$$\begin{aligned} a_3(N+1,x)&=3\sum_{k_2=0}^{N-2}x^{k_2}a_2(N-k_2,x)\\ &=3!\sum_{k_2=0}^{N-2}\sum_{k_1=0}^{N-k_2-2}x^{N-k_2-2}(N-k_2-k_1-1). \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} a_i(N+1,x)=i!\sum_{k_{i-1}=0}^{N-i+1}\sum_{k_{i-2}=0}^{N-k_{i-1}-i+1}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i+1}&x^{N-k_i-\cdots-k_2-i+1}\\ \times(N-k_{i-1}-k_{i-2}-\cdots-k_2-k_1-i+2).& \end{aligned} \tag{2.16}$$

Note that, here the matrix $a_i(j,x)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix} 1 & x & x^2 & x^3 & \cdots & x^{N+1}\\ 0 & 1! & 2x & \cdot & \cdots & (N+1)x^N\\ 0 & 0 & 2! & \cdot & \cdots & \cdot\\ 0 & 0 & 0 & 3! & \cdots & \cdot\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & (N+1)! \end{pmatrix}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 1. For $N = 0, 1, 2, \ldots$, the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N,x) \left(\frac{1}{1-t} \right)^i \right) F$$

has a solution

$$F = F(t,x) = \left(\frac{1}{1-t} \right) e^{xt},$$

where

$$\begin{aligned} &a_0(N,x) = x^N, \\ &a_N(N,x) = N!, \\ &a_i(N,x) = i! \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-k_{i-1}-i} \cdots \sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i} x^{N-k_{i-1}-\cdots-k_2-i} \\ &\qquad\qquad \times (N - k_{i-1} - k_{i-2} - \cdots - k_2 - k_1 - i + 1), \\ &(1 \le i \le N). \end{aligned}$$

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt} \right)^N F(t,x) = \sum_{k=0}^{\infty} \frac{(k+N)!}{k!} e_{k+N}(x) t^k. \tag{2.17}$$

From Theorem 1, (1.2), and (2.17), we can derive the following equation:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(k+N)!}{k!} e_{k+N}(x) t^k &= \left(\sum_{i=0}^{N} a_i(N,x) \left(\frac{1}{1-t} \right)^i \right) F \\ &= \sum_{i=0}^{N} a_i(N,x) \left(\frac{1}{1-t} \right)^{i+1} e^{xt} \\ &= \sum_{i=0}^{N} a_i(N,x) \left(\sum_{k=0}^{\infty} G_k^{(i)}(x) t^k \right) \\ &= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{N} a_i(N,x) G_k^{(i)}(x) \right) t^k. \end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2. For $k = 0, 1, \ldots,$ and $N = 0, 1, 2, \ldots$, we have

$$e_{k+N}(x) = \frac{k!}{(k+N)!} \sum_{i=0}^{N} a_i(N,x) G_k^{(i)}(x), \tag{2.19}$$

where

$$\begin{aligned} &a_0(N,x) = x^N, \\ &a_N(N,x) = N!, \\ &a_i(N,x) = i! \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-k_{i-1}-i} \cdots \sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i} x^{N-k_{i-1}-\cdots-k_2-i} \\ &\qquad\qquad \times (N - k_{i-1} - k_{i-2} - \cdots - k_2 - k_1 - i + 1), \\ &(1 \le i \le N). \end{aligned}$$

Let us take $k = 0$ in (2.19). Then, we have the following corollary.

Corollary 3. For $N = 0, 1, 2, \ldots,$ we have

$$e_N(x) = \frac{1}{N!}\sum_{i=0}^{N} a_i(N,x) G_0^{(i)}(x).$$

For $N = 1, 2, \ldots,$ the functional equation

$$F^{(N)} = \left(\sum_{i=0}^{N} a_i(N,x)\left(\frac{1}{1-t}\right)^i\right) F$$

has a solution

$$F = F(t,x) = \left(\frac{1}{1-t}\right) e^{xt}.$$

Here is a plot of the surface for this solution.

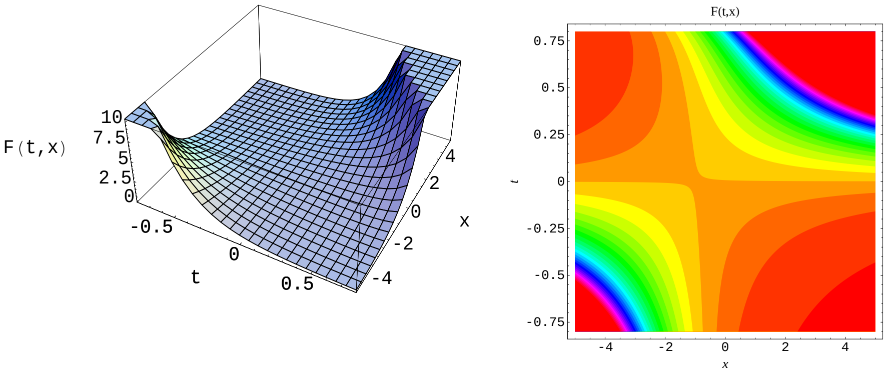

Figure 1: The surface for the solution $F(t,x)$

In Figure 1(left), we plot of the surface for this solution. In Figure 1(right), we shows a higher-resolution density plot of the solution.

3. Zeros of the truncated exponential polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the truncated exponential polynomials $e_n(x)$. By using computer, the truncated exponential polynomials $e_n(x)$ can be determined explicitly. We display the shapes of the truncated exponential polynomials $e_n(x)$ and investigate the zeros of the truncated exponential polynomials $e_n(x)$. We investigate the beautiful zeros of the truncated exponential polynomials $e_n(x)$ by using a computer. We plot the zeros of the $e_n(x)$ for $n = 20, 30, 40, 50$ and $x \in \mathbb{C}$(Figure 2). In Figure 2(top-left), we choose $n = 20$. In Figure 2(top-right), we choose $n = 30$. In Figure 2(bottom-left), we choose $n = 40$. In Figure 2(bottom-right), we choose $n = 50$.

Stacks of zeros of $e_n(x)$ for $1 \leq n \leq 40$, forming a 3D structure are presented(Figure 3). In Figure 3(top-left), we plot stacks of zeros of $e_n(x)$ for $1 \leq n \leq 40$. In Figure 3(top-right), we draw x and y axes but no z axis in three dimensions. In Figure 3(bottom-left), we draw y and z axes but no x axis in three dimensions. In Figure 3(bottom-right), we draw x and z axes but no y axis in three dimensions.

Our numerical results for approximate solutions of real zeros of the truncated exponential polynomials $e_n(x)$ are displayed(Tables 1, 2).

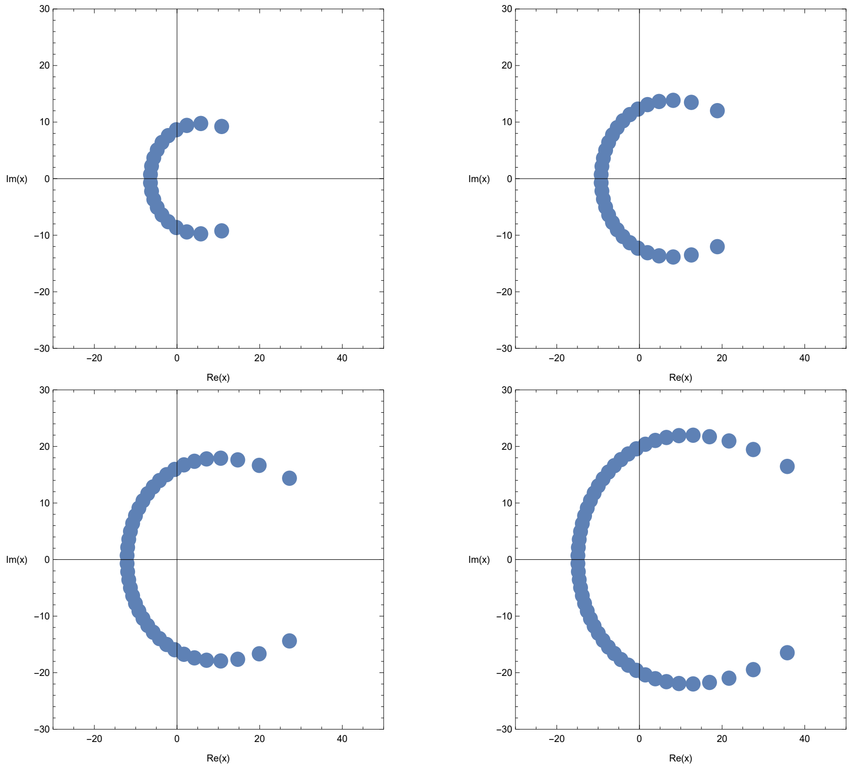

Figure 2: Zeros of $e_n(x)$

Table 1. Numbers of real and complex zeros of $e_n(x)$

degree n	real zeros	complex zeros
1	1	0
2	0	2
3	1	2
4	0	4
5	1	4
6	0	6
7	1	6
8	0	8
9	1	8
10	0	10
11	1	10
12	0	12
13	1	12
14	0	14

How many zeros does $e_n(x)$ have? We are not able to decide if $e_n(x)$ has n distinct solutions(see Table 1, Table 2). We would also like to know the number of complex zeros $C_{e_n(x)}$ of $e_n(x), Im(x) \neq 0$. Since n is the degree of the polynomial $e_n(x)$, the number of real zeros $R_{e_n(x)}$ lying on the real line $Im(x) = 0$ is then $R_{e_n(x)} = n - C_{e_n(x)}$, where $C_{e_n(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{e_n(x)}$ and $C_{e_n(x)}$.

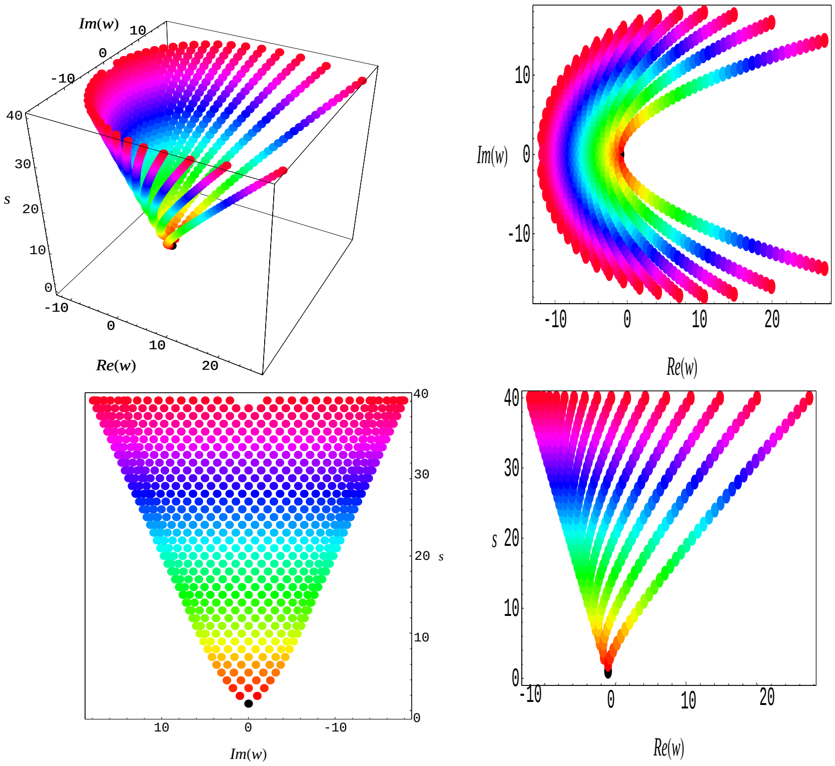

Figure 3: Stacks of zeros of $e_n(x)$ for $1 \leq n \leq 40$

Conjecture 5. Prove that $e_n(x) = 0$ has n distinct solutions.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n. Since n is the degree of the polynomial $e_n(x)$, the number of real zeros $R_{e_n(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{e_n(x)} = n - C_{e_n(x)}$, where $C_{e_n(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{e_n(x)}$ and $C_{e_n(x)}$.

Conjecture 6. Prove that the numbers of complex zeros $C_{e_n(x)}$ of $e_n(x), Im(x) \neq 0$ is

$$C_{e_n(x)} = 2\left[\frac{n}{2}\right],$$

where [ ] denotes taking the integer part.

Conjecture 7. For $n \in \mathbb{N}_0$, if $n \equiv 1 \pmod 2$, then $R_{e_n(x)} = 1$, if $n \equiv 0 \pmod 2$, then $R_{e_n(x)} = 0$.

The plot of real zeros of the truncated exponential polynomials $e_n(x)$ for $1 \leq n \leq 50$ structure are presented(Figure 4). It is expected that $e_n(x), x \in \mathbb{C}$, has $Im(x) = 0$ reflection symmetry analytic complex functions (see Figure 2, Figure 3, Figure 4). For $a \in \mathbb{R}$, we expect that $e_n(x), x \in \mathbb{C}$, has not $Re(x) = a$ reflection symmetry analytic complex functions. We observe a remarkable regular structure of the complex roots of the truncated exponential polynomials $e_n(x)$. We also hope to verify a remarkable regular structure of the complex roots of the truncated exponential polynomials $e_n(x)$(Table 1). Next, we calculated an approximate solution satisfying $e_n(x) = 0, x \in \mathbb{C}$. The

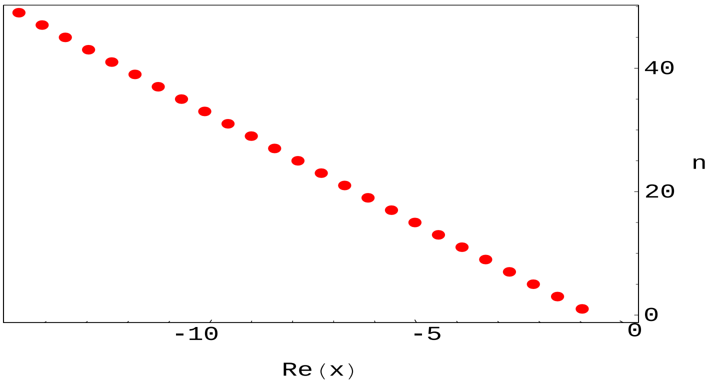

Figure 4: Real zeros of $e_n(x)$ for $1 \leq n \leq 50$

results are given in Table 2.

Table 2. Approximate solutions of $e_n(x) = 0, x \in \mathbb{C}$

degree n	x
1	-1.0000
2	$-1.0000 - 1.0000i$, $-1.0000 + 1.0000i$
3	-1.5961, $-0.7020 - 1.8073i$, $-0.7020 + 1.8073i$
4	$-1.7294 - 0.8890i$, $-1.7294 + 0.8890i$ $-0.2706 - 2.5048i$, $-0.2706 + 2.5048i$
5	-2.1806, $-1.6495 - 1.6939i$, $-1.6495 + 1.6939i$ $0.2398 - 3.1283i$, $0.2398 + 3.1283i$
6	$-2.3618 - 0.8384i$, $-2.3618 + 0.8384i$, $-1.4418 - 2.4345i$ $-1.4418 + 2.4345i$, $0.8036 - 3.6977i$, $0.8036 + 3.6977i$

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

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On generalized degenerate twisted (h, q)-tangent numbers and polynomials

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Abstract : We introduced the generalized twisted (h, q)-tangent numbers and polynomials. In this paper, our goal is to give generating functions of the generalized degenerate twisted (h, q)-tangent numbers and polynomials. We also obtain some explicit formulas for generalized degenerate twisted (h, q)-tangent numbers and polynomials.

Key words : Generalized tangent numbers and polynomials, degenerate generalized twisted (h, q)-tangent numbers and polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-16]). In [2], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, Feng Qi *et al.*[3] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. In this paper, we obtain some interesting properties for generalized degenerate tangent numbers and polynomials. Throughout this paper we use the following notations. Let p be a fixed odd prime number. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let r be a positive integer, and let ζ be rth root of 1. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$. Then the generalized twisted (h, q)-tangent numbers associated with associated with χ, $T^{(h)}_{n,\chi,q,\zeta}$, are defined by the following generating function

$$\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}\frac{t^n}{n!}. \tag{1.1}$$

We now consider the generalized twisted (h, q)-tangent polynomials associated with χ, $T^{(h)}_{n,\chi,q,\zeta}(x)$, are also defined by

$$\left(\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}\right)e^{xt} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x)\frac{t^n}{n!}. \tag{1.2}$$

When $\chi = \chi^0$, above (1.1) and (1.2) will become the corresponding definitions of the twisted (h, q)-tangent numbers $T^{(h)}_{n,q,w}$ and polynomials $T^{(h)}_{n,q,w}(x)$. If $q \to 1$, above (1.1) and (1.2) will become the corresponding definitions of the generalized twisted tangent numbers $T_{n,\chi,w}$ and polynomials $T_{n,\chi,w}(x)$. We recall that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations(see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{1.3}$$

We also have

$$\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.5}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.6}$$

2. On the generalized degenerate twisted (h,q)-tangent polynomials

In this section, we define the generalized degenerate twisted (h,q)-tangent numbers and polynomials, and we obtain explicit formulas for them. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 (\text{mod } 2)$, and let ζ be rth root of 1. For $h \in \mathbb{Z}$, the generalized degenerate twisted (h,q)-tangent polynomials associated with associated with χ, $T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)$, are defined by the following generating function

$$\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} \tag{2.1}$$

and their values at $x=0$ are called the generalized degenerate twisted (h,q)-tangent numbers and denoted $T^{(h)}_{n,\chi,q,\zeta}(\lambda)$.

From (2.1) and (1.2), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\lim_{\lambda\to 0} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \lim_{\lambda\to 0}\frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \left(\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}\right)e^{xt}\\ &= \sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, we get

$$\lim_{\lambda\to 0} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = T^{(h)}_{n,\chi,q,\zeta}(x), (n \geq 0).$$

From (2.1) and (1.6), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \left(\sum_{m=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l\frac{t^l}{l!}\right) \qquad (2.2)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T^{(h)}_{l,\chi,q,\zeta}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 1. For $n \geq 0$, we have

$$T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = \sum_{l=0}^{n}\binom{n}{l}T^{(h)}_{l,\chi,q,\zeta}(\lambda)(x|\lambda)_{n-l}.$$

For $\chi = \chi^0$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2}{\zeta q^h(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \sum_{m=0}^{\infty}T^{(h)}_{n,q,\zeta}(x|\lambda)\frac{t^m}{m!}.\end{aligned} \tag{2.3}$$

Theorem 2. For $n \geq 0$ and $\chi = \chi^0$, we have

$$T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = T^{(h)}_{n,q,\zeta}(x|\lambda).$$

For $d \in \mathbb{N}$ with $d \equiv 1(\mathrm{mod}2)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\\ &= \frac{2}{\zeta q^h(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\sum_{l=0}^{d-1}(-1)^l\chi(l)(1+\lambda t)^{2l/\lambda}\\ &= \sum_{n=0}^{\infty}\left(d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}|\frac{\lambda}{d}\right)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.4}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 3. Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1(\mathrm{mod}\ 2)$. Then we have

$$(1)\ T^{(h)}_{n,\chi,q,\zeta}(x|\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}|\frac{\lambda}{d}\right),$$

$$(2)\ T^{(h)}_{n,\chi,q,\zeta}(\lambda) = d^n\sum_{l=0}^{d-1}(-1)^l\chi(l)T^{(h)}_{n,q^d,\zeta^d}\left(\frac{2l+x}{d}\right).$$

For $m \in \mathbb{Z}_+$, we obtain we can derive the following relation:

$$\begin{aligned}&\sum_{m=0}^{\infty}\zeta^d q^{hd}T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda)\frac{t^m}{m!} + \sum_{m=0}^{\infty}T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda)\frac{t^m}{m!}\\ &= 2\sum_{l=0}^{d-1}(-1)^l\chi(l)\zeta^l q^{hl}(1+\lambda t)^{2l/\lambda}\\ &= \sum_{m=0}^{\infty}\left(2\sum_{l=0}^{d-1}(-1)^{n-1-l}\chi(l)\zeta^l q^{hl}(2l|\lambda)_m\right)\frac{t^m}{m!}.\end{aligned} \tag{2.5}$$

By comparing of the coefficients $\frac{t^m}{m!}$ on the both sides of (2.5), we have the following theorem.

Theorem 4. For $m \in \mathbb{Z}_+$, we have

$$\zeta^d q^{hd} T^{(h)}_{m,\chi,q,\zeta}(2d|\lambda) + T^{(h)}_{m,\chi,q,\zeta}(\lambda) = 2\sum_{l=0}^{d-1}(-1)^l \chi(l)\zeta^l q^{hl}(2l|\lambda)_m.$$

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x+y|\lambda)\frac{t^n}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{2a/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{(x+y)/\lambda}\\ &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{(2a+x)/\lambda}}{\zeta^d q^{dh}(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{y/\lambda}\\ &= \left(\sum_{n=0}^{\infty} T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n\frac{t^n}{n!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T^{(h)}_{l,\chi,q,\zeta}(x|\lambda)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

Therefore, by (2.6), we have the following theorem.

Theorem 5. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x+y|\lambda) = \sum_{k=0}^{n}\binom{n}{k}T^{(h)}_{k\chi,q,\zeta}(x|\lambda)(y|\lambda)_{n-k}.$$

From Theorem 5, we note that $T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)$ is a Sheffer sequence.

By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (2.1), we obtain

$$\begin{aligned}\frac{2\sum_{a=0}^{d-1}\chi(a)(-1)^a\zeta^a q^{ha}e^{2at}}{\zeta^d q^{hd}e^{2dt}+1}e^{xt} &= \sum_{n=0}^{\infty}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n\frac{1}{n!}\\ &= \sum_{n=0}^{\infty}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\lambda^m\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned} \tag{2.7}$$

Thus, by (2.7) and (1.2), we have the following theorem.

Theorem 6. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x) = \sum_{n=0}^{m}\lambda^{m-n}T^{(h)}_{n,\chi,q,\zeta}(x|\lambda)S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty}T^{(h)}_{n,\chi,q,\zeta}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} &= \frac{2\sum_{a=0}^{d-1}(-1)^a\chi(a)\zeta^a q^{ha}(1+\lambda t)^{(2a+x)/\lambda}}{\zeta^d q^{hd}(1+\lambda t)^{2d/\lambda}+1}\\ &= \sum_{m=0}^{\infty}T^{(h)}_{m,\chi,q,\zeta}(x|\lambda)\frac{t^m}{m!},\end{aligned} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty}T^{(h)}_{n,\chi,q,\zeta}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}T^{(h)}_{n,\chi,q,\zeta}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}. \tag{2.9}$$

Thus, by (2.8) and (2.9), we have the following theorem.

Theorem 8. For $n \in \mathbb{Z}_+$, we have

$$T^{(h)}_{m,\chi,q,\zeta}(x|\lambda) = \sum_{n=0}^{m} T^{(h)}_{n,\chi,q,\zeta}(x)\lambda^{m-n} S_1(m,n).$$

Acknowledgment

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

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New Oscillation Criteria of First Order Neutral Delay Difference Equations of Emden–Fowler Type

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Abstract

In this paper, we will establish some new sufficient condition for oscillation of solutions of a certain class of first-order neutral delay difference equations of the form

$$\Delta\left(x_n - p_n x_{n-1}\right) + q_n x_{n-\tau}^{\gamma} = 0,$$

where γ is a quotient of odd positive integers. We will consider the sublinear and super linear cases. The results will be obtained by using the oscillation theorems of second order delay difference equations.

2010 Mathematics Subject Classification: 34C10, 34K11, 34B05.

Keywords and phrases: Neutral difference equation, oscillation, Riccati technique.

1 Introduction

In recent decades there has been much research activity concerning oscillation and nonoscillation of first and second order delay and neutral delay difference equations, we refer the reader to the papers [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and the references cited therein. In the following, we recall some results of first order neutral delay difference equations of sublinear and super linear types that motivate the contents of this paper. Xiaoyan Lin in [12] studied the oscillatory behavior of solutions of the neutral difference equations with nonlinear neutral term of the form

$$\Delta\left(x_n - p_n x_{n-\sigma}^{\alpha}\right) + q_n x_{n-\tau}^{\beta} = 0, \text{ for } n \in \mathbb{N}_{n_0}, \tag{1.1}$$

where α and β are quotient of odd positive integers, τ and σ are nonnegative integers and $\{p_n\}$ and $\{q_n\}$ are two sequences of nonnegative real numbers. The authors obtained necessary and sufficient conditions for existence of oscillatory solutions and studied the two cases when $0 < \alpha < 1$ and when $\alpha > 1$. As usual, a nontrivial solution x_n of (1.1) is called nonoscillatory if it eventually positive or eventually negative, otherwise it is called oscillatory and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$ and $\mathbb{N}_i = \{i+1, i+2, ...\}$. Lalli [11] established several sufficient conditions for oscillation of the equation

$$\Delta\left(x_n + p x_{n-\delta k}\right) + q_n f\left(x_{n-\tau}\right) = F_n, \; n \geq n_0, \tag{1.2}$$

where $\delta = \pm 1$, p is a nonnegative real number, $k \in \mathbb{N} = \{1, 2, ...\}$, τ is a sequence of nonnegative integers with $\lim_{n\to\infty} \tau_n = \infty$, and $\{F_n\}$, $\{q_n\}$ are sequences of real numbers and f is a real valued function satisfying $xf(x) > 0$ for $x \neq 0$. El-Morshedy et al. [6] considered the equation

$$\Delta g\left(x_n + p_n x_{\sigma_n}\right) + f\left(n, x_{\tau_n}\right) = 0, \tag{1.3}$$

where $0 \leq p_n < p < 1$, σ_n and τ_n are sequences of integers such that $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} = \infty$ and $\sigma_{n+1} > \sigma_n$ for all $n \geq n_0$. They established several sufficient conditions for oscillation when the function f satisfies the condition

$$\frac{f(n, x)}{h(x)} \geq q_n, \ x \neq 0 \text{ and } n \geq n_0,$$

where $q_n \geq 0$ for $n \geq n_0$, $h \in C(\mathbb{R},\mathbb{R})$ and $xh(x) > 0$ for all $x \neq 0$. Recently Murugesan and Suganthi [13] discussed the oscillatory behavior of all solutions of the first order nonlinear neutral delay difference equation

$$[\Delta\left(r_n\left(a_n x_n - p_n x_{n-\tau}\right)\right)] + q_n x_{n-\sigma} = 0,$$

where r_n and a_n are sequences of positive real numbers p_n and q_n are sequences of non-negative real numbers, τ and σ are positive integers. Following this trend in this paper, we will consider the first order neutral delay difference equation

$$\Delta\left(x_n - p_n x_{n-1}\right) + q_n x_{n-\tau}^{\gamma} = 0, \text{ for } n \in \mathbb{N}_{n_0}, \tag{1.4}$$

Our aim in this paper is to establish some new sufficient conditions for oscillation of (1.4) by using a new technique when $0 < p_n \leq p \leq 1$ and we will consider the sublinear and the super linear cases. The new technique depends on the application of an invariant substitution which transforms the first nonlinear neutral difference equation to a second nonlinear difference equation. This allows us to obtain several sufficient conditions for oscillation of (1.4) by employing the oscillation conditions of second order delay difference equations by using the Riccati technique.

2 Main results

In this section, we prove the main results but before we do this, we apply an invariant substitution which transforms the first order neutral equation to a non-neutral second order difference equations. This substitution is given by

$$y_{n+1} = x_n \prod_{i=1}^{n} \frac{1}{p_i}, \quad \text{where} \quad \prod_{i=1}^{n} p_i = O(n), \tag{2.1}$$

This gives us that

$$x_n = y_{n+1} \prod_{i=1}^{n} p_i, \quad x_{n-1} = y_n \prod_{i=1}^{n-1} p_i, \quad \text{and } x_{n-\tau} = y_{n-\tau+1} \prod_{i=1}^{n-\tau} p_i. \tag{2.2}$$

From (2.2), we have

$$x_n - p_n x_{n-1} = \Delta y_n \prod_{i=1}^{n} p_i. \tag{2.3}$$

Substituting (2.3) into (1.4), we obtain

$$\Delta\left(\Delta y_n \prod_{i=1}^{n} p_i\right) + q_n \prod_{i=1}^{n-\tau} p_i y_{n-\tau+1} = 0. \tag{2.4}$$

Setting $d_n = \prod_{i=1}^{n} p_i$, and $Q_n = q_n d_{n-\tau}$ then (2.4) becomes

$$\Delta\left(d_n \Delta y_n\right) + Q_n y^{\gamma}_{n-(\tau-1)} = 0,\ n \in \mathbb{N}_0. \tag{2.5}$$

In this section, we intend to use the Riccati transformation technique for obtaining several new oscillation criteria for (1.4). First we state some fundamental lemmas for second order difference equations that will be used in the proofs of the main results (see [15]).

Lemma 2.1 *Assume that p_n is a real sequence with $0 < p_n \le p < 1$ for all $n \in \mathbb{N}$. Furthermore assume that*

$$\sum_{n=1}^{\infty} \frac{1}{d_n} = \infty. \tag{2.6}$$

Let y be a positive solution of (2.5). Then
(I). $\Delta y(n) \ge 0$, $y(n) \ge n\Delta y(n)$ for $n \ge N$,
(II). y is nondecreasing, while $y(n)/n$ is nonincreasing for $n \ge N$.

Lemma 2.2 *Assume that p_n is a real sequence with $0 < p_n \le p < 1$ for all $n \in \mathbb{N}$. Furthermore assume that (2.6) holds. If y_n be a nonoscillatory solution of (2.5) such that $y_n \ge 0,\ \Delta y_n \le 0$, then $\lim_{n\to\infty} y_n = 0$ and hence*

$$\lim_{n\to\infty} \frac{x_n}{d_n} = 0, \tag{2.7}$$

where x_n is a solution of (1.4).

Throughout this paper, we will assume that the real sequences $p_n,\ q_n$ are nonnegative, γ is a quotient of odd positive integers, τ is a nonnegative integer. Now, we state and prove the sufficient conditions which ensure that each solution of equation (1.4) is oscillatory or satisfies (2.7). We start with the case when $0 < \gamma \le 1$.

Theorem 2.3 *Assume that (H_1) holds and $\Delta d_n \ge 0$. Furthermore, assume that there exists a positive sequence ρ_n such that,*

$$\lim_{n\to\infty} \sup \sum_{i=n_0}^{n} \left[\rho_i Q_i - \frac{d_{i-\tau+1}\beta^{1-\gamma}\left(i+2-\tau\right)^{1-\gamma}\left(\Delta\rho_i\right)^2}{\rho_i}\right] = \infty, \tag{2.8}$$

where $d_n = \prod_{i=1}^{n} p_i$ and $Q_n = q_n d_{n-\tau}$. Then every solution of (1.4) oscillates for all $0 < \gamma \le 1$.

Proof. Assume to the contrary that x_n be a nonoscillatory solution of (1.4) such that $x_{n-1},\ x_{n-\tau},\ x_n > 0$ for all large $n \ge n_1 > n_0$ sufficiently large. We shall consider only this case, since the substitution $y_n = -x_n$ transforms equation (1.4) into an equation of the same form. From (2.1) we see that y_n is a positive solution of (2.5) such that $y_n > 0$ and $y_{n-\tau+1} > 0$ for $n > n_1 > n_0$ sufficiently large. From equation (2.5), we have

$$\Delta\left(d_n \Delta y_n\right) = -Q_n y^{\gamma}_{n-\tau+1} \le 0,\ n \ge n_1, \tag{2.9}$$

and then $d_n\Delta y_n$ is an eventually nonincreasing sequence. We first show that $d_n\Delta y_n \geq 0$ for $n \geq n_0$. In fact, if there exists an integer $n_1 \geq n_0$ such that $d_{n_1}\Delta y_{n_1} = c < 0$ then (2.9) implies that $d_n\Delta y_n \leq c$ for $n \geq n_1$ that is $\Delta y_n \leq c/d_n$, and hence

$$y_n \leq y_{n_1} + c\sum_{i=n_1}^{n-1}\frac{1}{d_i} \to -\infty, \text{ as } n \to \infty, \tag{2.10}$$

which contradicts the fact that $y_n > 0$ for $n \geq n_0$ then $d_n\Delta y_n \geq 0$. Also since $\Delta d_n \geq 0$, we can prove that $\Delta^2 y_n > 0$ for $n \geq n_1$. Therefore we have

$$y_n > 0, \ \Delta y_n \geq 0, \ \text{ and } \Delta^2 y_n \leq 0, \text{ for } n \geq n_1. \tag{2.11}$$

From (2.9) and (2.11)

$$d_{n-\tau+1}\Delta y_{n-\tau+1} \geq d_{n+1}\Delta\left(y_{n+1}\right) \text{ and } y_{n-\tau+1} \geq y_{n-\tau}. \tag{2.12}$$

Defining the sequence u_n by the Riccati substitution

$$u_n = \rho_n\frac{d_n\Delta y_n}{y^{\gamma}_{n-\tau+1}}, \quad \text{for } \ n > n_1. \tag{2.13}$$

This implies that $u_n > 0$, and

$$\Delta u_n = d_{n+1}\Delta y_{n+1}\Delta\left[\frac{\rho_n}{y^{\gamma}_{n-\tau+1}}\right] + \rho_n\frac{\Delta\left(d_n\Delta y_n\right)}{y^{\gamma}_{n-\tau+1}}.$$

Hence

$$\Delta u_n = d_{n+1}\Delta y_{n+1}\left[\frac{\Delta\rho_n\left(y^{\gamma}_{n-\tau+1}\right) - \rho_n\left(\Delta y^{\gamma}_{n-\tau+1}\right)}{y^{\gamma}_{n-\tau+1}y^{\gamma}_{n-\tau+2}}\right] + \rho_n\frac{\Delta\left(d_n\Delta y_n\right)}{y^{\gamma}_{n-\tau+1}}. \tag{2.14}$$

From this, (2.5) and (2.14) we see that

$$\Delta u_n \leq \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \left[\frac{d_{n+1}\Delta y_{n+1}\rho_n\Delta y^{\gamma}_{n-\tau+1}}{y^{\gamma}_{n-\tau+2}y^{\gamma}_{n-\tau+1}}\right] - \rho_n Q_n. \tag{2.15}$$

From (2.5) and (2.14), we have

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \frac{d_{n+1}\Delta y_{n+1}\rho_n\Delta y^{\gamma}_{n-\tau+1}}{y^{2\gamma}_{n-\tau+2}}. \tag{2.16}$$

By using the inequality (see [8]),

$$x^{\gamma} - y^{\gamma} \geq \gamma x^{\gamma-1}\left(x-y\right), \text{ for all } x \neq y > 0 \text{ where } 0 < \gamma \leq 1, \tag{2.17}$$

we have

$$\begin{aligned}\Delta y^{\gamma}_{n-\tau+1} &= \left(y^{\gamma}_{n+2-\tau} - y^{\gamma}_{n+1-\tau}\right) \geq \gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(y_{n-\tau+2} - y_{n-\tau+1}\right) \\ &= \gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(\Delta y_{n-\tau+1}\right).\end{aligned} \tag{2.18}$$

Substituting (2.18) into (2.16), we obtain that

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \rho_n d_{n+1}\frac{\gamma\left(y_{n+2-\tau}\right)^{\gamma-1}\left(\Delta y_{n-\tau+1}\right)\Delta y_{n+1}}{y^{2\gamma}_{n-\tau+2}}. \tag{2.19}$$

From (2.12) and (2.19), we have that

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma\rho_n d_{n+1}^2 \left(\Delta y_{n+1}\right)^2}{d_{n-\tau+1}\left(y_{n+2-\tau}\right)^{1-\gamma}\left(y_{n-\tau+2}^{\gamma}\right)^2}.$$

Hence,

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma\rho_n}{\left(\rho_{n+1}\right)^2 d_{n-\tau+1}\left(y_{n+2-\tau}\right)^{1-\gamma}} \left(u_{n+1}\right)^2. \tag{2.20}$$

From (2.11), we conclude that

$$y_n \leq y_{n_0} + \Delta y_{n_0}\left(n - n_0\right), \qquad n \geq n_1,$$

and consequently there exists a $n_2 \geq n_2$ and appropriate constant $\beta \geq 1$ such that

$$y_n \leq \beta n, \quad \text{for} \quad n \geq n_2,$$

and this implies that

$$y_{n+2-\tau} \leq \beta\left(n + 2 - \tau\right), \text{ for } \ n \geq n_3 = n_2 + \tau + 2,$$

and then

$$\frac{1}{\left(y_{n+2-\tau}\right)^{1-\gamma}} \geq \frac{1}{\left(\beta\left(n+2-\tau\right)\right)^{1-\gamma}}. \tag{2.21}$$

Substituting (2.21) into (2.20) we obtain

$$\Delta u_n \leq -\rho_n Q_n + \frac{\Delta\rho_n}{\rho_{n+1}} u_{n+1} - \frac{\gamma\rho_n}{\left(\rho_{n+1}\right)^2 d_{n-\tau+1}\beta^{1-\gamma}\left(n+2-\tau\right)^{1-\gamma}} \left(u_{n+1}\right)^2 \tag{2.22}$$

Hence

$$\begin{aligned} \Delta u_n \quad &\leq \quad -\rho_n Q_n + \frac{d_{n-\tau+1}\beta^{1-\gamma}\left(n+2-\tau\right)^{1-\gamma}\left(\Delta\rho_n\right)^2}{\rho_n} \\ &\quad - \left[\frac{\sqrt{\rho_n}}{\rho_{n+1}\sqrt{\left(\beta\left(n+2-\tau\right)\right)^{1-\gamma} d_{n-\tau+1}}} u_{n+1} - \frac{\Delta\rho_n\sqrt{d_{n-\tau+1}\beta^{1-\gamma}\left(n+2-\tau\right)^{1-\gamma}}}{2\rho_n}\right]^2 \end{aligned}$$

Then, we have

$$\Delta u_n \leq -\left[\rho_n Q_n - \frac{d_{n-\tau+1}\beta^{1-\gamma}\left(n+2-\tau\right)^{1-\gamma}\left(\Delta\rho_n\right)^2}{\rho_n}\right] \tag{2.23}$$

Summing (2.23) from n_3 to n we obtain

$$-u_{n_3} < u_{n+1} - u_{n_3} \leq -\sum_{i=n_3}^{n}\left[\rho_i Q_i - \frac{d_{i-\tau+1}\beta^{1-\gamma}\left(i+2-\tau\right)^{1-\gamma}\left(\Delta\rho_i\right)^2}{\rho_i}\right]$$

which yields

$$\sum_{i=n_3}^{n}\left[\rho_i Q_i - \frac{d_{i-\tau+1}\beta^{1-\gamma}\left(i+2-\tau\right)^{1-\gamma}\left(\Delta\rho_i\right)^2}{\rho_i}\right] < c_1,$$

for all large n, and this contrary to (2.8). The proof is complete. ■

From the Theorem 2.3, we can obtain different condition for oscillation of all solutions of (1.4) by different choices of ρ_n. For example if we take $\rho_n = n^{\lambda}$, $n \geq n_0$ and $\lambda > 1$ is a constant we have the following result.

Corollary 2.4 *Assume that all the assumptions of Theorem 2.3 hold, except that the condition (2.8) is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\left[s^{\lambda}Q_s-\frac{d_{s-\tau+1}\beta^{1-\gamma}\left(s+2-\tau\right)^{1-\gamma}\left(\Delta s^{\lambda}\right)^2}{s^{\lambda}}\right]=\infty.$$

Then every solution of (1.4) oscillates for all $0<\gamma\leq 1$.

Remark 2.5 *When* $\gamma=1$ *the equation (1.4) reduced to linear delay difference equation*

$$\Delta\left(x_n-p_nx_{n-1}\right)+q_nx_{n-\tau}=0,\ \textit{for } n\in\mathbb{N}_{n_0},$$

and the condition (2.8) in Theorem 2.3 reduced to

$$\lim_{n\to\infty}\sup\sum_{i=n_0}^{n}\left[\rho_iQ_i-\frac{d_{i-\tau+1}\left(\Delta\rho_i\right)^2}{\rho_i}\right]=\infty, \tag{2.24}$$

where $d_n=\prod_{i=1}^{n}p_i$ *and* $Q_n=q_nd_{n-\tau}$ *for all* $0<\gamma\leq 1$.

Now, we consider the case when $\gamma\geq 1$.

Theorem 2.6 *Assume that (2.6) holds. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=1}^{\infty}$ *such that for every positive constant* M,

$$\lim_{n\to\infty}\sup\sum_{l=n_0}^{n}\left[\rho_lq_l-\frac{\left(d_{l-\sigma}\right)^{\gamma}\left(\Delta\rho_l\right)^2}{2^{3-\gamma}\left(M\right)^{\gamma-1}\left(d_{l+1}\right)^{2\gamma-2}\rho_l}\right]=\infty, \tag{2.25}$$

where $\sigma=\tau-1$. *Then every solution of (1.4) oscillates for all* $\gamma\geq 1$.

Proof. Suppose to the contrary that x_n is a nonoscillatory solution of (1.4). Without loss of generality, we may assume that x_n is an eventually positive solution of (1.4) such that $x_{n-1},\ x_{n-\tau},\ x_n>0$ for all large $n\geq n_1>n_0$ sufficiently large. We shall consider only this case, since the substitution $y_n=-x_n$ transforms equation (1.4) into an equation of the same form. As in the proof of Theorem 2.3, we have by (2.6) that

$$y_n>0,\ \Delta y_n\geq 0,\ \Delta(d_n\left(\Delta y_n\right))\leq 0,\ n\geq n_1. \tag{2.26}$$

Define the sequence u_n by

$$u_n:=\rho_n\frac{d_n\Delta y_n}{y_{n-\sigma}^{\gamma}}. \tag{2.27}$$

Then $u_n>0$, and

$$\Delta u_n=d_{n+1}\Delta y_{n+1}\Delta\left[\frac{\rho_n}{y_{n-\sigma}^{\gamma}}\right]+\frac{\rho_n\Delta(d_n\Delta y_n)}{y_{n-\sigma}^{\gamma}}. \tag{2.28}$$

In view of (2.5), (2.28), we have

$$\Delta u_n\leq-\rho_nq_n+\frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1}-\frac{\rho_nd_{n+1}\Delta y_{n+1}\Delta y_{n-\sigma}^{\gamma}}{y_{n+1-\sigma}^{\gamma}y_{n-\sigma}^{\gamma}}. \tag{2.29}$$

From (2.26), we see that

$$d_{n-\sigma}\Delta y_{n-\sigma}\geq d_{n+1}\Delta y_{n+1},\quad\text{and}\quad y_{n+1-\sigma}\geq y_{n-\sigma}. \tag{2.30}$$

Substituting (2.30) into (2.29), we have

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{\rho_n d_{n+1} \Delta y_{n+1} \Delta y_{n-\sigma}^{\gamma}}{\left(y_{n+1-\sigma}^{\gamma}\right)^2}. \tag{2.31}$$

Now, by using the inequality

$$x^\gamma - y^\gamma \geqslant 2^{1-\gamma}(x-y)^\gamma, \text{ for all } x \geqslant y > 0 \text{ and } \gamma \geqslant 1,$$

we find that

$$\Delta y_{n-\sigma}^{\gamma} = y_{n+1-\sigma}^{\gamma} - y_{n-\sigma}^{\gamma} \geqslant 2^{1-\gamma}(y_{n+1-\sigma} - y_{n-\sigma})^\gamma = 2^{1-\gamma}\left(\Delta y_{n-\sigma}\right)^\gamma. \tag{2.32}$$

Substituting (2.32) into (2.31), we have

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - 2^{1-\gamma} \rho_n d_{n+1} \frac{\Delta y_{n+1} \left(\Delta y_{n-\sigma}\right)^\gamma}{\left(y_{n+1-\sigma}^{\gamma}\right)^2}. \tag{2.33}$$

From (2.30) and (2.33), we obtain

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - 2^{1-\gamma} \rho_n \frac{\left(d_{n+1}\right)^{\gamma+1}}{\left(d_{n-\sigma}\right)^\gamma} \frac{\left(\Delta y_{n+1}\right)^{\gamma+1}}{\left(y_{n+1-\sigma}^{\gamma}\right)^2}. \tag{2.34}$$

Hence,

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{\left(d_{n+1}\right)^{\gamma+1}}{\left(d_{n-\sigma}\right)^\gamma} \frac{2^{1-\gamma} \rho_n \left(\Delta y_{n+1}\right)^{2\gamma}}{\left(y_{n+1-\sigma}^{\gamma}\right)^2 \left(\Delta y_{n+1}\right)^{\gamma-1}}. \tag{2.35}$$

From the definition of u_n, we get that

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \frac{2^{1-\gamma} \rho_n}{\left(\rho_{n+1}\right)^2} \frac{\left(d_{n+1}\right)^{\gamma-1}}{\left(d_{n-\sigma}\right)^\gamma} \frac{u_{n+1}^2}{\left(\Delta y_{n+1}\right)^{\gamma-1}}. \tag{2.36}$$

Since $\{d_n \left(\Delta y_n\right)\}$ is a positive and nonincreasing sequence, there exists a $n_2 \geq n_1$ sufficiently large such that $d_n \left(\Delta y_n\right) \leq 1/M$ for some positive constant M and $n \geq n_1$, and hence by (2.26), we have

$$\frac{1}{\left(\Delta y_{n+1}\right)^{\gamma-1}} \geqslant \left(M d_{n+1}\right)^{\gamma-1}.$$

Substituting the last inequality into (2.36), we obtain

$$\Delta u_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} u_{n+1} - \left(\frac{M}{2}\right)^{\gamma-1} \frac{\rho_n \left(d_{n+1}\right)^{2\gamma-2}}{\left(\rho_{n+1}\right)^2} \frac{1}{\left(d_{n-\sigma}\right)^\gamma} u_{n+1}^2, \tag{2.37}$$

so that

$$\begin{aligned} \Delta u_n \quad \leq \quad & -\rho_n q_n + \frac{\left(d_{n-\sigma}\right)^\gamma \left(\Delta \rho_n\right)^2}{2^{3-\gamma} \left(M\right)^{\gamma-1} \left(d_{n+1}\right)^{2\gamma-2} \rho_n} - \\ & \left[\frac{\sqrt{\left(\frac{M}{2}\right)^{\gamma-1} \left(d_{n+1}\right)^{2\gamma-2} \rho_n}}{\rho_{n+1} \sqrt{\left(d_{n-\sigma}\right)^\gamma}} u_{n+1} - \frac{\sqrt{\left(d_{n-\sigma}\right)^\gamma} \Delta \rho_n}{2\sqrt{\left(\frac{M}{2}\right)^{\gamma-1} \left(d_{n+1}\right)^{2\gamma-2} \rho_n}}\right]^2 \\ < \quad & -\left[\rho_n q_n - \frac{\left(d_{n-\sigma}\right)^\gamma \left(\Delta \rho_n\right)^2}{2^{3-\gamma} \left(M\right)^{\gamma-1} \left(d_{n+1}\right)^{2\gamma-2} \rho_n}\right] \end{aligned}$$

Then, we have

$$\Delta u_n < -\left[\rho_n q_n - \frac{(d_{n-\sigma})^{\gamma}(\Delta\rho_n)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}\right]. \tag{2.38}$$

Summing (2.38) from n_2 to n, we obtain

$$-u_{n_2} < u_{n+1} - u_{n_2} < -\sum_{l=n_2}^{n}\left[\rho_l q_l - \frac{(d_{l-\sigma})^{\gamma}(\Delta\rho_l)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{l+1})^{2\gamma-2}\rho_l}\right],$$

which yields

$$\sum_{l=n_2}^{n}\left[\rho_l q_l - \frac{(d_{l-\sigma})^{\gamma}(\Delta\rho_l)^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{l+1})^{2\gamma-2}\rho_l}\right] < c_1,$$

for all large n. This contradicts (2.25). The proof is complete. ■

From Theorem 2.6, we can obtain different conditions for oscillation of all solutions of (1.4) when (2.6) holds by different choices of $\{\rho_n\}$. For example, let $\rho_n = n^{\lambda}$, $n \geq n_0$ and $\lambda > 1$ is a constant. From Theorem 2.6 we have the following result.

Corollary 2.7 *Assume that all the assumptions of Theorem 2.6 hold, except the condition (2.25) is replaced by*

$$\lim_{n\to\infty}\sup\sum_{s=n_0}^{n}\left[s^{\lambda}q_s - \frac{(d_{s-\sigma})^{\gamma}((s+1)^{\lambda}-s^{\lambda})^2}{2^{3-\gamma}(M)^{\gamma-1}(d_{s+1})^{2\gamma-2}s^{\lambda}}\right] = \infty. \tag{2.39}$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

As a variant of the Riccati transformation technique used above, we will derive some oscillation criterion which can be considered as a discrete analogy of the Philos condition for oscillation of second order differential equation by introducing the following class of sequences that will be used in this chapter and later. Let

$$\pounds_0 = \{(m,n):\ m > n \geq n_0\},\quad \pounds = \{(m,n):\ m \geq n \geq n_0\}.$$

The double sequence $H_{m,n} \in \Sigma$ if:

(I). $H(m,m) = 0$ on $\pounds$,

(II). $H(m,n) > 0$ on $\pounds_0$;

(III). $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *for* $m \geq n \geq 0$, and there exists a double sequence $h_{m,n}$ such that

$$h_{m,n} = -\frac{\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}},\ \text{for}\ \ m > n \geq 0.$$

Theorem 2.8 *Assume that (2.6) hold. Let* $\{\rho_n\}_{n=1}^{\infty}$ *be a positive sequence and* $H_{m,n} \in \Sigma$. *If*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left[H_{m,n}\rho_n q_n - B_n\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] = \infty, \tag{2.40}$$

where

$$B_n := \frac{(d_{n-\sigma})^{\gamma}\rho_{n+1}^2}{2^{3-\gamma}M^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}.$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

Proof. We proceed as in the proof of Theorem 2.6, we may assume that (1.4) has a nonoscillatory solution x_n such that $x_n > 0$. As in the proof of Theorem 2.6 we get that (2.26) holds. Define $\{u_n\}$ by (2.27) as before, then we have $u_n > 0$ and there is some $M > 0$ such that (2.37) holds. For the sake of convenience, let us set

$$\bar{\rho}_n = \frac{2^{1-\gamma}(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}{(d_{n-\sigma})^{\gamma}}.$$

Then, we have from (2.37) that

$$\rho_n q_n \leq -\Delta u_n + \frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \frac{\bar{\rho}_n}{\left(\rho_{n+1}\right)^2}u_{n+1}^2. \tag{2.41}$$

Therefore, we get

$$\sum_{n=n_1}^{m-1} H_{m,n}\rho_n q_n \leq -\sum_{n=n_1}^{m-1} H_{m,n}\Delta u_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}u_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{\bar{\rho}_n u_{n+1}^2}{\left(\rho_{n+1}\right)^2}. \tag{2.42}$$

The rest of the proof is similar to the proof of [15, Theorem 2.3.6]. ■

As an immediate consequence of Theorem 2.8, we get the following:

Corollary 2.9 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1} H_{m,n}\rho_n q_n = \infty,$$

$$\lim_{m\to\infty}\sup\frac{1}{H_{m,0}}\sum_{n=n_0}^{m-1}\frac{(d_{n-\sigma})^{\gamma}\rho_{n+1}^2}{(M)^{\gamma-1}(d_{n+1})^{2\gamma-2}\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2 < \infty.$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

By choosing the sequence $H_{m,n}$ in appropriate manners, we can derive several oscillation criteria for (1.4). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$\left.\begin{array}{l} H_{m,n} = (m-n)^{\lambda}, \quad \lambda \geq 1, m \geq n \geq 0, \\ H_{m,n} = \left(\log\frac{m+1}{n+1}\right)^{\lambda}, \lambda \geq 1, m \geq n \geq 0, \\ H_{m,n} = (m-n)^{(\lambda)} \quad \lambda > 2,\ m \geq n \geq 0, \end{array}\right\} \tag{2.43}$$

where $(m-n)^{(\lambda)} = (m-n)(m-n+1)...(m-n+\lambda-1)$, and

$$\Delta_2(m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda(m-n)^{(\lambda-1)}.$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Hence we have the following result which gives new sufficient conditions for the oscillation of (1.4) of Kamenev type.

Corollary 2.10 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{\lambda}}\sum_{n=0}^{m-1}\left[(m-n)^{\lambda}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n}V_{m,n}^2\right] = \infty, \tag{2.44}$$

where

$$V_{m,n} := \left(\lambda(m-n)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{(m-n)^{\lambda}}\right).$$

Then every solution of (1.4) oscillates for all $\gamma \geq 1$.

Corollary 2.11 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{(\log(m+1))^{\lambda}}\sum_{n=0}^{m-1}\left[\left(\log\frac{m+1}{n+1}\right)^{\lambda}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n}^{2} R_{m,n}^2\right] = \infty, \tag{2.45}$$

where

$$R_{m,n} = \left(\frac{\lambda}{n+1}\left(\log\frac{m+1}{n+1}\right)^{\frac{\lambda-2}{2}} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{\left(\log\frac{m+1}{n+1}\right)^{\lambda}}\right).$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

Corollary 2.12 *Assume that all the assumptions of Theorem 2.8 hold, except that the condition (2.40) is replaced by*

$$\lim_{m\to\infty}\sup\frac{1}{m^{(\lambda)}}\sum_{n=0}^{m-1}(m-n)^{(\lambda)}\left[\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n}U_n^2\right] = \infty, \tag{2.46}$$

where

$$U_n := \left(\frac{\lambda}{m-n+\lambda-1} - \frac{\Delta\rho_n}{\rho_{n+1}}\right)^2.$$

Then, every solution of (1.4) oscillates for all $\gamma \geq 1$.

In the following theorem, we consider the case when $0 < \gamma < 1$.

Theorem 2.13 *Assume that (2.6) holds and* $\Delta d_n \geq 0$. *If*

$$\sum_{n=n_0}^{\infty}\left(\frac{n-\sigma}{d_n}\right)^{\gamma} q_n = \infty. \tag{2.47}$$

Then every solution of (1.4) oscillates for all $0 < \gamma < 1$.

Proof. Proceeding as in Theorem *2.6*, we assume that (1.4) has a nonoscillatory solution, say $x_n > 0$ and $x_{n-\tau} > 0$ for all $n \geq n_0$. From the proof of Theorem 2.6 we know that $\Delta y_n > 0$, then y_n is nondecreasing sequence. Since $\Delta d_n \geq 0$ we obtain that $\Delta^2 y_n \leq 0$ and then Δy_n is a nonincreasing for all $n \geq n_1 \geq n_0$. Then, we have $y_n \geq (n-n_1)\Delta y_n$ which implies that $y_n \geq \frac{n}{2}\Delta y_n$ for $n \geq n_2 \geq 2n_1 + 1$. Then

$$y_{n-\sigma} \geq \frac{n-\sigma}{2}\Delta y_{n-\sigma} \geq \frac{n-\sigma}{2}\Delta y_{n+1}, \quad \textit{for } n \geq N = n_2 + \sigma. \tag{2.48}$$

From (2.5) and (2.48) by dividing by $z_{n+1} = (d_n \Delta y_{n+1})^{\gamma} > 0$ and summing from $2N$ to k, we obtain

$$\sum_{n=2N}^{k}\left(\frac{n-\sigma}{2d_n}\right)^{\gamma} q_n \leq -\sum_{n=2N}^{k}\frac{\Delta(z_n)}{(z_{n+1})^{\gamma}}, \quad k \geq 2N. \tag{2.49}$$

Since

$$y^{\gamma}-z^{\gamma}\leq\gamma y^{\gamma-1}(y-z)\text{ for }\gamma<1\text{ and }y>z>0,$$

we see that

$$\Delta\left(z_n^{1-\gamma}\right)=\left(z_{n+1}^{1-\gamma}\right)-\left(z_n^{1-\gamma}\right)\leq(1-\gamma)(z(n+1))^{-\gamma}\Delta z(n).$$

Substituting in (2.49), we see that

$$\begin{aligned}\sum_{n=2N}^{k}\left(\frac{n-\sigma}{2d_n}\right)^{\gamma}q_n &\leq -\sum_{n=2N}^{k}\frac{\Delta\left(z_n^{1-\gamma}\right)}{(1-\gamma)}=-\frac{\left(z_{k-1}^{1-\gamma}\right)}{(1-\gamma)}+\frac{\left(z_{2N}^{1-\gamma}\right)}{(1-\gamma)}\\ &< \frac{\left(z_{2N}^{1-\gamma}\right)}{(1-\gamma)}<\infty,\text{ as }n\to\infty\end{aligned}$$

which contradicts (2.47). The proof is complete. ■

Now, we consider the case when

$$\sum_{n=0}^{\infty}\left(\frac{1}{d_n}\right)<\infty, \tag{2.50}$$

holds and establish some oscillation criteria for (1.4) in the sublinear and superlinear cases.

Theorem 2.14 *Assume that (2.50) holds and there exist positive sequences $\{\rho_n\}_{n=1}^{\infty}$ such that (2.25) holds for every positive constant M, and*

$$\sum_{n=0}^{\infty}\left(\frac{1}{d_n}\sum_{i=n_0}^{n-1}q_i\right)=\infty. \tag{2.51}$$

Then every solution of (1.4) oscillates or $\lim_{n\to\infty}x_n/d_n=0$ for all $\gamma\geq1$.

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of (1.4). Without loss of generality we may assume that $\{x_n\}$ is eventually positive. From (2.5), we have

$$\Delta(d_n\Delta y_n)\leq-q_ny_{n-\sigma}^{\gamma}\leq0,\qquad n\geq n_0, \tag{2.52}$$

and so $\{d_n(\Delta y_n)\}$ is an eventually nonincreasing sequence. Since $\{q_n\}$ has a positive subsequence, either $\{\Delta y_n\}$ is eventually negative or eventually positive. If $\{\Delta y_n\}$ is eventually positive, we are then back to the case where (2.26) holds. Thus the proof of Theorem 2.6 goes through, and we may conclude that $\{y_n\}$ cannot be eventually positive, which is not possible. If $\{\Delta y_n\}$ is eventually negative, then $\lim_{n\to\infty}y_n=b\geq0$. We assert that $b=0$. If not then $y_{n-\sigma}^{\gamma}\to b^{\gamma}>0$ as $n\to\infty$, and hence there exists $n_1\geq n_0>0$ such that $y_{n-\sigma}^{\gamma}\geq b^{\gamma}$. Therefore from (2.52) we have

$$\Delta(d_n\Delta y_n)\leq-q_nb^{\gamma}.$$

The rest of the proof is similar to the proof of [15, Theorem 2.3.7] and hence is omitted. ■

By choosing $\{\rho_n\}_{n=1}^{\infty}$ in appropriate manners, we may obtain different oscillation criteria. For instance, let $\rho_n=n^{\lambda}$ for $n\geq0$ and $\lambda>1$. Then we have the following oscillation conditions of all solutions of (1.4) when (2.50) holds.

Corollary 2.15 *Assume that all assumptions of Theorem 2.14 hold, except that the condition (2.25) is replaced by (2.39). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Theorem 2.16 *Assume that (2.50) and (2.51) hold. Furthermore, assume that there exists a double sequence* $H_{m,n} \in \Sigma$ *such that (2.40) holds. Then every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Indeed, suppose that $\{x_n\}$ is an eventually positive solution of (1.4). Then as seen in the proof of Theorem *2.3*, either $\{\Delta x_n\}$ is eventually positive or is eventually negative. In the case when $\{\Delta y_n\}$ is eventually positive, we may follow the proof of Theorem *2.8* and obtain a contradiction. If $\{\Delta y_n\}$ is eventually negative, then we may follow the proof of Theorem *2.14* to show that $\{y_n\}$ converges to zero.

By choosing $H_{m,n}$ in appropriate manners, we can derive several oscillation criteria for (2.5) when (2.50) holds. For instance, let us consider the double sequence $H_{m,n}$ defined again by (2.43). Hence we have the following results.

Corollary 2.17 *Assume that all the assumptions of Theorem 2.16 hold, except that the condition (2.40) is replaced by (2.44). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Corollary 2.18 *Assume that all the assumptions of Theorem 2.16 hold, except that the condition (2.40) is replaced by(2.45) or (2.46). Then, every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all* $\gamma \geq 1$.

Theorem 2.19 *Assume that (2.50) and (2.47) hold. Let* $\{\rho_n\}_{n=1}^{\infty}$ *such that (2.51) holds. Then every solution of (1.4) oscillates or* $\lim_{n\to\infty} x_n/d_n = 0$ *for all 0* $<\gamma < 1$.

Indeed, suppose that $\{x_n\}$ is an eventually positive solution of (1.4). Then as seen in the proof of Theorem 2.6, either $\{\Delta y_n\}$ is eventually positive or is eventually negative. In the case when $\{\Delta y_n\}$ is eventually positive , we may follow the proof of Theorem 2.13 and obtain a contradiction. If $\{\Delta y_n\}$ is eventually negative, then we may follow the proof of Theorem 2.14 to show that $\{x_n/d_n\}$ converges to zero.

From Theorem 2.14 if $\rho_n = 1$, we see that the Riccati inequality associated with the equation (1.4) is given by

$$\Delta u_n + \rho_n q_n + \frac{1}{a_n} u_{n+1}^2 \leq 0, \tag{2.53}$$

where

$$A_n = \frac{2^{\gamma-1} (d_{n-\sigma})^{\gamma}}{(M)^{\gamma-1} (d_{n+1})^{2\gamma-2}} > 0, \tag{2.54}$$

for every positive constant $M > 0$. Using the inequality (2.53) and proceeding as in the proof [15, Theorem 2.3.8], we can prove the following Hille and Nehari type results.

Theorem 2.20 *Assume that* (H_1) *holds and* $\Delta d_n \geq 0$. *Furthermore, assume that*

$$\liminf_{n\to\infty} \frac{n}{A_n} \sum_{n+1}^{\infty} q(s) > \frac{1}{4},$$

or

$$\liminf_{n\to\infty} \frac{n}{A_n} \sum_{n+1}^{\infty} q_s + \liminf_{n\to\infty} \frac{1}{n} \sum_{N}^{n-1} \frac{s^2}{A_n} q_s > 4.$$

Then every solution of (1.4) is oscillatory.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

RICCATI TECHNIQUE AND OSCILLATION OF SECOND ORDER NONLINEAR NEUTRAL DELAY DYNAMIC EQUATIONS

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Abstract. In this paper, by using the Riccati technique which reduces the higher order dynamic equations to a Riccati dynamic inequality, we will establish some new sufficient conditions for oscillation of the second order nonlinear neutral dynamic equation

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\alpha}(\delta(t))+v(t)x^{\beta}(\eta(t))=0,$$

on time scales where γ, α β are quotient of odd positive integers.

Mathematics Subject Classification(2010): 34C10, 34K11, 39A21, 34A40, 34N05.

Keywords: Oscillation, nonoscillation, neutral, delay dynamic equations, time scales, neutral delay equations

1. Introduction

The theory of time scales has been introduced by Stefan Hilger in [14] in 1988 in his Ph.D thesis in order to unify continuous and discrete analysis. In the last decades the subject is going fast and simultaneously extending to the other areas of research and many researchers have contributed on different aspects of this new theory, see the survey paper by Agarwal et al. [1] and the references cited therein. In the last few years, there has been an increasing interest in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of different classes of dynamic equations on a time scale $\mathbb{T}$ which may be an arbitrary closed subset of real numbers $\mathbb{R}$, and as special cases contains the continuous and the discrete results as well, we refer the reader to papers ([3],[6], [7], [21]) and the references cited therein.

Following this trend, in this paper, we are concerned with oscillation of a certain class of nonlinear neutral delay dynamic equations of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\alpha}(\delta(t))+v(t)x^{\beta}(\eta(t))=0,\ \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where γ, α, β are quotient of odd positive integers, $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ and p, $q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ with $0\le p(t)<1$, $q(t)$, $v(t)\ge 0$ and τ, δ, $\eta\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and $\tau(t)\le t$, $\delta(t)\le t$, $\eta(t)\le t$ with $\lim_{t\to\infty}\tau(t)=\infty=\lim\limits_{t\to\infty}\delta(t)=\infty=\lim_{t\to\infty}\eta(t)$. By a solution of (1.1), we mean a nontrivial real-valued function $x(t)\in C^1_{rd}([T_x,\infty),\mathbb{R})$, $T_x\ge t_0$ which has the properties that $r(z^{\Delta})^{\gamma})^{\Delta}\in C^1_{rd}([T_x,\infty),\mathbb{R})$ such that $x(t)$ satisfies (1.1) for all $[T_x,\infty)_{\mathbb{T}}$.

We mention here that the neutral delay differential equations appear in modelling of the networks containing lossless transmission lines (as in high-speed computers where the lossless transmission lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, as the Euler equation in some variational problems, theory of automatic control and in neuromechanical systems in which inertia plays an important role, we refer the reader to the papers by Boe and Chang [4], Brayton and Willoughby [8] and to the books by Driver [9], Hale [13] and Popov [16] and reference cited therein.

For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [5], [6] which summarize and organize much of the time scale calculus. Throughout the paper, we will denote the time scale by the symbol $\mathbb{T}$. For example, the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$ and the natural numbers $\mathbb{N}$ are time scales. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. A time-scale $\mathbb{T}$ equipped with the order topology is metrizable and is a K_σ −space; i.e. it is a union of at most countably many compact sets. The metric on $\mathbb{T}$ which generates the order topology is given by $d(r; s) := |\mu(r; s)|$, where $\mu(.) = \mu(.; \tau)$ for a fixed $\tau \in \mathbb{T}$ is defined as follows: The mapping $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ such that $\mu(t) := \sigma(t) - t$ is called graininess.

When $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$ and $\mu(t) \equiv 0$ for all $t \in \mathbb{T}$. If $\mathbb{T} = \mathbb{N}$, then $\sigma(t) = t+1$ and $\mu(t) \equiv 1$ for all $t \in \mathbb{T}$. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$. The mapping : $\nu : \mathbb{T} \to \mathbb{R}_0^+$ such that $\nu(t) = t - \rho(t)$ is called the backward graininess. If $\sigma(t) > t$, we say that t is right-scattered , while if $\rho(t) < t$, we say that t is left-scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. A function $f : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous ($rd-$continuous) if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, we define the derivative f^Δ as follows: Let $t \in \mathbb{T}$. If there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon|\sigma(t) - s|,$$

for all $s \in U$, then f is said to be differentiable at t, and we call α the delta derivative of f at t and denote it by $f^\Delta(t)$. For example, if $\mathbb{T} = \mathbb{R}$, then

$$f^\Delta(t) = f^{'}(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \text{ for all } t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{N}$, then $f^\Delta(t) = f(t + 1) - f(t)$ for all $t \in \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$ (the range $\mathbb{R}$ of f may be actually replaced by any Banach space) the (delta) derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t},$$

if f is continuous at t and t is right–scattered. If t is not right–scattered then the derivative is defined by

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{t \to \infty} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. A function $f : [a, b] \to \mathbb{R}$ is said to be right–dense continuous ($rd-$continuous) if it is right continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. The space of $rd-$continuous functions is denoted by $C_r(\mathbb{T}, \mathbb{R})$. A useful formula is

$$f^\sigma = f + \mu f^\Delta, \quad \text{where} f^\sigma := f \circ \sigma.$$

A time scale $\mathbb{T}$ is said to be regular if the following two conditions are satisfied simultaneously:
(a). For all $t \in \mathbb{T}$, $\sigma(\rho(t)) = t$,
(b). For all $t \in \mathbb{T}$, $\rho(\sigma(t)) = t$.

Remark 1.1. *If $\mathbb{T}$ is a regular time scale, then both operators and are invertible with $\sigma^{-1} = \rho$ and $\rho^{-1} = \sigma$.*

The following formulae give the product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable function f and g. Assume f; $g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}$, then

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \tag{1.2}$$

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{1.3}$$

The chain rule formula that we will use in this paper is

$$(x^{\gamma}(t))^{\Delta}=\gamma\int\limits_{0}^{1}\left[hx^{\sigma}+(1-h)x\right]^{\gamma-1}dhx^{\Delta}(t), \tag{1.4}$$

which is a simple consequence of Keller's chain rule [5, Theorem 1.90]. Note that when $\mathbb{T}=\mathbb{R}$, we have

$$\sigma(t)=t,\ \mu(t)=0,\ \ f^{\Delta}(t)=f^{'}(t),\ \ \int_a^b f(t)\Delta t=\int_a^b f(t)dt.$$

When $\mathbb{T}=\mathbb{Z}$, we have

$$\sigma(t)=t+1,\ \mu(t)=1,\ f^{\Delta}(t)=\Delta f(t),\ \int_a^b f(t)\Delta t=\sum_{t=a}^{b-1}f(t).$$

When $\mathbb{T}$ =$h\mathbb{Z}$, $h>0$, we have $\sigma(t)=t+h$, $\mu(t)=h$,

$$f^{\Delta}(t)=\Delta_h f(t)=\frac{(f(t+h)-f(t))}{h},\ \int_a^b f(t)\Delta t=\sum_{k=0}^{\frac{b-a-h}{h}}f(a+kh)h.$$

When $\mathbb{T}=\{t: t=q^k,\ k\in\mathbb{N}_0,\ q>1\}$, we have $\sigma(t)=qt$, $\mu(t)=(q-1)t$,

$$f^{\Delta}(t)=\Delta_q f(t)=\frac{(f(q\,t)-f(t))}{(q-1)\,t},\ \int_{t_0}^{\infty}f(t)\Delta t=\sum_{k=0}^{\infty}f(q^k)\mu(q^k).$$

When $\mathbb{T}=\mathbb{N}_0^2=\{t^2: t\in\mathbb{N}\}$, we have $\sigma(t)=(\sqrt{t}+1)^2$ and

$$\mu(t)=1+2\sqrt{t},\ f^{\Delta}(t)=\Delta_0 f(t)=(f((\sqrt{t}+1)^2)-f(t))/1+2\sqrt{t}.$$

When $\mathbb{T}=\mathbb{T}_n=\{t_n: n\in\mathbb{N}\}$ where $(t_n\}$ is the harmonic numbers that are defined by $t_0=0$ and $t_n=\sum_{k=1}^{n}\frac{1}{k}, n\in\mathbb{N}_0$, we have

$$\sigma(t_n)=t_{n+1},\ \mu(t_n)=\frac{1}{n+1},\ f^{\Delta}(t)=\Delta_1 f(t_n)=(n+1)f(t_n).$$

When $\mathbb{T}_2$=$\{\sqrt{n}: n\in\mathbb{N}\}$, we have $\sigma(t)=\sqrt{t^2+1}$,

$$\mu(t)=\sqrt{t^2+1}-t,\ f^{\Delta}(t)=\Delta_2 f(t)=\frac{(f(\sqrt{t^2+1})-f(t))}{\sqrt{t^2+1}-t}.$$

When $\mathbb{T}_3$=$\{\sqrt[3]{n}: n\in\mathbb{N}\}$, we have $\sigma(t)=\sqrt[3]{t^3+1}$ and

$$\mu(t)=\sqrt[3]{t^3+1}-t,\ f^{\Delta}(t)=\Delta_3 f(t)=\frac{(f(\sqrt[3]{t^3+1})-f(t))}{\sqrt[3]{t^3+1}-t}.$$

Now, we pass to the antiderivative and the integration on time scales for detla differentiable functions. For $a,b\in\mathbb{T}$, and a delta differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t=f(b)-f(a).$$

An integration by parts formula reads

$$\int_a^b f(t)g^{\Delta}(t)\Delta t=\ f(t)g(t)|_a^b-\int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t, \tag{1.5}$$

and infinite integrals are defined as

$$\int_a^{\infty}f(t)\Delta t=\lim_{b\to\infty}\int_a^b f(t)\Delta t.$$

It is well known that $rd-$continuous functions possess antiderivative. If f is $rd-$continuous and $F^{\Delta} = f$, then

$$\int_{t}^{\sigma(t)} f(s)\Delta s = F(\sigma(t)) - F(t) = \mu(t)F^{\Delta}(t) = \mu(t)f(t).$$

Note that the integration formula on a discrete time scale is defined by

$$\int_{a}^{b} f(t)\Delta t = \sum_{t\in(a,b)} f(t)\mu(t).$$

We say that a solution x of (1.1) has a generalized zero at t if $x\,(t) = 0$ and has a generalized zero in $(t, \sigma(t))$ in case $x\,(t)\, x^{\sigma}\,(t) < 0$ and $\mu(t) > 0$. To investigate the oscillation properties of (1.1) it is proper to use the notions such as conjugacy and disconjugacy of the equation (1.1). Equation (1.1) is disconjugate on the interval $[t_0, b]_{\mathbb{T}}$, if there is no nontrivial solution of (1.1) with two (or more) generalized zeros in $[t_0, b]_{\mathbb{T}}$.

Equation (1.1) is said to be nonoscillatory on $[t_0, \infty]_{\mathbb{T}}$ if there exists $c \in [t_0, \infty]_{\mathbb{T}}$ such that this equation is disconjugate on $[c, d]_{\mathbb{T}}$ for every $d > c$. In the opposite case (1.1) is said to be oscillatory on $[t_0, \infty]_{\mathbb{T}}$. A solution $x\,(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is oscillatory. We say that (1.1) is right disfocal (left disfocal) on $[a, b]_{\mathbb{T}}$ if the solutions of (1.1) such that $x^{\Delta}\,(a) = 0$ $(x^{\Delta}\,(b) = 0)$ have no generalized zeros in $[a, b]_{\mathbb{T}}$.

In recent two decades some authors have been studied the oscillation of the second order nonlinear neutral delay dynamic equations on time scales and established several sufficient conditions for oscillation of some different types of equations by employing the Riccati transformation technique. For example, Saker [18] has studied the oscillation of second order neutral delay dynamic equations of Emden-fowler type of the form

$$[a(t)(y(t) + r(t)y(\tau(t))]^{\Delta} + p(t)|y(\delta(t))|^{\gamma} signy(\delta(t))) = 0,$$

on time scale $\mathbb{T}$, where, $\gamma > 1$, $a(t)$, $p(t)$, $r(t)$ and $\delta(t)$ are real-valued function defined on $\mathbb{T}$. Also Saker [19] studied the oscillation of the superlinear and sublinear neutral delay dynamic equations of the form

$$[a(t)([y(t) + p(t)y(\tau(t)))]^{\Delta})^{\gamma}]^{\Delta} + q(t)y^{\gamma}(\delta(t))) = 0,$$

on time scale, where $\gamma > 0$ is a quotient of odd positive integers. The main results has been obtained under the conditions $\tau(t) : \mathbb{T} \to \mathbb{T}$, $\delta(t) : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$, $\delta(t) \leq t$ for all $t \in \mathbb{T}$ and $\lim\limits_{t\to\infty} \tau(t) = \lim\limits_{t\to\infty} \delta(t) = \infty$, $\int_{t_0}^{\infty} \frac{1}{a(t)}^{\frac{1}{\gamma}} \Delta t = \infty$, $a^{\Delta}(t) \geq t$ and $0 \leq p(t) < 1$.

Thandapani et. al [24] studied the oscillation of second order nonlinear neutral dynamic equations on time scale of the form

$$(r(t)((y(t) + p(t)y(t - \tau))^{\Delta})^{\gamma})^{\Delta} + q(t)y^{\beta}(t - \delta) = 0,\ t \in \mathbb{T},$$

where $\mathbb{T}$ is a time scales. They obtained their results under the conditions $\gamma \geq 1$ and $\beta > 0$ are quotients of odd positive integers, τ, δ are fixed nonnegative constants such that the delay function $\tau(t) = t - \tau < t$ and $\delta(t) = t - \delta < t$ satisfying $\tau : \mathbb{T} \to \mathbb{T}$ and $\delta : \mathbb{T} \to \mathbb{T}$ for all $t \in \mathbb{T}$, $q(t)$ and $\tau(t)$ real valued rd-continuous functions defined on $\mathbb{T}$, $p(t)$ is a positive and rd-continuous function $\mathbb{T}$ such that $0 \leq p(t) < 1$.

Sun et al. [22] studied the oscillation of a second order quasiliniear neutral delay dynamic equation on time scales of the form

$$(r(t)((x(t) + p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta} + q_1(t)x^{\alpha}(\tau_1(t)) + q_2(t)x^{\beta}(\tau_2(t)) = 0,$$

on time scale $\mathbb{T}$, where α, β, γ are quotients of odd positive integers, r, p, q_1, q_2 are rd-continuous function on $\mathbb{T}$ and r, q_1, q_2 are positive, $-1 < -p_0 \leq p(t) < 1$, $p_0 > 0$, the delay functions $\tau_i : \mathbb{T} \to \mathbb{T}$ satisfying $\tau_i(t) \leq t$ for $t \in \mathbb{T}$ and $\tau_i(t) \to \infty$ as $t \to \infty$, for $i = 0, 1, 2$ and there exists a function $\tau : \mathbb{T} \to \mathbb{T}$ which satisfying $\tau(t) \leq \tau_1(t)$, $\tau(t) \leq \tau_2(t)$, $\tau(t) \to \infty$ as $t \to \infty$.

Gao et al. [12] established some oscillation theorems for second order neutral functional dynamic equations on time scale of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q_1(t)x^{\alpha}(\delta(t))+q_2(t)x^{\beta}(\eta(t))=0,$$

where γ, α, β are ratios of odd positive integers by using the comparison theorems for oscillation. Sethi [26] considered the second order sublinear neutral delay dynamic equations of the form

$$(r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+q(t)x^{\gamma}(\alpha(t))+v(t)x^{\gamma}(\eta(t))=0,$$

under the assumptions:

(H_0) $\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t=+\infty,$

(H_1). $\int_0^{\infty}\left(\frac{1}{r(t)}\right)^{\frac{1}{\gamma}}\Delta t<\infty,$

where $0<\gamma\leq 1$ is a quotient of odd positive integers, q, $v\to[0,\infty)$ and p, q, $v:\mathbb{T}\to\mathbb{T}$ are rd-continuous functions and τ, σ, $\eta:\mathbb{T}\to\mathbb{T}$ are positive rd-continuous functions such that $\lim_{t\to\infty}\tau(t)=\infty=\lim\limits_{t\to\infty}\alpha(t)=\infty=\lim_{t\to\infty}\eta(t)$ and obtained some sufficient conditions for oscillation. Our aim in this paper is to establish some new sufficient conditions for oscillation of the equation (1.1) by employing the Riccati technique and some basic lemmas studied the behavior of nonoscillatory solutions. Our motivation of the present work has come under two ways. First is due to the work in [17] and [22] and second is due to the work in [10].

2. Main Results

In this section, we establish some sufficient conditions for oscillation of all solutions of (1.1) under the hypothesis (H_0). Throughout the paper, we use the notation

$$z(t)=x(t)+p(t)x(\tau(t)). \tag{2.1}$$

Lemma 2.1. [2] *Assume that* (H_0) *holds and* $r(t)\in C^1_{rd}([(a,\infty),\mathbb{R}^+)$ *such that* $r^{\Delta}(t)\geq 0$. *Let* $x(t)$ *be an eventually positive real valued function such that* $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}\leq 0$, *for* $t\geq t_1>t_0$. *Then* $x^{\Delta}(t)>0$ *and* $x^{\Delta\Delta}(t)<0$ *for* $t\geq t_1>t_0$.

Lemma 2.2. [2] *Assume that the assumptions of Lemma 2.1 holds and let* $\tau(t)$ *be a positive continuous function such that* $\tau(t)\leq t$ *and* $\lim\limits_{t\to\infty}\tau(t)=\infty$. *Then there exists* $t_l>t_1$ *such that for each* $l\in(0,1)$

$$\frac{x(\tau(t))}{x(\delta(t))}\geq l\frac{\tau(t)}{\delta(t)}.$$

Proof. Indeed, for $t\geq t_1$

$$u(\delta(t))-u(\tau(t))=\int_{\tau(t)}^{\delta(t)}u^{\Delta}(s)\Delta s\leq(\delta(t)-\tau(t)))u^{\Delta}(\tau(t),$$

which implies that

$$\frac{u(\delta(t))}{u(\tau(t))}\leq 1+(\delta(t)-\tau(t)))\frac{u^{\Delta}(\tau(t)}{u(\tau(t))}.$$

On the other hand, it follows that

$$u(\tau(t))-u(t_1))=\int_{t_1}^{\tau(t)}u^{\Delta}(s)\Delta s\geq(u(t)-t_1)u^{\Delta}(\tau(t)).$$

That is for each $l\in(0,1)$, there exists a $t_l>t_1$ such that

$$l(\tau(t))\leq\frac{u(\tau(t))}{u^{\Delta}(\tau(t))},\ t\geq t_l.$$

Consequently,

$$\frac{u(\delta(t))}{u(\tau(t))}\leq 1+(\delta(t)-\tau(t)))\frac{u^{\Delta}(\tau(t))}{u(\tau(t))}\leq\frac{\delta(t)}{l\tau(t)}.$$

The proof is complete. ☐

In the following, for simplicity, we denote

$$a_1(t) := \int_t^\infty [q(s)(1-p(\delta(s))] \left(\frac{l\delta(s)}{\sigma(s)}\right)^\alpha \Delta s + \int_t^\infty [v(s)(1-p(\delta(s)))] \left(\frac{l\delta(s)}{\delta(s)}\right)^\alpha \Delta s,$$

and

$$A_1(t, K_1) := \left[a_1(t) + K_1 \int_t^\infty \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (a_1^\delta(s))^{1+\frac{1}{\gamma}} \Delta s\right]^{\frac{1}{\gamma}}, \text{ for } t \in [t_0, \infty)_{\mathbb{T}},$$

where $K_1 > 0$ is an arbitrary constant.

Theorem 2.1. *Assume that* (H_0) *holds and let* $0 \leq p(t) \leq a < 1$, $r^\Delta(t) > 0$ *and* $\gamma < \alpha < \beta$, $\eta(t) \geq \delta(t)$ *and* $\delta^\Delta(t) \geq 1$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$. *If*

(H_1). $\limsup_{t\to\infty} a_1(t) < \infty$,

(H_2). $\int_{t_0}^\infty (\frac{1}{r(s)})^{\frac{1}{\gamma}} A_1^\sigma(s, K_1)\Delta s = \infty$.

Then every solution of (1.1) oscillates on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Suppose the contrary that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0$. Hence there exists $t \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ and $x(\eta(t)) > 0$ for $t \geq t_1$. Using (2.1), we see that (1.1) becomes

$$(r(t)(z^\Delta(t))^\gamma)^\Delta = -q(t)x^\alpha(\delta(t)) - v(t)x^\beta(\eta(t)) \leq 0, \text{ for } t \geq t_2. \tag{2.2}$$

So $r(t)(z^\Delta(t))^\gamma$ is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$, that is, either $z^\Delta(t) > 0$ or $z^\Delta(t) < 0$. By Lemma 2.1, it follows that $z^\Delta(t) > 0$ for $t \geq t_2$. Hence there exists $t_3 > t_2$ such that

$$\begin{aligned} z(t) - p(t)z(\tau(t)) &= x(t) + p(t)x(\tau(t)) - p(t)x(\tau(t)) \\ &\quad -p(t)p(\tau(t))p(\tau(\tau(t))) \\ &= x(t) - p(t)p(\tau(t))p(\tau(\tau(t))) \leq x(t), \end{aligned}$$

which implies that

$$x(t) \geq (1 - p(t))z(t), \text{ for } t \in [t_3, \infty)_{\mathbb{T}}.$$

Therefore (1.1) can be written as

$$(r(t)(z^\Delta(t))^\gamma)^\Delta + q(t)(1 - p(\delta(t)))^\alpha z^\alpha(\delta(t)) + v(t)(1 - p(\eta(t)))^\alpha z^\alpha(\eta(t)) \leq 0,$$

where $\gamma < \alpha < \beta$. Define $w(t)$ by the Riccati transformation

$$w(t) = r(t)\frac{(z^\Delta(t))^\gamma}{z^\alpha(t)}, \quad \text{for} \quad t \in [t_3, \infty)_{\mathbb{T}}. \tag{2.3}$$

By using the product and quotient rules, we see that

$$w^\Delta(t) = \frac{(r(z^\Delta)^\gamma)^\Delta}{(z^\sigma)^\alpha} - \frac{(r(z^\Delta)^\gamma)^\sigma (z^\alpha)^\Delta}{z^\alpha (z^\sigma)^\alpha}, \quad \text{for } t \in [t_3, \infty)_{\mathbb{T}}. \tag{2.4}$$

Now, since $\eta(t) > \delta(t)$ and due to (2.3) and (2.4), we have

$$w^\Delta(t) \leq -q(1-p^\delta)^\alpha - v(1-p^\delta)^\alpha \frac{(z^\delta)^\alpha}{(z^\sigma)^\alpha} - \frac{w^\sigma (z^\alpha)^\Delta}{z^\alpha}, \text{ for } t \in [t_3, \infty)_{\mathbb{T}},$$

Now, by using the chain rule [6], we get that

$$\begin{aligned} (z^\alpha(t))^\Delta &= \alpha \int_0^1 [(1-h)z(t) + hz(\sigma(t))]^{\alpha-1} dh z^\Delta(t) \\ &\geq \begin{cases} \alpha(z(t))]^{\alpha-1} z^\Delta(t), \ \alpha > 1, \\ \alpha(z(\sigma(t)))]^{\alpha-1} z^\Delta(t), \ 0 < \alpha \leq 1. \end{cases} \end{aligned}$$

Since $z(t)$ is nondecreasing function on $[t_3,\infty)_{\mathbb{T}}$, then for $t \geq t_3$,

$$\frac{(z^{\alpha}(t))^{\Delta}}{z^{\alpha}(t)} \geq \begin{cases} \alpha\frac{z^{\Delta}(t)}{z(t)}, & \text{for} \quad \alpha > 1 \\ \alpha\frac{(z(\sigma(t)))^{\alpha-1}}{z^{\alpha}(t)}z^{\Delta}(t), & for\ 0 < \alpha \leq 1. \end{cases}$$

Using the fact that $t \leq \sigma(t)$, we have

$$\frac{(z^{\alpha})^{\Delta}}{z^{\alpha}} \geq \alpha\frac{z^{\Delta}}{z^{\sigma}},\ \alpha > 0 \quad \text{on} \quad [t_3,\infty)_{\mathbb{T}}.$$

Therefore (2.4) yields that

$$w^{\Delta} \leq -q(1-p^{\delta})^{\alpha} - v(1-p^{\delta})^{\alpha}\frac{(z^{\sigma})^{\alpha}}{(z^{\delta})^{\alpha}} - \alpha w^{\sigma}\frac{z^{\Delta}}{z^{\sigma}},\ t \geq t_3. \tag{2.5}$$

Now, since $\left(r^{\frac{1}{\gamma}}z^{\Delta}\right)$ is nonincreasing on $[t_3,\infty)_{\mathbb{T}}$, then for $t \leq \sigma(t)$, we have that

$$z^{\Delta} \geq r^{-\frac{1}{\gamma}}(w^{\sigma})^{\frac{1}{\gamma}}(z^{\sigma})^{\frac{\alpha}{\gamma}},\ t \geq t_3. \tag{2.6}$$

Substituting (2.6) into (2.5), we get

$$w^{\Delta} \leq -q(1-p^{\delta})^{\alpha}\frac{(z^{\delta})^{\alpha}}{(z^{\sigma})^{\alpha}} - v(1-p^{\delta})^{\alpha}\frac{(z^{\delta})^{\alpha}}{(z^{\sigma})^{\alpha}} - \alpha r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}(z^{\sigma})\frac{\alpha}{\gamma} - 1,\ t \geq t_3.$$

Since $z(t)$ is nondecreasing on $[t_3,\infty)_{\mathbb{T}}$, then there exists $t_4 > t_3$ and $C > 0$ such that

$$(z(\sigma(t)))^{\frac{\alpha}{\gamma}-1} \geq (z(t))^{\frac{\alpha}{\gamma}-1} \geq C, \quad \text{for} \quad t \geq t_4.$$

By using Lemma 2.2, it follows from the last inequality that

$$\begin{aligned} w^{\Delta}(t) \quad &\leq \quad -q(1-p(\delta(t)))^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} - v(1-p(\delta(t)))^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} \\ -\alpha Cr^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}},\ t \quad &\geq \quad t_l > t_4. \end{aligned}$$

Integrating the above inequality from t to u $(t < u)$ for $t,\ u \in [t_4,\infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} -w(t) \quad &\leq \quad w(u) - w(t) \\ &\leq \quad -\int_t^u \left[q(1-p^{\delta})^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} + v(1-p^{\delta})^{\alpha}\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha} + \alpha Cr^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}}\right]\Delta s, \end{aligned}$$

that is,

$$w(t) \geq a_1(t) + K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s,\ t \geq t_1,$$

where $K_{1\alpha} = C\alpha$. Indeed, $w(t) > a_1(t)$ implies that

$$w(t) \geq a_1(t) + K_1\int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1(\sigma(s)))^{1+\frac{1}{\gamma}}\Delta s = A_1^{\gamma}(t,K_1).$$

Since $t \leq \sigma(t)$ we see

$$r(z^{\Delta})^{\gamma} \geq (r(z^{\Delta})^{\gamma})^{\sigma},$$

which implies that

$$\frac{r(z^{\Delta})^{\gamma}}{(z^{\sigma})^{\alpha}} \geq \frac{(r(z^{\Delta})^{\gamma})^{\sigma}}{(z^{\sigma})^{\alpha}} = w^{\sigma} \geq (A_1^{\gamma}(t,k_1))^{\sigma},$$

that is,

$$(z^{\sigma})^{\delta}z^{\Delta} \geq r^{-\frac{1}{\gamma}}(A_1^{\sigma}(t,k_1)),\ t \in [t_5,\infty]_{\mathbb{T}},$$

where $\delta = (\frac{\alpha}{\gamma}) > 1$. Using the chain rule, we have

$$\begin{aligned} (z^{1-\delta}(t))^{\Delta} &= (1-\delta)\int_0^1 [(1-h)z(t) + hz(\sigma(t))]^{\delta}dhz^{\Delta}(t) \\ &\leq (1-\delta)(z(\sigma(t)))^{-\delta}z^{\Delta}(t), \end{aligned}$$

that is,

$$\frac{(z^{1-\delta}(\sigma(t)))^\Delta}{1-\delta} \geq z(\sigma(t))^{-\delta} z^\Delta(\sigma(t)).$$

Hence

$$\frac{(z^{1-\delta}(t))^\Delta}{1-\delta} \geq (z(\sigma(t)))^{-\delta} z^\Delta(t),$$

and then due to (2.6), we see that

$$\frac{(z^{1-\delta}(t))^\Delta}{1-\delta} \geq r^{-\frac{1}{\gamma}}(t)(A_1^\sigma(t,k_1)),\ t \in [t_5,\infty)_{\mathbb{T}}.$$

Integrating above inequality from t_5 to t, we get

$$\int_{t_5}^t r(s)^{-\frac{1}{\gamma}} \Big(A_1^\sigma(s,K_1)\Big)^{\frac{1}{\gamma}} \Delta s < \infty,$$

which is a contradiction to (H_2). The proof is complete. ☐

Theorem 2.2. *Let $0 \leq p(t) \leq p(t) \leq 1$, $r^\Delta(t) \geq 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $\gamma = \alpha = \beta$, $\eta(t) \geq \sigma(t)$ and assume that (H_0), and (H_1) hold. Furthermore assume that*

(H_3). $\limsup\limits_{t\to\infty} \left(\int_{t_0}^t r^{-\frac{1}{\gamma}}(s) A_1(s,K_1)\Delta s\right) > 1.$

Then every solution of (1.1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.1, we have

$$w(t) \geq A_1^\gamma(t,K_1)\ for\ t \in [t_4,\infty)_{\mathbb{T}}.$$

Using the fact that $r^{\frac{1}{\gamma}} z^\Delta$ is nonincreasing on $[t_4,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} z(t) = z(t_4) + \int_{t_4}^t z^\Delta(s)\Delta s &= z(t_4) + \int_{t_4}^t r^{-\frac{1}{\gamma}}(s)\Big(r(s)^{-\frac{1}{\gamma}} z^\Delta(s)\Big)\Delta s \\ &\geq r^{\frac{1}{\gamma}}(t) z^\Delta(t) r^{-\frac{1}{\gamma}}(s)\Delta s, \end{aligned}$$

that is,

$$\frac{r(t)^{\frac{1}{\gamma}} z^\Delta(t)}{z(t)} \leq \Big(\int_{t_4}^t r(s)^{-\frac{1}{\gamma}}\Delta s\Big)^{-1},\ t \geq t_4, \tag{2.7}$$

Consequently,

$$A_1(t,K_1) \leq w^{\frac{1}{\gamma}}(t) = \frac{r(t)^{\frac{1}{\gamma}} z'(t)}{z(t} \leq \Big(\int_{t_2}^t r^{-\frac{1}{\gamma}}(s)\Delta s\Big)^{-1},$$

implies that

$$\Big(\int_{t_4}^t r^{-\frac{1}{\gamma}}(s)\Delta s\Big) A_1(t,K_1) \leq 1,$$

which contradicts (H_3). Hence the theorem is proved. ☐

Theorem 2.3. *Let $0 \leq p(t) \leq p(t) \leq 1$, $r^\Delta(t) \geq 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ and $\gamma > \alpha > \beta$, $\eta(t) \geq \sigma(t)$ and assume that (H_0) and (H_2) hold. Furthermore assume that*

(H_4). $\limsup\limits_{t\to\infty} (a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \left(\int_{t_0}^t r^{-\frac{1}{\gamma}}(s)\Delta s\right) \left[a_1(t) + K_1 \int_t^\infty \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} (a_1^\sigma(s))^{1+\frac{1}{\gamma}}\Delta s\right]^{\frac{1}{\gamma}} = \infty.$

Then every solution of (1.1) oscillates.

Proof. Proceeding as in the proof of Theorem 2.1, we obtain (2.2) and (2.3) and hence $w(t) > a_1(t)$, for $t \in [t_4,\infty)$. Consequently, it follows from (2.3) that

$$r^{\frac{1}{\gamma}} z^\Delta > z^{\frac{\alpha}{\gamma}} a_1^{\frac{1}{\gamma}},\quad \text{for} \quad t \geq t_4.$$

We have $(rz^{\Delta})^{\gamma})^{\Delta} \leq 0$ implies that there exists a constant $C > 0$ and $t_5 > t_4$ such that $r^{\frac{1}{\gamma}} z^{\Delta} \leq C$, for $t \geq t_5$, that is $C \geq r^{\frac{1}{\gamma}} z^{\Delta} > z^{\frac{\alpha}{\gamma}} a_1^{\frac{1}{\gamma}}$ and hence

$$z(t) \leq C^{\frac{\gamma}{\alpha}} a_1(t)^{-\frac{1}{\alpha}}, \text{ for } t \in [t_5, \infty)_{\mathbb{T}}, \tag{2.8}$$

which implies that

$$(z^{\sigma})^{\frac{(\alpha-\gamma)}{\gamma}} \geq C^{\frac{(\alpha-\gamma)}{\alpha}} (a_1^{\sigma})^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \text{ for } t \in [t_5, \infty)_{\mathbb{T}}. \tag{2.9}$$

Due to (2.5), (2.6) and using Lemma 2.2, we have that

$$\begin{aligned} w^{\Delta}(t) \quad \leq \quad & -q(1-p(\delta(t)))^{\alpha}\Big(\frac{l\delta(t)}{\sigma(t)}\Big)^{\alpha} - v(1-p(\delta(t)))^{\alpha}\Big(\frac{l\delta(t)}{\sigma(t)}\Big)^{\alpha} \\ & -\alpha C r^{-\frac{1}{\gamma}}(t)(w^{\sigma}(t))^{1+\frac{1}{\gamma}}(z^{\sigma}(t))^{\frac{(\alpha-\gamma)}{\alpha}}. \end{aligned}$$

Integrating the last inequality as in the proof of Theorem 2.1 and using (2.8), we obtain for $t \geq t_1 \geq t_5$ that

$$w(t) \geq a_1(t) + K_3 \int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1(s))^{1+\frac{1}{\gamma}} \Delta s, \ for\ t \in [t_l, \infty)_{\mathbb{T}}, \tag{2.10}$$

where $K_1 = \alpha C^{\frac{(\alpha-\gamma)}{\gamma}}$. Substitute (2.9) into (2.3), it is easy to verify that

$$(z(t)^{\frac{(\alpha-\gamma)}{\gamma}} \frac{r^{\frac{1}{\gamma}}(t) z^{\Delta}(t)}{z(t)} \geq \Big[a_1(t) + K_1 \int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\Big]^{\frac{1}{\gamma}}. \tag{2.11}$$

Using (2.7) and (2.9) in (2.11), we can find

$$\begin{aligned} C^{\frac{\alpha-\gamma}{\alpha}} a_1(t)^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \Big(\int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big)^{-1} \quad & \geq \quad \Big[a_1(t) \\ +K_1 \int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\Big]^{\frac{1}{\gamma}}, \text{ for } t \quad & \in \quad [t_1, \infty)_{\mathbb{T}}. \end{aligned}$$

Therefore, for $t \geq t_1$ we have

$$(a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \Big(\int_{t_2}^{t} r^{-\frac{1}{\gamma}}(s)\Delta s\Big) \Big[a_1(t) + K_1 \int_t^{\infty} r^{-\frac{1}{\gamma}}(s)(a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s\Big]^{\frac{1}{\gamma}} \leq C^{\frac{\alpha-\gamma}{\alpha}},$$

which contradicts (H_4). This completes the proof of theorem. ☐

Theorem 2.4. *Let* $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$ *and* $\gamma < \beta < \alpha$, $\eta(t) \geq \sigma(t)$. *If* (H_0), (H_2) *and* (H_3) *hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1. Hence the details are omitted. ☐

Theorem 2.5. *Let* $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$ *and* $\alpha > \gamma > \beta$, $\eta(t) \geq \sigma(t)$. *If* (H_0), (H_2) *and* (H_3) *hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1. ☐

Theorem 2.6. *Let* $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$ *and* $\alpha < \beta < \gamma$, $\eta(t) \geq \sigma(t)$. *If* (H_0), (H_1) *and* (H_4) *hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1 and Theorem 2.3. ☐

Theorem 2.7. *Let* $0 \leq p(t) \leq 1$, $r^{\Delta}(t) \geq 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$ *and* $\alpha < \gamma < \beta$, $\eta(t) \geq \sigma(t)$. *If* (H_0), (H_1) *and* (H_4) *hold. Then every solution of (1.1) oscillates.*

Proof. The proof of the theorem follows from Theorem 2.1 and Theorem 2.3. ☐

In the following theorems we will denote

$$a_2(t)=\int_t^{\infty}\Big[\lambda Q(s)\left(\frac{l\delta(t)}{\delta(t)}\right)^{\alpha}+\mu V(s)\left(\frac{l\delta(t)}{\sigma(t)}\right)^{\alpha}\Big]\Delta s,\ t\in[t_0,\infty)_{\mathbb{T}},$$

and

$$A_2(t,K_2)=\left[\frac{\lambda}{1+a^{\alpha}}a_2(\tau^{-1}(t))+\frac{\mu a_2(\tau^{-1}(t))}{1+a^{\alpha}}+K_2\int_{\tau^{-1}(t)}^{\infty}\left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}}((a_2(\tau^{-1\delta}(s)))^{1+\frac{1}{\gamma}}\Delta s\right]^{\frac{1}{\gamma}},$$

where K_2 is an arbitrary positive constant and $a>0$ $\lambda,\mu>0$ are positive constants, $Q(t)=min\{q(t),q(\tau(t))\}$, $V(t)=min\{v(t),v(\tau(t))\}$. From the definitions of τ, δ, η, we see that τ^{-1}, δ^{-1}, $\eta^{-1}:\mathbb{T}\to\mathbb{T}$ and τ^{-1}, δ^{-1}, η^{-1} are rd-continuous functions and $\tau^{-1}(t)\geq t,\delta^{-1}(t)\geq t$ and $\eta^{-1}(t)\geq t$.

Theorem 2.8. *Let* $1\leq p(t)\leq p<\infty$, $r^{\Delta}(t)\geq 0$ $\tau(\delta(t))=\delta(\tau(t))$, $\tau(\eta(t))=\eta(\tau(t))$ *and* $\gamma<\alpha<\beta$, $\eta(t)\geq\delta(t)$ *and If* (H_0) *holds and the following conditions hold:*
(H_5). $\limsup\limits_{t\to\infty}a_2(t)<\infty$,
(H_6). $\int_{t_0}^{\infty}(\frac{1}{r(s)})^{\frac{1}{\gamma}}A_2^{\sigma}(s,K_2)\Delta s=\infty$,
Then every solution of (1.1) oscillates.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1) such that $x(t)>0$ for $t\geq t_0$. Proceeding as in the proof of Theorem 2.1, we get (2.2) for $t\in[t_2,\infty)$, that is either $z^{\Delta}(t)>0$ or $z^{\Delta}(t)<0$. By lemma 2.1, it follows that $z^{\Delta}(t)>0$. From (1.1), it is easy to see for $t\geq t_1$, that

$$\begin{aligned}&(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+p^{\beta}(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta}+q(t)x^{\alpha}(\delta(t))\\&+p^{\beta}q(\tau(t))x^{\alpha}(\delta(\tau(t))+v(t)x^{\beta}(\eta(t))+p^{\beta}v(\tau(t))x^{\beta}(\eta(\tau(t))=0.\end{aligned}\tag{2.12}$$

By assuming that there exists $\lambda>0$ such that $u^{\gamma}(x)+u^{\gamma}(y)\geq\lambda u^{\gamma}(x+y)$, x, $y\in\mathbb{R}^{+}$, and there exists $\mu>0$ such that $u^{\gamma}(x)+u^{\gamma}(y)\geq\mu u^{\gamma}(x+y)$, x, $y\in\mathbb{R}^{+}$, we obtain (note that $\gamma<\alpha<\beta$) that

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+p^{\alpha}(r(\tau(t))(z^{\Delta}(\tau(t))^{\gamma})^{\Delta}+\lambda Q(t)z^{\alpha}(\delta(t))+\mu V(t)z^{\alpha}(\eta(t))\leq 0.$$

for $t\in[t_2,\infty)_{\mathbb{T}}$, where $z(t)\leq x(t)+px(\tau(t))$. Define $w(t)$ as in (2.3). Upon using the fact that

$$w^{\Delta}(t)=\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\delta})^{\alpha}}-\frac{r(z^{\Delta})^{\gamma})^{\delta}(z^{\alpha})^{\Delta}}{z^{\alpha}(z^{\delta})^{\alpha}}$$

and

$$\frac{(z^{\alpha})^{\Delta}}{(z^{\sigma})^{\alpha}}\geq\alpha\frac{(z^{\Delta})}{z^{\sigma}},\ \alpha>0\ for\ t\in[t_3,\infty)_{\mathbb{T}}.$$

By using the fact that $z(t)$ is nondecreasing and using (2.12) into (2.11) we obtain

$$w^{\Delta}\leq\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}-\alpha w^{\sigma}\frac{z^{\Delta}}{z^{\rho}},\ t\geq t_3.$$

Due to (2.6) and $(z(\sigma(t)))^{\frac{\alpha}{\gamma}}\geq C$, there exists $t_4>t_3$ such that, for $t\in[t_4,\infty)_{\mathbb{T}}$,

$$w^{\Delta}\leq\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}-\alpha Cr^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}.\tag{2.13}$$

From (2.13), we find

$$w^{\Delta}+a^{\alpha}w^{\tau\Delta}\leq\frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}-\alpha Cr^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}+a^{\alpha}\frac{(r(z^{\Delta})^{\gamma})^{\tau\Delta}}{(z^{\sigma\Delta})^{\alpha}}-\alpha C(r^{\tau})^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}},$$

that is,

$$\begin{aligned} w^{\Delta}+a^{\alpha}w^{\tau\Delta} &\leq \frac{(r(z^{\Delta})^{\gamma})^{\Delta}}{(z^{\sigma})^{\alpha}}+a^{\alpha}\frac{(r(z^{\Delta})^{\gamma})^{\tau\Delta}}{(z^{\sigma\Delta})^{\alpha}} \\ &\quad -\alpha C\left[r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}+a^{\alpha}(r^{\tau})^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}}\right].\end{aligned}$$

Applying Lemma 2.2 on the above inequality, we get

$$\begin{aligned} w^{\Delta}+a^{\alpha}w^{\tau\Delta} \quad &\leq \quad -\lambda Q\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\mu V\left(\frac{l\delta}{\sigma}\right)^{\alpha} \\ &\quad -\alpha C\left[r^{-\frac{1}{\gamma}}(w^{\sigma})^{1+\frac{1}{\gamma}}+a^{\alpha}(r^{\tau})^{-\frac{1}{\gamma}}(w^{\sigma\tau})^{1+\frac{1}{\gamma}}\right]\end{aligned}$$

for $t\in[t_1,\infty)_{\mathbb{T}}$, that is

$$w^{\Delta}+a^{\alpha}w^{\tau\Delta}\leq-\lambda Q(t)\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\mu V(t)\left(\frac{l\delta}{\sigma}\right)^{\alpha}-\alpha Cr^{-\frac{1}{\gamma}}(1+a^{\alpha})(w^{\sigma})^{1+\frac{1}{\gamma}}, \tag{2.14}$$

where we used the fact that $r^{\Delta}(t)\geq 0$ and $w(t)$ is a decreasing function due to (2.6) and (2.14) on $[t_1,\infty)_{\mathbb{T}}$. Integrating (2.14) from t to v for $t,\, v\in[t_1,\infty)_{\mathbb{T}}$, it is easy to verify that

$$\begin{aligned} w^{\Delta}+a^{\alpha}w^{\tau(t)} &\geq \int_t^{\infty}\lambda Q(s)\left(\frac{l\delta}{\sigma}\right)^{\alpha}\Delta s+\int_t^{\infty}\mu V(s)\left(\frac{l\delta}{\sigma}\right)^{\alpha}\Delta s \\ &\quad +\alpha C(1+a^{\alpha})\int_t^{\infty}\left[r(s)^{-\frac{1}{\gamma}}w(\sigma(s))^{1+\frac{1}{\gamma}}\right]\Delta s,\end{aligned}$$

that is,

$$w^{\Delta}+a^{\alpha}w^{\tau(t)}=a_2(t)+\alpha C(1+a^{\alpha})\int_t^{\infty}\left[r(s)^{-\frac{1}{\gamma}}w(\sigma(s))^{1+\frac{1}{\gamma}}\right]\Delta s,$$

which implies that

$$(1+a^{\alpha})w(\tau(t))\geq a_2(t)+\alpha C(1+a^{\alpha})\int_t^{\infty}r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s. \tag{2.15}$$

Then (H_2) and (2.15) yield that

$$w(t)\geq\frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+\alpha C\int_{\tau^{-1}(t)}^{\infty}r^{-\frac{1}{\gamma}}(s)w(\sigma(s))^{1+\frac{1}{\gamma}}\Delta s.$$

Indeed

$$w(t)\geq\frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}.$$

Hence the last inequality becomes

$$\begin{aligned} w(t) \quad &\geq \quad \frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+\alpha C\int_{\tau^{-1}(t)}^{\infty}\left[r^{-\frac{1}{\gamma}}(s)\left(\frac{1}{1+a^{\alpha}}\right)^{1+\frac{1}{\gamma}}(a_2(\tau^{-1}(\sigma(s)))^{1+\frac{1}{\gamma}}\right]\Delta s \\ &= \quad \frac{(a_2(\tau^{-1}(t))}{(1+a^{\alpha})}+K_2\int_{\tau^{-1}(t)}^{\infty}\left[r^{-\frac{1}{\gamma}}(s)(a_2(\tau^{-1}(\sigma(s)))^{1+\frac{1}{\gamma}}\right]\Delta s \\ &= \quad A_2^{\gamma}(t,K_2),K_2=\alpha C\left(\frac{1}{1+a^{\alpha}}\right)^{1+\frac{1}{\gamma}}.\end{aligned}$$

Proceeding as in the proof of theorem 2.1, we obtain

$$\int_{t_4}^{t}r^{-\frac{1}{\gamma}}(s)A_2^{\sigma}(s,K_2)\Delta s<\infty,$$

a contradiction due to (H_6). The proof is complete. □

Theorem 2.9. *Let* $1 \leq p(t) \leq p < \infty$, $r^{\Delta}(t) \geq 0$ *for* $t \in [t_0, \infty)_{\mathbb{T}}$, $\tau(\delta(t)) = \delta(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$ *and* $\gamma = \alpha = \beta$, $\eta(t) \geq \delta(t)$. *If* (H_0), $(H_5) - (H_7)$ *and*

(H_7). $\limsup\limits_{t\to\infty} \left(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s) A_2(s, K_2) \Delta s \right) > 1.$

Then every solution of (1.1) oscillates.

Proof. The proof of the theorem follows from Theorem 2.2 and Theorem 2.8. Hence the details are omitted. ☐

Theorem 2.10. *Let* $1 \leq p(t) \leq a < \infty$, $r^{\Delta}(t) \geq 0$ $\tau(\sigma(t)) = \sigma(\tau(t))$, $\tau(\eta(t)) = \eta(\tau(t))$, $\gamma > \alpha > \beta$, $\eta(t) \geq \delta(t)$. *If* (H_0), (H_2), $(H_5) - (H_7)$ *and*

(H_8). $\limsup\limits_{t\to\infty} (a_1(t))^{\frac{(\gamma-\alpha)}{\alpha\gamma}} \left(\int_{t_0}^{t} r^{-\frac{1}{\gamma}}(s) \Delta s \right) \left[a_1(t) + K_3 \int_t^{\infty} \left(\frac{1}{r(s)} \right)^{\frac{1}{\gamma}} (a_1^{\sigma}(s))^{1+\frac{1}{\gamma}} \Delta s \right]^{\frac{1}{\gamma}} = \infty.$

Then every solution of (1.1) oscillates.

ACKNOWLEDGEMENT

The second author is supported by Rajiv Gandhi National Fellowship (UGC), New Delhi, India, through the letter No. F1-17.1/2013-14/RGNF-2013-14-SC-ORI-42425, dated Feb. 6, 2014.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Semilocal Convergence of a Newton-Secant Solver for Equations with a Decomposition of Operator

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May 4, 2019

Abstract. We provide the semilocal convergence analysis of the Newton-Secant solver with a decomposition of a nonlinear operator under classical Lipschitz conditions for the first order Fréchet derivative and divided differences. We have weakened the sufficient convergence criteria, and obtained tighter error estimates. We give numerical experiments that confirm theoretical results. The same technique without additional conditions can be used to extend the applicability of other iterative solvers using inverses of linear operators. The novelty of the paper is that the improved results are obtained using parameters which are special cases of the ones in earlier works. Therefore, no additional information is needed to establish these advantages.

Keywords: Newton-Secant solver; semilocal convergence analysis; Fréchet derivative; divided differences; decomposition of nonlinear operator

AMS Classification: 45B05, 47J05, 65J15, 65J22

1 Introduction

One of the important problems in Computational Mathematics including Mathematical Biology, Chemistry, Economic, Physics, Engineering and other disciplines is finding solutions of nonlinear equations and systems of nonlinear equations [1-14]. For most of these problems, to find the exact solution is difficult or impossible. Therefore, the development and research of numerical methods for solving nonlinear problems is an urgent task.

A popular solver for dealing with nonlinear equations is Newton's [2, 3, 4]. But it is not applicable, if functions are nondifferentiable. In this case, we can apply solvers with divided differences [1, 2, 3, 7, 8, 10, 11]. If it is possible to decompose into differentiable and nondifferentiable parts, it is advisable to use combined methods [2, 3, 5, 6, 12, 13, 14].

Consider a nonlinear equation

$$F(x) + G(x) = 0, \tag{1}$$

where the operators F and G are defined on a open convex set D of a Banach space E_1 with values in a Banach space E_2 , F is a Fréchet differentiable operator, G is a continuous operator for which differentiability is not assumed. It is necessary to find an approximate solution $x_* \in D$ that satisfies equation (1).

In this paper, we consider the Newton-Secant solver

$$x_{n+1} = x_n - [F'(x_n) + G(x_{n-1}, x_n)]^{-1}(F(x_n) + G(x_n)), \quad n = 0, 1, ... \tag{2}$$

This iterative process was proposed in [6] and studied in [2, 3, 13], and the convergence order $\frac{1+\sqrt{5}}{2}$ was established. It is shown that (2) converges faster than the Secant solver.

In this paper, we study solver (2) under the classical Lipschitz conditions for first-order Fréchet derivative and divided differences. Our technique allows to get the weaker convergence criteria, and tighter error estimates. This way, we extended the applicability of the results obtained in [13].

2 Convergence Analysis

Let $L(E_1, E_2)$ be a space of linear bounded operators from E_1 into E_2. Set $S(x, \tau) = \{y \in E_1 : \|y - x\| < \tau\}$ and let $\bar{S}(x, \tau)$ denote its closure.

Define quadratic polynomial φ by

$$\varphi(t) = \alpha_1 t^2 + \alpha_2 t + \alpha_3$$

and parameters r, and r_1 by

$$r = \frac{1 - (q_0 + \bar{q}_0)a}{p_0 + q_0 + 2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0},$$

$$r_1 = \frac{1 - \bar{q}_0 a}{2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0},$$

where

$$\alpha_1 = p_0 + q_0 + 2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0,$$

$$\alpha_2 = -[1 - (q_0 + \bar{q}_0)a + (2\bar{p}_0 + \bar{q}_0 + \bar{\bar{q}}_0)c]$$

and

$$\alpha_3 = (1 - \bar{q}_0 a)c,$$

where p_0, $\bar{p}_0$, q_0, $\bar{q}_0$, $\bar{\bar{q}}_0$, a and c are nonnegative numbers.

Suppose that $(q_0+\bar{q}_0)a<1$ and $\varphi\big(\frac{1}{2}r\big)\le 0$. Then, it is simple algebra to show, function φ has a unique root $\bar{r}_0\in(0,\frac{r}{2}]$, and

$$r\le r_1,$$

$$\bar{\gamma}=\frac{p_0\bar{r}_0+q_0(\bar{r}_0+a)}{1-\bar{q}_0a-(2\bar{p}_0+\bar{q}_0+\bar{\bar{q}}_0)\bar{r}_0}\in[0,1)$$

and

$$\bar{r}_0\ge\frac{c}{1-\bar{\gamma}}.$$

Set $D_0=D\cap S(x_0,r_1)$.

Definition 2.1. *We call an operator that acts from E_1 into E_2 and is denoted by $G(x,y)$ a first-order divided difference for the operator G by fixed points x and y $(x\neq y)$, if the equality*

$$G(x,y)(x-y)=G(x)-G(y)$$

is satisfied.

Theorem 2.2. *Suppose that:*

1) *F and G are nonlinear operators on an open convex set D of a Banach space E_1 into a Banach space E_2;*

2) *F is a Fréchet-differentiable operator, and let G is a continuous operator;*

3) *$G(\cdot,\cdot)$ is the first-order divided differences of the operator G defined on the set D;*

4) *the linear operator $A_0=F'(x_0)+G(x_{-1},x_0)$, where x_{-1}, x_0 $\in D$, is invertible;*

5) *the following conditions are satisfied for all $x,y,\in D$*

$$\|A_0^{-1}(F'(x_0)-F'(x))\|\le 2\bar{p}_0\|x_0-x\|, \tag{3}$$

$$\|A_0^{-1}(G(x_{-1},x_0)-G(x,x_0))\|\le\bar{q}_0\|x_{-1}-x\|, \tag{4}$$

$$\|A_0^{-1}(G(x,x_0)-G(x,y))\|\le\bar{\bar{q}}_0\|x_0-y\|, \tag{5}$$

and for all $x,y,u\in D_0$

$$\|A_0^{-1}(F'(x)-F'(y))\|\le 2p_0\|x-y\|, \tag{6}$$

$$\|A_0^{-1}(G(x,y)-G(u,y))\|\le q_0\|x-u\|; \tag{7}$$

6) *a, c are nonnegative numbers such that*

$$\|x_0-x_{-1}\|\le a,\ \|A_0^{-1}(F(x_0)+G(x_0))\|\le c,\ c>a, \tag{8}$$

$$(q_0+\bar{q}_0)a<1,\quad \varphi\Big(\frac{1}{2}r\Big)\le 0; \tag{9}$$

7) $\bar{S}(x_0, \bar{r}_0) \subset D$.

Then, the solver (2) is well-defined and the sequence generated by it converges to the solution x_* *of equation (1), so that for each* $n \in \{-1, 0, 1, 2, ...\}$, *the following inequalities are satisfied*

$$\|x_n - x_{n+1}\| \le t_n - t_{n+1}, \tag{10}$$

$$\|x_n - x_*\| \le t_n - \bar{t}_*, \tag{11}$$

where sequence $\{t_n\}_{n\ge -1}$ *defined by the formulas*

$$t_{-1} = \bar{r}_0 + a,\ t_0 = \bar{r}_0,\ t_1 = \bar{r}_0 - c,$$

$$t_{n+1} - t_{n+2} = \bar{\gamma}_n(t_n - t_{n+1}),\ n \ge 0,$$

$$\bar{\gamma}_n = \frac{\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})},\ 0 \le \bar{\gamma}_n < \bar{\gamma} \tag{12}$$

is decreasing, nonnegative, and converges to $\bar{t}_*$, *so that* $\bar{r}_0 - c/(1-\bar{\gamma}) \le \bar{t}_* < t_0$, *where*

$$\tilde{p}_0 = \begin{cases} \bar{p}_0,\ n = 0 \\ p_0,\ n > 0 \end{cases},\quad \tilde{q}_0 = \begin{cases} \bar{q}_0,\ n = 0 \\ q_0,\ n > 0. \end{cases}$$

Proof. We use mathematical induction to show that, for each $k \ge 0$ the following inequalities are satisfied

$$t_{k+1} \ge t_{k+2} \ge \bar{r}_0 - \frac{1 - \bar{\gamma}^{k+2}}{1 - \bar{\gamma}} c \ge \bar{r}_0 - \frac{c}{1 - \bar{\gamma}} \ge 0, \tag{13}$$

$$t_{k+1} - t_{k+2} \le \bar{\gamma}(t_k - t_{k+1}). \tag{14}$$

Setting $k = 0$ in (12), we get

$$t_1 - t_2 = \frac{\tilde{p}_0(t_0 - t_1) + \tilde{q}_0(t_{-1} - t_1)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_1) - \bar{\bar{q}}_0(t_0 - t_1)}(t_0 - t_1) \le \bar{\gamma}(t_0 - t_1),$$

$$t_0 \ge t_1,\ t_1 \ge t_2 \ge t_1 - \bar{\gamma}(t_0 - t_1) \ge \bar{r}_0 - (1+\bar{\gamma})c = \bar{r}_0 - \frac{(1-\bar{\gamma}^2)c}{1-\bar{\gamma}} \ge \bar{r}_0 - \frac{c}{1-\bar{\gamma}} \ge 0.$$

Suppose that (13) and (14) are true for $k = 0, 1, ..., n-1$. Then, for $k = n$, we obtain

$$\begin{aligned} t_{n+1} - t_{n+2} &= \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} \\ &\le \frac{\tilde{p}_0 t_n + \tilde{q}_0 t_{n-1}}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}(t_n - t_{n+1}) \le \bar{\gamma}(t_n - t_{n+1}), \end{aligned}$$

$$t_{n+1} \ge t_{n+2} \ge t_{n+1} - \bar{\gamma}(t_n - t_{n+1}) \ge \bar{r}_0 - \frac{1 - \bar{\gamma}^{n+2}}{1 - \bar{\gamma}} c \ge \bar{r}_0 - \frac{c}{1 - \bar{\gamma}} \ge 0.$$

Thus, $\{t_n\}_{n\geq 0}$ is a decreasing nonnegative sequence, and converges to $\bar{t}_* \geq 0$.

Let us prove that the method (2) is well-defined, and for each $n \geq 0$ the inequality (10) is satisfied.

Since $t_{-1} - t_0 = a$, $t_0 - t_1 = c$ and conditions (8) are fulfilled then $x_1 \in S(x_0, \bar{r}_0)$ and (10) is satisfied for $n \in \{-1, 0\}$. Let conditions (8) be satisfied for $k = 0, 1, ..., n$. Let us prove that the method (2) is well-defined for $k = n + 1$.

Denote $A_n = F'(x_n) + G(x_{n-1}, x_n)$. Using the Lipschitz conditions (3) – (5), we have

$$\begin{gathered}
\|I - A_0^{-1}A_{n+1}\| = \|A_0^{-1}(A_0 - A_{n+1})\| \leq \|A_0^{-1}(F'(x_0) - F'(x_{n+1}))\| \\
+\|A_0^{-1}(G(x_{-1}, x_0) - G(x_n, x_0) + G(x_n, x_0) - G(x_n, x_{n+1}))\| \\
\leq 2\bar{p}_0\|x_0 - x_{n+1}\| + \bar{q}_0(\|x_{-1} - x_0\| + \|x_0 - x_n\|) + \bar{\bar{q}}_0\|x_0 - x_{n+1}\| \\
\leq 2\bar{p}_0\|x_0 - x_{n+1}\| + \bar{q}_0 a + \bar{q}_0\|x_0 - x_n\| + \bar{\bar{q}}_0\|x_0 - x_{n+1}\| \\
\leq \bar{q}_0 a + 2\bar{p}_0(t_0 - t_{n+1}) + \bar{q}_0(t_0 - t_n) + \bar{\bar{q}}_0(t_0 - t_{n+1}) \\
\leq \bar{q}_0 a + 2\bar{p}_0\bar{r}_0 + \bar{q}_0\bar{r}_0 + \bar{\bar{q}}_0\bar{r}_0 < 1.
\end{gathered}$$

According to the Banach lemma on inverse operators [2] A_{n+1} is invertible, and

$$\|A_{n+1}^{-1}A_0\| \leq (1 - \bar{q}_0 a - 2\bar{p}_0\|x_0 - x_{n+1}\| - \bar{q}_0\|x_0 - x_n\| + \bar{\bar{q}}_0\|x_0 - x_{n+1}\|)^{-1}.$$

By the definition of the divided difference and conditions (6), (7), we obtain

$$\begin{gathered}
\|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\
= \|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}) - F(x_n) - G(x_n) - A_n(x_n - x_{n+1}))\| \\
\leq \|A_0^{-1}(\textstyle\int_0^1\{F'(x_{n+1} + t(x_n - x_{n+1})) - F'(x_n)\}dt)\|\|x_n - x_{n+1}\| \\
+\|A_0^{-1}(G(x_{n+1}, x_n) - G(x_{n-1}, x_n))\|\|x_n - x_{n+1}\| \\
\leq (\tilde{p}_0\|x_n - x_{n+1}\| + \tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|))\|x_n - x_{n+1}\|.
\end{gathered}$$

In view of condition (10), we have

$$\begin{gathered}
\|x_{n+1} - x_{n+2}\| = \|A_{n+1}^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\
\leq \|A_{n+1}^{-1}A_0\|\|A_0^{-1}(F(x_{n+1}) + G(x_{n+1}))\| \\
\leq \frac{\tilde{p}_0\|x_n - x_{n+1}\| + \tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|)}{1 - \bar{q}_0 a - 2\bar{p}_0\|x_0 - x_{n+1}\| - \bar{q}_0\|x_0 - x_{n+1}\| + \bar{\bar{q}}_0\|x_0 - x_n\|}\|x_n - x_{n+1}\| \\
\leq \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_{n-1} - t_{n+1})\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} = t_{n+1} - t_{n+2}.
\end{gathered}$$

Thus, the method (2) is well-defined for each $n \geq 0$. Hence it follows that

$$\|x_n - x_k\| \leq t_n - t_k, \quad -1 \leq n \leq k. \tag{15}$$

Therefore, the sequence $\{x_n\}_{n\geq 0}$ is fundamental, so it converges to some $x_* \in \bar{S}(x_0, \bar{r}_0)$. Inequality (11) is obtained from (15) for $k \to \infty$. Let us show that x_* solves the equation $F(x) + G(x) = 0$. Indeed, we get in turn that

$$A_0^{-1}(F(x_{n+1}) + G(x_{n+1})) \leq \big(\tilde{p}_0\|x_n - x_{n+1}\|$$
$$+\tilde{q}_0(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\|)\big)\|x_n - x_{n+1}\| \to 0, \;\; n \to \infty.$$

Hence, $F(x_*) + G(x_*) = 0$. □

Remark 2.3. *The order of convergence of method (2) is equal to* $\dfrac{1+\sqrt{5}}{2}$.

Proof. In view of $t_n - t_{n+1} \leq \bar{\gamma}(t_{n-1} - t_n)$, and (12), we obtain

$$\begin{aligned} t_{n+1} - t_{n+2} &= \frac{\big(\tilde{p}_0(t_n - t_{n+1}) + \tilde{q}_0(t_n - t_{n+1} + t_{n-1} - t_n)\big)(t_n - t_{n+1})}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} \\ &\leq \frac{\tilde{p}_0\bar{\gamma}(t_{n-1} - t_n) + \tilde{q}_0(1+\bar{\gamma})(t_{n-1} - t_n)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})}(t_n - t_{n+1}) \\ &= \frac{\big(\bar{p}_0\bar{\gamma} + \bar{q}_0(1+\bar{\gamma})\big)(t_n - t_{n+1})(t_{n-1} - t_n)}{1 - \bar{q}_0 a - 2\bar{p}_0(t_0 - t_{n+1}) - \bar{q}_0(t_0 - t_n) - \bar{\bar{q}}_0(t_0 - t_{n+1})} \\ &\leq \frac{\tilde{p}_0\bar{\gamma} + \tilde{q}_0(1+\bar{\gamma})}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}(t_n - t_{n+1})(t_{n-1} - t_n). \end{aligned}$$

Denote $\bar{C} = \dfrac{\bar{p}_0\bar{\gamma} + \bar{q}_0(1+\bar{\gamma})}{1 - \bar{q}_0 a - 2\bar{p}_0 t_0 - \bar{q}_0 t_0 - \bar{\bar{q}}_0 t_0}$. Clearly,

$$t_{n+1} - t_{n+2} \leq \bar{C}(t_{n-1} - \bar{t}_*)(t_n - \bar{t}_*). \tag{16}$$

Since, for each $k > 2$, the estimate is satisfied

$$t_{n+k-1} - t_{n+k} \leq \bar{\gamma}^{k-2}(t_{n+1} - t_{n+2}),$$

we get

$$\begin{aligned} t_{n+1} - t_{n+k} &= t_{n+1} - t_{n+2} + t_{n+2} - t_{n+3} + \ldots + t_{n+k-1} - t_{n+k} \\ &\leq (1 + \bar{\gamma} + \ldots + \bar{\gamma}^{k-2})(t_{n+1} - t_{n+2}) \\ &= \frac{1 - \bar{\gamma}^{k-1}}{1 - \bar{\gamma}}(t_{n+1} - t_{n+2}) \leq \frac{1}{1 - \bar{\gamma}}(t_{n+1} - t_{n+2}). \end{aligned}$$

In view of (16), for $k \to \infty$, we have

$$t_{n+1} - \bar{t}_* \leq \frac{\bar{C}}{1 - \bar{\gamma}}(t_{n-1} - \bar{t}_*)(t_n - \bar{t}_*)$$

Hence, it follows that the order of convergence of the sequence $\{t_n\}_{n\geq 0}$ is equal to $\dfrac{1+\sqrt{5}}{2}$, and, according (11), the sequence $\{x_n\}_{n\geq 0}$ converges with the same order. □

Remark 2.4. *(a) The following conditions were used for each* $x, y, u, v \in D$ *in [13]*

$$\|A_0^{-1}(F'(y) - F'(x))\| \leq 2P_0\|y - x\|, \tag{17}$$

$$\|A_0^{-1}(G(x,y) - G(u,v))\| \leq Q_0(\|x - u\| + \|y - v\|), \tag{18}$$

$$\begin{aligned} r_0 &\geq \frac{c}{1-\gamma},\ Q_0 a + 2P_0 r_0 + 2Q_0 r_0 < 1, \\ \gamma &= \frac{P_0 r_0 + Q_0(r_0 + a)}{1 - Q_0 a - 2P_0 r_0 - 2Q_0 r_0},\ 0 \leq \gamma < 1. \end{aligned} \tag{19}$$

But, then we have

$$\begin{aligned} \bar{p}_0 &\leq P_0, \\ \bar{q}_0 &\leq Q_0, \\ \bar{\bar{q}}_0 &\leq Q_0, \end{aligned}$$

since $D_0 \subseteq D$, *(3) and (4), (5), (7) are weaker than (17) and (18) respectively for* $\bar{r}_0 \leq r_0$. *Notice that sufficient convergence criteria (9) imply (19) but not necessarily vice versa, unless if* $\bar{p}_0 = P_0$, $\bar{q}_0 = \bar{\bar{q}}_0 = Q_0$ *and* $\bar{r}_0 = r_0$.

A simple inductive argument shows that

$$\bar{\gamma}_n \leq \gamma_n, \tag{20}$$

$$t_n - t_{n+1} \leq s_n - s_{n+1}, \tag{21}$$

where

$$s_{-1} = r_0 + a,\ s_0 = r_0,\ s_1 = r_0 - c,$$

$$s_{n+1} - s_{n+2} = \gamma_n(s_n - s_{n+1}),\ n \geq 0,$$

$$\gamma_n = \frac{P_0(s_n - s_{n+1}) + Q_0(s_{n-1} - s_{n+1})}{1 - Q_0 a - 2P_0(s_0 - s_{n+1}) - Q_0(2s_0 - s_n - s_{n+1})},\ 0 \leq \gamma_n \leq \gamma.$$

Notice that the corresponding quadratic polynomial φ_1 *to* φ *is defined similarly by*

$$\varphi_1(t) = b_1 t^2 + b_2 t + b_3$$

where

$$b_1 = 3P_0 + 3Q_0,$$

$$b_2 = -[1 - 2Q_0 a + (2P_0 + 2Q_0)c]$$

and

$$b_3 = (1 - Q_0 a)c.$$

We have by these definitions that

$$\alpha_1 < b_1,\ \alpha_2 < b_2,\ but\ \alpha_3 > b_3.$$

Therefore, we cannot tell, if $r_0 < \bar{r}_0$ *or* $\bar{r}_0 < r_0$ *or* $r_0 = \bar{r}_0$*. But, we have*

$$\begin{aligned} \gamma \le \bar{\gamma} &\Rightarrow r_0 \le \bar{r}_0, \\ s_n &\le t_n, \\ s_* \le \bar{t}_* &= \lim_{n\to\infty} t_n \end{aligned} \tag{22}$$

and

$$\begin{aligned} \bar{\gamma} \le \gamma \Rightarrow \bar{r}_0 \le r_0 &\Rightarrow \bar{C} \le C, \\ t_n &\le s_n, \\ \bar{t}_* \le s_* &= \lim_{n\to\infty} s_n, \end{aligned} \tag{23}$$

It is simple algebra to show that $\varphi(r) \ge 0$*, and for* $r_{min} = -\frac{\alpha_2}{2\alpha_1}$ *(solving* $\varphi'(t) = 0$*),* $r_{min} \ge \frac{r}{2}$*,* $r_{min} \le \frac{r_1}{2}$*. Hence, one may replace the second inequation in (9) by* $\varphi(\lambda r) \le 0$ *for some* $\lambda \in (0, \frac{1}{2}]$ *to obtain a better information about the location of* $\bar{r}_0$*, if* $\lambda \ne \frac{1}{2}$*, especially in the case when we do not actually need to compute* $\bar{r}_0$*.*

(b) The Lipschitz parameters $\bar{p}_0$*,* $\bar{q}_0$*,* $\bar{\bar{q}}_0$ *can become even smaller, if we define the set* $D_1 = D \cap S(x_1, r_1 - c)$ *for* $r_1 > c$ *to replace* D_0 *in Theorem 2.2., since* $D_1 \subseteq D_0$*.*

3 Numerical experiments

Let us define function $F + G : R \to R$, where

$$F(x) = e^{x-0.5} + x^3 - 1.3,\ G(x) = 0.2x|x^2 - 2|.$$

The exact solution of $F(x) + G(x) = 0$ is $x_* = 0.5$. Let $D = (0, 1)$. Then

$$F'(x) = e^{x-0.5} + 3x^2,$$

$$G(x, y) = \frac{0.2x(2 - x^2) - 0.2y(2 - y^2)}{x - y} = 0.2(1 - x^2 - xy - y^2).$$

$$A_0 = e^{x_0-0.5} + 3x_0^2 + 0.2(1 - x_{-1}^2 - x_{-1}x_0 - x_0^2),$$

$$|A_0^{-1}(F'(x) - F'(y))| \le \frac{e^{0.5} + 3|x + y|}{|A_0|}|x - y|,$$

$$|A_0^{-1}(G(x, y) - G(u, v))| = \frac{0.2}{|A_0|}|(u + x + y)(u - x) + (v + y + u)(v - y)|.$$

Let $x_0 = 0.57$, $x_{-1} = 0.571$. Then, we have $a = 0.001$, $c \approx 0.0660157$, $\bar{p}_0 \approx 1.4118406$, $\bar{q}_0 \approx 0.1901483$, $\bar{\bar{q}}_0 \approx 0.2282491$, $r_1 \approx 0.3083854$,

$$D_0 = D \cap S(x_0, r_1) = (0.2616146, 0.8783854),$$

$p_0 \approx 1.5362481$, $q_0 \approx 0.2340358$, $P_0 \approx 1.6982621$, $Q_0 \approx 0.2664386$, and $r \approx 0.1994221$, $\varphi(\frac{1}{2}r) \approx -0.0051722 < 0$. So, $\bar{p}_0 < P_0$, $\bar{q}_0 < Q_0$, $\bar{\bar{q}}_0 < Q_0$.

By solving inequalities $\varphi(t) \leq 0$ and $\varphi_1(t) \leq 0$, we get

$$t \in [0.0824903,\ 0.1596319] \Rightarrow \bar{r}_0^{(1)} \approx 0.0824903,\ \bar{r}_0^{(2)} \approx 0.1596319,$$

$$t \in [0.0924062,\ 0.1211750] \Rightarrow r_0^{(1)} \approx 0.0924062,\ r_0^{(2)} \approx 0.1211750.$$

Then $\bar{r}_0 = \bar{r}_0^{(1)} \approx 0.0824903$, $r_0 = r_0^{(1)} \approx 0.0924062$, and

$$S(x_0, \bar{r}_0) = (0.4875097,\ 0.6524903),\ \bar{\gamma} \approx 0.1997151 < 1,\ \bar{C} \approx 0.8023108,$$

$$S(x_0, r_0) = (0.4775938,\ 0.6624062),\ \gamma \approx 0.2855916 < 1,\ C \approx 1.2998717.$$

In Table 1, there are results that confirm estimates (10), (11) and (21). Table 2 shows that sequences $\{t_n\}$ and $\{s_n\}$ converge to $\bar{t}_* \approx 0.0073550$ and $s_* \approx 0.0144209$, respectively, and confirms (20) and (23).

Table 1: Obtained results for $\varepsilon = 10^{-7}$

n	$\lvert x_{n-1} - x_n\rvert$	$t_{n-1} - t_n$	$s_{n-1} - s_n$	$\lvert x_n - x_*\rvert$	$t_n - \bar{t}_*$	$s_n - s_*$
1	0.0660157	0.0660157	0.0660157	0.0039843	0.0091195	0.0119695
2	0.0040123	0.0087609	0.0113203	0.0000281	0.0003586	0.0006492
3	0.0000281	0.0003573	0.0006452	1.761e-08	0.0000013	0.0000040
4	1.761e-08	0.0000040	0.0000040	7.438e-14	1.440e-10	1.033e-09

Table 2: Obtained results for $\varepsilon = 10^{-7}$

n	t_n	s_n	$\bar{\gamma}_{n-2}$	γ_{n-2}
-1	0.0834903	0.0934062		
0	0.0824903	0.0924062		
1	0.0164746	0.0263904		
2	0.0077136	0.0150701	0.1327096	0.1714793
3	0.0077136	0.0144249	0.0407873	0.0569927
4	0.0073550	0.0144209	0.0035475	0.0061771
5	0.0073550	0.0144209	0.0001136	0.0002592

4 Conclusions

We investigated the semilocal convergence of Newton-Secant solver under classical center and restricted Lipschitz conditions. This technique weakens the sufficient convergence criteria without adding more conditions and uses constants that are specializations of earlier ones. Moreover, tighter estimate errors are obtained. The theoretical results are confirmed by numerical experiments. Our technique can be used to extend the applicability of other iterative methods using inverses of linear operators [1-14] along the same lines.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Global behavior of a nonlinear higher-order rational difference equation

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Abstract

In this paper, we investigate the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \sum_{i=1}^{k} x_{n-2m_i}^{p-1} \prod_{j=1}^{k} x_{n-2m_j}}, \quad n = 0, 1, 2, ...$$

with positive parameters and non-negative initial conditions.

Keywords: Recursive sequences; Global asymptotic stability; Oscillation; Period two solutions; Semicycles.

Mathematics Subject Classification: 39A10.

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1. INTRODUCTION

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. More importantly, difference equations also appear in the study of discretization methods for differential equations. Several results in the theory of difference equations have been obtained as more or less natural discrete analogues of corresponding results of differential equations.

The study of these equations is quite challenging and rewarding and is still in its infancy.

We believe that the nonlinear rational difference equations are of paramount importance in their own right, and furthermore, that results about such equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations.

Recently there has been a lot of interest in studying the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations.

El-Owaidy et al [1] investigated the global asymptotic behavior and the periodic character of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}, \qquad n = 0, 1, 2, ...$$

where the parameters $\alpha,\ \beta,\ \gamma$ and p are non-negative real numbers.

Other related results on rational difference equations can be found in refs. [2-15].

In this paper, we investigate the global asymptotic behavior and the periodic character of the solutions of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma \sum\limits_{i=1}^{k} x_{n-2m_i}^{p-1} \prod\limits_{j=1}^{k} x_{n-2m_j}}, \qquad n = 0, 1, 2, ... \tag{1.1}$$

where the parameters $\alpha,\ \beta,\ \gamma$ and p are positive real numbers, $k \in \{1, 2, ...\}$, $\{m_i\}_{i=1}^{k}$ be positive integers such that $m_i > m_{i-1}$; $i = 2, ...k$ and the initial conditions $x_{-2m_k}, x_{-2m_k+1}, ..., x_0$ are non-negative real numbers.

The results in this work are consistent with the results in [1] when $k = 1$ and $m_1 = 1$.

The results in this work are consistent with the results in [3] when $k = 2$, $m_1 = 1$ and $m_2 = 2$.

We need the following definitions.

Definition 1. *Let I be an interval of real numbers and let*

$$f : I^{k+1} \to I$$

be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.2}$$

with $x_{-k}, x_{-k+1}, ..., x_0 \in I$. Let $\overline{x}$ be the equilibrium point of Eq.(1.2). The linearized equation of Eq.(1.2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + ... + c_{k+1} y_{n-k} \tag{1.3}$$

where
$c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, \overline{x}, ..., \overline{x})$, $c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, \overline{x}, ..., \overline{x}), ..., c_{k+1} = \frac{\partial f}{\partial x_{n-k}}(\overline{x}, \overline{x}, ..., \overline{x})$.
The characteristic equation of Eq.(1.3) is

$$\lambda^{k+1} - \sum_{i=1}^{k+1} c_i \lambda^{k-i+1} = 0. \tag{1.4}$$

(i) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta,$$

we have

$$|x_n - \overline{x}| < \epsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable if $\overline{x}$ is locally stable and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I$ with

$$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma,$$

we have

$$\lim_{n \to \infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ of Eq.(1.2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_{-1}$, $x_0 \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iv) The equilibrium point $\overline{x}$ of Eq.(1.2) is globally asymptotically stable if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of Eq.(1.2).
(v) The equilibrium point $\overline{x}$ of Eq.(1.2) is unstable if $\overline{x}$ is not locally stable.

Definition 2. *A positive semicycle of $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.2) consists of a 'string' of terms $\{x_l, x_{l+1}, ..., x_m\}$, all greater than or equal to $\overline{x}$, with $l \geq -k$ and $m < \infty$ and such that either $l = -k$ or $l > -k$ and $x_{l-1} < \overline{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} < \overline{x}$.*

A negative semicycle of $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.2) consists of a 'string' of terms $\{x_l, x_{l+1}, ..., x_m\}$, all less than $\overline{x}$, with $l \geq -k$ and $m < \infty$ and such that either $l = -k$ or $l > -k$ and $x_{l-1} \geq \overline{x}$ and either $m = \infty$ or $m < \infty$ and $x_{m+1} \geq \overline{x}$.

Definition 3. *A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(1.2) is called nonoscillatory if there exists $N \geq -k$ such that either*

$$x_n \geq \overline{x} \ \ \forall n \geq N \quad \text{or} \quad x_n < \overline{x} \ \ \forall n \geq N \ ,$$

and it is called oscillatory if it is not nonoscillatory.

(a) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if

$$x_{n+p} = x_n \ \text{ for all } n \geq -k. \tag{1.5}$$

(b) A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if it is periodic with period p and p is the least positive integer for which (1.5) holds.

We need the following theorem.

Theorem 1.1. *(i) If all roots of Eq.(1.4) have absolute value less than one, then the equilibrium point $\overline{x}$ of Eq.(1.2) is locally asymptotically stable.*

(ii) If at least one of the roots of Eq.(1.4) has absolute value greater than one, then $\overline{x}$ is unstable.

The equilibrium point $\overline{x}$ of Eq.(1.2) is called a saddle point if Eq.(1.4) has roots both inside and outside the unit disk.

2. Main results

In this section, we investigate the dynamics of Eq.(1.1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = \left(\frac{\beta}{\gamma}\right)^{\frac{1}{p+k-1}} y_n$ reduces Eq.(1.1) to the difference equation

$$y_{n+1} = \frac{ry_{n-1}}{1+\sum\limits_{i=1}^{k} y_{n-2m_i}^{p-1} \prod\limits_{j=1}^{k} y_{n-2m_j}}, \qquad n = 0,1,2,... \tag{2.1}$$

where $r = \frac{\alpha}{\beta} > 0$.

Note that $\overline{y_1} = 0$ is always an equilibrium point of Eq.(2.1). When $r > 1$, Eq.(2.1) also possesses the unique positive equilibrium $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$.

Theorem 2.1. *The following statements are true*

(i) If $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is locally asymptotically stable.

(ii) If $r > 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is a saddle point.

(iii) When $r > 1$, then the positive equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ of Eq.(2.1) is unstable.

Proof: The linearized equation of Eq.(2.1) about the equilibrium point $\overline{y_1} = 0$ is

$$z_{n+1} = rz_{n-1}, \qquad n = 0,1,2,...$$

so, the characteristic equation of Eq.(2.1) about the equilibrium point $\overline{y_1} = 0$ is

$$\lambda^{2m_k+1} - r\lambda^{2m_k-1} = 0,$$

and hence, the proof of (i) and (ii) follows from Theorem A.

For (iii), we assume that $r > 1$, then the linearized equation of Eq.(2.1) about the equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ has the form

$$z_{n+1} = z_{n-1} - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_1} - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_2} - ... - \frac{(r-1)(p+k-1)}{rk} z_{n-2m_k}, \quad n = 0,1,2,...$$

so, the characteristic equation of Eq.(2.1) about the equilibrium point $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ is

$$f(\lambda) = \lambda^{2m_k+1} - \lambda^{2m_k-1} + \frac{(r-1)(p+k-1)}{rk} \sum_{i=1}^{k} \lambda^{2m_k-2m_i} = 0,$$

It is clear that $f(\lambda)$ has a root in the interval $(-\infty, -1)$, and so, $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ is an unstable equilibrium point.

This completes the proof.

Theorem 2.2. *Assume that $r < 1$, then the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is globally asymptotically stable.*

Proof: We know by Theorem 2.1 that the equilibrium point $\overline{y_1} = 0$ of Eq.(2.1) is locally asymptotically stable. So, let $\{y_n\}_{n=-2m_k}^{\infty}$ be a solution of Eq.(2.1). It suffices to show that $\lim_{n\to\infty} y_n = 0$. Since

$$0 \le y_{n+1} = \frac{r y_{n-1}}{1 + \sum\limits_{i=1}^{k} y_{n-2m_i}^{p-1} \prod\limits_{j=1}^{k} y_{n-2m_j}} \le r y_{n-1} < y_{n-1}.$$

So, the even terms of this solution decrease to a limit (say $L_1 \ge 0$), and the odd terms decrease to a limit (say $L_2 \ge 0$), which implies that

$$L_1 = \frac{rL_1}{1 + kL_2^{k+p-1}} \qquad \text{and} \qquad L_2 = \frac{rL_2}{1 + kL_1^{k+p-1}}.$$

If $L_1 \ne 0 \Rightarrow L_2^{k+p-1} = \frac{r-1}{k} < 0$, which is a contradiction, so $L_1 = 0$, which implies that $L_2 = 0$.

So, $\lim_{n\to\infty} y_n = 0$, which the proof is complete.

Theorem 2.3. *Assume that $r = 1$, then Eq.(2.1) possesses the prime period two solution*

$$..., \phi, 0, \phi, 0, ... \tag{2.2}$$

with $\phi > 0$. Furthermore, every solution of Eq.(2.1) converges to a period two solution (2.2) with $\phi \ge 0$.

Proof: Let

$$..., \phi, \psi, \phi, \psi, ...$$

be period two solutions of Eq.(2.1). Then

$$\phi = \frac{r\phi}{1 + k\psi^{k+p-1}}, \quad \text{and} \quad \psi = \frac{r\psi}{1 + k\phi^{k+p-1}},$$

so,

$$k\phi\psi = \frac{(r-1)(\phi - \psi)}{\psi^{k+p-2} - \phi^{k+p-2}} \geq 0,$$

If $k + p > 2$, then we have $r - 1 \leq 0$.

If $r < 1$, then this implies that $\phi < 0$ or $\psi < 0$, which is impossible, so $r = 1$.

If $k + p < 2$, then we have $r - 1 \geq 0$.

If $r > 1$, then we have either $\phi = \psi = 0$, which is impossible or $\phi = \psi = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$, which is impossible, so $r = 1$.

If $k + p = 2$, then we have$(r-1)(\phi - \psi) = 0$, which implies that $r = 1$.

To complete the proof, assume that $r = 1$ and let $\{y_n\}_{n=-2k}^{\infty}$ be a solution of Eq.(2.1), then

$$y_{n+1} - y_{n-1} = \frac{-y_{n-1}\left(\sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}\right)}{1 + \sum_{i=1}^{k} y_{n-2m_i}^{p-1} \prod_{j=1}^{k} y_{n-2m_j}} \leq 0, \qquad n = 0, 1, 2, ...$$

So, the even terms of this solution decrease to a limit (say $\Phi \geq 0$), and the odd terms decrease to a limit (say $\Psi \geq 0$). Thus,

$$\Phi = \frac{\Phi}{1 + k\Psi^{k+p-1}} \text{ and } \Psi = \frac{\Psi}{1 + k\Phi^{k+p-1}},$$

which implies that $k\Phi\Psi^{k+p-1} = 0$ and $k\Phi^{k+p-1}\Psi = 0$. Then the proof is complete.

Theorem 2.4. *Assume that* $r > 1$, *and let* $\{y_n\}_{n=-2m_k}^{\infty}$ *be a solution of Eq.(2.1) such that*

$$y_{-2m_k}, y_{-2m_k+2}, ..., y_0 \geq \overline{y_2} \text{ and } y_{-2m_k+1}, y_{-2m_k+3}, ..., y_{-1} < \overline{y_2}, \tag{2.3}$$

or

$$y_{-2m_k}, y_{-2m_k+2}, ..., y_0 < \overline{y_2} \text{ and } y_{-2m_k+1}, y_{-2m_k+3}, ..., y_{-1} \geq \overline{y_2}. \tag{2.4}$$

Then $\{y_n\}_{n=-2m_k}^{\infty}$*oscillates about* $\overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}$ *with a semicycle of length one.*

Proof: Assume that (2.3) holds. (The case where (2.4) holds is similar and will be omitted.) Then,

$$y_1 = \frac{ry_{-1}}{1 + \sum\limits_{i=1}^{k} y_{-2m_i}^{p-1} \prod\limits_{j=1}^{k} y_{-2m_j}} < \frac{r\overline{y_2}}{1 + k\overline{y_2}^{k+p-1}} = \overline{y_2}$$

and

$$y_2 = \frac{ry_0}{1 + \sum\limits_{i=1}^{k} y_{-2m_i+1}^{p-1} \prod\limits_{j=1}^{k} y_{-2m_j+1}} > \frac{r\overline{y_2}}{1 + k\overline{y_2}^{k+p-1}} = \overline{y_2}$$

and then the proof follows by induction.

Theorem 2.5. *Assume that* $r > 1$, *then Eq.(2.1) possesses an unbounded solution.*

Proof: From Theorem 2.4, we can assume without loss of generality that the solution $\{y_n\}_{n=-2k}^{\infty}$of Eq.(2.1) is such that

$$y_{2n-1} < \overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}} \quad \text{and} \quad y_{2n} > \overline{y_2} = \left(\frac{r-1}{k}\right)^{\frac{1}{k+p-1}}, \quad \text{for } n \geq -m_k+1.$$

Then

$$y_{2n+1} = \frac{ry_{2n-1}}{1 + \sum\limits_{i=1}^{k} y_{2n-2m_i}^{p-1} \prod\limits_{j=1}^{k} y_{2n-2m_j}} < \frac{ry_{2n-1}}{1 + k\overline{y_2}^{k+p-1}} = y_{2n-1}$$

and

$$y_{2n+2} = \frac{ry_{2n}}{1 + \sum\limits_{i=1}^{k} y_{2n-2m_i+1}^{p-1} \prod\limits_{j=1}^{k} y_{2n-2m_j+1}} > \frac{ry_{2n}}{1 + k\overline{y_2}^{k+p-1}} = y_{2n}$$

from which it follows that

$$\lim_{n\to\infty} y_{2n} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} y_{2n+1} = 0.$$

Then, the proof is complete.

Acknowledgement:

The author would like to express his gratitude to the anonymous referees of Journal of Computational Analysis and Applications for their interesting remarks.

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Weighted composition operator acting between some classes of analytic function spaces

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Abstract

In this paper, we define some general classes of weighted analytic function spaces in the unit disc. For the new classes, we investigate boundedness and compactness of the weighted composition operator uC_ϕ under some mild conditions on the weighted functions of the classes.

1 Introduction

Let $\mathbb{H}(\mathbb{D})$ denote the class of analytic functions in the unit disk $\mathbb{D}$. As usual, two quantities L_f and M_f, both depending on analytic function f on the unit disk $\mathbb{D}$, are said to be equivalent, and written in the form $L_f \approx M_f$, if there exists a positive constant C such that

$$\frac{1}{C} M_f \leq L_f \leq C\, M_f.$$

The notation $A \lesssim B$ means that there exists a positive constant C_1 such that $A \leq C_1 B$.
For $0 < \alpha < \infty$. The weighted type space H_α^∞ is the space of all $f \in \mathbb{H}(\mathbb{D})$ such that

$$\|f\|_{H_\alpha^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty.$$

and $H_{\alpha,\,0}^\infty$ denotes the closed subspace of H_α^∞ such that $f \in H_\alpha^\infty$ satisfies

$$(1 - |z|^2)^\alpha |f(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

Let the Green's function $g(z, a) = \ln |\frac{1-\bar{a}z}{a-z}| = \ln \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{1-\bar{a}z}{a-z}$ stands for Möbius transformation. The following classes of weighted function spaces are defined in [7]:

Definition 1.1 *Let* $K : [0, \infty) \to [0, \infty)$ *be a nondecreasing function and let* f *be an analytic function in* $\mathbb{D}$ *then* $f \in \mathcal{N}_K$ *if*

$$\|f\|_{\mathcal{N}_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 K(g(z, a)) dA(z) < \infty,$$

AMS 2010 classification: 30H30, 30C45, 46E15.
Key words and phrases: analytic classes, weighted composition operators.

where $dA(z)$ defines the normalized area measure on $\mathbb{D}$, so that $A(\mathbb{D}) \equiv 1$.
Now, if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f(z)|^2 K(g(z,a))dA(z) = 0,$$

then f is said to belong to the class $\mathcal{N}_{K,0}$.

Clearly, if $K(t) = t^p$, then $\mathcal{N}_K = \mathcal{N}_p$ (see [19]) , since $g(z,a) \approx (1 - |\varphi_a(z)|^2)$. For $K(t) = 1$ it gives the Bergman space $\mathcal{A}^2$ (see [17]).
It is easy to check that $\|\cdot\|_{\mathcal{N}_K}$ is a complete semi-norm on $\mathcal{N}_K$ and it is Möbius invariant in the sense that

$$\|f \circ \varphi_a\|_{\mathcal{N}_K} = \|f\|_{\mathcal{N}_K}, \ \ a \in \mathbb{D},$$

whenever $f \in \mathcal{N}_K$ and $\varphi_a \in \mathrm{Aut}(\mathbb{D})$ is the group of all Möbius maps of $\mathbb{D}$. If $\mathcal{N}_K$ consists of just the constant functions, we say that it is trivial.

We assume from now that all $K : [0,\infty) \to [0,\infty)$ to appear in this paper is right-continuous and nondecreasing function such that the integral

$$\int_0^{1/e} K(\log(1/\rho))\rho \, d\rho \ = \int_1^{\infty} K(t)e^{-2t} \, dt < \infty.$$

From a change of variables we see that the coordinate function z belongs to $\mathcal{N}_K$ space if and only if

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} \ K\big(\log(1/|z|)\big) \, d\,A(z) < \infty.$$

Simplifying the above integral in polar coordinates, we conclude that $\mathcal{N}_K$ space is nontrivial if and only if

$$\sup_{t\in(0,1)} \int_0^1 \frac{(1-t)^2}{(1-tr^2)^3} K\big(\log(1/r)\big) r dr < \infty. \tag{1}$$

An important tool in the study of $\mathcal{N}_K$ space is the auxiliary function ϕ_K defined by

$$\phi_K(s) = \sup_{0<t<1} \frac{K(st)}{K(t)}, \ \ 0 < s < \infty.$$

The following condition has played a crucial role in the study of $\mathcal{N}_K$ space:

$$\int_1^{\infty} \phi_K(s) \frac{ds}{s^2} < \infty, \tag{2}$$

and

$$\int_0^1 \phi_K(s) \frac{ds}{s} < \infty. \tag{3}$$

The test function in $\mathcal{N}_K$ can be stated as follows (see [7]):

Lemma 1.1 *For $w \in \mathbb{D}$ we define*

$$h_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^2}.$$

Suppose that condition (1) is satisfied. Then $h_w \in \mathcal{N}_K$ and

$$\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_K} \le 1.$$

2 Analytic $\mathcal{N}_{K,\,\omega}$ and $H^\infty_{\alpha,\,\omega}$-spaces

Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$ be any reasonable and right continuous nondecreasing function and $K : [0,\ \infty) \longrightarrow [0,\ \infty)$ be right continuous nondecreasing function too. Then, we give the following definitions.

Definition 2.1 *Let $0 < \alpha < \infty$. The weighted type space $H^\infty_{\alpha,\,\omega}$ is defined by*

$$H^\infty_{\alpha,\omega} := \{f \in \mathbb{H}(\mathbb{D})\ :\ \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{\omega(1-|z|^2)}|f(z)| < \infty\}.$$

If

$$\lim_{|z|\longrightarrow 1} \frac{(1-|z|^2)^\alpha}{\omega(1-|z|^2)}|f(z)| = 0,$$

we say that f belongs to $H^\infty_{\alpha,\,\omega,0}$.

Definition 2.2 *The analytic $\mathcal{N}_{K,\omega}$ -space is defined by*

$$\mathcal{N}_{K,\omega} := \{f \in \mathbb{H}(\mathbb{D})\ :\ \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)} dA(z) < \infty\}.$$

If

$$\lim_{|a|\longrightarrow 1} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)} dA(z) = 0,$$

we say that f belongs to the class $\mathcal{N}_{K,\omega,0}$.

Clearly, if $K(t) = t^p$ and $\omega \equiv 1$, then $\mathcal{N}_{K,1} = \mathcal{N}_p$.
For $K(t) = 1$ and $\omega \equiv 1$, it gives the Bergman space $\mathcal{A}^2$.

In the study of the space $\mathcal{N}_{K,\omega}$, we assume the following condition holds:

$$\int_0^1 \frac{K(\log\frac{1}{r})}{\omega^2(1-r^2)}\, r\, dr < \infty. \tag{4}$$

Throughout this paper, we always assume that condition (4) is satisfied, so that the $\mathcal{N}_{K,\omega}$ space we study is not trivial.

Remark 2.1 *It should be remarked that the weight function $\omega(1-|z|)$ is used to define and study some general classes of function spaces, see [10, 15, 21] and others.*

For a point $a \in \mathbb{D}$ and $0 < r < 1$, let $D(a,r)$ denote an Euclidean disk with center $\frac{(1-r^2)a}{1-r^2|a|^2}$ and radius $\frac{(1-|a|^2)r}{1-r^2|a|^2}$ (see [20]). Suppose also that $E(a,r) = \{z \in \mathbb{D}\ :\ |z-a| < r(1-|a|)\ \}$.
Now, we will prove the following lemma:

Lemma 2.1 *Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$ be any reasonable and right continuous nondecreasing function and let $K : [0,\infty) \to [0,\infty)$ be right continuous nondecreasing function. Then*

$$\mathcal{N}_{K,\omega} \subset H^\infty_{1,\,\omega}.$$

Proof: Suppose that $f \in \mathcal{N}_{K,\,\omega}$, and let C be a constant such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)}\ dA(z) = C < \infty.$$

By the fact that K is nondecreasing, for all $r, 0 < r < 1$, we have

$$\begin{aligned} C &\geq \int_{\mathbb{D}} |f(z)|^2 \frac{K(g(z,a))}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \int_{D(a,r)} |f(z)|^2 \frac{K\big(\log \frac{1}{|\varphi_a(z)|}\big)}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \int_{E(a,r)} |f(z)|^2 \frac{K\big(\log \frac{1}{|\varphi_a(z)|}\big)}{\omega^2(1-|z|^2)}\, dA(z) \\ &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} \int_{E(a,r)} |f(z)|^2\, dA(z). \end{aligned}$$

Since $|f(z)|^2$ is subharmonic it follows that,

$$\begin{aligned} C &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} \int_{E(a,r)} |f(z)|^2\, dA(z) \\ &\geq \frac{K\big(\log \frac{1}{r}\big)}{\omega^2(1-|a|^2)} 2r^2\pi(1-|a|^2)^2|f(a)|^2. \end{aligned}$$

For $r_0 \in (0,1)$, there exists a constant λ such that

$$\begin{aligned} \frac{|f(a)|^2(1-|a|^2)^2}{\omega^2(1-|a|^2)} &\leq \lambda \int_{D(a,r_0)} |f(z)|^2 dA(z) \\ &\leq \frac{C\,\lambda}{K(\log 1/r_0)}. \end{aligned}$$

Since r_0 is fixed, then

$$\sup_{a\in\mathbb{D}} \frac{|f(a)|(1-|a|^2)}{\omega(1-|a|^2)} \leq \sqrt{\frac{C\,\lambda}{K(\log 1/r_0)}}\ .$$

Thus $f \in H^\infty_{1,\,\omega}$ in $\mathbb{D}$. Hence, $\mathcal{N}_{K,\,\omega} \subset H^\infty_{1,\,\omega}$.

We will prove the following lemmas on $\mathcal{N}_{K,\,\omega}$- spaces:

Lemma 2.2 *Let $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ and $X, Y \in \{H^\infty_{\alpha,\,\omega}, \mathcal{N}_{K,\,\omega}\}$. Suppose that $uC_\phi(X) \subset Y$. Then $uC_\phi : X \to Y$ is compact if and only if for every bounded sequence $\{f_j\} \in X$ which converges to 0 uniformly on compact subset of $\mathbb{D}$, we have*

$$\lim_{j\to\infty} \|uC_\phi f_j\|_Y = 0.$$

Proof: This is an extension of a well-known result on the compactness of the composition operator on the Hardy spaces (see [9], Proposition 3.11). We see that any bounded sequence in $H^\infty_{\alpha,\,\omega}$ forms a normal family. Also by Lemma 2.1 we have the relation

$$\|f\|_{H^\infty_{1,\,\omega}} \leq \|f\|_{\mathcal{N}_{K,\,\omega}}$$

and the growth estimate for $f \in H^\infty_{1,\,\omega}$ imply that any bounded sequence in $\mathcal{N}_{K,\,\omega}$ forms a normal family. Hence a similar argument by using Montel's theorem also proves this lemma, and so we omit its proof.

Now, for $\alpha \in (0,\infty)$, $\theta \in [0,2\pi)$ and $r \in (0,1]$, we put

$$f_{\theta,r}(z) := \sum_{k=0}^{\infty} 2^{\alpha k}(re^{i\theta})^{2^k} z^{2^k} \quad (z \in \mathbb{D}).$$

Using the function $f_{\theta,r}$, we have the following result:

Lemma 2.3 *The function* $f_{\theta,r}(z)$ *belongs to* $H^{\infty}_{\alpha,\,\omega}$ *and* $\|f_{\theta,r}\|_{H^{\infty}_{\alpha,\,\omega}} \lesssim 1$ *which is independent of* θ *and* r. *In particular,* $f_{\theta,r} \in H^{\infty}_{\alpha,\,\omega.\,0}$ *if* $r \in (0,1)$.

Proof: The proof is similar to the corresponding results in [27], with some simple modifications, so it will be omitted.

Lemma 2.4 *Let* $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ *with* $\omega(kt) = k\omega(t)\,,\ k > 0$. *Suppose that condition (4) is satisfied. For all* $z, w \in \mathbb{D}$, *we define the function* $h_w(z)$ *by*

$$h_w(z) = \frac{(1-|w|^2)(1-|z|)}{(1-\overline{w}z)^2}.$$

Then $h_w(z) \in \mathcal{N}_{K,\omega}$ *and* $\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_{K,\omega}} \lesssim 1$.
Proof: First, we have that

$$\|h_w\|_{\mathcal{N}_{K,\,\omega}} = \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \left|\frac{(1-|w|^2)}{(1-\overline{w}z)^2}\right|^2 \frac{(1-|z|)^2}{\omega^2(1-|z|^2)} K(g(z,w)dA(z).$$

Since, $1-|w| \le |1-\bar{w}z| \le 1+|w| < 2$ and $1-|z| \le |1-\bar{w}z| \le 1+|z| < 2$ where $z, w \in \mathbb{D}$, then

$$\|h_w\|_{\mathcal{N}_{K,\,\omega}} \le 4 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(g(z,w)}{\omega^2(1-|z|^2)} dA(z) = 4 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|\varphi_w(z)|})}{\omega^2(1-|z|^2)} dA(z).$$

Now, let $z = \varphi_w(z)$, then

$$\begin{aligned} \|h_w\|_{\mathcal{N}_{K,\,\omega}} &\le 4 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{\omega^2(1-|\varphi_w(z)|^2)} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, dA(z) \\ &\le 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2 \omega^2(1-|\varphi_w(z)|^2)}\, dA(z). \end{aligned}$$

Since,

$$1-|\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2}.$$

Then, we obtain that

$$\begin{aligned} \|h_w\|_{\mathcal{N}_{K,\,\omega}} &\le 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2\ \omega^2\big(\frac{(1-|w|^2)(1-|z|^2)}{|1-\overline{w}z|^2}\big)}\, dA(z) \\ &= 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\, \omega^2\big(\frac{(1-|z|^2)}{|1-\overline{w}z|^2}\big)}\, dA(z) \\ &\le 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\, \omega^2\big(\frac{(1-|z|^2)}{(1-|w|)^2}\big)}\, dA(z) \\ &= 16 \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|)^4 K(\log\frac{1}{|z|})}{|1-\overline{w}z|^2(1-|w|^2)^2\, \omega^2(1-|z|^2)}\, dA(z) \\ &\le c \int_0^1 \frac{K(\log\frac{1}{r})}{\omega^2(1-r^2)}\, r\, dr < \infty, \end{aligned}$$

where c is a positive constant. Then,

$$\sup_{w\in\mathbb{D}} \|h_w\|_{\mathcal{N}_{K,\omega}} \lesssim 1.$$

This completes the proof.

3 Weighted composition operator on $H^{\infty}_{\alpha,\,\omega}$ and $\mathcal{N}_{K,\,\omega}$ spaces

Let ϕ be an analytic self-map of the unit disk $\mathbb{D}$. For any $u \in \mathbb{H}(\mathbb{D})$ and $\phi : \mathbb{D} \to \mathbb{D}$, the weighted composition operator $uC_\phi : \mathbb{H}(\mathbb{D}) \to \mathbb{H}(\mathbb{D})$ is defined by $uC_\phi f = u.(f \circ \phi)$. This class of operators has been appeared in the studies of isometries of many holomorphic function spaces. In fact, many isometries of holomorphic function spaces are described as weighted composition operators. For more information and various studies on weighted composition operators, we refer to [8, 12, 13, 16, 18, 25, 27, 28, 29] and others.

In this section we study weighted composition operators acting on $\mathcal{N}_{K,\,\omega}$-space.
Let $\phi \in \mathbb{H}(\mathbb{D})$ to denoted a non-constant function satisfying $\phi(\mathbb{D}) \subset \mathbb{D}$. First, in the following result, we describe boundedness for the $\mathcal{N}_{K,\,\omega}$-class. The results in this section generalizing some results in [19].

Theorem 3.1 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\phi(z)|^2)\omega(1-|z|^2))} < \infty. \tag{5}$$

Proof: First assume that (5) holds. Then

$$\begin{aligned}
\|uC_\phi f\|_{H^{\infty}_{\alpha,\,\omega}} &= \sup_{z\in\mathbb{D}} |u(z)|f(\phi(z))\left|\frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|^2)}\right| \\
&\lesssim \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\phi_a(z)|^2)\omega(1-|z|^2)} \sup_{z\in\mathbb{D}} |f(\phi(z))| \frac{(1-|\phi_a(z)|^2)}{\omega(1-|\phi(z)|^2)} \\
&\lesssim \|f\|_{H^{\infty}_{1,\,\omega}} \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\phi_a(z)|^2)\omega(1-|\phi_a(z)|^2)} \\
&\le \lambda\|f\|_{\mathcal{N}_{K,\,\omega}},
\end{aligned}$$

where λ is a positive constant.

Conversely, assume that $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is bounded, then

$$\|uC_\phi f\|_{H^{\infty}_{\alpha,\,\omega}} \lesssim \|f\|_{\mathcal{N}_{K,\,\omega}}.$$

Fix a $z_0 \in \mathbb{D}$, and let h_w be the test function in Lemma 2.4 with $w = \phi(z_0)$. Then

$$\begin{aligned}
1 \gtrsim \|h_w\|_{\mathcal{N}_{K,\,\omega}} &\ge \lambda_1 \|uC_\phi h_w\|_{H^{\infty}_{\alpha,\,\omega}} \\
&\ge \frac{|u(z_0)|(1-|w|^2)}{|1-\overline{w}\phi_a(z_0)|^2)\omega(1-|z_0|^2)}(1-|z_0|^2)^{\alpha} \\
&= \frac{|u(z_0)|(1-|z_0|^2)^{\alpha}}{(1-|\phi_a(z_0|^2))\omega(1-|z_0|^2)},
\end{aligned}$$

where λ_1 is a positive constant. The proof of Theorem 3.1 is therefore established.

Theorem 3.2 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then the weighted composition operator $uC_\phi : H^{\infty}_{\alpha,\,\omega} \to \mathcal{N}_{K,\,\omega}$ is bounded if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}(\omega^2(1-|z|^2))} K(g(z,a))dA(z) < \infty. \tag{6}$$

Proof: First we assume that condition (6) holds and let

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}(\omega^2(1-|z|^2))} K(g(z,a))dA(z) < C,$$

where C is a positive constant. If $f \in H^{\infty}_{\alpha,\,\omega}$, then for all $a \in \mathbb{D}$ we have

$$\begin{aligned}\|uC_\phi f\|_{\mathcal{N}_{K,\,\omega}} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}} |u(z)|^2|f(\phi(z)|^2\frac{K(g(z,a)}{\omega^2(1-|z|^2)}dA(z)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z)|^2}{\omega^2(1-|z|^2)}\frac{(1-|\phi(z)|^2)^{2\alpha}|f(\phi(z)|^2}{\omega^2(1-|\phi(z)|^2)}.\frac{\omega^2(1-|\phi(z)|^2)K(g(z,a))}{(1-|\phi(z)|^2)^{2\alpha}}dA(z)\\ &\le \|f\|^2_{H^{\infty}_{\alpha,\,\omega}}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z)|^2\omega^2(1-|\phi(z)|^2)}{(1-|\phi(z)|^2)^{2\alpha}\,\omega^2(1-|z|^2)}K(g(z,a))dA(z)\\ &\le C\|f\|^2_{H^{\infty}_{\alpha,\,\omega}}.\end{aligned}$$

Conversely, assume that $uC_\phi : H^{\infty}_{\alpha,\,\omega} \to \mathcal{N}_{K,\,\omega}$ is bounded, then

$$\|uC_\phi f\|^2_{\mathcal{N}_{K,\,\omega}} \lesssim \|f\|^2_{H^{\infty}_{\alpha,\,\omega}}.$$

fixing a point $z_0 \in \mathbb{D}$, with $w = \phi(z_0)$ then we set that

$$f_w(z) = \frac{\omega\big(1-\overline{w}\phi(z_0)\big)}{(1-\overline{w}z)^\alpha},$$

it is easy to check that $\|f_w\|_{H^{\infty}_{\alpha,\,\omega}} \lesssim 1$. Then,

$$\begin{aligned}\|uC_\phi f_w\|^2_{\mathcal{N}_{K,\,\omega}} &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z_0)|^2\omega^2(1-|\phi(z_0)|^2)K(g(z_0,a))}{\big(1-\overline{\phi(z_0)}\phi(z_0)\big)^{2\alpha}\omega^2(1-|z_0|^2)}dA(z_0)\\ &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{|u(z_0)|^2\omega^2(1-|\phi(z_0)|^2)K(g(z_0,a))}{\big(1-|\phi(z_0)|^2\big)^{2\alpha}\omega^2(1-|z_0|^2)}dA(z_0)\\ &\lesssim \|f_w\|^2_{H^{\infty}_{\alpha,\,\omega}}.\end{aligned}$$

Theorem 3.3 *Let $u \in H(\mathbb{D})$, suppose that $\omega : (0,\ 1] \longrightarrow [0,\ \infty)$, $K : [0,\infty) \to [0,\infty)$ be nondecreasing right continuous functions with $\omega(kt) = k\omega(t)$, $k > 0$, also suppose that condition (4) is satisfied and $\alpha \in (0,\infty)$. Then, the operator $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is compact if and only if*

$$\lim_{r\to 1}\sup_{|\phi(z)|>r}\frac{|u(z)|(1-|z|^2)^\alpha}{(1-|\phi(z)|^2)\omega(1-|z|^2)} = 0. \tag{7}$$

Proof: First assume that $uC_\phi : \mathcal{N}_{K,\,\omega} \to H^{\infty}_{\alpha,\,\omega}$ is compact and suppose that there exists $\varepsilon_0 > 0$ a sequence $(z_n) \subset \mathbb{D}$ such that

$$\frac{|u(z_n)|(1-|z_n|^2)^\alpha}{(1-|\phi(z_n)|^2)\omega(1-|z_n|^2)} \ge \varepsilon_0 \quad \text{whenever} \quad |\phi(z_n)| > 1-\frac{1}{n}.$$

Clearly, we can assume that

$$w_n = \phi(z_n) \longrightarrow w_0 \in \partial\mathbb{D} \ \text{ as } \ n\to\infty.$$

Let $h_{w_n} = \dfrac{(1-|w_n|^2)}{(1-\overline{w_n}z)^2}$ be the test function in Lemma 2.4. Then $h_{w_n} \to h_{w_0}$ with respect to the compact open topology. Define $f_n = h_{w_n} - h_{w_0}$. Then $\|f_n\|_{\mathcal{N}_{K,\,\omega}} \le 1$ (see Lemma 2.4) and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Thus, $uf_n \circ \phi \to 0$ in $H^{\infty}_{\alpha,\,\omega}$ by assumption. But, for n big enough, we obtain

$$\begin{aligned}\|uC_\phi f_n\|_{H^{\infty}_{\alpha,\,\omega}} &\ge |u(z_n)|\big|h_{w_n}(\phi(z_n)) - h_{w_0}(\phi(z_n))\big|\frac{(1-|z_n|^2)^\alpha}{\omega(1-|z_n|^2)}\\ &\ge \underbrace{\frac{|u(z_n)|(1-|z_n|^2)^\alpha}{(1-|\phi(z_n)|^2)\omega(1-|z_n|^2)}}_{\ge\varepsilon_0}\ \underbrace{\left|1-\frac{(1-|w_n|^2)(1-|w_0|^2)}{\big|1-\overline{w_0}\,w_n\big|}\right|}_{=1},\end{aligned}$$

which is a contradiction.

Conversely, assume that for all $\varepsilon > 0$ there exists $r \in (0,1)$ such that

$$\frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\phi(z)|^2)\omega(1-|z|^2)} < \varepsilon \quad \text{whenever} \quad |\phi(z)| > r.$$

Let $(f_n)_n$ be a bounded sequence in $\mathcal{N}_{K,\,\omega}$ norm which converges to zero on compact subsets of $\mathbb{D}$. Clearly, we may assume that $|\phi(z)| > r$. Then

$$\begin{aligned} \|uC_\phi f_n\|_{H^{\infty}_{\alpha,\,\omega}} &= \sup_{z\in\mathbb{D}} |u(z)||f_n(\phi(z))| \frac{(1-|z|^2)^{\alpha}}{\omega(1-|z|^2)} \\ &= \sup_{z\in\mathbb{D}} \frac{|u(z)|(1-|z|^2)^{\alpha}}{(1-|\phi(z)|^2)\omega(1-|z|^2)} \ \big|f_n(\phi(z))\big|(1-|\phi(z)|^2). \end{aligned}$$

It is not hard to show that

$$||.||_{H^{\infty}_{1,\,\omega}} \lesssim ||.||_{\mathcal{N}_{K,\omega}}.$$

Thus, we obtain that

$$\|uC_\phi f_n\|_{H^{\infty}_{\alpha,\,\omega}} \ \leq \ \varepsilon \ \|f_n\|_{H^{\infty}_{1,\,\omega}} \ \leq \ \varepsilon \ \|f_n\|_{\mathcal{N}_{K,\,\omega}} \ \leq \ \varepsilon.$$

It follows that uC_ϕ is a compact operator. This completes the proof of the theorem.

Remark 3.1 *It is still an open problem to extend the results of this paper in Clifford analysis, for several studies of function spaces in Clifford analysis, we refer to [1, 2, 3, 4, 5, 6] and others.*

Remark 3.2 *It is still an open problem to study properties for differences of weighted composition operators between $\mathcal{N}_{K,\,\omega}$ and $H^{\infty}_{\alpha,\,\omega}$ classes. For more information of studying differences of weighted composition operators, we refer to [14, 22, 23, 26] and others.*

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1

HERMITE-HADAMARD TYPE INEQUALITIES FOR THE ABK-FRACTIONAL INTEGRALS

ARTION KASHURI

ABSTRACT. The author introduced the new fractional integral operator called ABK-fractional integral and proved four identities for this type. By applying the established identities, some integral inequalities connected with the right hand side of the Hermite-Hadamard type inequalities for the ABK-fractional integrals are given. Various special cases have been identified. The ideas of this paper may stimulate further research in the field of integral inequalities.

1. INTRODUCTION

The class of convex functions is well known in the literature and is usually defined in the following way:

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $f : I \longrightarrow \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda e_1 + (1-\lambda)e_2) \leq \lambda f(e_1) + (1-\lambda)f(e_2) \tag{1.1}$$

holds for all $e_1, e_2 \in I$ and $\lambda \in [0,1]$. Also, we say that f is concave, if the inequality in (1.1) holds in the reverse direction.

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $e_1, e_2 \in I$ with $e_1 < e_2$. Then the following inequality holds:*

$$f\left(\frac{e_1+e_2}{2}\right) \leq \frac{1}{e_2-e_1}\int_{e_1}^{e_2} f(x)dx \leq \frac{f(e_1)+f(e_2)}{2}. \tag{1.2}$$

This inequality (1.2) is also known as trapezium inequality.

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.2) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [2],[4]-[20],[22]-[27].

In [8], Dragomir and Agarwal proved the following results connected with the right part of (1.2).

Lemma 1.3. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $e_1, e_2 \in I^\circ$ with $e_1 < e_2$. If $f' \in L[e_1, e_2]$, then the following equality holds:*

$$\frac{f(e_1)+f(e_2)}{2} - \frac{1}{e_2-e_1}\int_{e_1}^{e_2} f(x)dx = \frac{(e_2-e_1)}{2}\int_0^1 (1-2t)f'(te_1+(1-t)e_2)dt. \tag{1.3}$$

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[1] 2010 Mathematics Subject Classification: Primary: 26A09; Secondary: 26A33, 26D10, 26D15, 33E20.

Key words and phrases. Hermite-Hadamard inequality, Hölder inequality, power mean inequality, Katugampola fractional integral, AB-fractional integrals.

Theorem 1.4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $e_1, e_2 \in I^\circ$ with $e_1 < e_2$. If $|f'|$ is convex on $[e_1, e_2]$, then the following inequality holds:*

$$\left| \frac{f(e_1) + f(e_2)}{2} - \frac{1}{e_2 - e_1} \int_{e_1}^{e_2} f(x)dx \right| \leq \frac{(e_2 - e_1)}{8} \left(|f'(e_1)| + |f'(e_2)| \right). \tag{1.4}$$

Now, let us recall the following definitions.

Definition 1.5. $X_c^p(e_1, e_2)$ $(c \in \mathbb{R})$, $1 \leq p \leq \infty$ denotes the space of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p} < \infty$, where the norm $\|\cdot\|_{X_c^p}$ is defined by

$$\|f\|_{X_c^p} = \left(\int_{e_1}^{e_2} \left| t^c f(t) \right|^p \frac{dt}{t} \right)^{\frac{1}{p}} \quad (1 \leq p < \infty)$$

and for $p = \infty$

$$\|f\|_{X_c^\infty} = ess \sup_{e_1 \leq t \leq e_2} \left| t^c f(t) \right|.$$

Recently, in [12], Katugampola introduced a new fractional integral operator which generalizes the Riemann-Liouville and Hadamard fractional integrals as follows:

Definition 1.6. Let $[e_1, e_2] \subset \mathbb{R}$ be a finite interval. Then, the left and right side Katugampola fractional integrals of order $\alpha\,(> 0)$ of $f \in X_c^p(e_1, e_2)$ are defined by

$${}^\rho I_{e_1^+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{e_1}^{x} \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt, \quad x > e_1 \tag{1.5}$$

and

$${}^\rho I_{e_2^-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{x}^{e_2} \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt, \quad x < e_2, \tag{1.6}$$

where $\rho > 0$, if the integrals exist.

In [3], Atangana and Baleanu produced two new fractional derivatives based on the Caputo and the Riemann-Liouville definitions of fractional order derivatives. They declared that their fractional derivative has a fractional integral as the antiderivative of their operators. The Atangana-Baleanu (AB) fractional order derivative is known to possess nonsingularity as well as nonlocality of the kernel, which adopts the generalized Mittag-Leffler function, see [15],[21].

Definition 1.7. The fractional AB-integral of the function $f \in H^*(e_1, e_2)$ is given by

$${}_{e_1}^{AB}I_t^\nu f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{t} (t-u)^{\nu-1} f(u)du, \quad t > e_1, \tag{1.7}$$

where $e_1 < e_2$, $0 < \nu < 1$ and $\mathbb{B}(\nu) > 0$ satisfies the property $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Similarly, we give the definition of the (1.7) opposite side is given by

$${}_{e_2}^{AB}I_t^\nu f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{t}^{e_2} (u-t)^{\nu-1} f(u)du, \quad t < e_2.$$

Here, $\Gamma(\nu)$ is the Gamma function. Since the normalization function $\mathbb{B}(\nu) > 0$ is positive, it immediately follows that the fractional AB-integral of a positive function is positive. It should be noted that, when the order $\nu \to 1$, we recover the classical integral. Also, the initial function is recovered whenever the fractional order $\nu \to 0$.

Motivated by the above literatures, the main objective of this paper is to establish some new estimates for the right hand side of Hermite-Hadamard type integral inequalities for new fractional integral operator called the ABK-fractional integral operator. Various special cases will be identified. The ideas of this paper may stimulate further research in the field of integral inequalities.

2. Hermite-Hadamard inequalities for ABK-fractional integrals

Now, we are in position to introduce the left and right side ABK-fractional integrals as follows.

Definition 2.1. Let $[e_1, e_2] \subset \mathbb{R}$ be a finite interval. Then, the left and right side ABK-fractional integrals of order $\nu \in (0,1)$ of $f \in X_c^p(e_1, e_2)$ are defined by

$$ {}^{ABK}_{e_1^+}{}^{\rho} I_t^{\nu} f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\rho^{1-\nu}\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{t} \frac{u^{\rho-1}}{(t^{\rho}-u^{\rho})^{1-\nu}} f(u)du, \quad t > e_1 \geq 0 \tag{2.1}$$

and

$$ {}^{ABK}_{e_2^-}{}^{\rho} I_t^{\nu} f(t) = \frac{1-\nu}{\mathbb{B}(\nu)} f(t) + \frac{\rho^{1-\nu}\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{t}^{e_2} \frac{u^{\rho-1}}{(u^{\rho}-t^{\rho})^{1-\nu}} f(u)du, \quad t < e_2, \tag{2.2}$$

where $\rho > 0$ and $\mathbb{B}(\nu) > 0$ satisfies the property $\mathbb{B}(0) = \mathbb{B}(1) = 1$.

Remark 2.2. Since the normalization function $\mathbb{B}(\nu) > 0$ is positive, it immediately follows that the fractional ABK-integral of a positive function is positive. It should be noted that, when the $\rho \to 1$, we recover the AB-fractional integral. Also, using the same idea as in [12], the ABK-fractional integral operators are well-defined on $X_c^p(e_1, e_2)$. Finally, using the same idea as in [1], the interested reader can find new nonlocal fractional derivative of it with Mittag-Leffler nonsingular kernel, several formulae and many applications.

Let represent Hermite-Hadamard's inequalities in the ABK-fractional integral forms as follows:

Theorem 2.3. *Let $\nu \in (0,1)$ and $\rho > 0$. Let $f : [e_1^{\rho}, e_2^{\rho}] \to \mathbb{R}$ be a function with $0 \leq e_1 < e_2$ and $f \in X_c^p(e_1^{\rho}, e_2^{\rho})$. If f is a convex function on $[e_1^{\rho}, e_2^{\rho}]$, then the following inequalities for the ABK-fractional integrals hold:*

$$\begin{aligned} & \frac{2\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu}}{\mathbb{B}(\nu)\Gamma(\nu+1)\rho^{2-\nu}} f\left(\frac{e_1^{\rho}+e_2^{\rho}}{2}\right) + \frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1^{\rho}) + f(e_2^{\rho})\right] \\ \leq & \left[{}^{ABK}_{e_1^+}{}^{\rho} I_{e_2^{\rho}}^{\nu} f(e_2^{\rho}) + {}^{ABK}_{e_2^-}{}^{\rho} I_{e_1^{\rho}}^{\nu} f(e_1^{\rho}) \right] \\ \leq & \left(\frac{(e_2^{\rho}-e_1^{\rho})^{\nu} + \rho(1-\nu)\Gamma(\nu)}{\rho\mathbb{B}(\nu)\Gamma(\nu)} \right)\left[f(e_1^{\rho}) + f(e_2^{\rho})\right]. \end{aligned} \tag{2.3}$$

Proof. Let $t \in [0,1]$. Consider x^{ρ}, $y^{\rho} \in [e_1^{\rho}, e_2^{\rho}]$, defined by $x^{\rho} = t^{\rho}e_1^{\rho} + (1-t^{\rho})e_2^{\rho}$, $y^{\rho} = (1-t^{\rho})e_1^{\rho} + t^{\rho}e_2^{\rho}$. Since f is a convex function on $[e_1^{\rho}, e_2^{\rho}]$, we have

$$f\left(\frac{x^{\rho}+y^{\rho}}{2}\right) \leq \frac{f(x^{\rho}) + f(y^{\rho})}{2}.$$

Then, we get

$$2f\left(\frac{e_1^{\rho}+e_2^{\rho}}{2}\right) \leq f\left(t^{\rho}e_1^{\rho} + (1-t^{\rho})e_2^{\rho}\right) + f\left((1-t^{\rho})e_1^{\rho} + t^{\rho}e_2^{\rho}\right). \tag{2.4}$$

Multiplying both sides of (2.4) by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}t^{\rho\nu-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} & \frac{2}{\rho\mathbb{B}(\nu)\Gamma(\nu)} f\left(\frac{e_1^{\rho}+e_2^{\rho}}{2}\right) \\ \leq & \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} f\left(t^{\rho}e_1^{\rho} + (1-t^{\rho})e_2^{\rho}\right) dt + \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 t^{\rho\nu-1} f\left((1-t^{\rho})e_1^{\rho} + t^{\rho}e_2^{\rho}\right) dt \\ = & \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)} \int_{e_1}^{e_2} \left(\frac{e_2^{\rho}-x^{\rho}}{e_2^{\rho}-e_1^{\rho}}\right)^{\nu-1} f(x^{\rho}) \frac{x^{\rho-1}}{e_2^{\rho}-e_1^{\rho}} dx \end{aligned}$$

$$+\quad \frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_{e_1}^{e_2}\left(\frac{y^\rho-e_1^\rho}{e_2^\rho-e_1^\rho}\right)^{\nu-1} f(y^\rho)\frac{y^{\rho-1}}{e_2^\rho-e_1^\rho}dy$$

Therefore, it follows that

$$\begin{aligned}&\frac{2\left(e_2^\rho-e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+1)\rho^{2-\nu}}f\left(\frac{e_1^\rho+e_2^\rho}{2}\right)+\frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1^\rho)+f(e_2^\rho)\right]\\ \leq\ &\left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho}f(e_2^\rho)+{}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho}f(e_1^\rho)\right]\end{aligned}$$

and the left hand side inequality of (2.3) is proved.
For the proof of the right hand side inequality of (2.3) we first note that if f is a convex function, then

$$f\left(t^\rho e_1^\rho+(1-t^\rho)e_2^\rho\right)\leq t^\rho f(e_1^\rho)+(1-t^\rho)f(e_2^\rho)$$

and

$$f\left((1-t^\rho)e_1^\rho+t^\rho e_2^\rho\right)\leq (1-t^\rho)f(e_1^\rho)+t^\rho f(e_2^\rho).$$

By adding these inequalities, we have

$$f\left(t^\rho e_1^\rho+(1-t^\rho)e_2^\rho\right)+f\left((1-t^\rho)e_1^\rho+t^\rho e_2^\rho\right)\leq f(e_1^\rho)+f(e_2^\rho). \tag{2.5}$$

Then multiplying both sides of (2.5) by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}t^{\rho\nu-1}$ and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}&\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}f\left(t^\rho e_1^\rho+(1-t^\rho)e_2^\rho\right)dt+\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}f\left((1-t^\rho)e_1^\rho+t^\rho e_2^\rho\right)dt\\ \leq\ &\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\left[f(e_1^\rho)+f(e_2^\rho)\right]\int_0^1 t^{\rho\nu-1}dt\end{aligned}$$

i.e.

$$\left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho}f(e_2^\rho)+{}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho}f(e_1^\rho)\right]\ \leq\ \left(\frac{(e_2^\rho-e_1^\rho)^\nu+\rho(1-\nu)\Gamma(\nu)}{\rho\mathbb{B}(\nu)\Gamma(\nu)}\right)\left[f(e_1^\rho)+f(e_2^\rho)\right].$$

The proof of this theorem is complete. □

Corollary 2.4. *If we take $\rho\to 1$ in Theorem 2.3, then the following Hermite-Hadamard's inequalities for the AB-fractional integrals hold:*

$$\begin{aligned}&\frac{2\left(e_2-e_1\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+1)}f\left(\frac{e_1+e_2}{2}\right)+\frac{1-\nu}{\mathbb{B}(\nu)}\left[f(e_1)+f(e_2)\right]\\ \leq\ &\left[{}^{AB}_{e_1}I^{\nu}_{e_2}f(e_2)+{}^{AB}_{e_2}I^{\nu}_{e_1}f(e_1)\right]\\ \leq\ &\left(\frac{(e_2-e_1)^\nu+(1-\nu)\Gamma(\nu)}{\mathbb{B}(\nu)\Gamma(\nu)}\right)\left[f(e_1)+f(e_2)\right].\end{aligned}\tag{2.6}$$

Remark 2.5. If in Corollary 2.4, we let $\nu\to 1$, then the inequalities (2.6) become the inequalities (1.2).

3. The ABK-fractional inequalities for convex functions

For establishing some new results regarding the right side of Hermite-Hadamard type inequalities for the ABK-fractional integrals we need to prove the following four lemmas.

Lemma 3.1. *Let $\nu\in(0,1)$ and $\rho>0$ and $f:[e_1^\rho,e_2^\rho]\to\mathbb{R}$ be a differentiable mapping on (e_1^ρ,e_2^ρ) with $0\leq e_1<e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1^\rho)+f(e_2^\rho)\right]-\left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho}f(e_2^\rho)+{}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho}f(e_1^\rho)\right]$$

$$= \frac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} \int_0^1 \left[(1-t^\rho)^\nu - t^{\rho\nu}\right] t^{\rho-1} f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt. \tag{3.1}$$

Proof. Integrating by parts, we get

$$\begin{aligned} I_1 &= \int_0^1 (1-t^\rho)^\nu t^{\rho-1} f'\left(t^\rho e_1^\rho + (1-t^\rho)\, e_2^\rho\right) dt \\ &= \frac{(1-t^\rho)^\nu}{\rho(e_1^\rho - e_2^\rho)} f\left(t^\rho e_1^\rho + (1-t^\rho)\, e_2^\rho\right)\Big|_0^1 - \frac{\nu}{e_1^\rho - e_2^\rho}\int_0^1 (1-t^\rho)^{\nu-1} t^{\rho-1} f\left(t^\rho e_1^\rho + (1-t^\rho)\, e_2^\rho\right) dt \\ &= \frac{f(e_2^\rho)}{\rho(e_2^\rho - e_1^\rho)} - \frac{\nu}{e_1^\rho - e_2^\rho}\int_0^1 (1-t^\rho)^{\nu-1} t^{\rho-1} f\left(t^\rho e_1^\rho + (1-t^\rho)\, e_2^\rho\right) dt. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^1 t^{\rho(\nu+1)-1} f'\left(t^\rho e_1^\rho + (1-t^\rho)\, e_2^\rho\right) dt \\ &= \frac{t^{\rho(\nu+1)-1}}{\rho(e_1^\rho - e_2^\rho)} f\left(t^\rho e_1^\rho + (1-t^\rho)\, e_2^\rho\right)\Big|_0^1 - \frac{\nu}{e_1^\rho - e_2^\rho}\int_0^1 t^{\rho(\nu+1)} f\left(t^\rho e_1^\rho + (1-t^\rho)\, e_2^\rho\right) dt \\ &= -\frac{f(e_1^\rho)}{\rho(e_2^\rho - e_1^\rho)} - \frac{\nu}{e_1^\rho - e_2^\rho}\int_0^1 t^{\rho(\nu+1)} f\left(t^\rho e_1^\rho + (1-t^\rho)\, e_2^\rho\right) dt. \end{aligned}$$

Thus, by multiplying I_1 and I_2 with $\dfrac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}$, using definition of the ABK-fractional integrals and subtracting them, we get the result. □

Remark 3.2. If in Lemma 3.1, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned} &\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1) + f(e_2)\right] - \left[\, {}^{AB}_{e_1}I^{\nu}_{e_2} f(e_2) + {}^{AB}_{e_2}I^{\nu}_{e_1} f(e_1)\right] \\ = \; &\frac{(e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 \left[(1-t)^\nu - t^\nu\right] f'\left(te_1 + (1-t)e_2\right) dt. \end{aligned} \tag{3.2}$$

Remark 3.3. If in Lemma 3.1, we let $\rho,\, \nu \to 1$, then we obtain the equality (1.3).

Lemma 3.4. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned} &\left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1^\rho) + f(e_2^\rho)\right] - \left[\, {}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho} f(e_1^\rho)\right] \\ = \; &\frac{\left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho(\nu+1)-1}\left[f'\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) - f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\right] dt. \end{aligned} \tag{3.3}$$

Proof. The proof is similarly as Lemma 3.1, so we omit it. □

Remark 3.5. If in Lemma 3.4, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned} &\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1) + f(e_2)\right] - \left[\, {}^{AB}_{e_1}I^{\nu}_{e_2} f(e_2) + {}^{AB}_{e_2}I^{\nu}_{e_1} f(e_1)\right] \\ = \; &\frac{(e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^\nu\left[f'\left((1-t)e_1 + te_2\right) - f'\left(te_1 + (1-t)e_2\right)\right] dt. \end{aligned} \tag{3.4}$$

Lemma 3.6. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned}
&\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{\;e_1^+}I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{\;e_2^-}I^{\nu}_{e_1^\rho} f(e_1^\rho)\right] \\
= \;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{ \int_0^1 \left[1 - t^{\rho(\nu+1)}\right] t^{\rho-1} f''\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right) dt \right. \\
- \;& \left. \int_0^1 t^{\rho(\nu+2)-1} f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt \right\}.
\end{aligned}$$

Proof. By using twice integration by parts the proof is similarly as Lemma 3.1, so we omit it. ☐

Remark 3.7. If in Lemma 3.6, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned}
&\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[{}^{AB}_{\;e_1}I^{\nu}_{e_2} f(e_2) + {}^{AB}_{\;e_2}I^{\nu}_{e_1} f(e_1)\right] \\
= \;& \frac{\nu (e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times \;& \left\{ \int_0^1 \left[1 - t^{\nu+1}\right] f''\left((1-t)e_1 + te_2\right) dt - \int_0^1 t^{\nu+1} f''\left(te_1 + (1-t)e_2\right) dt \right\}.
\end{aligned}$$

Lemma 3.8. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. Then the following equality for the ABK-fractional integrals exist:*

$$\begin{aligned}
&\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{\;e_1^+}I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{\;e_2^-}I^{\nu}_{e_1^\rho} f(e_1^\rho)\right] \\
= \;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+2}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu+2)} \int_0^1 \left[1 - (1-t^\rho)^{\nu+1} - t^{\rho(\nu+1)}\right] t^{\rho-1} f''\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right) dt. \quad (3.5)
\end{aligned}$$

Proof. By using twice integration by parts and Lemma 3.1, we get the desired result. ☐

Remark 3.9. If in Lemma 3.8, we let $\rho \to 1$, then we get the following equality for the AB-fractional integrals:

$$\begin{aligned}
&\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1) + f(e_2)] - \left[{}^{AB}_{\;e_1}I^{\nu}_{e_2} f(e_2) + {}^{AB}_{\;e_2}I^{\nu}_{e_1} f(e_1)\right] \\
= \;& \frac{\nu (e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \int_0^1 \left[1 - (1-t)^{\nu+1} - t^{\nu+1}\right] f''\left(te_1 + (1-t)e_2\right) dt. \quad (3.6)
\end{aligned}$$

Using Lemmas 3.1, 3.4, 3.6 and 3.8, we can obtain the following the ABK-fractional integral inequalities.

Theorem 3.10. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ be a differentiable mapping on (e_1^ρ, e_2^ρ) with $0 \le e_1 < e_2$. If $|f'|^q$ is convex on $[e_1^\rho, e_2^\rho]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho) + f(e_2^\rho)] - \left[{}^{ABK\,\rho}_{\;e_1^+}I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{\;e_2^-}I^{\nu}_{e_1^\rho} f(e_1^\rho)\right]\right| \\
\le \;& \frac{\nu (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu+\frac{1}{q}}\mathbb{B}(\nu)\Gamma(\nu)} \times \sqrt[p]{D(p,\rho,\nu)}\sqrt[q]{\frac{|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q}{2}}, \quad (3.7)
\end{aligned}$$

where

$$\begin{aligned}D(p,\rho,\nu) &:= \int_0^{\frac{1}{2}}\Big[(1-t^{\rho})^{p\nu}-t^{p\rho\nu}\Big]t^{\rho-1}dt+\int_{\frac{1}{2}}^{1}\Big[t^{p\rho\nu}-(1-t^{\rho})^{p\nu}\Big]t^{\rho-1}dt\\ &= \frac{2}{\rho(p\nu+1)}\left\{1-\left(1-\frac{1}{2^{\rho}}\right)^{p\nu+1}-\frac{1}{2^{\rho(p\nu+1)}}\right\}.\end{aligned}$$

Proof. Using Lemma 3.1, convexity of $|f'|^{q}$, Hölder inequality and properties of the modulus, we have

$$\begin{aligned}&\left|\left(\frac{(e_2^{\rho}-e_1^{\rho})^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^{\rho})+f(e_2^{\rho})]-\Big[{}_{e_1^{+}}^{ABK\,\rho}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho})+{}_{e_2^{-}}^{ABK\,\rho}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\Big]\right|\\
\leq\ & \frac{\nu\,(e_2^{\rho}-e_1^{\rho})^{\nu+1}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}\\
\times\ & \left(\int_0^1\Big|(1-t^{\rho})^{\nu}-t^{\rho\nu}\Big|^{p}t^{\rho-1}dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{\rho-1}\Big|f'\left(t^{\rho}e_1^{\rho}+(1-t^{\rho})e_2^{\rho}\right)\Big|^{q}dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{\nu\,(e_2^{\rho}-e_1^{\rho})^{\nu+1}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}\left(\int_0^{\frac{1}{2}}\Big[(1-t^{\rho})^{p\nu}-t^{p\rho\nu}\Big]t^{\rho-1}dt+\int_{\frac{1}{2}}^{1}\Big[t^{p\rho\nu}-(1-t^{\rho})^{p\nu}\Big]t^{\rho-1}dt\right)^{\frac{1}{p}}\\
\times\ & \left(\int_0^1 t^{\rho-1}\left(t^{\rho}\big|f'(e_1^{\rho})\big|^{q}+(1-t^{\rho})\big|f''(e_2^{\rho})\big|^{q}\right)dt\right)^{\frac{1}{q}}\\
=\ & \frac{\nu\,(e_2^{\rho}-e_1^{\rho})^{\nu+1}}{\rho^{\nu+\frac{1}{q}}\mathbb{B}(\nu)\Gamma(\nu)}\times\sqrt[p]{D(p,\rho,\nu)}\sqrt[q]{\frac{\big|f'(e_1^{\rho})\big|^{q}+\big|f'(e_2^{\rho})\big|^{q}}{2}}.\end{aligned}$$

The proof of this theorem is complete. ☐

Corollary 3.11. *With the notations in Theorem 3.10, if we take* $|f'|\leq K$*, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{(e_2^{\rho}-e_1^{\rho})^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^{\rho})+f(e_2^{\rho})]-\Big[{}_{e_1^{+}}^{ABK\,\rho}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho})+{}_{e_2^{-}}^{ABK\,\rho}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\Big]\right|\\
&\leq\frac{\nu K\,(e_2^{\rho}-e_1^{\rho})^{\nu+1}}{\rho^{\nu+\frac{1}{q}}\mathbb{B}(\nu)\Gamma(\nu)}\times\sqrt[p]{D(p,\rho,\nu)}.\end{aligned}\tag{3.8}$$

Corollary 3.12. *With the notations in Theorem 3.10, if we take* $\rho\to 1$*, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{(e_2-e_1)^{\nu}}{\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)]-\Big[{}_{e_1}^{AB}I^{\nu}_{e_2}f(e_2)+{}_{e_2}^{AB}I^{\nu}_{e_1}f(e_1)\Big]\right|\\
\leq\ & \frac{\nu\,(e_2-e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)}\times\sqrt[p]{D(p,1,\nu)}\sqrt[q]{\frac{\big|f'(e_1)\big|^{q}+\big|f'(e_2)\big|^{q}}{2}}.\end{aligned}\tag{3.9}$$

Theorem 3.13. *Let* $\nu\in(0,1)$ *and* $\rho>0$ *and* $f:[e_1^{\rho},e_2^{\rho}]\to\mathbb{R}$ *be a differentiable mapping on* (e_1^{ρ},e_2^{ρ}) *with* $0\leq e_1<e_2$*. If* $|f'|^{q}$ *is convex on* $[e_1^{\rho},e_2^{\rho}]$ *for* $q\geq 1$*, then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}&\left|\left(\frac{(e_2^{\rho}-e_1^{\rho})^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^{\rho})+f(e_2^{\rho})]-\Big[{}_{e_1^{+}}^{ABK\,\rho}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho})+{}_{e_2^{-}}^{ABK\,\rho}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\Big]\right|\\
&\leq\frac{\nu\,(e_2^{\rho}-e_1^{\rho})^{\nu+1}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}\,[D(1,\rho,\nu)]^{1-\frac{1}{q}}\end{aligned}\tag{3.10}$$

$$\times \quad \Big\{E(\rho,\nu)\big|f'(e_1^\rho)\big|^q + (F(\rho,\nu)-E(\rho,\nu))\big|f'(e_2^\rho)\big|^q$$

$$+ \quad G(\rho,\nu)\big|f'(e_1^\rho)\big|^q + (F(\rho,\nu)-G(\rho,\nu))\big|f'(e_2^\rho)\big|^q\Big\}^{\frac{1}{q}},$$

where

$$E(\rho,\nu) := \int_0^{\frac{1}{2}}\Big[(1-t^\rho)^\nu - t^{\rho\nu}\Big]t^{2\rho-1}dt = \frac{1}{\rho}\left[\beta\left(\frac{1}{2^\rho};2,\nu+1\right) - \frac{1}{2^{\rho(\nu+2)}(\nu+2)}\right];$$

$$F(\rho,\nu) := \int_0^{\frac{1}{2}}\Big[(1-t^\rho)^\nu - t^{\rho\nu}\Big]t^{\rho-1}dt = \int_{\frac{1}{2}}^1\Big[t^{\rho\nu} - (1-t^\rho)^\nu\Big]t^{\rho-1}dt$$

$$= \frac{1}{\rho(\nu+1)}\left[1-\left(1-\frac{1}{2^\rho}\right)^{\nu+1} - \frac{1}{2^{\rho(\nu+1)}}\right];$$

$$G(\rho,\nu) := \int_{\frac{1}{2}}^1\Big[t^{\rho\nu} - (1-t^\rho)^\nu\Big]t^{2\rho-1}dt = \frac{1}{\rho}\left[\frac{1-\frac{1}{2^{\rho(\nu+2)}}}{\nu+2} + \beta\left(\frac{1}{2^\rho};2,\nu+1\right) - \beta(2,\nu+1)\right],$$

where $\beta(\cdot\,;\cdot,\cdot)$, $\beta(\cdot,\cdot)$ *are respectively the incomplete and complete beta functions and* $D(1,\rho,\nu)$ *is defined as in Theorem 3.10 for value* $p=1$.

Proof. Using Lemma 3.1, convexity of $|f'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^\nu_{e_2^\rho}f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^\nu_{e_1^\rho}f(e_1^\rho)\right]\right|$$

$$\leq \quad \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}\left(\int_0^1\Big|(1-t^\rho)^\nu - t^{\rho\nu}\Big|t^{\rho-1}dt\right)^{1-\frac{1}{q}}$$

$$\times \quad \left(\int_0^1\Big|(1-t^\rho)^\nu - t^{\rho\nu}\Big|t^{\rho-1}\Big|f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\Big|^q dt\right)^{\frac{1}{q}}$$

$$\leq \quad \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}[D(1,\rho,\nu)]^{1-\frac{1}{q}}$$

$$\times \quad \left\{\int_0^{\frac{1}{2}}\Big[(1-t^\rho)^\nu - t^{\rho\nu}\Big]t^{\rho-1}\left(t^\rho\big|f'(e_1^\rho)\big|^q + (1-t^\rho)\big|f'(e_2^\rho)\big|^q\right)dt\right.$$

$$+ \quad \left.\int_{\frac{1}{2}}^1\Big[t^{\rho\nu} - (1-t^\rho)^\nu\Big]t^{\rho-1}\left(t^\rho\big|f'(e_1^\rho)\big|^q + (1-t^\rho)\big|f'(e_2^\rho)\big|^q\right)dt\right\}^{\frac{1}{q}}$$

$$= \quad \frac{\nu(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}[D(1,\rho,\nu)]^{1-\frac{1}{q}}$$

$$\times \quad \Big\{E(\rho,\nu)\big|f'(e_1^\rho)\big|^q + (F(\rho,\nu)-E(\rho,\nu))\big|f'(e_2^\rho)\big|^q$$

$$+ \quad G(\rho,\nu)\big|f'(e_1^\rho)\big|^q + (F(\rho,\nu)-G(\rho,\nu))\big|f'(e_2^\rho)\big|^q\Big\}^{\frac{1}{q}}.$$

The proof of this theorem is complete. □

Corollary 3.14. *With the notations in Theorem 3.13, if we take* $|f'| \leq K$, *the following inequality for the* ABK*-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho} f(e_1^\rho)\right]\right| \\ \leq\ & \frac{\nu K (e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)}\left[D(1,\rho,\nu)\right]. \end{aligned} \tag{3.11}$$

Corollary 3.15. *With the notations in Theorem 3.13, if we take* $\rho \to 1$, *the following inequality for the* AB*-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2 - e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)] - \left[{}^{AB}_{e_1}I^{\nu}_{e_2} f(e_2) + {}^{AB}_{e_2}I^{\nu}_{e_1} f(e_1)\right]\right| \\ \leq\ & \frac{\nu (e_2 - e_1)^{\nu+1}}{\mathbb{B}(\nu)\Gamma(\nu)}\left[D(1,1,\nu)\right]^{1-\frac{1}{q}} \\ \times\ & \Bigg\{E(1,\nu)\big|f'(e_1)\big|^q + (F(1,\nu)-E(1,\nu))\big|f'(e_2)\big|^q \qquad (3.12) \\ +\ & G(1,\nu)\big|f'(e_1)\big|^q + (F(1,\nu)-G(1,\nu))\big|f'(e_2)\big|^q\Bigg\}^{\frac{1}{q}}. \qquad (3.13) \end{aligned}$$

Theorem 3.16. *Let* $\nu \in (0,1)$ *and* $\rho > 0$ *and* $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ *be a differentiable mapping on* (e_1^ρ, e_2^ρ) *with* $0 \leq e_1 < e_2$. *If* $|f'|^q$ *is convex on* $[e_1^\rho, e_2^\rho]$ *for* $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality for the* ABK*-fractional integrals holds:*

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho} f(e_1^\rho)\right]\right| \\ &\qquad \leq \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}}\,\frac{1}{\sqrt[q]{\rho+1}} \qquad (3.14) \\ &\qquad \times\left\{\sqrt[q]{|f'(e_1^\rho)|^q + \rho|f'(e_2^\rho)|^q} + \sqrt[q]{\rho|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q}\right\}. \end{aligned}$$

Proof. Using Lemma 3.4, convexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\begin{aligned} &\left|\left(\frac{(e_2^\rho - e_1^\rho)^\nu}{\rho^\nu \mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho} f(e_1^\rho)\right]\right| \\ \leq\ & \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{p(\rho(\nu+1)-1)}dt\right)^{\frac{1}{p}} \\ \times\ & \left\{\left(\int_0^1 \left|f'\left(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|f'\left((1-t^\rho)e_1^\rho + t^\rho e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\ \leq\ & \frac{(e_2^\rho - e_1^\rho)^{\nu+1}}{\rho^{\nu-1} \mathbb{B}(\nu)\Gamma(\nu)} \times \left(\int_0^1 t^{p(\rho(\nu+1)-1)}dt\right)^{\frac{1}{p}} \\ \times\ & \left\{\left(\int_0^1 \left(t^\rho\big|f'(e_1^\rho)\big|^q + (1-t^\rho)\big|f'(e_2^\rho)\big|^q\right)dt\right)^{\frac{1}{q}}\right. \\ +\ & \left.\left(\int_0^1 \left((1-t^\rho)\big|f'(e_1^\rho)\big|^q + t^\rho\big|f'(e_2^\rho)\big|^q\right)dt\right)^{\frac{1}{q}}\right\} \end{aligned}$$

$$= \frac{\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}}\frac{1}{\sqrt[q]{\rho+1}}$$

$$\times\left\{\sqrt[q]{|f'(e_1^{\rho})|^q+\rho|f'(e_2^{\rho})|^q}+\sqrt[q]{\rho|f'(e_1^{\rho})|^q+|f'(e_2^{\rho})|^q}\right\}.$$

The proof of this theorem is complete. □

Corollary 3.17. *With the notations in Theorem 3.16, if we take* $|f'| \leq K$, *the following inequality for the* ABK*-fractional integrals holds:*

$$\left|\left(\frac{\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu}}{\rho^{\nu}\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)}+\frac{1-\nu}{\mathbb{B}\left(\nu\right)}\right)\left[f(e_1^{\rho})+f(e_2^{\rho})\right]-\left[\ {}^{ABK\,\rho}_{e_1^+}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho})+\ {}^{ABK\,\rho}_{e_2^-}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\right]\right|$$

$$\leq \frac{2K\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)} \times \frac{1}{\sqrt[p]{p(\rho(\nu+1)-1)+1}}. \tag{3.15}$$

Corollary 3.18. *With the notations in Theorem 3.16, if we take* $\rho \to 1$, *the following inequality for the* AB*-fractional integrals holds:*

$$\left|\left(\frac{\left(e_2-e_1\right)^{\nu}}{\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)}+\frac{1-\nu}{\mathbb{B}\left(\nu\right)}\right)\left[f(e_1)+f(e_2)\right]-\left[\ {}^{AB}_{e_1}I^{\nu}_{e_2}f(e_2)+\ {}^{AB}_{e_2}I^{\nu}_{e_1}f(e_1)\right]\right|$$

$$\leq \frac{2\left(e_2-e_1\right)^{\nu+1}}{\sqrt[p]{p\nu+1}\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)} \times \sqrt[q]{\frac{|f'(e_1)|^q+|f'(e_2)|^q}{2}}. \tag{3.16}$$

Theorem 3.19. *Let* $\nu \in (0,1)$ *and* $\rho > 0$ *and* $f : [e_1^{\rho}, e_2^{\rho}] \to \mathbb{R}$ *be a differentiable mapping on* (e_1^{ρ}, e_2^{ρ}) *with* $0 \leq e_1 < e_2$. *If* $|f'|^q$ *is convex on* $[e_1^{\rho}, e_2^{\rho}]$ *for* $q \geq 1$, *then the following inequality for the* ABK*-fractional integrals holds:*

$$\left|\left(\frac{\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu}}{\rho^{\nu}\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)}+\frac{1-\nu}{\mathbb{B}\left(\nu\right)}\right)\left[f(e_1^{\rho})+f(e_2^{\rho})\right]-\left[\ {}^{ABK\,\rho}_{e_1^+}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho})+\ {}^{ABK\,\rho}_{e_2^-}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\right]\right|$$

$$\leq \frac{\nu\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+1}}{\rho^{\nu}\sqrt[q]{\nu+2}\mathbb{B}\left(\nu\right)\Gamma\left(\nu+2\right)} \tag{3.17}$$

$$\times\left\{\sqrt[q]{|f'(e_1^{\rho})|^q+(\nu+1)|f'(e_2^{\rho})|^q}+\sqrt[q]{(\nu+1)|f'(e_1^{\rho})|^q+|f'(e_2^{\rho})|^q}\right\}.$$

Proof. Using Lemma 3.4, convexity of $|f'|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\left|\left(\frac{\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu}}{\rho^{\nu}\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)}+\frac{1-\nu}{\mathbb{B}\left(\nu\right)}\right)\left[f(e_1^{\rho})+f(e_2^{\rho})\right]-\left[\ {}^{ABK\,\rho}_{e_1^+}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho})+\ {}^{ABK\,\rho}_{e_2^-}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\right]\right|$$

$$\begin{aligned}
\leq\ & \frac{\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)} \times \left(\int_0^1 t^{\rho(\nu+1)-1}dt\right)^{1-\frac{1}{q}} \\
\times\ & \left\{\left(\int_0^1 t^{\rho(\nu+1)-1}\left|f'\left(t^{\rho}e_1^{\rho}+(1-t^{\rho})e_2^{\rho}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
+\ & \left.\left(\int_0^1 t^{\rho(\nu+1)-1}\left|f'\left((1-t^{\rho})e_1^{\rho}+t^{\rho}e_2^{\rho}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq\ & \frac{\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+1}}{\rho^{\nu-1}\mathbb{B}\left(\nu\right)\Gamma\left(\nu\right)} \times \left(\int_0^1 t^{\rho(\nu+1)-1}dt\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times \quad \left\{ \left(\int_0^1 t^{\rho(\nu+1)-1} \left(t^\rho \left| f'(e_1^\rho) \right|^q + (1-t^\rho) \left| f'(e_2^\rho) \right|^q \right) dt \right)^{\frac{1}{q}} \right.$$

$$+ \quad \left. \left(\int_0^1 t^{\rho(\nu+1)-1} \left((1-t^\rho) \left| f'(e_1^\rho) \right|^q + t^\rho \left| f'(e_2^\rho) \right|^q \right) dt \right)^{\frac{1}{q}} \right\}$$

$$= \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \sqrt[q]{\nu+2}\,\mathbb{B}(\nu)\,\Gamma(\nu+2)}$$

$$\times \left\{ \sqrt[q]{|f'(e_1^\rho)|^q + (\nu+1)|f'(e_2^\rho)|^q} + \sqrt[q]{(\nu+1)|f'(e_1^\rho)|^q + |f'(e_2^\rho)|^q} \right\}.$$

The proof of this theorem is complete. □

Corollary 3.20. *With the notations in Theorem 3.19, if we take* $|f'| \leq K$, *the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\,\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1^\rho) + f(e_2^\rho) \right] - \left[{}^{ABK\,\rho}_{\;e_1^+} I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{\;e_2^-} I^{\nu}_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{2\nu K \left(e_2^\rho - e_1^\rho\right)^{\nu+1}}{\rho^\nu \mathbb{B}(\nu)\,\Gamma(\nu+2)}. \tag{3.18}$$

Corollary 3.21. *With the notations in Theorem 3.19, if we take* $\rho \to 1$, *the following inequality for the AB-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2 - e_1\right)^\nu}{\mathbb{B}(\nu)\,\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1) + f(e_2) \right] - \left[{}^{AB}_{\;e_1} I^{\nu}_{e_2} f(e_2) + {}^{AB}_{\;e_2} I^{\nu}_{e_1} f(e_1) \right] \right|$$

$$\leq \frac{\nu \left(e_2 - e_1\right)^{\nu+1}}{\sqrt[q]{\nu+2}\,\mathbb{B}(\nu)\,\Gamma(\nu+2)} \tag{3.19}$$

$$\times \left\{ \sqrt[q]{|f'(e_1)|^q + (\nu+1)|f'(e_2)|^q} + \sqrt[q]{(\nu+1)|f'(e_1)|^q + |f'(e_2)|^q} \right\}.$$

Theorem 3.22. *Let* $\nu \in (0,1)$ *and* $\rho > 0$ *and* $f : [e_1^\rho, e_2^\rho] \to \mathbb{R}$ *be a twice differentiable mapping on* (e_1^ρ, e_2^ρ) *with* $0 \leq e_1 < e_2$. *If* $|f''|^q$ *is convex on* $[e_1^\rho, e_2^\rho]$ *for* $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then the following inequality for the ABK-fractional integrals holds:*

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\,\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1^\rho) + f(e_2^\rho) \right] - \left[{}^{ABK\,\rho}_{\;e_1^+} I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{\;e_2^-} I^{\nu}_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu)\,\Gamma(\nu+2)} \times \left\{ \frac{1}{\rho} \sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + |f''(e_2^\rho)|^q}{2}} \right. \tag{3.20}$$

$$\left. + \frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}} \sqrt[q]{\frac{|f''(e_1^\rho)|^q + \rho|f''(e_2^\rho)|^q}{\rho+1}} \right\}.$$

Proof. Using Lemma 3.6, convexity of $|f''|^q$, Hölder inequality and properties of the modulus, we have

$$\left| \left(\frac{\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho^\nu \mathbb{B}(\nu)\,\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) \left[f(e_1^\rho) + f(e_2^\rho) \right] - \left[{}^{ABK\,\rho}_{\;e_1^+} I^{\nu}_{e_2^\rho} f(e_2^\rho) + {}^{ABK\,\rho}_{\;e_2^-} I^{\nu}_{e_1^\rho} f(e_1^\rho) \right] \right|$$

$$\leq \quad \frac{\nu \left(e_2^\rho - e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1} \mathbb{B}(\nu)\,\Gamma(\nu+2)}$$

$$
\begin{aligned}
\times & \left\{ \left(\int_0^1 \left|1-t^{\rho(\nu+1)}\right|^p t^{\rho-1}dt\right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1}\left|f''\left((1-t^{\rho})e_1^{\rho}+t^{\rho}e_2^{\rho}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
+ & \left.\left(\int_0^1 t^{p(\rho(\nu+2)-1)}dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f''\left(t^{\rho}e_1^{\rho}+(1-t^{\rho})e_2^{\rho}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq & \frac{\nu\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times & \left\{ \left(\int_0^1 \left|1-t^{\rho(\nu+1)}\right|^p t^{\rho-1}dt\right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1}\left((1-t^{\rho})\left|f''(e_1^{\rho})\right|^q+t^{\rho}\left|f''(e_2^{\rho})\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\
+ & \left.\left(\int_0^1 t^{p(\rho(\nu+2)-1)}dt\right)^{\frac{1}{p}} \left(\int_0^1 \left(t^{\rho}\left|f''(e_1^{\rho})\right|^q+(1-t^{\rho})\left|f''(e_2^{\rho})\right|^q\right)dt\right)^{\frac{1}{q}}\right\} \\
& = \frac{\nu\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{\frac{1}{\rho}\sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}}\sqrt[q]{\frac{|f''(e_1^{\rho})|^q+|f''(e_2^{\rho})|^q}{2}}\right. \\
& \qquad \left.+\frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}}\sqrt[q]{\frac{|f''(e_1^{\rho})|^q+\rho|f''(e_2^{\rho})|^q}{\rho+1}}\right\}.
\end{aligned}
$$

The proof of this theorem is complete. □

Corollary 3.23. *With the notations in Theorem 3.22, if we take* $|f''|\leq K$, *the following inequality for the* ABK*-fractional integrals holds:*

$$
\begin{aligned}
& \left|\left(\frac{\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1^{\rho})+f(e_2^{\rho})\right]-\left[{}^{ABK\,\rho}_{\;e_1^+}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho})+{}^{ABK\,\rho}_{\;e_2^-}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\right]\right| \\
& \leq \frac{\nu K\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \times \left\{\frac{1}{\rho}\sqrt[p]{\frac{p(\nu+1)}{p(\nu+1)+1}}+\frac{1}{\sqrt[p]{p(\rho(\nu+2)-1)+1}}\right\}.
\end{aligned}
\tag{3.21}
$$

Corollary 3.24. *With the notations in Theorem 3.22, if we take* $\rho\to 1$, *the following inequality for the* AB*-fractional integrals holds:*

$$
\begin{aligned}
& \left|\left(\frac{\left(e_2-e_1\right)^{\nu}}{\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1)+f(e_2)\right]-\left[{}^{AB}_{\;e_1}I^{\nu}_{e_2}f(e_2)+{}^{AB}_{\;e_2}I^{\nu}_{e_1}f(e_1)\right]\right| \\
& \leq \frac{\nu\left(e_2-e_1\right)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \times \frac{\left[\sqrt[p]{p(\nu+1)}+1\right]}{\sqrt[p]{p(\nu+1)+1}}\sqrt[q]{\frac{|f''(e_1)|^q+|f''(e_2)|^q}{2}}
\end{aligned}
\tag{3.22}
$$

Theorem 3.25. *Let* $\nu\in(0,1)$ *and* $\rho>0$ *and* $f:[e_1^{\rho},e_2^{\rho}]\to\mathbb{R}$ *be a twice differentiable mapping on* (e_1^{ρ},e_2^{ρ}) *with* $0\leq e_1<e_2$. *If* $|f''|^q$ *is convex on* $[e_1^{\rho},e_2^{\rho}]$ *for* $q\geq 1$, *then the following inequality for the* ABK*-fractional integrals holds:*

$$
\begin{aligned}
& \left|\left(\frac{\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)\left[f(e_1^{\rho})+f(e_2^{\rho})\right]-\left[{}^{ABK\,\rho}_{\;e_1^+}I^{\nu}_{e_2^{\rho}}f(e_2^{\rho})+{}^{ABK\,\rho}_{\;e_2^-}I^{\nu}_{e_1^{\rho}}f(e_1^{\rho})\right]\right| \\
& \qquad \leq \frac{\nu\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)} \\
& \times \left\{\left(\frac{\nu+1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}}\sqrt[q]{\frac{(\nu+1)(\nu+4)}{2\rho(\nu+2)(\nu+3)}\left|f''(e_1^{\rho})\right|^q+\frac{(\nu+1)}{2\rho(\nu+3)}\left|f''(e_2^{\rho})\right|^q}\right.
\end{aligned}
\tag{3.23}
$$

$$+\quad\left(\frac{1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}}\sqrt[q]{\frac{1}{\rho(\nu+3)}\left|f''(e_1^\rho)\right|^q+\frac{1}{\rho(\nu+2)(\nu+3)}\left|f''(e_2^\rho)\right|^q}\Bigg\}.$$

Proof. Using Lemma 3.6, convexity of $|f''|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK\,\rho}_{\;e_1^+}I^{\nu}_{e_2^\rho}f(e_2^\rho)+{}^{ABK\,\rho}_{\;e_2^-}I^{\nu}_{e_1^\rho}f(e_1^\rho)\right]\right|\\
\le\quad&\frac{\nu\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)}\\
\times\quad&\Bigg\{\left(\int_0^1\left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}\left|f''\left((1-t^\rho)e_1^\rho+t^\rho e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\\
+\quad&\left(\int_0^1 t^{\rho(\nu+2)-1}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\rho(\nu+2)-1}\left|f''\left(t^\rho e_1^\rho+(1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}\\
\le\quad&\frac{\nu\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)}\times\Bigg\{\left(\int_0^1\left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}}\\
\times\quad&\left(\int_0^1\left[1-t^{\rho(\nu+1)}\right]t^{\rho-1}\left((1-t^\rho)\left|f''(e_1^\rho)\right|^q+t^\rho\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}}\\
+\quad&\left(\int_0^1 t^{\rho(\nu+2)-1}dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{\rho(\nu+2)-1}\left(t^\rho\left|f''(e_1^\rho)\right|^q+(1-t^\rho)\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}}\Bigg\}\\
=\quad&\frac{\nu\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^{\nu-1}\mathbb{B}(\nu)\Gamma(\nu+2)}\\
\times\quad&\Bigg\{\left(\frac{\nu+1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}}\sqrt[q]{\frac{(\nu+1)(\nu+4)}{2\rho(\nu+2)(\nu+3)}\left|f''(e_1^\rho)\right|^q+\frac{(\nu+1)}{2\rho(\nu+3)}\left|f''(e_2^\rho)\right|^q}\\
+\quad&\left(\frac{1}{\rho(\nu+2)}\right)^{1-\frac{1}{q}}\sqrt[q]{\frac{1}{\rho(\nu+3)}\left|f''(e_1^\rho)\right|^q+\frac{1}{\rho(\nu+2)(\nu+3)}\left|f''(e_2^\rho)\right|^q}\Bigg\}.
\end{aligned}$$

The proof of this theorem is complete. □

Corollary 3.26. *With the notations in Theorem 3.25, if we take $|f''|\le K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK\,\rho}_{\;e_1^+}I^{\nu}_{e_2^\rho}f(e_2^\rho)+{}^{ABK\,\rho}_{\;e_2^-}I^{\nu}_{e_1^\rho}f(e_1^\rho)\right]\right|\\
\le\quad&\frac{\nu K\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu+2)}.
\end{aligned}\tag{3.24}$$

Corollary 3.27. *With the notations in Theorem 3.25, if we take $\rho\to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2-e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)]-\left[\,{}^{AB}_{e_1}I^{\nu}_{e_2}f(e_2)+{}^{AB}_{e_2}I^{\nu}_{e_1}f(e_1)\right]\right|\\
\le\quad&\frac{\nu\left(e_2-e_1\right)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)}
\end{aligned}$$

$$\begin{aligned}
&\times \quad \left\{ \left(\frac{\nu+1}{\nu+2}\right) \sqrt[q]{\frac{(\nu+4)\big|f''(e_1)\big|^q + (\nu+2)\big|f''(e_2)\big|^q}{2(\nu+3)}} \right.\\
&+ \quad \left. \frac{1}{(\nu+2)\sqrt[q]{\nu+3}} \sqrt[q]{(\nu+2)\big|f''(e_1)\big|^q + \big|f''(e_2)\big|^q} \right\}.
\end{aligned}$$

Theorem 3.28. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^{\rho}, e_2^{\rho}] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^{ρ}, e_2^{ρ}) with $0 \le e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^{\rho}, e_2^{\rho}]$ for $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left| \left(\frac{(e_2^{\rho} - e_1^{\rho})^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^{\rho}) + f(e_2^{\rho})] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^{\rho}} f(e_2^{\rho}) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^{\rho}} f(e_1^{\rho}) \right] \right| \\
\le\quad & \frac{\nu\,(e_2^{\rho} - e_1^{\rho})^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\big|f''(e_1^{\rho})\big|^q + \big|f''(e_2^{\rho})\big|^q}{2}}. \qquad (3.25)
\end{aligned}$$

Proof. Using Lemma 3.8, convexity of $|f''|^q$, Hölder inequality and properties of the modulus, we have

$$\begin{aligned}
&\left| \left(\frac{(e_2^{\rho} - e_1^{\rho})^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^{\rho}) + f(e_2^{\rho})] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^{\rho}} f(e_2^{\rho}) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^{\rho}} f(e_1^{\rho}) \right] \right| \\
\le\quad & \frac{\nu\,(e_2^{\rho} - e_1^{\rho})^{\nu+2}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu+2)} \\
\times\quad & \left(\int_0^1 \left| 1 - (1-t^{\rho})^{\nu+1} - t^{\rho(\nu+1)} \right|^p t^{\rho-1} dt \right)^{\frac{1}{p}} \left(\int_0^1 t^{\rho-1} \left| f''\left(t^{\rho}e_1^{\rho} + (1-t^{\rho})e_2^{\rho}\right) \right|^q dt \right)^{\frac{1}{q}} \\
\le\quad & \frac{\nu\,(e_2^{\rho} - e_1^{\rho})^{\nu+2}}{\rho^{\nu+\frac{1}{p}}\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \times \left(\int_0^1 t^{\rho-1} \left(t^{\rho}\big|f''(e_1^{\rho})\big|^q + (1-t^{\rho})\big|f''(e_2^{\rho})\big|^q \right) dt \right)^{\frac{1}{q}} \\
=\quad & \frac{\nu\,(e_2^{\rho} - e_1^{\rho})^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\big|f''(e_1^{\rho})\big|^q + \big|f''(e_2^{\rho})\big|^q}{2}}.
\end{aligned}$$

The proof of this theorem is complete. □

Corollary 3.29. *With the notations in Theorem 3.28, if we take $|f''| \le K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left| \left(\frac{(e_2^{\rho} - e_1^{\rho})^{\nu}}{\rho^{\nu}\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1^{\rho}) + f(e_2^{\rho})] - \left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^{\rho}} f(e_2^{\rho}) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^{\rho}} f(e_1^{\rho}) \right] \right| \\
\le\quad & \frac{\nu K\,(e_2^{\rho} - e_1^{\rho})^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}}. \qquad (3.26)
\end{aligned}$$

Corollary 3.30. *With the notations in Theorem 3.28, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}
&\left| \left(\frac{(e_2 - e_1)^{\nu}}{\mathbb{B}(\nu)\Gamma(\nu)} + \frac{1-\nu}{\mathbb{B}(\nu)} \right) [f(e_1) + f(e_2)] - \left[{}^{AB}_{e_1}I^{\nu}_{e_2} f(e_2) + {}^{AB}_{e_2}I^{\nu}_{e_1} f(e_1) \right] \right| \\
\le\quad & \frac{\nu\,(e_2 - e_1)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)} \sqrt[p]{\frac{p(\nu+1)-1}{p(\nu+1)+1}} \sqrt[q]{\frac{\big|f''(e_1)\big|^q + \big|f''(e_2)\big|^q}{2}}. \qquad (3.27)
\end{aligned}$$

Theorem 3.31. *Let $\nu \in (0,1)$ and $\rho > 0$ and $f : [e_1^{\rho}, e_2^{\rho}] \to \mathbb{R}$ be a twice differentiable mapping on (e_1^{ρ}, e_2^{ρ}) with $0 \le e_1 < e_2$. If $|f''|^q$ is convex on $[e_1^{\rho}, e_2^{\rho}]$ for $q \ge 1$, then the following inequality* *for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK\,\rho}_{\ e_1^+}I^\nu_{e_2^\rho}f(e_2^\rho)+{}^{ABK\,\rho}_{\ e_2^-}I^\nu_{e_1^\rho}f(e_1^\rho)\right]\right|\\
\leq\ & \frac{\nu\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}} \qquad (3.28)\\
\times\ & \sqrt[q]{C(\rho,\nu)\left|f''(e_1^\rho)\right|^q+\left(\frac{\nu}{\rho(\nu+2)}-C(\rho,\nu)\right)\left|f''(e_2^\rho)\right|^q},
\end{aligned}$$

where

$$C(\rho,\nu):=\frac{1}{\rho}\left(\frac{\nu+1}{2(\nu+3)}-\beta(2,\nu+2)\right).$$

Proof. Using Lemma 3.8, convexity of $|f''|^q$, the well-known power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK\,\rho}_{\ e_1^+}I^\nu_{e_2^\rho}f(e_2^\rho)+{}^{ABK\,\rho}_{\ e_2^-}I^\nu_{e_1^\rho}f(e_1^\rho)\right]\right|\\
\leq\ & \frac{\nu\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu+2)}\\
\times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}dt\right)^{1-\frac{1}{q}}\\
\times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}\left|f''\left(t^\rho e_1^\rho+(1-t^\rho)e_2^\rho\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq\ & \frac{\nu\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}}\\
\times\ & \left(\int_0^1\left[1-(1-t^\rho)^{\nu+1}-t^{\rho(\nu+1)}\right]t^{\rho-1}\left(t^\rho\left|f''(e_1^\rho)\right|^q+(1-t^\rho)\left|f''(e_2^\rho)\right|^q\right)dt\right)^{\frac{1}{q}}\\
=\ & \frac{\nu\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\rho(\nu+2)}\right)^{1-\frac{1}{q}}\sqrt[q]{C(\rho,\nu)\left|f''(e_1^\rho)\right|^q+\left(\frac{\nu}{\rho(\nu+2)}-C(\rho,\nu)\right)\left|f''(e_2^\rho)\right|^q}.
\end{aligned}$$

The proof of this theorem is complete. □

Corollary 3.32. *With the notations in Theorem 3.31, if we take $|f''|\leq K$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2^\rho-e_1^\rho)^\nu}{\rho^\nu\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1^\rho)+f(e_2^\rho)]-\left[\,{}^{ABK\,\rho}_{\ e_1^+}I^\nu_{e_2^\rho}f(e_2^\rho)+{}^{ABK\,\rho}_{\ e_2^-}I^\nu_{e_1^\rho}f(e_1^\rho)\right]\right|\\
\leq\ & \frac{\nu^2K\left(e_2^\rho-e_1^\rho\right)^{\nu+2}}{\rho^{\nu+1}\mathbb{B}(\nu)\Gamma(\nu+3)}. \qquad (3.29)
\end{aligned}$$

Corollary 3.33. *With the notations in Theorem 3.31, if we take $\rho\to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}
&\left|\left(\frac{(e_2-e_1)^\nu}{\mathbb{B}(\nu)\Gamma(\nu)}+\frac{1-\nu}{\mathbb{B}(\nu)}\right)[f(e_1)+f(e_2)]-\left[\,{}^{AB}_{e_1}I^\nu_{e_2}f(e_2)+{}^{AB}_{e_2}I^\nu_{e_1}f(e_1)\right]\right|\\
\leq\ & \frac{\nu\left(e_2-e_1\right)^{\nu+2}}{\mathbb{B}(\nu)\Gamma(\nu+2)}\left(\frac{\nu}{\nu+2}\right)^{1-\frac{1}{q}}
\end{aligned}$$

$$\times \sqrt[q]{C(1,\nu)\left|f''(e_1)\right|^q + \left(\frac{\nu}{\nu+2} - C(1,\nu)\right)\left|f''(e_2)\right|^q}.$$

Theorem 3.34. *Let $\nu \in (0,1)$ and $\rho > 0$. Let f and g be real valued, nonnegative and convex functions on $[e_1^\rho, e_2^\rho]$, where $0 \le e_1 < e_2$. Then the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}&\left[{}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^\rho} f(e_2^\rho)g(e_2^\rho) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^\rho} f(e_1^\rho)g(e_1^\rho)\right]\\ \le\ &\left(\frac{1-\nu}{\mathbb{B}(\nu)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2^\rho - e_1^\rho\right)^\nu}{\rho\mathbb{B}(\nu)\Gamma(\nu+3)}\right)M(e_1^\rho, e_2^\rho) + \frac{2\nu^2\left(e_2^\rho - e_1^\rho\right)^\nu}{\mathbb{B}(\nu)\Gamma(\nu+3)}N(e_1^\rho, e_2^\rho), \quad (3.30)\end{aligned}$$

where

$$M(e_1^\rho, e_2^\rho) = f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)$$

and

$$N(e_1^\rho, e_2^\rho) = f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho).$$

Proof. Since f and g are convex on $[e_1^\rho, e_2^\rho]$, then

$$f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \le t^\rho f(e_1^\rho) + (1-t^\rho)f(e_2^\rho) \quad (3.31)$$

and

$$g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \le t^\rho g(e_1^\rho) + (1-t^\rho)g(e_2^\rho). \quad (3.32)$$

From (3.31) and (3.32), we get

$$\begin{aligned}f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) \ &\le\ t^{2\rho} f(e_1^\rho)g(e_1^\rho) + (1-t^\rho)^2 f(e_2^\rho)g(e_2^\rho)\\ &+\ t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].\end{aligned}$$

Similarly,

$$\begin{aligned}f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho) \ &\le\ (1-t^\rho)^2 f(e_1^\rho)g(e_1^\rho) + t^{2\rho} f(e_2^\rho)g(e_2^\rho)\\ &+\ t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].\end{aligned}$$

By adding the above two inequalities, it follows that

$$\begin{aligned}&f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho) + f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)\\ \le\ &(2t^{2\rho} - 2t^\rho + 1)[f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)] + 2t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)].\end{aligned}$$

Multiplying both sides of above inequality by $\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}t^{\rho\nu-1}$ and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}&\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1} f(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)g(t^\rho e_1^\rho + (1-t^\rho)e_2^\rho)dt\\ &+\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1} f((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)g((1-t^\rho)e_1^\rho + t^\rho e_2^\rho)dt\\ \le\ &\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}(2t^{2\rho} - 2t^\rho + 1)[f(e_1^\rho)g(e_1^\rho) + f(e_2^\rho)g(e_2^\rho)]dt\\ +\ &\frac{\nu}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1} 2t^\rho(1-t^\rho)[f(e_1^\rho)g(e_2^\rho) + f(e_2^\rho)g(e_1^\rho)]dt\\ =\ &\frac{\nu M(e_1^\rho, e_2^\rho)}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}(2t^{2\rho} - 2t^\rho + 1)dt + \frac{2\nu N(e_1^\rho, e_2^\rho)}{\mathbb{B}(\nu)\Gamma(\nu)}\int_0^1 t^{\rho\nu-1}t^\rho(1-t^\rho)dt\\ =\ &\frac{\nu(\nu^2+\nu+2)}{\rho\mathbb{B}(\nu)\Gamma(\nu+3)}M(e_1^\rho, e_2^\rho) + \frac{2\nu^2}{\mathbb{B}(\nu)\Gamma(\nu+3)}N(e_1^\rho, e_2^\rho).\end{aligned}$$

By the change of variables and with simple integral calculations, we get the desired result. □

Corollary 3.35. *With the notations in Theorem 3.34, if we choose $f = g$, the following inequality for the ABK-fractional integrals holds:*

$$\begin{aligned}&\left[\, {}^{ABK}_{e_1^+}{}^{\rho}I^{\nu}_{e_2^{\rho}} f^2(e_2^{\rho}) + {}^{ABK}_{e_2^-}{}^{\rho}I^{\nu}_{e_1^{\rho}} f^2(e_1^{\rho})\right] \\ \leq\ & \left(\frac{1-\nu}{\mathbb{B}\left(\nu\right)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu}}{\rho\mathbb{B}\left(\nu\right)\Gamma\left(\nu+3\right)}\right) M_1(e_1^{\rho}, e_2^{\rho}) + \frac{2\nu^2\left(e_2^{\rho}-e_1^{\rho}\right)^{\nu}}{\mathbb{B}\left(\nu\right)\Gamma\left(\nu+3\right)} N_1(e_1^{\rho}, e_2^{\rho}), \quad (3.33)\end{aligned}$$

where

$$M_1(e_1^{\rho}, e_2^{\rho}) = f^2(e_1^{\rho}) + f^2(e_2^{\rho}), \quad N_1(e_1^{\rho}, e_2^{\rho}) = 2f(e_1^{\rho})f(e_2^{\rho}).$$

Corollary 3.36. *With the notations in Theorem 3.34, if we take $\rho \to 1$, the following inequality for the AB-fractional integrals holds:*

$$\begin{aligned}&\left[\, {}^{AB}_{e_1}I^{\nu}_{e_2} f(e_2)g(e_2) + {}^{AB}_{e_2}I^{\nu}_{e_1} f(e_1)g(e_1)\right] \\ \leq\ & \left(\frac{1-\nu}{\mathbb{B}\left(\nu\right)} + \frac{\nu\left(\nu^2+\nu+2\right)\left(e_2-e_1\right)^{\nu}}{\mathbb{B}\left(\nu\right)\Gamma\left(\nu+3\right)}\right) M(e_1, e_2) + \frac{2\nu^2\left(e_2-e_1\right)^{\nu}}{\mathbb{B}\left(\nu\right)\Gamma\left(\nu+3\right)} N(e_1, e_2). \quad (3.34)\end{aligned}$$

Remark 3.37. With the notations in our theorems given in Section 3, if we take ρ, $\nu \to 1$, then we get some classical integral inequalities.

Acknowledgements

The author would like to thank the referee for valuable comments and suggestions.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

A unified convergence analysis for single step-type methods for non-smooth operators

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May 7, 2019

Abstract

This paper is devoted to the approximation of solutions for nonlinear equations by using iterative methods. We present a unified convergence analysis for some Newton-type methods. We consider both semilocal and local analysis. In the first one, the hypotheses are imposed on the initial guess and in the second on the solution. The results can be applied for smooth and non-smooth operators. In the numerical section we study two applications, first one, it is devoted to a nonlinear integral equation of Hammerstein type and in second one, we approximate the solution of a nonlinear PDE related to image denoising.

1 Introduction

There are several situations in which the modeling of a problem leads us to calculate a solution of an equation

$$F(x) = 0. \tag{1}$$

This equation can represent a differential equation, ordinary or partial, an integral equation, an integro-differential equation or a simple system of equations. In general, mathematical methods that obtain exact solutions of (1) are not known, so that iterative methods are usually used to solve (1) [9, 10, 1, 2, 3, 4, 5, 7, 12]. For a greater generality,

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in this study, we consider $F : D \subset X \to Y$, where X, Y are Banach spaces and D is a nonempty, open and convex set. And we pay attention to F is continuous and Fréchet non-differentiable. In this case, to approximate a solution of (1), iterative methods using divided differences are usually applied instead of using derivatives [12]-[11]. It is common to approximate derivatives by divided differences for obtaining derivative free iterative schemes. So, given an operator $G : D \subset X \to Y$, let us denote by $\mathfrak{L}(X, Y)$ the space of bounded linear operators from X into Y, an operator $[x, y; G] \in \mathfrak{L}(X, Y)$ is called a first order divided difference for the operator G on the points x and y $(x \neq y)$ in D if

$$[x, y; G](x - y) = G(x) - G(y). \tag{2}$$

Steffensen's method [13] is the most used iterative method using divided differences in the algorithm, which is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [x_n, x_n + F(x_n); F]^{-1} F(x_n), \quad n \geq 0. \end{cases} \tag{3}$$

As we can see in [14], Steffensen's method has a problem of accessibility that can be solved by using a procedure of decomposition ([15]) for operator F, the Fréchet differentiable part and the non-differentiable part. So, we consider

$$F(x) = F_1(x) + F_2(x) \tag{4}$$

where $F_1, F_2 : D \subset X \to Y$, F_1 is Fréchet differentiable and F_2 is continuous and Fréchet non-differentiable. Thus, in [14], we consider the method of Newton-Steffensen, given by the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - \left(F_1'(x_n) + [x_n, x_n + F(x_n); F_2]\right)^{-1} (F_1(x_n) + F_2(x_n)), \quad n \geq 0, \end{cases} \tag{5}$$

with $X = Y$, which improves significantly the accessibility of method (3) and has quadratic convergence.

By using this procedure of decomposition for operator F, we see that we can also consider the application of iterative methods that use derivatives when F is non-differentiable. So, for example, we can consider the well-known Newton's method, which algorithm is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F'(x_n)]^{-1} F(x_n), \quad n \geq 0, \end{cases} \tag{6}$$

Obviously, Newton's method is not applicable, under form (6), when F is not Fréchet differentiable. However, if we consider decomposition of F given in (4), we can use the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F_1'(x_n)]^{-1} (F_1(x_n) + F_2(x_n)), \quad n \geq 0, \end{cases} \tag{7}$$

which is known as method of Zincenko [17].

The main aim of this paper consists of defining one-point iterative methods of Newton-type, as we can see previously, to obtain a general study for the convergence, local and semilocal, for these type of iterative methods. Moreover, in view of the last two considerations, with these one point iterative methods we can to improve the accessibility of one-point iterative methods that use divided differences and, in addition, to extend the application of iterative methods that use derivatives when F is Fréchet non-differentiable. For this aim, we consider the one-point iterative methods of Newton-type given by the following algorithm

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - L_n^{-1}(F_1(x_n) + F_2(x_n)), \quad n \geq 0, \end{cases} \tag{8}$$

where $L_n := L(x_n)$ with $L(.) : D \to \mathfrak{L}(X, Y)$. Clearly, method (8) can be used to solve equations containing a nondifferentiable term.

There are a lot of iterative methods that can be written as algorithm (8), in addition to modifications of Steffensen and Newton given in (5) and (7), where $L(x) = F_1'(x) + [x, x + F_2(x); F_2]$ and $L(x) = F_1'(x)$, respectively. At the same time, we can also consider two interesting cases. Firstly, the generalized Steffensen methods [6], that are very used in the approximation of solutions of non-differentiable operators equations and the algorithm is

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [x_n - aF(x_n), x_n + bF(x_n); F]^{-1}F(x_n), \quad n \geq 0. \end{cases}$$

Then, it is clear that we can define the generalized Newton-Steffensen method from 8) with $L(x) = F_1'(x) + [x - aF_2(x), x + bF_2(x); F_2]$, so we have the final iterative function given as:

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - (F_1'(x_n) + [x_n - aF_2(x_n), x_n + bF_2(x_n); F_2])^{-1}F(x_n), \quad n \geq 0. \end{cases} \tag{9}$$

where $a, b \in \mathbb{R}$.

In the same way as Newton's method, from Stirling method [16],

$$\begin{cases} x_0 \text{ given in } D, \\ x_{n+1} = x_n - [F_1'(x_n - F(x_n))]^{-1}F(x_n), \quad n \geq 0, \end{cases} \tag{10}$$

we can define a modification of Newton-type, that can be applied to Fréchet non-differentiable operators. For this, just consider (8) with $L(x) = F_1'(x - F(x))$. In both cases, we choose $X = Y$. Obviously, we can include a lot of iterative methods in (8) if F is Fréchet differentiable.

So, in this paper, we study the convergence of algorithm (8). We analyze the semilocal and local convergences, so that we have a study of convergence of a lot of iterative methods that are usually used and can be written by algorithm (8).

Section 2 is devoted to the theoretical analysis about local and semilocal convergence for a very general single step Newton-like methods. In Section 3 we make a comparison for the behavior of some of these methods by solving a non-differentiable problem. In Section 4, we consider an application related to image denoising. Finally, in Section 5 we give some conclusions.

2 Convergence Analysis for single step Newton-like methods

In this section, we present both semilocal and local convergence analysis. In the first one, the hypotheses are imposed on the initial guess and in the second on the solution. The results can be applied for smooth and non-smooth operators.

2.1 Local Convergence Analysis

In this section, we first present the local followed by the semilocal convergence of method (8). Let $v_0 : [0,+\infty) \to [0,+\infty)$ be a nondecreasing continuous function with $v_0(0) = 0$. Suppose that the equation

$$v_0(t) = 1 \tag{11}$$

has at least one positive root r_0. Let also $v : [0, r_0) \to [0,+\infty)$ be a nondecreasing continuous function. Define function $\bar{v}$ on the interval $[0, r_0)$ by $\bar{v}(t) = \frac{v(t)}{1-v_0(t)} - 1$.

Suppose equation

$$\bar{v}(t) = 0 \tag{12}$$

has at least one positive root. Denote by r the smallest such root. It follows that for each $t \in [0, r)$

$$0 \leq v_0(t) < 1 \tag{13}$$

and

$$0 \leq \bar{v}(t) < 1. \tag{14}$$

The local convergence analysis of method (8) uses the conditions (A):

- (a_1) There exist a solution $x^* \in D$ of equation (4), and $B \in \mathfrak{L}(X,Y)$ such that $B^{-1} \in \mathfrak{L}(Y,X)$.
- (a_2) Condition (11) holds and for each $x \in D$

  $$\|B^{-1}(L(x) - B)\| \leq v_0(\|x - x^*\|),$$

  where v_0 is defined previously and r_0 is given in (11).

  Set $D_0 = D \cap \bar{U}(x*, r_0)$.

- (a_3) For $L : D_0 \to \mathfrak{L}(X, Y)$, any solution y of equation (4) and each $x \in D_0$

  $$\|B^{-1}(F_1(x) + F_2(x) - L(x)(x - y))\| \leq v(\|x - y\|)\|x - y\|,$$

  where v is defined previously.

- (a_4) $\bar{U}(x^*, r) \subset D$, where r is given in (12).

- (a_5) There exist $r^* \geq r$ such that

  $$\xi := \frac{v(r^*)}{1 - v_0(r)} \in [0, 1).$$

  Set $D_1 = D \cap \bar{U}(x*, r^*)$.

Remark 1

- *Condition (a_3) can be replaced by the stronger: for each $x, y, z \in D_0$*

  $$\|B^{-1}(F_1(x) + F_2(x) - L(x)(x - y))\| \leq v_1(\|x - y\|)\|x - y\|,$$

  where function v_1 is as v. But for each $t \geq 0$

  $$v(t) \leq v_1(t).$$

- *Linear operator B does not necessarily depend on the solution x^*. It is used to determine the invertibility of linear operator $L(\cdot)$ appearing in the method. The invertibility of B can be assured by an additional condition of the form $||I - B|| < 1$ or some other way. A possible choice for B is $B = B(x^*)$ or $B = F_1'(x^*)$.*

- *It follows from the definition of r_0 and r that $r_0 \geq r$.*

We can present the local convergence analysis of method (8) based on the aforementioned conditions (A).

Theorem 2 *Suppose that the conditions (A) hold. Then, sequence x_k generated by method (8) for $x_0 \in U(x^*, r) - x^*$ is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ and converges to x^*. Moreover, the following estimates hold.*

$$\|x_{k+1} - x^*\| \leq \frac{v(\|x_k - x^*\|)}{1 - v_0(\|x_k - x^*\|)}\|x_k - x^*\| \leq \|x_k - x^*\| < r. \tag{15}$$

The vector x^* is the only solution of equation (4) in D_1, where D_1 is given in $(a5)$.

Proof We base the proof on k and mathematical induction. Let $x \in U(x^*, r)$. Using (8), $(a1)$ and $(a2)$, we have in turn that

$$\|B^{-1}(L(x) - B)\| \leq v_0(\|x - x^*\|) \leq v_0(r) < 1. \tag{16}$$

It follows by (16) and the Banach lemma on invertible operators [] that $L(x)^{-1} \in \mathfrak{L}(Y, X)$ and

$$\|L(x)^{-1}B\| \leq \frac{1}{1 - v_0(\|x - x^*\|)}. \tag{17}$$

In particular, estimate (17) holds for $x = x_0$, so x_1 is well defined by method (8) for $k = 0$. We also get by method (8) (for $k = 0$), $(a1)$, $(a3)$, (14) and (17) (for $k = 0$) that

$$\begin{aligned} \|x_1 - x^*\| &= \|x_0 - x^* - L(x_0)^{-1}(F_1(x_0) + F_2(x_0))\| \\ &= \|[-L(x_0)^{-1}B][B^{-1}(F_1(x_0) + F_2(x_0) - L(x_0)(x_0 - x^*))]\| \\ &\leq \|L(x_0)^{-1}B\|\|B^{-1}(F_1(x_0) + F_2(x_0) - L(x_0)(x_0 - x^*))\| \\ &\leq \frac{v(\|x_0 - x^*\|)}{1 - v_0(\|x_0 - x^*\|)}\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \end{aligned} \tag{18}$$

which shows estimate (15) for $k = 0$, and $x_1 \in U(x^*, r)$.

Simply, replace x_0, x_1 by x_i, x_{i+1} in the preceding estimates to complete the induction for estimate (15). Then, in view of the estimate

$$\|x_{i+1} - x^*\| \leq \xi\|x_i - x^*\| < r, \tag{19}$$

where

$$\xi = \frac{v(\|x_0 - x^*\|)}{1 - v_0(\|x_0 - x^*\|)} \in [0, 1),$$

we deduce that $\lim_{i\to+\infty} x_i = x^*$ and $x_{i+1} \in U(x^*, r)$. Moreover, to show the uniqueness part, let $y^* \in D_1$ with $F_1(y^*) + F_2(y^*) = 0$. Using $(a3)$, $(a5)$ and estimate (18), we obtain in turn that

$$\begin{aligned} \|x_{i+1} - y^*\| &\leq \|L(x_i)^{-1}B\|\|B^{-1}(F_1(x_i) + F_2(x_i) - L(x_i)(x_i - y^*))\| \\ &\leq \frac{v(\|x_i - y^*\|)}{1 - v_0(\|x_i - x^*\|)}\|x_i - y^*\| \\ &\leq \xi\|x_i - y^*\| < \xi^{i+1}\|x_0 - y^*\|, \end{aligned} \tag{20}$$

which shows $\lim_{i\to+\infty} x_i = y^*$. But, we showed $\lim_{i\to+\infty} x_i = x^*$. Hence, we conclude that $x^* = y^*$.

□

2.2 Semilocal Convergence Analysis

As in the local case it is convenient to define some functions and parameters for the semilocal analysis. Let $w_0 : [0, +\infty) \to [0, +\infty)$ be a continuous and nondecreasing function.

Suppose that equation

$$w_0(t) = 1. \tag{21}$$

has at least one positive root. Denote by ρ_0 the smallest such root. Let also $w : [0, \rho_0) \times [0, \rho_0) \to [0, +\infty)$ be a nondecreasing continuous function. Moreover, for $\eta \geq 0$, define parameters C_1 and C_2 by

$$\begin{aligned} C_1 &= \frac{w(\eta, 0)}{1 - w_0(\eta)}, \\ C_2 &= \frac{w(\frac{\eta}{1-C_1}, \eta)}{1 - w_0(\frac{\eta}{1-C_1})} \end{aligned}$$

and function $C : [0, \rho_0) \to [0, +\infty)$ by $C(t) = \frac{w(t,t)}{1-w_0(t)}$. Suppose that equation

$$(\frac{C_1 C_2}{1 - C(t)} + C_1 + 1)\eta - t = 0 \tag{22}$$

has as least one positive root. Denote by ρ the smallest such root.

Next, we show the semilocal convergence analysis of method (8) in an analogous way, under the conditions (H):

- $(h1)$ There exists $x_0 \in D$ and $B \in \mathfrak{L}(X, Y)$ such that $B^{-1} \in \mathfrak{L}(Y, X)$.
- $(h2)$ Condition (21) holds and for each $x \in D$

  $$\|B^{-1}(L(x) - B)\| \leq w_0(\|x - x_0\|),$$

  where w_0 is as defined previously, and ρ_0 is given in (21).

  Set $D_2 = D \bigcap \bar{U}(x_0, \rho_0)$.
- $(h3)$ For $L(\cdot) : D_2 \to \mathfrak{L}(X, Y)$, and each $x, y \in D_2$

  $$\begin{aligned} &\|B^{-1}(F_1(y) - F_1(x) + F_2(y) - F_2(x) - L(x)(y - x))\| \\ &\qquad \leq w(\|y - x_0\|, \|x - x_0\|)\|y - x\|, \end{aligned}$$

  where w is as defined previously.
- $(h4)$ $\bar{U}(x_0, \rho) \subseteq D$ and condition (22) holds for ρ, where $\|x_1 - x_0\| \leq \eta$.
- $(h5)$ There exists $\rho^* \geq \rho$ such that

  $$\xi_0 := \frac{w(\rho, \rho^*)}{1 - w_0(\rho)} \in [0, 1).$$

  Set $D_2 = D \bigcap \bar{U}(x^*, \rho^*)$.

Then, as in the local case but using the (H) instead of the (A) conditions, we have in turn the estimates:

$$
\begin{aligned}
\|x_2-x_1\| &\le \frac{w(\|x_1-x_0\|,\|x_0-x_0\|)}{1-w_0(\|x_1-x_0\|)} = C_1\|x_1-x_0\|,\\
\|x_2-x_0\| &\le \|x_2-x_1\|+\|x_1-x_0\| \le (1+C_1)\|x_1-x_0\|\\
&= \frac{1-C_1^2}{1-C_1}\|x_1-x_0\|\\
&< \frac{\|x_1-x_0\|}{1-C_1}\eta < \rho,\\
\|x_3-x_2\| &\le \frac{w(\|x_2-x_0\|,\|x_1-x_0\|)}{1-w_0(\|x_2-x_0\|)}\|x_2-x_1\|\\
&\le \frac{w(\frac{\eta}{1-C_1},\eta)}{1-w_0(\frac{\eta}{1-C_1})}\|x_2-x_1\| = C_2\|x_2-x_1\|\\
\|x_3-x_0\| &\le \|x_3-x_2\|+\|x_2-x_1\|+\|x_1-x_0\|\\
&\le C_2\|x_2-x_1\|+C_1\|x_1-x_0\|+\|x_1-x_0\|\\
&\le (C_2C_1+C_1+1)\|x_1-x_0\|,\\
\|x_4-x_3\| &\le \frac{w(\|x_3-x_0\|,\|x_2-x_0\|)}{1-w_0(\|x_3-x_0\|)}\|x_3-x_2\|\\
&\le C(\rho)\|x_3-x_2\| \le C(\rho)C_2\|x_2-x_1\|\\
&\le C(\rho)C_2C_1\|x_1-x_0\|,\\
&\quad \ldots\\
\|x_{i+1}-x_i\| &\le C(\rho)\|x_i-x_{i-1}\| \le C(\rho)^{i-2}\|x_3-x_2\|\\
\|x_{i+1}-x_0\| &\le \|x_{i+1}-x_i\|+\ldots+\|x_4-x_3\|+\|x_3-x_0\|\\
&\le C(\rho)\|x_i-x_{i-1}\|+\ldots+C(\rho)\|x_3-x_2\|\\
&\quad +(C_2C_1+C_1+1)\|x_1-x_0\|\\
&\le C(\rho)^{i-2}\|x_3-x_2\|+\ldots+C(\rho)\|x_3-x_2\|\\
&\quad +(C_2C_1+C_1+1)\|x_1-x_0\|\\
&\le (\frac{1-C(\rho)^{i-1}}{1-C(\rho)}C_2C_1+C_1+1)\|x_1-x_0\|\\
&< (\frac{C_1C_2}{1-C(\rho)}+C_1+1)\eta \le \rho, \qquad (23)
\end{aligned}
$$

$$\begin{aligned}\|x_{i+j}-x_i\| &\leq \|x_{i+j}-x_{i+j-1}\|+\|x_{i+j-1}-x_{i+j-2}\|+\ldots+\|x_{i+1}-x_i\|\\ &\leq (C(\rho)^{i+j-3}+\ldots+C(\rho)^{i-2})\|x_3-x_2\|\\ &\leq C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}\|x_3-x_2\|\\ &\leq C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}C_2C_1\|x_1-x_0\|\\ &\leq C(\rho)^{i-2}\frac{1-C(\rho)^{j-1}}{1-C(\rho)}C_2C_1\eta. \qquad (24)\end{aligned}$$

It follows from (23) that $x_i \in U(x_0,\rho)$ and from (24) that sequence x_i is complete in X and as such it converges to some $x^* \in \bar{U}(x_0,\rho)$. By letting $i \to +\infty$ in the estimate

$$\|B^{-1}(F_1(x_i)+F_2(x_i))\| = \|B^{-1}(F_1(x_i)+F_2(x_i)-F_1(x_{i-1})-F_2(x_{i-1})-B_{i-1}(x_i-x_{i-1}))\|$$

$$\leq \frac{w(\|x_i-x_0\|,\|x_{i-1}-x_0\|)\|\|x_i-x_{i-1}\|}{1-w_0(\|x_i-x_0\|)} \leq \frac{w(\rho,\rho)}{1-w_0(\rho)}\|x_i-x_{i-1}\|,$$

we obtain $F_1(x^*)+F_2(x^*)=0$. The uniqueness part is omitted as identical to the one in the local convergence case.

Hence, we arrived at the semilocal convergence result for method (8).

Theorem 3 *Suppose that the conditions (H) hold. Then, sequence x_k generated by method (8) for $x_0 \in D$ is well defined in $U(x_0,\rho)$ remains in $U(x_0,\rho)$ and converges $x^* \in \bar{U}(x_0,\rho)$ to a solution of equation (4). Moreover, the vector x^* is the only solution of equation (4) in D_3, where D_3 is defined previously.*

The same comments introduced in the previous remark are valid.

We emphasize the theoretical importance of this theorem because it presents a unified studied of the local and semilocal convergence of a big variety of Newton-Type methods and Steffensen type methods, so the study is applicable to differentiable an non differentiable equations.

3 Numerical Experiments

In this section, we consider a nonlinear integral equation of Hammerstein type, which can be used to describe applied problems in the fields of electro-magnetics, fluid dynamics, in the kinetic theory of gases and, in general, in the reformulation of boundary value problems. These equations are of the form:

$$x(s) = f(s) - \int_a^b K(s,t)\Phi(x(t))dt, \quad a \leq s \leq b, \qquad (25)$$

where $x(s), f(s) \in C[a,b]$, with $-\infty < a < b < \infty$, and Φ is a polynomial function. One of the most used techniques to solve this kind of equations consists of expressing them as a nonlinear operator in a Banach space and solving the following operator equation:

$$F(x)(s) = x(s) - f(s) + \int_a^b K(s,t)\Phi(x(t))dt = 0, \tag{26}$$

where $F : D \subseteq C[a,b] \to C[a,b]$ with D a non-empty open convex subset of $C[a,b]$ with the max-norm $\|\nu\| = \max_{s\in[a,b]} |\nu(s)|$.

We consider (25), where K is the Green function in $[a,b] \times [a,b]$, and then use a discretization process to transform equation (26) into a finite dimensional problem by approximating the integral by an adequate quadrature formula

$$\int_a^b q(t)\,dt \simeq \sum_{i=1}^{p} w_i q(t_i),$$

where the nodes t_i and the weights w_i are known.

If we denote the approximations of $x(t_i)$ and $f(t_i)$ by x_i and f_i, respectively, with $i = 1, 2, \ldots, p$, then equation (26) is equivalent to the following system of nonlinear equations:

$$x_i = f_i + \sum_{j=1}^{p} a_{ij}\,\Phi(x_j), \quad j = 1, 2, \ldots, p, \tag{27}$$

where

$$a_{ij} = w_j K(t_i, t_j) = \begin{cases} w_j \frac{(b-t_i)(t_j-a)}{b-a}, & j \le i, \\ w_j \frac{(b-t_j)(t_i-a)}{b-a}, & j > i. \end{cases}$$

Now, system (27) can be written as

$$\mathbb{F}(\mathbf{x}) \equiv \mathbf{x} - \mathbf{f} - A\,\mathbf{z} = 0, \qquad \mathbb{F} : \Delta \subseteq \mathbb{R}^p \longrightarrow \mathbb{R}^p, \tag{28}$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_p)^T, \qquad \mathbf{f} = (f_1, f_2, \ldots, f_p)^T, \qquad A = (a_{ij})_{i,j=1}^{p},$$
$$\mathbf{z} = (\Phi(x_1), \Phi(x_2), \ldots, \Phi(x_p))^T.$$

After that, we choose $a = 0$, $b = 1$, $K(s,t)$ as the Green function in $[0,1] \times [0,1]$ and $\Phi(x(t)) = x(t)^3 + |x(t)|$ in (25). Then, the system of nonlinear equations given in (28) is of the form

$$\mathbb{F}(\mathbf{x}) = \mathbf{x} - \mathbf{f} - A\,(\mathbf{v}_{\mathbf{x}} + \mathbf{w}_{\mathbf{x}}) = 0, \qquad \mathbb{F} : \mathbb{R}^p \longrightarrow \mathbb{R}^p, \tag{29}$$

where

$$\mathbf{v}_{\mathbf{x}} = (x_1^3, x_2^3, \ldots, x_p^3)^T, \qquad \mathbf{w}_{\mathbf{x}} = (|x_1|, |x_2|, \ldots, |x_p|)^T.$$

It is obvious that the function $\mathbb{F}$ defined in (29) is nonlinear and non-differentiable. So, we consider $\mathbb{F}(\mathbf{x}) = \mathbb{F}_1(\mathbf{x}) + \mathbb{F}_2(\mathbf{x})$ where:

$$\mathbb{F}_1(\mathbf{x}) = \mathbf{x} - \mathbf{f} - A\mathbf{v}_{\mathbf{x}} \quad \text{and} \quad \mathbb{F}_2(\mathbf{x}) = -\mathrm{A}\mathbf{w}_{\mathbf{x}}.$$

As in $\mathbb{R}^p$ we can consider divided difference of first order that do not need that the function $\mathbb{F}$ is differentiable (see [16]), we use the divided difference of first order given by $[\mathbf{u},\mathbf{v};\mathbb{G}] = ([\mathbf{u},\mathbf{v};\mathbb{G}]_{ij})_{i,j=1}^{p} \in \mathcal{L}(\mathbb{R}^p,\mathbb{R}^p)$, where

$$[\mathbf{u},\mathbf{v};\mathbb{G}]_{ij} = \frac{1}{u_j - v_j}\left(\mathbb{G}_i(u_1,\ldots,u_j,v_{j+1},\ldots,v_p) - \mathbb{G}_i(u_1,\ldots,u_{j-1},v_j,\ldots,v_p)\right), \tag{30}$$

if $u_j \neq v_j$, in other case $[\mathbf{u},\mathbf{v};\mathbb{G}]_{ij} = 0$, for $\mathbf{u} = (u_1,u_2,\ldots,u_p)^T$ and $\mathbf{v} = (v_1,v_2,\ldots,v_p)^T$.

Now, to compare the behavior of different methods we consider the case $\mathbf{f} = \mathbf{0}$ in (29). Obviously, for this problem, $\mathbf{x}^* = \mathbf{0}$ is a solution of $\mathbb{F}(\mathbf{x}) = \mathbf{0}$. Then, the system of nonlinear equations given in (29) is of the form

$$\mathbb{F}(\mathbf{x}) = \mathbf{x} - A\,\mathbf{z}, \qquad z_j = x_j^3 + |x_j|,\ j = 1,\ldots,p. \tag{31}$$

The numerical results are obtained by using MATLAB 2018 and working with variable precision arithmetic with 100 digits. In Table 1 we can see the results obtained by using the methods mentioned in our study. First of all we take nodes and weights of Trapezoidal rule with $n = 10$ subintervals for approximatting the integral and starting guess $x_0(t) = 1/2\ \forall t \in [0,1]$. We compare the distance between consecutive iterates of the first 7 iterations of each method. In the case of the Newton-Steffensen General method (9), the parameters involved are $a = 0.5$ and $b = 1.5$.

	Stirling (10)	Zincenko (7)	Steffensen (3)	New-Steff. (5)	New-Steff. Gen.(9)
1	1.5887	1.1637	7.4375	2.9044	2.9044
2	$6.0578e-01$	$3.0210e-01$	$2.7350e-01$	1.3867	1.3867
3	$4.7941e-01$	$1.2065e-01$	$1.8235e-02$	$3.2041e-01$	$1.2942e-01$
4	$4.1942e-01$	$4.9511e-02$	$5.5411e-05$	$2.8725e-04$	$2.8725e-04$
5	$3.5456e-01$	$2.0403e-02$	$2.8134e-09$	$1.3552e-12$	$1.3552e-12$
6	$1.9024e-01$	$8.4133e-03$	$3.0173e-18$	$3.1538e-37$	$3.3246e-37$
7	$2.9676e-02$	$3.4697e-03$	$3.9490e-36$	$1.7796e-111$	$2.1782e-111$

Table 1: Results with different methods in the first iterations.

In Table 2 we work with same conditions, we obtain the iterations that each method needs to satisfy the stopping criterion $||x_{k+1} - x_k|| \leq 10^{-40}$. It should be noted that the first two methods never meet the required tolerance because they are not convergent and, therefore, the methods end when the required iterations are completed (in this case 15 iterations at most). Second, we observe a good approximation to the order of convergence of each method p in case the method converges. In the last two rows of Table 2 we compare

	Stirling (10)	Zincenko (7)	Steffensen (3)	New-Steff. (5)	New-Steff. Gen.(9)
k	15	15	8	7	7
p	1.0000	1.0000	1.9994	3.0142	3.0148
$\|\|x_{k-1} - x_k\|\|$	$2.3258e-04$	$2.9041e-06$	$6.9382e-72$	$1.7796e-111$	$2.1782e-111$
$\|\|F(x_k)\|\|$	$9.5985e-05$	$1.1977e-06$	$1.2745e-107$	$7.8863e-219$	$6.8587e-219$

Table 2: Numerical results for comparing the proposed methods.

the difference between the last iterates of each method and we also see the norm of the function evaluted in the last iteration.

Now, we also want to use the Gauss-Legendre quadrature to approximate the integral of equation (25). Moreover, by using the Newton-Steffensen method we compare two different possibilities for implementing the divided differences given in (30), that is, in Tables 1 and 2 we obtain the divided difference like $[x_n, x_n + F_1(x_n) + F_2(x_n), F_2]$ but we want to compare with $[x_n, x_n + F_2(x_n), F_2]$. The results in Table 3 show that the use of first form used for obtaining the divided differences gives better residual errors, which was expected because $F_1(x_n) + F_2(x_n)$ tends to zero quicker than $F_2(x_n)$. Even in some different example the value $F_2(x_n)$ could not tend to zero, in this case only first form of obtaining the divided differences considered would work. In Table 3 we have also included the computational time, as can be observed in the last row, notice that the use of Gauss-Legedre quadrature needs much more time than the trapezoidal rule although in some cases reaches better accuracy.

	$\|\|x_n - x_{n-1}\|\|$			
Iterations	*Trapezoidal rule*		*Gauss − Legendre*	
n	$[x, x + F_1 + F_2, F_2]$	$[x, x + F_2, F_2]$	$[x, x + F_1 + F_2, F_2]$	$[x, x + F_2, F_2]$
1	2.9044	2.9044	2.7204	2.7204
2	1.3867	1.3867	1.1355	1.1355
3	$3.2041e-01$	$1.2942e-01$	$6.6978e-02$	$6.6978e-02$
4	$2.8725e-04$	$2.8725e-04$	$3.4608e-05$	$3.4608e-05$
5	$1.3552e-12$	$1.3552e-12$	$2.1448e-15$	$2.1448e-15$
6	$3.1538e-37$	$3.3489e-28$	$1.124e-45$	$1.124e-45$
7	$1.7796e-111$	$1.3651e-43$	$8.0773e-137$	$7.8571e-137$

Table 3: Results with Trapezoidal rule and Gauss-Legendre method by using different form of obtaining the divided differences.

	$Trapezoidal \quad rule$		$Gauss - Legendre$	
n	$[x, x+F_1+F_2, F_2]$	$[x, x+F_2, F_2]$	$[x, x+F_1+F_2, F_2]$	$[x, x+F_2, F_2]$
k	7	8	7	7
p	3.0142	$unstable$	3.0099	3.0103
$\|\|x_{k-1} - x_k\|\|$	$1.7796e-111$	$4.3463e-59$	$8.0772e-137$	$7.8571e-137$
$\|\|F(x_k)\|\|$	$7.8863e-219$	$1.0160e-74$	$1.3057e-243$	$1.5367e-138$
$time$	17.796129	20.6134	282.5403	309.3090

Table 4: Numerical results and computational time for comparing the proposed methods.

4 Approximating the solution of a nonlinear PDE related to image denoising

In some steps of the manipulation of an image, some random noise is usually introduced. This noise makes the later steps of processing the image difficult and inaccurate.

In many applications like astrophysics, astronomy or meteorology we have to manipulate images contaminated by noise. The image processing becomes difficult and inaccurate. For these reasons, usually some image denoising strategies are developed. In this paper, we center our attention in the PDE framework.

Let $f\colon \Omega \to \mathbb{R}$ be a signal or image which we would like to denoise.

The usual PDE frameworks start with constrained optimization problems like

$$\begin{aligned} &\text{Minimize in } u\colon R(u) \\ &\text{subject to } \|u-f\|^2_{L^2(\Omega)} = |\Omega|\sigma^2. \end{aligned}$$

where $n = u - f$ denotes the noise. If there is no good estimate of the variance of the noise, then we may consider the unconstrained optimization problem.

Different linear regularization functionals $R(u)$ can be consider, the most used is $\|\nabla u\|_{L^2}$. This type of functionals introduce diffusion near the edges of the images, this is their main limitation.

The TV norm does not penalize discontinuities in u, and thus allows us to improve the approximation near the edges.

$$\int_\Omega |\nabla u(x)| dx.$$

For the linear model its Euler–Lagrange equation, with Neumann's boundary conditions for u, is

$$-\triangle u + \lambda(u-f) = 0, \tag{32}$$

which comes from the corresponding unconstrained problem with the norm $\|\nabla u\|^2_{L^2(\Omega)}$ and where the positive parameter λ determines the relative importance of the smoothness of u and the quality of the approximation to the given signal f..

For the TV- model we have

$$-\nabla \cdot (\frac{\nabla u}{|\nabla u|}) + \lambda(u - f) = 0. \tag{33}$$

In practice, the term $|\nabla u|$ is replaced by $\sqrt{|\nabla u|^2 + \epsilon}$, but even after this regularization, Newton's method does not work satisfactorily in the sense that its domain of convergence is very small. This is especially true if the regularizing parameter ϵ is small.

On the other hand, while the singularity and nondifferentiability of the term $w = \nabla u/|\nabla u|$ is the source of numerical problems, w itself is usually smooth because it is in fact the unit vector normal to the level sets of u. The numerical difficulties arise only because we linearize it the wrong way.

Thus we should introduce a new variable w; namely

$$w = \frac{\nabla u}{\sqrt{|\nabla u|^2}},$$

and replace (33) by the equivalent system of nonlinear PDEs:

$$\begin{aligned} -\nabla \cdot w + \lambda(u - f) &= 0, \\ w\sqrt{|\nabla u|^2} - \nabla u &= 0. \end{aligned}$$

Without the inclusion of the above regularization parameter ϵ, this system is nonlinear and nondifferentiable .

4.1 Discretization and numerical implementation

We present a comparison between the nonlinear model and the linear model using a simple finite difference discretization procedure.

For a regular mesh of size $h = 1/m$, $m \in \mathbb{N}$ $(x_i = i \cdot h,\ i = 0, \ldots, m)$, if in each iteration k we approximate the divergence and the gradient operators (these operators are the same in 1D) by

$$\nabla \cdot v(x_i) = \nabla v(x_i) \approx \frac{v_i - v_{i-1}}{h},$$

we obtain a nonlinear system for the unknowns w_i and u_i.

That is,

$$\begin{aligned} -\frac{w_i - w_{i-1}}{h} - \lambda(u_i - f_i) &= 0, \quad w_1 = w_m = 0, \\ w_i \cdot \sqrt{(\frac{u_i - u_{i-1}}{h})^2 - \frac{u_i - u_{i-1}}{h}} &= 0, \quad u_0 = f_0, u_m = f_m, \end{aligned}$$

for $i = 1, \dots, m-1$.

We then consider the nonlinear and nondifferentiable operator

$$\begin{aligned} F_{2i-1}(u, w, \lambda_h) &= w_i - w_{i-1} + \lambda_h(u_i - f_i) \ = \ 0, \\ F_{2i}(u, w, \lambda_h) &= w_i\sqrt{(u_i - u_{i-1})^2 +} - (u_i - u_{i-1}) \ = \ 0, \quad 1 \leq i \leq m-1, \end{aligned}$$

with $\lambda_h = h\,\lambda$, $w_0 = w_m = 0$, $u_0 = f_0$ and $u_m = f_m$.

For the discretization of the linear model we can consider the system

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \lambda(u_i - f_i) \ = \ 0, \quad u_0 = f_0, u_m = f_m,$$

for $i = 1, \dots, m-1$.

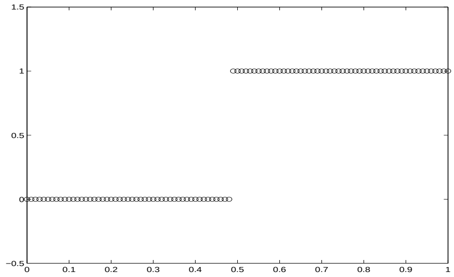

Figure 1: Original signal with a jump singularity.

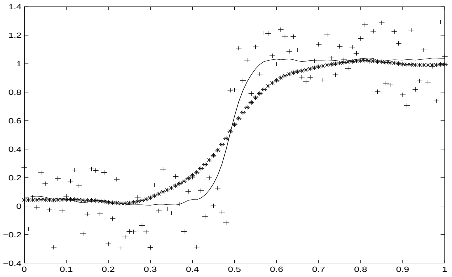

Figure 2: Solid lines = nonlinear model, starred lines = linear model and + lines = signal with noise. Noise level = 0.3, $\lambda = 10$.

In Figure 2, the solid lines are the function reconstructed by the nonlinear model approximated by the linearization based on a dual variable, solving the nonlinear system of equations by Steffensen's method 3 and the starred lines are given by the standard linear model, solving the associated linear system of equations by Gauss's method. The line with '+' is the noisy signal. The linear model introduces too much diffusion, giving a continuous function.

5 Conclusions

We have to point out the generalization of this study in which we have analyzed the local and semilocal convergence for Newton type methods and Steffensen like methods, so we can consider Newton-Steffensen's methods. The main idea it is to apply these kind of study to non-differentiable equations by taking in to account the advantages of consider the decomposition of the nonlinear equation into a sum of the differentiable part and the one non-differentiable.

Acknowledgements

Research of the first and third authors supported in part by Programa de Apoyo a la investigación de la fundación Séneca-Agencia de Ciencia y Tecnología de la Región de Murcia 19374/PI/14 and by MTM2015-64382-P. Research of the fourth and fifth authors supported in part by the project MTM2014-52016-C2-1-2-P of the Spanish Ministry of Science and Innovation and by the project of Generalitat Valenciana Prometeo/2016/089.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On the Localization of Factored Fourier Series

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Abstract

In the present paper, a theorem concerning local property of $|A, p_n|_k$ summability of factored Fourier series, which generalizes a result dealing with $|\bar{N}, p_n|_k$ summability of factored Fourier series, has been obtained. Also, some results have been given.

2010 AMS Mathematics Subject Classification : 26D15, 40D15, 40F05, 40G99, 42A24.
Keywords and Phrases :Absolute matrix summability, Fourier series, Hölder inequality, Infinite series, Local property, Minkowski inequality, Summability factors.

1 Introduction

Let $\sum a_n$ be an infinite series with its partial sums (s_n) and (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1) .$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

The series $\sum a_n$ is said to be summable $|A,p_n|_k$, $k\geq 1$, if (see [21])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|A_n(s)-A_{n-1}(s)|^k<\infty.$$

If we take $a_{nv}=\frac{p_v}{P_n}$, then $|A,p_n|_k$ summability reduces to $|\bar{N},p_n|_k$ summability (see [2]). If we take $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all values of n (resp. $a_{nv}=\frac{p_v}{P_n}$ and $k=1$), $|A,p_n|_k$ summability reduces to $|C,1|_k$ summability (see [11]) (resp. $|\bar{N},p_n|$) summability. Also, if we take $p_n=1$ for all values of n, then $|A,p_n|_k$ summability reduces to $|A|_k$ summability (see [22]). Furthermore, if we take $a_{nv}=\frac{p_v}{P_n}$, then $|A|_k$ summability reduces to $|R,p_n|_k$ summability (see [4]).

A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n\geq 0$ for every positive integer n, where $\Delta^2\lambda_n=\Delta(\Delta\lambda_n)$ and $\Delta\lambda_n=\lambda_n-\lambda_{n+1}$ (see [24]).

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi,\pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi}f(t)dt=0$$

and

$$f(t)\sim\sum_{n=1}^{\infty}(a_n cosnt+b_n sinnt) \quad = \quad \sum_{n=1}^{\infty}C_n(t),$$

where (a_n) and (b_n) denote the Fourier coefficients. It is well known that the convergence of the Fourier series at $t=x$ is a local property of the generating function f (i.e. it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t=x$ by any regular linear summability method is also a local property of the generating function f (see [23]).

2 Known Results

There are many different applications of Fourier series. Some of them can be find in [1], [5]-[10], [12]-[20]. Furthermore, Bor [3] has proved the following theorem.

Theorem 1 *Let $k \geq 1$ and (p_n) be a sequence such that*

$$P_n = O(np_n), \tag{1}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{2}$$

Then the summability $|\bar{N}, p_n|_k$ of the series $\sum \frac{C_n(t)\lambda_n P_n}{np_n}$ at a point can be ensured by local property, where (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.

3 Main Result

The purpose of this paper is to generalize Theorem 1 by using the definition of $|A, p_n|_k$ summability. Now, let us introduce some further notations. Let $A = (a_{nv})$ be a normal matrix, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{3}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{4}$$

and it is well known that

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{5}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{6}$$

Now, we will prove the following theorem.

Theorem 2 *Let $k \geq 1$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \ n = 0, 1, ..., \tag{7}$$

$$a_{n-1,v} \geq a_{nv}, \ for \ n \geq v + 1, \tag{8}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{9}$$

$$|\hat{a}_{n,v+1}| = O\left(v\left|\Delta_v \hat{a}_{nv}\right|\right), \tag{10}$$

where $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1}$. *Let the sequence* (p_n) *be such that the conditions (1) and (2) of Theorem 1 are satisfied. Then the summability* $|A, p_n|_k$ *of the series* $\sum \frac{C_n(t)\lambda_n P_n}{np_n}$ *at a point can be ensured by local property, where* (λ_n) *is as in Theorem 1.*

Here, if we take $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1.

We should give the following lemmas for the proof of Theorem 2.

Lemma 3 ([13]) *If the sequence* (p_n) *is such that the conditions (1) and (2) of Theorem 1 are satisfied, then*

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{11}$$

Lemma 4 ([10]) *If* (λ_n) *is a convex sequence such that* $\sum n^{-1}\lambda_n$ *is convergent, then* (λ_n) *is non-negative and decreasing, and* $n\Delta\lambda_n \to 0 \quad as \quad n \to \infty$.

Lemma 5 *Let* $k \geq 1$ *and let the sequence* (p_n) *be such that the conditions (1) and (2) of Theorem 1 are satisfied. If* (s_n) *is bounded and the conditions (7)-(10) are satisfied, then the series*

$$\sum_{n=1}^{\infty} \frac{a_n \lambda_n P_n}{np_n} \tag{12}$$

is summable $|A, p_n|_k$, *where* (λ_n) *is as in Theorem 1.*

Remark 6 *Since* (λ_n) *is a convex sequence, therefore* $(\lambda_n)^k$ *is also convex sequence and*

$$\sum \frac{1}{n}(\lambda_n)^k < \infty. \tag{13}$$

4 Proof of Lemma 5

Let (M_n) denotes the A-transform of the series $\sum \frac{a_n\lambda_n P_n}{np_n}$. Then, we have

$$\bar{\Delta} M_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v \lambda_v P_v}{vp_v}$$

by (5) and (6).

Now, we get

$$\begin{aligned}
\bar{\Delta} M_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v P_v}{vp_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{np_n} \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv}\lambda_v P_v}{vp_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{np_n} s_n \\
&= \frac{a_{nn} P_n \lambda_n}{np_n} s_n + \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v(\hat{a}_{nv})}{vp_v} s_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \Delta\lambda_v P_v}{vp_v} s_v \\
&+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left(\frac{P_v}{vp_v} \right) s_v \\
&= M_{n,1} + M_{n,2} + M_{n,3} + M_{n,4}
\end{aligned}$$

by applying Abel's transformation. For the proof of Lemma 5, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |M_{n,r}|^k < \infty, \quad for \quad r = 1, 2, 3, 4.$$

First, we have

$$\begin{aligned}
\sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} |M_{n,1}|^k &= \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} \left| \frac{a_{nn} P_n \lambda_n}{np_n} s_n \right|^k \\
&= O(1) \sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} \left(\frac{p_n}{P_n} \right)^k \frac{1}{n^k} \left(\frac{P_n}{p_n} \right)^k (\lambda_n)^k |s_n|^k \\
&= O(1) \sum_{n=1}^{m} \frac{1}{n} (\lambda_n)^k = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by (9), (1) and (13).

From Hölder's inequality, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v\Delta_v(\hat{a}_{nv})}{vp_v}s_v\right|^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)|\Delta_v(\hat{a}_{nv})|\,(\lambda_v)|s_v|\right\}^k\\ &\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k|s_v|^k\right\}\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.\end{aligned}$$

By (4) and (3), we have that

$$\begin{aligned}\Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv}-\hat{a}_{n,v+1}\\ &= \bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}\\ &= a_{nv}-a_{n-1,v}. \qquad (14)\end{aligned}$$

Thus using (8), (3) and (7)

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|=\sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})\leq a_{nn}. \qquad (15)$$

Hence, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k\right\}\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|.\end{aligned}$$

Here, from (14) and (8), we obtain

$$\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|=\sum_{n=v+1}^{m+1}(a_{n-1,v}-a_{nv})\leq a_{vv}.$$

Then,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,2}|^k = O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k a_{vv}$$

$$\begin{aligned}
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}(\lambda_v)^k\\
&= O(1)\sum_{v=1}^{m} v^{k-1}\frac{1}{v^k}(\lambda_v)^k\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_v)^k = O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by (9), (1) and (13).

Now, by (1) and Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_v P_v}{vp_v}s_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v|s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v|s_v|^k\right\}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}^{k-1}.
\end{aligned}$$

Now, (4), (3), (7) and (8) imply that

$$\begin{aligned}
\hat{a}_{n,v+1} = \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1} &= \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i}\\
&= \sum_{i=0}^{n} a_{ni} - \sum_{i=0}^{v} a_{ni} - \sum_{i=0}^{n-1} a_{n-1,i} + \sum_{i=0}^{v} a_{n-1,i}\\
&= 1 - \sum_{i=0}^{v} a_{ni} - 1 + \sum_{i=0}^{v} a_{n-1,i}\\
&= \sum_{i=0}^{v}(a_{n-1,i} - a_{ni}) \geq 0 \qquad (16)
\end{aligned}$$

and from this, using (4), (3) and (8), we have

$$\begin{aligned}
|\hat{a}_{n,v+1}| &= \bar{a}_{n,v+1} - \bar{a}_{n-1,v+1}\\
&= \sum_{i=v+1}^{n} a_{ni} - \sum_{i=v+1}^{n-1} a_{n-1,i}\\
&= a_{nn} + \sum_{i=v+1}^{n-1}(a_{ni} - a_{n-1,i})\\
&\leq a_{nn}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\left\{\sum_{v=1}^{n-1}\Delta\lambda_v\right\}^{k-1}\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}\\ &= O(1)\sum_{v=1}^{m}\Delta\lambda_v\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.\end{aligned}$$

Now, by (16), (3) and (7), we find

$$\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|\leq 1. \tag{17}$$

Thus,

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,3}|^k = O(1)\sum_{v=1}^{m}\Delta\lambda_v = O(1) \quad as \quad m\to\infty,$$

by Lemma 4.

Since $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$ by Lemma 3 and also by using (10), we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\Delta\left(\frac{P_v}{vp_v}\right)s_v\right|^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})|s_v|\right\}^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k|s_v|^k\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.\end{aligned}$$

From (15) and (9),

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k\\ &= O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_{v+1})^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.\end{aligned}$$

Again using (17),

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|M_{n,4}|^k = O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_{v+1})^k = O(1) \quad as \quad m\to\infty,$$

by (13). Hence the proof of Lemma 5 is completed.

5 Proof of Theorem 2

The convergence of the Fourier series at $t = x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of f. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 2 is a consequence of Lemma 5.

6 Conclusions

For $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then we get a result concerning $|C, 1|_k$ summability factors of Fourier series. If we take $a_{nv} = \frac{p_v}{P_n}$ and $k = 1$, then we get a result concerning $|\bar{N}, p_n|$ summability factors of Fourier series (see [13]).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Analysis of Solutions of Some Discrete Systems of Rational Difference Equations

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Abstract

The major objective of this article is to determine and formulate the analytical solutions of the following systems of rational recursive equations:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(\pm 1\mp x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(\mp 1\pm y_{n-1}x_{n-3}\right)},\quad n=0,1,...,$$

where the initial conditions $x_{-3},\ x_{-2},\ x_{-1},\ x_0,\ y_{-3},\ y_{-2},\ y_{-1}$ and y_0 are required to be arbitrary non-zero real numbers. We also introduce some graphs describing these exact solutions under a suitable choice of some initial conditions.

Keywords: difference equations, system of recursive equations, periodicity, local stability, global stability.
Mathematics Subject Classification: 39A10.

1 Introduction

The global interest in exploring the qualitative behaviours of discrete systems of recursive equations has been recently emerged due to the significance of difference equations in modelling a considerable number of discrete phenomena. More specifically, recursive equations are utilized in describing some real life problems that originate in genetics in biology, queuing problems, enegineering, physics, etc. Some experts put effort to analyse dynamical systems of difference equations. Take, for instance, the following ones. Almatrafi et al. [1] studied the local stability, global attractivity, periodicity and solutions for a special case for the difference equation

$$x_{n+1}=ax_{n-1}-\frac{bx_{n-1}}{cx_{n-1}-dx_{n-3}}.$$

Clark and Kulenovic [6] investigated the global attractivity of the system

$$x_{n+1}=\frac{x_n}{a+cy_n},\quad y_{n+1}=\frac{y_n}{b+dx_n}.$$

The author in [8] explored the equilibrium points and the stability of a discrete Lotka-Volterra model shown as follows:

$$x_{n+1} = \frac{\alpha x_n - \beta x_n y_n}{1 + \gamma x_n}, \quad y_{n+1} = \frac{\delta y_n + \epsilon x_n y_n}{1 + \eta y_n}.$$

The positive solutions of the system

$$u_{n+1} = \frac{\alpha u_{n-1}}{\beta + \gamma v_n^p v_{n-2}^q}, \quad v_{n+1} = \frac{\alpha_1 v_{n-1}}{\beta_1 + \gamma_1 u_n^{p_1} u_{n-2}^{q_1}}.$$

were obtained in [14] by Gűműş and Őcalan. Moreover, Kurbanli et al. [18] solved the dynamical systems of recursive equations given by

$$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1}, \quad z_{n+1} = \frac{x_n}{y_n z_{n-1}}.$$

In [19] Mansour et al. presented the analytical solutions of the system

$$x_{n+1} = \frac{x_{n-1}}{\alpha - x_{n-1} y_n}, \quad y_{n+1} = \frac{y_{n-1}}{\beta + \gamma y_{n-1} x_n}.$$

Finally, the author in [23] demonstrated the dynamics of the system

$$x_{n+1} = \frac{x_{n-2}}{B + y_n y_{n-1} y_{n-2}}, \quad y_{n+1} = \frac{y_{n-2}}{A + x_n x_{n-1} x_{n-2}}.$$

To attain more information on the qualitative behaviours of dynamical difference equations, one can refer to refs [1–5, 7, 9–13, 15–17, 20–22]

In this paper, the rational solutions of the following discrete systems of difference equations will be discovered and given in four different theorems:

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1} \left(\pm 1 \mp x_{n-1} y_{n-3}\right)}, \quad y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1} \left(\mp 1 \pm y_{n-1} x_{n-3}\right)}, \quad n = 0, 1, ...,$$

where the initial values are as described previously.

2 Main Results

2.1 First System $x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1}(1 - x_{n-1} y_{n-3})}, \quad y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1}(1 - y_{n-1} x_{n-3})}$

This subsection concentrates on obtaining the solutions of a dynamical system of fourth order difference equations given by the form:

$$x_{n+1} = \frac{x_{n-1} y_{n-3}}{y_{n-1} \left(1 - x_{n-1} y_{n-3}\right)}, \; y_{n+1} = \frac{y_{n-1} x_{n-3}}{x_{n-1} \left(1 - y_{n-1} x_{n-3}\right)}, \; n = 0, \; 1, \ldots, \tag{1}$$

where the initial values are as shown previously. The following fundamental theorem presents the solutions of system (1).

Theorem 1 *Assume that* $\{x_n, y_n\}$ *is a solution to system (1) and let* $x_{-3} = \alpha,\ x_{-2} = \beta,\ x_{-1} = \gamma,\ x_0 = \delta,\ y_{-3} = \epsilon,\ y_{-2} = \eta,\ y_{-1} = \mu$ *and* $y_0 = \omega$. *Then, for* $n = 0,\ 1,\ \ldots$ *we have*

$$\begin{aligned}
x_{4n-3} &= \frac{\gamma^n \epsilon^n \prod\limits_{i=0}^{n-1} \left[(2i)\,\alpha\mu - 1\right]}{\alpha^{n-1}\mu^n \prod\limits_{i=0}^{n-1} \left[(2i+1)\,\gamma\epsilon - 1\right]}, \quad x_{4n-2} = \frac{\delta^n \eta^n \prod\limits_{i=0}^{n-1} \left[(2i)\,\beta\omega - 1\right]}{\beta^{n-1}\omega^n \prod\limits_{i=0}^{n-1} \left[(2i+1)\,\delta\eta - 1\right]}, \\
x_{4n-1} &= \frac{\gamma^{n+1} \epsilon^n \prod\limits_{i=0}^{n-1} \left[(2i+1)\,\alpha\mu - 1\right]}{\alpha^{n}\mu^n \prod\limits_{i=0}^{n-1} \left[(2i+2)\,\gamma\epsilon - 1\right]}, \quad x_{4n} = \frac{\delta^{n+1} \eta^n \prod\limits_{i=0}^{n-1} \left[(2i+1)\,\beta\omega - 1\right]}{\beta^{n}\omega^n \prod\limits_{i=0}^{n-1} \left[(2i+2)\,\delta\eta - 1\right]}.
\end{aligned}$$

And

$$\begin{aligned}
y_{4n-3} &= \frac{\alpha^n \mu^n \prod\limits_{i=0}^{n-1} \left[(2i)\,\gamma\epsilon - 1\right]}{\gamma^{n}\epsilon^{n-1} \prod\limits_{i=0}^{n-1} \left[(2i+1)\,\alpha\mu - 1\right]}, \quad y_{4n-2} = \frac{\beta^n \omega^n \prod\limits_{i=0}^{n-1} \left[(2i)\,\delta\eta - 1\right]}{\delta^{n}\eta^{n-1} \prod\limits_{i=0}^{n-1} \left[(2i+1)\,\beta\omega - 1\right]}, \\
y_{4n-1} &= \frac{\alpha^{n} \mu^{n+1} \prod\limits_{i=0}^{n-1} \left[(2i+1)\,\gamma\epsilon - 1\right]}{\gamma^{n}\epsilon^n \prod\limits_{i=0}^{n-1} \left[(2i+2)\,\alpha\mu - 1\right]}, \quad y_{4n} = \frac{\beta^{n} \omega^{n+1} \prod\limits_{i=0}^{n-1} \left[(2i+1)\,\delta\eta - 1\right]}{\delta^{n}\eta^n \prod\limits_{i=0}^{n-1} \left[(2i+2)\,\beta\omega - 1\right]}.
\end{aligned}$$

Proof. For $n = 0$, our results hold. Next, let $n > 1$ and suppose that the relations hold for $n - 1$. That is

$$\begin{aligned}
x_{4n-7} &= \frac{\gamma^{n-1} \epsilon^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i)\,\alpha\mu - 1\right]}{\alpha^{n-2}\mu^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+1)\,\gamma\epsilon - 1\right]}, \quad x_{4n-6} = \frac{\delta^{n-1} \eta^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i)\,\beta\omega - 1\right]}{\beta^{n-2}\omega^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+1)\,\delta\eta - 1\right]}, \\
x_{4n-5} &= \frac{\gamma^{n} \epsilon^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+1)\,\alpha\mu - 1\right]}{\alpha^{n-1}\mu^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+2)\,\gamma\epsilon - 1\right]}, \quad x_{4n-4} = \frac{\delta^{n} \eta^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+1)\,\beta\omega - 1\right]}{\beta^{n-1}\omega^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+2)\,\delta\eta - 1\right]}.
\end{aligned}$$

And

$$\begin{aligned}
y_{4n-7} &= \frac{\alpha^{n-1} \mu^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i)\,\gamma\epsilon - 1\right]}{\gamma^{n-1}\epsilon^{n-2} \prod\limits_{i=0}^{n-2} \left[(2i+1)\,\alpha\mu - 1\right]}, \quad y_{4n-6} = \frac{\beta^{n-1} \omega^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i)\,\delta\eta - 1\right]}{\delta^{n-1}\eta^{n-2} \prod\limits_{i=0}^{n-2} \left[(2i+1)\,\beta\omega - 1\right]}, \\
y_{4n-5} &= \frac{\alpha^{n-1} \mu^{n} \prod\limits_{i=0}^{n-2} \left[(2i+1)\,\gamma\epsilon - 1\right]}{\gamma^{n-1}\epsilon^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+2)\,\alpha\mu - 1\right]}, \quad y_{4n} = \frac{\beta^{n-1} \omega^{n} \prod\limits_{i=0}^{n-2} \left[(2i+1)\,\delta\eta - 1\right]}{\delta^{n-1}\eta^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+2)\,\beta\omega - 1\right]}.
\end{aligned}$$

Now, it can be obviously observed from system (1) that

$$\begin{aligned}
x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(1-x_{4n-5}y_{4n-7}\right)} \\
&= \frac{\frac{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\frac{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}\prod\limits_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}}{\frac{\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\left[1-\frac{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\frac{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}\prod\limits_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}\right]} \\
&= \frac{\frac{\gamma\epsilon\prod\limits_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}}{\frac{\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\left[1-\frac{\gamma\epsilon\prod\limits_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\right]} \\
&= \frac{\gamma^{n}\epsilon^{n}\prod\limits_{i=0}^{n-2}\left[\left(2i\right)\gamma\epsilon-1\right]\prod\limits_{i=0}^{n-2}\left[\left(2i+2\right)\alpha\mu-1\right]}{\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-2}\left[\left(2i+1\right)\gamma\epsilon-1\right]\left[\prod\limits_{i=0}^{n-2}\left[\left(2i+2\right)\gamma\epsilon-1\right]-\gamma\epsilon\prod\limits_{i=0}^{n-2}\left[\left(2i\right)\gamma\epsilon-1\right]\right]} \\
&= -\frac{\gamma^{n}\epsilon^{n}\prod\limits_{i=0}^{n-2}\left[\left(2i+2\right)\alpha\mu-1\right]}{\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-1}\left[\left(2i+1\right)\gamma\epsilon-1\right]} = \frac{\gamma^{n}\epsilon^{n}\prod\limits_{i=0}^{n-1}\left[\left(2i\right)\alpha\mu-1\right]}{\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-1}\left[\left(2i+1\right)\gamma\epsilon-1\right]}.
\end{aligned}$$

Now, system (1) gives us that

$$\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left[1-y_{4n-5}x_{4n-7}\right]} \\
&= \frac{\frac{\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}}{\frac{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[1-\frac{\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}\right]} \\
&= \frac{\frac{\alpha\mu\prod\limits_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}}{\frac{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[1-\frac{\alpha\mu\prod\limits_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\right]} \\
&= \frac{\alpha^{n}\mu^{n}\prod\limits_{i=0}^{n-2}\left[\left(2i\right)\alpha\mu-1\right]\prod\limits_{i=0}^{n-2}\left[\left(2i+2\right)\gamma\epsilon-1\right]}{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}\left[\left(2i+1\right)\alpha\mu-1\right]\left[\prod\limits_{i=0}^{n-2}\left[\left(2i+2\right)\alpha\mu-1\right]-\alpha\mu\prod\limits_{i=0}^{n-2}\left[\left(2i\right)\alpha\mu-1\right]\right]}
\end{aligned}$$

$$= -\frac{\alpha^{n}\mu^{n}\prod\limits_{i=0}^{n-2}\left[(2i+2)\gamma\epsilon-1\right]}{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-1}\left[(2i+1)\alpha\mu-1\right]} = \frac{\alpha^{n}\mu^{n}\prod\limits_{i=0}^{n-1}\left[(2i)\gamma\epsilon-1\right]}{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-1}\left[(2i+1)\alpha\mu-1\right]}.$$

Hence, the rest of the results can be similarly proved.

2.2 Second System $x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(-1+x_{n-1}y_{n-3})}$, $y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(-1+y_{n-1}x_{n-3})}$

Our leading duty in this subsection is to determine the solutions of the following discrete systems:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1+x_{n-1}y_{n-3}\right)}, \quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1+y_{n-1}x_{n-3}\right)}. \tag{2}$$

The initial values of this system are arbitrary real numbers.

Theorem 2 *Suppose that $\{x_n, y_n\}$ is a solution to system (2) and assume that $x_{-3}=\alpha,\ x_{-2}=\beta,\ x_{-1}=\gamma,\ x_0=\delta,\ y_{-3}=\epsilon,\ y_{-2}=\eta,\ y_{-1}=\mu$ and $y_0=\omega$. Then, for $n=0,\ 1,\ \ldots$ we have*

$$\begin{aligned} x_{4n-3} &= \frac{\gamma^{n}\epsilon^{n}}{\alpha^{n-1}\mu^{n}\left(\gamma\epsilon-1\right)^{n}}, \quad x_{4n-2}=\frac{\delta^{n}\eta^{n}}{\beta^{n-1}\omega^{n}\left(\delta\eta-1\right)^{n}}, \\ x_{4n-1} &= \frac{\gamma^{n+1}\epsilon^{n}\left(\alpha\mu-1\right)^{n}}{\alpha^{n}\mu^{n}}, \quad x_{4n}=\frac{\delta^{n+1}\eta^{n}\left(\beta\omega-1\right)^{n}}{\beta^{n}\omega^{n}}. \end{aligned}$$

And

$$\begin{aligned} y_{4n-3} &= \frac{\alpha^{n}\mu^{n}}{\gamma^{n}\epsilon^{n-1}\left(\alpha\mu-1\right)^{n}}, \quad y_{4n-2}=\frac{\beta^{n}\omega^{n}}{\delta^{n}\eta^{n-1}\left(\beta\omega-1\right)^{n}}, \\ y_{4n-1} &= \frac{\alpha^{n}\mu^{n+1}\left(\gamma\epsilon-1\right)^{n}}{\gamma^{n}\epsilon^{n}}, \quad y_{4n}=\frac{\beta^{n}\omega^{n+1}\left(\delta\eta-1\right)^{n}}{\delta^{n}\eta^{n}}. \end{aligned}$$

Proof. It is obvious that all solutions are satisfied for $n=0$. Next, we suppose that $n>1$ and assume that the solutions hold for $n-1$. That is

$$\begin{aligned} x_{4n-7} &= \frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\left(\gamma\epsilon-1\right)^{n-1}}, \quad x_{4n-6}=\frac{\delta^{n-1}\eta^{n-1}}{\beta^{n-2}\omega^{n-1}\left(\delta\eta-1\right)^{n-1}}, \\ x_{4n-5} &= \frac{\gamma^{n}\epsilon^{n-1}\left(\alpha\mu-1\right)^{n-1}}{\alpha^{n-1}\mu^{n-1}}, \quad x_{4n-4}=\frac{\delta^{n}\eta^{n-1}\left(\beta\omega-1\right)^{n-1}}{\beta^{n-1}\omega^{n-1}}. \end{aligned}$$

And

$$\begin{aligned} y_{4n-7} &= \frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2}\left(\alpha\mu-1\right)^{n-1}}, \quad y_{4n-6}=\frac{\beta^{n-1}\omega^{n-1}}{\delta^{n-1}\eta^{n-2}\left(\beta\omega-1\right)^{n-1}}, \\ y_{4n-5} &= \frac{\alpha^{n-1}\mu^{n}\left(\gamma\epsilon-1\right)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}, \quad y_{4n-4}=\frac{\beta^{n-1}\omega^{n}\left(\delta\eta-1\right)^{n-1}}{\delta^{n-1}\eta^{n-1}}. \end{aligned}$$

We now turn to illustrate the first result. System (2) leads to

$$\begin{aligned}
x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}\left(-1+x_{4n-5}y_{4n-7}\right)} \\
&= \frac{\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}}{\frac{\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\left[-1+\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\frac{\alpha^{n-1}\mu^{n-1}}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}\right]} \\
&= \frac{\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\left(\gamma\epsilon-1\right)^{n-1}\left[-1+\gamma\epsilon\right]} = \frac{\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\left(\gamma\epsilon-1\right)^{n}}.
\end{aligned}$$

Similarly, it is easy to see from system (2) that

$$\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(-1+y_{4n-5}x_{4n-7}\right)} \\
&= \frac{\frac{\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}}{\frac{\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}}\left[-1+\frac{\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}}\frac{\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}\right]} \\
&= \frac{\alpha^n\mu^n}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^{n-1}\left[-1+\alpha\mu\right]} = \frac{\alpha^n\mu^n}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^{n}}.
\end{aligned}$$

The remaining solutions of system (2) can be clearly justified in a similar technique. Thus, the proof is complete.

2.3 Third System $x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(1-x_{n-1}y_{n-3})}$, $y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(-1+y_{n-1}x_{n-3})}$

The central point of this subsection is to resolve a system of fourth order rational recursive equations given by the form:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(1-x_{n-1}y_{n-3}\right)}, \quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(-1+y_{n-1}x_{n-3}\right)}, \tag{3}$$

where the initial values are as described previously.

Theorem 3 *Let* $\{x_n, y_n\}$ *be a solution to system (3) and suppose that* $x_{-3}=\alpha,\ x_{-2}=\beta,\ x_{-1}=\gamma,\ x_0=\delta,\ y_{-3}=\epsilon,\ y_{-2}=\eta,\ y_{-1}=\mu$ *and* $y_0=\omega$. *Then, for* $n=0,\ 1,\ ...$ *we have*

$$\begin{aligned}
x_{4n-3} &= \frac{\left(-1\right)^n\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod\limits_{i=0}^{n-1}\left[\left(2i+1\right)\gamma\epsilon-1\right]}, \quad x_{4n-2}=\frac{\left(-1\right)^n\delta^n\eta^n}{\beta^{n-1}\omega^n\prod\limits_{i=0}^{n-1}\left[\left(2i+1\right)\delta\eta-1\right]}, \\
x_{4n-1} &= \frac{\left(-1\right)^n\gamma^{n+1}\epsilon^n\left(\alpha\mu-1\right)^n}{\alpha^n\mu^n\prod\limits_{i=0}^{n-1}\left[\left(2i+2\right)\gamma\epsilon-1\right]}, \quad x_{4n}=\frac{\left(-1\right)^n\delta^{n+1}\eta^n\left(\beta\omega-1\right)^n}{\beta^n\omega^n\prod\limits_{i=0}^{n-1}\left[\left(2i+2\right)\delta\eta-1\right]}.
\end{aligned}$$

And

$$
\begin{aligned}
y_{4n-3} &= \frac{(-1)^{n}\alpha^{n}\mu^{n}\prod\limits_{i=0}^{n-1}\left[(2i)\gamma\epsilon-1\right]}{\gamma^{n}\epsilon^{n-1}(\alpha\mu-1)^{n}}, \quad y_{4n-2}=\frac{(-1)^{n}\beta^{n}\omega^{n}\prod\limits_{i=0}^{n-1}\left[(2i)\delta\eta-1\right]}{\delta^{n}\eta^{n-1}(\beta\omega-1)^{n}},\\
y_{4n-1} &= \frac{(-1)^{n}\alpha^{n}\mu^{n+1}\prod\limits_{i=0}^{n-1}\left[(2i+1)\gamma\epsilon-1\right]}{\gamma^{n}\epsilon^{n}}, \quad y_{4n}=\frac{(-1)^{n}\beta^{n}\omega^{n+1}\prod\limits_{i=0}^{n-1}\left[(2i+1)\delta\eta-1\right]}{\delta^{n}\eta^{n}}.
\end{aligned}
$$

Proof. The results are true for $n=0$. Next, we suppose that $n>1$ and assume that the relations hold for $n-1$. That is

$$
\begin{aligned}
x_{4n-7} &= \frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\prod\limits_{i=0}^{n-2}\left[(2i+1)\gamma\epsilon-1\right]}, \quad x_{4n-6}=\frac{(-1)^{n-1}\delta^{n-1}\eta^{n-1}}{\beta^{n-2}\omega^{n-1}\prod\limits_{i=0}^{n-2}\left[(2i+1)\delta\eta-1\right]},\\
x_{4n-5} &= \frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}\left[(2i+2)\gamma\epsilon-1\right]}, \quad x_{4n-4}=\frac{(-1)^{n-1}\delta^{n}\eta^{n-1}(\beta\omega-1)^{n-1}}{\beta^{n-1}\omega^{n-1}\prod\limits_{i=0}^{n-2}\left[(2i+2)\delta\eta-1\right]}.
\end{aligned}
$$

And

$$
\begin{aligned}
y_{4n-7} &= \frac{(-1)^{n-1}\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}\left[(2i)\gamma\epsilon-1\right]}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}, \quad y_{4n-6}=\frac{(-1)^{n-1}\beta^{n-1}\omega^{n-1}\prod\limits_{i=0}^{n-2}\left[(2i)\delta\eta-1\right]}{\delta^{n-1}\eta^{n-2}(\beta\omega-1)^{n-1}},\\
y_{4n-5} &= \frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-2}\left[(2i+1)\gamma\epsilon-1\right]}{\gamma^{n-1}\epsilon^{n-1}}, \quad y_{4n-4}=\frac{(-1)^{n-1}\beta^{n-1}\omega^{n}\prod\limits_{i=0}^{n-2}\left[(2i+1)\delta\eta-1\right]}{\delta^{n-1}\eta^{n-1}}.
\end{aligned}
$$

Now, we establish the proofs of two relations. Firstly, system (3) gives us that

$$
\begin{aligned}
x_{4n-3} &= \frac{x_{4n-5}y_{4n-7}}{y_{4n-5}(1-x_{4n-5}y_{4n-7})}\\
&= \frac{\frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}}{\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}}\left[1-\frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n-1}\prod\limits_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-2}(\alpha\mu-1)^{n-1}}\right]}\\
&= \frac{\frac{\gamma\epsilon\prod\limits_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}}{\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}\prod\limits_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}}\left[1-\frac{\gamma\epsilon\prod\limits_{i=0}^{n-2}[(2i)\gamma\epsilon-1]}{\prod\limits_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\right]}
\end{aligned}
$$

$$\begin{aligned}
&= \frac{(-1)^{-n+1}\gamma^n\epsilon^n\prod_{i=0}^{n-2}\left[(2i)\gamma\epsilon-1\right]}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}\left[(2i+1)\gamma\epsilon-1\right]\left[\prod_{i=0}^{n-2}\left[(2i+2)\gamma\epsilon-1\right]-\gamma\epsilon\prod_{i=0}^{n-2}\left[(2i)\gamma\epsilon-1\right]\right]} \\
&= \frac{-(-1)^{-n+1}\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}\left[(2i+1)\gamma\epsilon-1\right]} = \frac{(-1)^{n}\gamma^n\epsilon^n}{\alpha^{n-1}\mu^n\prod_{i=0}^{n-1}\left[(2i+1)\gamma\epsilon-1\right]}.
\end{aligned}$$

Next, it can be noticed from system (3) that

$$\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(-1+y_{4n-5}x_{4n-7}\right)} \\
&= \frac{\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}}{\frac{(-1)^{n-1}\gamma^n\epsilon^{n-1}(\alpha\mu-1)^{n-1}}{\alpha^{n-1}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+2)\gamma\epsilon-1]}\left[-1+\frac{(-1)^{n-1}\alpha^{n-1}\mu^n\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}{\gamma^{n-1}\epsilon^{n-1}}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}}{\alpha^{n-2}\mu^{n-1}\prod_{i=0}^{n-2}[(2i+1)\gamma\epsilon-1]}\right]} \\
&= \frac{(-1)^{-n+1}\alpha^n\mu^n\prod_{i=0}^{n-2}\left[(2i+2)\gamma\epsilon-1\right]}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^{n-1}\left[-1+\alpha\mu\right]} = \frac{-(-1)^{n-1}\alpha^n\mu^n\prod_{i=0}^{n-1}\left[(2i)\gamma\epsilon-1\right]}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^{n}} \\
&= \frac{(-1)^{n}\alpha^n\mu^n\prod_{i=0}^{n-1}\left[(2i)\gamma\epsilon-1\right]}{\gamma^n\epsilon^{n-1}\left(\alpha\mu-1\right)^{n}}.
\end{aligned}$$

The proofs of the remaining relations can be likewise achieved. Therefore, they are omitted.

2.4 Fourth System $x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}(-1+x_{n-1}y_{n-3})},\ y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}(1-y_{n-1}x_{n-3})}$

Our fundamental task in this subsection is to develop fractional solutions to the system of recursive equations given by the form:

$$x_{n+1}=\frac{x_{n-1}y_{n-3}}{y_{n-1}\left(-1+x_{n-1}y_{n-3}\right)},\quad y_{n+1}=\frac{y_{n-1}x_{n-3}}{x_{n-1}\left(1-y_{n-1}x_{n-3}\right)}, \tag{4}$$

where the initial conditions are required to be non-zero real numbers.

Theorem 4 *Assume that* $\{x_n, y_n\}$ *is a solution to system (4) and suppose that* $x_{-3}=\alpha,\ x_{-2}=\beta,\ x_{-1}=\gamma,\ x_0=\delta,\ y_{-3}=\epsilon,\ y_{-2}=\eta,\ y_{-1}=\mu$ *and* $y_0=\omega$. *Then, for* $n=0,\ 1,\ \ldots$ *we have*

$$\begin{aligned}
x_{4n-3} &= \frac{(-1)^n\gamma^n\epsilon^n\prod_{i=0}^{n-1}\left[(2i)\alpha\mu-1\right]}{\alpha^{n-1}\mu^n\left(\gamma\epsilon-1\right)^n},\quad x_{4n-2}=\frac{(-1)^n\delta^n\eta^n\prod_{i=0}^{n-1}\left[(2i)\beta\omega-1\right]}{\beta^{n-1}\omega^n\left(\delta\eta-1\right)^n}, \\
x_{4n-1} &= \frac{(-1)^n\gamma^{n+1}\epsilon^n\prod_{i=0}^{n-1}\left[(2i+1)\alpha\mu-1\right]}{\alpha^n\mu^n},\quad x_{4n}=\frac{(-1)^n\delta^{n+1}\eta^n\prod_{i=0}^{n-1}\left[(2i+1)\beta\omega-1\right]}{\beta^n\omega^n}.
\end{aligned}$$

And

$$
\begin{aligned}
y_{4n-3} &= \frac{(-1)^n \alpha^n \mu^n}{\gamma^n \epsilon^{n-1} \prod\limits_{i=0}^{n-1} \left[(2i+1)\alpha\mu - 1\right]}, \quad y_{4n-2} = \frac{(-1)^n \beta^n \omega^n}{\delta^n \eta^{n-1} \prod\limits_{i=0}^{n-1} \left[(2i+1)\beta\omega - 1\right]}, \\
y_{4n-1} &= \frac{(-1)^n \alpha^n \mu^{n+1} (\gamma\epsilon - 1)^n}{\gamma^n \epsilon^n \prod\limits_{i=0}^{n-1} \left[(2i+2)\alpha\mu - 1\right]}, \quad y_{4n} = \frac{(-1)^n \beta^n \omega^{n+1} (\delta\eta - 1)^n}{\delta^n \eta^n \prod\limits_{i=0}^{n-1} \left[(2i+2)\beta\omega - 1\right]}.
\end{aligned}
$$

Proof. The relations hold for $n = 0$. Next, we let $n > 1$ and assume that the formulas hold for $n - 1$. That is

$$
\begin{aligned}
x_{4n-7} &= \frac{(-1)^{n-1} \gamma^{n-1} \epsilon^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i)\alpha\mu - 1\right]}{\alpha^{n-2} \mu^{n-1} (\gamma\epsilon - 1)^{n-1}}, \quad x_{4n-6} = \frac{(-1)^{n-1} \delta^{n-1} \eta^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i)\beta\omega - 1\right]}{\beta^{n-2} \omega^{n-1} (\delta\eta - 1)^{n-1}}, \\
x_{4n-5} &= \frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+1)\alpha\mu - 1\right]}{\alpha^{n-1} \mu^{n-1}}, \quad x_{4n-4} = \frac{(-1)^{n-1} \delta^n \eta^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+1)\beta\omega - 1\right]}{\beta^{n-1} \omega^{n-1}}.
\end{aligned}
$$

And

$$
\begin{aligned}
y_{4n-7} &= \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod\limits_{i=0}^{n-2} \left[(2i+1)\alpha\mu - 1\right]}, \quad y_{4n-6} = \frac{(-1)^{n-1} \beta^{n-1} \omega^{n-1}}{\delta^{n-1} \eta^{n-2} \prod\limits_{i=0}^{n-2} \left[(2i+1)\beta\omega - 1\right]}, \\
y_{4n-5} &= \frac{(-1)^{n-1} \alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1}}{\gamma^{n-1} \epsilon^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+2)\alpha\mu - 1\right]}, \quad y_{4n-4} = \frac{(-1)^{n-1} \beta^{n-1} \omega^n (\delta\eta - 1)^{n-1}}{\delta^{n-1} \eta^{n-1} \prod\limits_{i=0}^{n-2} \left[(2i+2)\beta\omega - 1\right]}.
\end{aligned}
$$

We now turn to verify the proof of two relations. It can be obviously seen from system (4) that

$$
\begin{aligned}
x_{4n-3} &= \frac{x_{4n-5} y_{4n-7}}{y_{4n-5} \left(-1 + x_{4n-5} y_{4n-7}\right)} \\
&= \frac{\frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}{\alpha^{n-1} \mu^{n-1}} \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}}{\frac{(-1)^{n-1} \alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1}}{\gamma^{n-1} \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+2)\alpha\mu - 1]} \left[-1 + \frac{(-1)^{n-1} \gamma^n \epsilon^{n-1} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}{\alpha^{n-1} \mu^{n-1}} \frac{(-1)^{n-1} \alpha^{n-1} \mu^{n-1}}{\gamma^{n-1} \epsilon^{n-2} \prod_{i=0}^{n-2} [(2i+1)\alpha\mu - 1]}\right]} \\
&= \frac{(-1)^{-n+1} \gamma^n \epsilon^n \prod\limits_{i=0}^{n-2} \left[(2i+2)\alpha\mu - 1\right]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^{n-1} \left[-1 + \gamma\epsilon\right]} = \frac{-(-1)^{n-1} \gamma^n \epsilon^n \prod\limits_{i=0}^{n-1} \left[(2i)\alpha\mu - 1\right]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^n} \\
&= \frac{(-1)^n \gamma^n \epsilon^n \prod\limits_{i=0}^{n-1} \left[(2i)\alpha\mu - 1\right]}{\alpha^{n-1} \mu^n (\gamma\epsilon - 1)^n}.
\end{aligned}
$$

Further, it can be attained from system (4) that

$$\begin{aligned}
y_{4n-3} &= \frac{y_{4n-5}x_{4n-7}}{x_{4n-5}\left(1-y_{4n-5}x_{4n-7}\right)} \\
&= \frac{\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}}{\frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}}\left[1-\frac{(-1)^{n-1}\alpha^{n-1}\mu^{n}(\gamma\epsilon-1)^{n-1}}{\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\frac{(-1)^{n-1}\gamma^{n-1}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\alpha^{n-2}\mu^{n-1}(\gamma\epsilon-1)^{n-1}}\right]} \\
&= \frac{\frac{\alpha\mu\prod\limits_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}}{\frac{(-1)^{n-1}\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}[(2i+1)\alpha\mu-1]}{\alpha^{n-1}\mu^{n-1}}\left[1-\frac{\alpha\mu\prod\limits_{i=0}^{n-2}[(2i)\alpha\mu-1]}{\prod\limits_{i=0}^{n-2}[(2i+2)\alpha\mu-1]}\right]} \\
&= \frac{(-1)^{-n+1}\alpha^{n}\mu^{n}\prod\limits_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]}{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-2}\left[(2i+1)\alpha\mu-1\right]\left[\prod\limits_{i=0}^{n-2}\left[(2i+2)\alpha\mu-1\right]-\alpha\mu\prod\limits_{i=0}^{n-2}\left[(2i)\alpha\mu-1\right]\right]} \\
&= \frac{(-1)^{n}\alpha^{n}\mu^{n}}{\gamma^{n}\epsilon^{n-1}\prod\limits_{i=0}^{n-1}\left[(2i+1)\alpha\mu-1\right]}.
\end{aligned}$$

Other results can be proved in a similar way. Thus, the remaining proofs are omitted.

2.5 Numerical Examples

This subsection aims to present graphical confirmations to the whole solutions obtained in the previous subsections. Here, we plot the solutions (by using MATLAB software) under specific selections of some initial conditions.

Example 1. This example shows the paths of the solutions of system (1). The initial conditions of this example are given as follows: $x_{-3}=3,\ x_{-2}=1,\ x_{-1}=5,\ x_{0}=2,\ y_{-3}=1,\ y_{-2}=3,\ y_{-1}=5$ and $y_{0}=5$. See Figure 1.

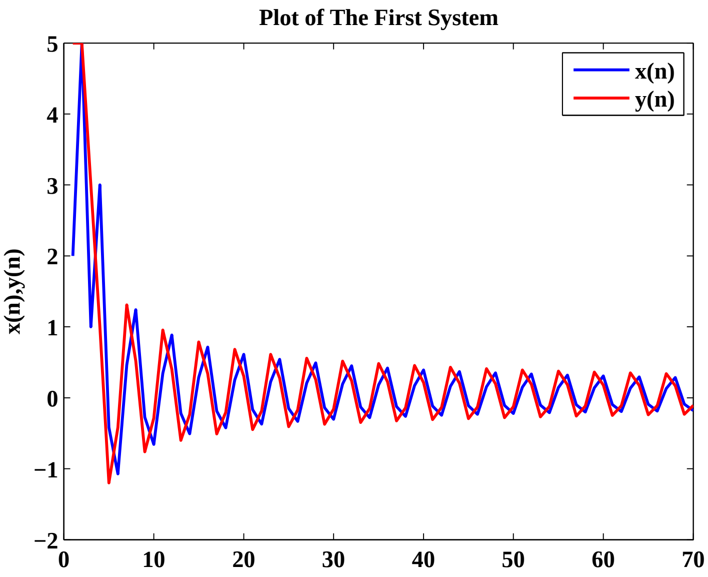

Figure 1: The behaviour of the solution of system (1).

Example 2. In Figure 2, we illustrate the behaviour of the solution of system (2) under the following selection of initial conditions: $x_{-3} = 3.4,\ x_{-2} = 0.7,\ x_{-1} = 2,\ x_0 = 3,\ y_{-3} = 1.5,\ y_{-2} = 1.5,\ y_{-1} = 0.5$ and $y_0 = 1.22$.

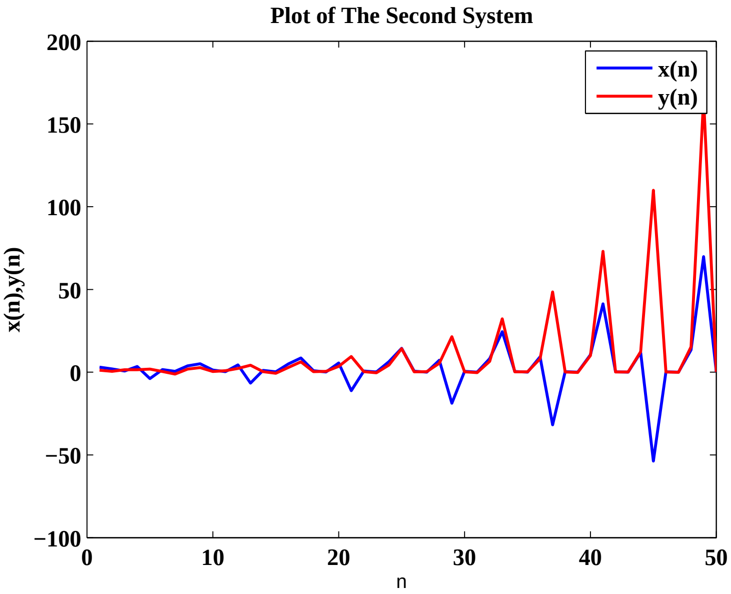

Figure 2: The behaviour of the solution of system (2).

Example 3. Figure 3 illustrates the curves of the solutions of system (3) when we assume that $x_{-3} = 0.7,\ x_{-2} = 2.1,\ x_{-1} = 1,\ x_0 = 0.5,\ y_{-3} = 0.1,\ y_{-2} = 0.2,\ y_{-1} = 2.2$ and $y_0 = 0.5$.

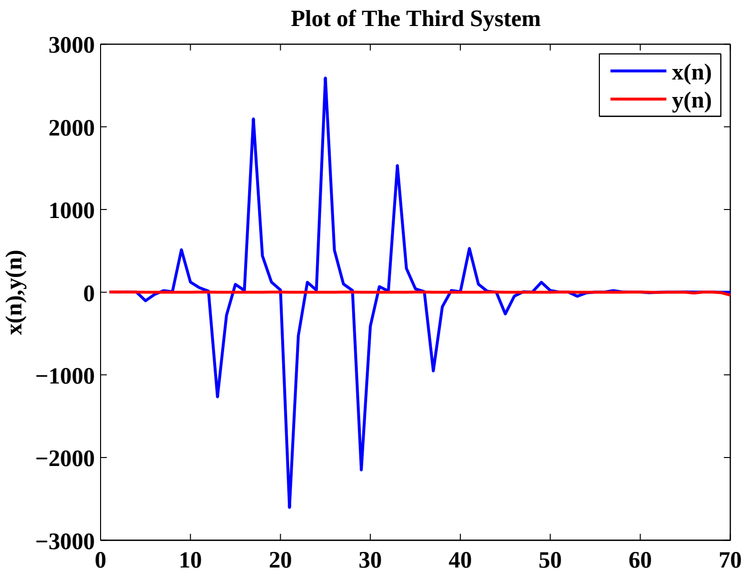

Figure 3: The behaviour of the solution of system (3).

Example 4. The solutions of system (4) are depicted in Figure 4 under the following initial data: $x_{-3} = 0.2,\ x_{-2} = 1,\ x_{-1} = 0.3,\ x_0 = 0.2,\ y_{-3} = 3,\ y_{-2} = 1,\ y_{-1} = 2$ and $y_0 = 0.3$.

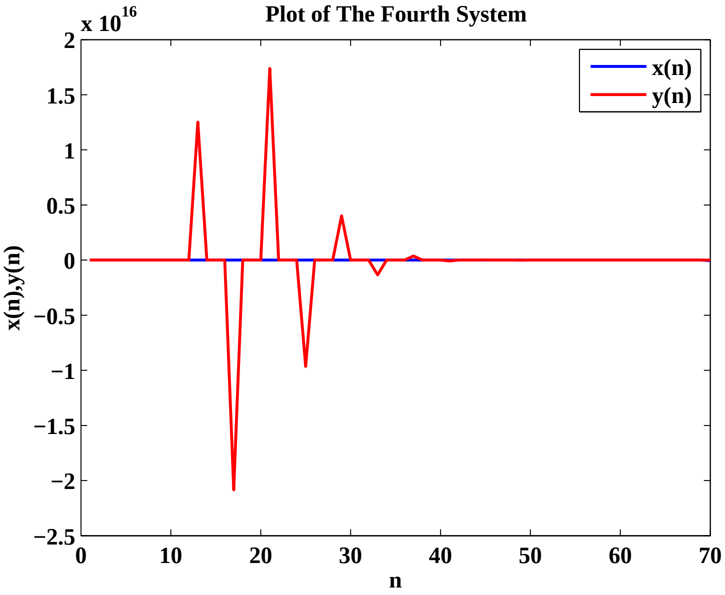

Figure 4: The behaviour of the solution of system (4).

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The ELECTRE multi-attribute group decision making method based on interval-valued intuitionistic fuzzy sets

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Abstract

In this paper, based on the ELECTRE method and new ranking for the interval-valued intuitionistic fuzzy set (IVIFS), the IVIF ELECTRE method to solve multi-attribute group decision-making problems with interval-valued intuitionistic fuzzy input data is proposed, it is extending the intuitionistic fuzzy set (IF) ELECTRE method. This method firstly use AHP (Analytic hierarchy process) to find the weights of attribute, and use new ranking method for IVIFS and similarity measure between IVIFS to determine the weights of decision makers (DMs), then give the concordance set, midrange concordance set, weak concordance set and cosponging discordance set, midrange discordance set, weak discordance set. From this, the concordance matrix, discordance index, concordance dominance matrix and discordance dominance matrix are proposed. Finally, the ranking order of all the alternatives $A_i(i = 1, 2, \ldots, n)$ and the best alternative are obtained. A numerical example is taken to illustrate the feasibility and practicability of the proposed method.

Keywords: Interval-valued intuitionistic fuzzy sets; ELECTRE method; Multi-attribute group decision making

1 Introduction

Since the multi-attribute decision making (MADM) was introduced in 1960's, it has been a hot topic in decision making and systems engineering, and been proven as a useful tool due to its broad applications in a number of practical problems. But in some real-life situations, a decision maker (DM) may not be able to accurately express his/her preferences for alternatives due to that (1) the DM may not possess a precise or sufficient level of knowledge of the problem; (2) the DM is unable to discriminate explicitly the degree to which one alternative are better than others. In order to handle inexact and imprecise data, in 1965 Zadeh [38] introduced fuzzy set (FS) theory. In 1983 Atanassov [1,2] generalized FS to intuitionistic fuzzy set (IFS) by using two characteristic functions to express the degree of membership and the degree of non-membership of elements of the universal set. Since IFS tackled the drawback of the single membership value in FS theory, IFS has been widely applied to the multi-attribute decision making (MADM) [4,7,8,10-14,20,22,23,28] and multi-attribute group decision making (MAGDM) [18,19,21].

In 1989 Atanassov and Gargov [3] further generalized the IFS in the spirit of the ordinary interval-valued fuzzy set (IVFS) and defined the concept of interval-valued intuitionistic fuzzy set (IVIFS), which enhances greatly the representation ability of uncertainty than IFS. Similar to the IFS, IVIFSs were also

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used in the problems of MADM [6,15-17,28,32] and MAGDM [29,31,33,34]. In these researches, some are extension of classic decision making methods in IVIFS environment. For example, Li [15] developed the closeness coefficient-based nonlinear-programming method for interval-valued intuitionistic fuzzy MADM with incomplete preference information, Li [16] proposed the TOPSIS-based nonlinear-programming methodology for MADM with IVIFSs, Li [17] proposed the linear-programming method for MADM with IVIFSs. These decision methods under interval-valued intuitionistic fuzzy environments also generalize the classic decision making methods, such as TOPSIS and LINMAP. In [32], Wang et al. proposed a expect to apply ELECTRE and PROMETHEE motheds to MADM and MAGDM with IVIFS.

In this paper, based on the new ranking method of interval in [27] and similarity measure of IVIFSs in [35, 37], the IF ELECTRE [30] method is applied to MAGDM with IVIFS, and obtain IVIFS ELECTRE method for solving MAGDM problems under IVIF environments.

This paper is organized as follows. Section 2 briefly reviews the analytic hierarchy process (AHP). Section 3 and Section 4 introduce the new ranking method of intervals and similarity measure between IVIFSs, respectively. Section 5 formulates an MAGDM problem in which the evaluation of alternatives in each attribute is expressed by IVIF sets, and also develops an extended ELECTRE method. Section 6 demonstrates the feasibility and applicability of the proposed method by applying it to the MAGDM problem of the air-condition. Finally, Section 7 presents the conclusions.

2 Analytic hierarchy process (AHP)

AHP was introduced for the first time in 1980 by Thomas L. Saaty [24]. For years, AHP has been used in various fields such as social sciences, health planning and management. Many researchers have preferred to use AHP to find the weights of attribute [25,26]. Due to the fact that attribute weights in the decision-making problems are various, it is not correct to assign all of them as equalled [5]. To solve the problem of indicating the weights, some methods like AHP, eigenvector, entropy analysis, and weighted least square methods were used. For the calculation of attribute weight in AHP the following steps are used:

(i) Arrange the attribute in $n \times n$ square matrix form as rows and columns.

(ii) Using pairwise comparisons, the relative importance of one attribute over another can be expressed as follow:

If two attribute have equal importance in pairwise comparison enter 1; if one of them is moderately more important than the other enter 3 and for the other enter 1/3; if one of them is strongly more important enter 5 and for the other enter 1/5; if one of them is very strongly more important enter 7 and for the other enter 1/7, and if one of them is extremely important enter 9 and for the other enter 1/9. 2, 4, 6 and 8 can be entered as intermediate values. Thus, pairwise comparison matrix is obtained as a result of the pairwise comparisons. Note that all elements in the comparison matrix are positive, in other words $a_{ij} > 0$ $(i, j = 1, 2, \ldots, n)$.

(a) To find the maximum eigenvalue λ of the comparison matrix.

(b) Calculate consistency index $CI = \frac{\lambda - n}{n-1}$ and consistency ratio $CR = \frac{CI}{RI}$, where RI is the random consistency index given by Saaty.(Table 1)

(c) If $CR \geq 0.1$, then adjusts elements a_{ij} $(i, j = 1, 2, \ldots, n)$ of the comparison matrix, (a) and (b) choices are done iteratively until $CR < 0.1$.

(d) Compute eigenvector of the maximum eigenvalue of the comparison matrix.

(e) Normalized eigenvector.

Table1:Random consistency index RI.

n	1	2	3	4	5	6	7	8	9	10	11
RI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51

3 Ranking method for intervals

Let $x = [a, b] \subseteq [0, 1]$ and $y = [c, d] \subseteq [0, 1]$ be two intervals. Since the location relations between $x = [a, b]$ and $y = [c, d]$ include the following six cases, Wan and Dong [27] calculated the occurrence probability for the fuzzy(or random) event $x \geq y$, denoted by $P(x \geq y)$, under different cases.

Case1: $a < b \leq c < d$,

$$P(x \geq y) = 0. \tag{1}$$

Case2: $a \leq c < b < d$ or $a < c < b \leq d$,

$$P(x \geq y) = \frac{(b-c)^2}{2(b-a)(d-c)}. \tag{2}$$

Case3: $a \leq c < d < b$ or $a < c < d \leq b$ or $a \leq c < d \leq b$,

$$P(x \geq y) = \frac{2b-d-c}{2(b-a)}. \tag{3}$$

Case4: $c \leq a < b < d$ or $c < a < b \leq d$,

$$P(x \geq y) = \frac{b+a-2c}{2(d-c)}. \tag{4}$$

Case5: $c \leq a < d < b$ or $c < a < d \leq b$,

$$P(x \geq y) = \frac{2bd+2ac-2bc-a^2-d^2}{2(b-a)(d-c)}. \tag{5}$$

Case6: $c \leq d \leq a < b$ or $c < a < b \leq d$,

$$P(x \geq y) = 1. \tag{6}$$

In order to rank intervals $\tilde{a}_i = [a_i, b_i]$ $(i = 1, 2, \ldots, n)$, Wang and Dong [27] construct the matrix of possibility degree as $P = (P_{ij})_{n\times n}$, where $P_{ij} = P(\tilde{a}_i \geq \tilde{a}_j)$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$. Then, the ranking vector $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is derived as follows:

$$\omega_i = (\sum_{j=1}^{n} P_{ij} + \frac{n}{2} - 1)/(n(n-1)) \ (i = 1, 2, \cdots, n). \tag{7}$$

The larger the value of ω_i, the bigger the corresponding intervals $\tilde{a}_i = [a_i, b_i]$. In other words, for the two intervals $\tilde{a}_i = [a_i, b_i]$ and $\tilde{a}_j = [a_j, b_j]$, if $\omega_i \geq \omega_j$, then $[a_i, b_i] \geq [a_j, b_j]$.

4 Similarity measure between IVIFSs

Definition 1.[3] An IVIFS A in the universe set of discourse X is defined as

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \,|x \in X\},$$

where $\mu_A(x) \subseteq [0, 1]$ and $\nu_A(x) \subseteq [0, 1]$ denote respectively the membership degree interval and the non-membership degree interval of x to A,with the condition:

$\sup\mu_A(x) + \sup\nu_A(x) \leq 1, \forall x \in X$.

Since IVIFS is composed of two ordered interval pairs, Xu [31,32] called them interval-valued intuitionistic fuzzy numbers(IVIFNs) and simply denoted by $G = ([a,b],[c,d])$, where $[a,b] \subseteq [0,1], [c,d] \subseteq [0,1]$ and $b + d \leq 1$.

Definition 2.[37] Let $G_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2)$ be two IVIFNs, the normalized Hamming distance between G_1 and G_2 can be defined as:

$$d(G_1, G_2) = \frac{1}{4}(|a_1 - a_2| + |b_1 - b_2| + |c_1 - c_2| + |d_1 - d_2| + \left|\pi_1' - \pi_2'\right| + \left|\pi_1'' - \pi_2''\right|), \tag{8}$$

where $\pi_{G_i} = [\pi_i', \pi_i''] = [1 - b_i - d_i, 1 - a_i - c_i]$ $(i = 1, 2)$ is called the degree of indeterminacy or called the degree of hesitancy of the IVIFN G_i.

Definition 3.[35, 37] Let $G_i = ([a_i, b_i], [c_i, d_i])$ $(i = 1, 2)$ be two IVIFNs, then

$$s(G_1, G_2) = \begin{cases} 1, & if\ G_1 = G_2 = G_2^c, \\ \frac{d(G_1, G_2^c)}{d(G_1, G_2) + d(G_1, G_2^c)}, & otherwise \end{cases} \tag{9}$$

is called the degree of similarity between G_1 and G_2, where $G_2^c = ([c_2, d_2], [a_2, b_2])$ is denoted as the complement of G_2.

Definition 4.[37] Let A and B be two IVIFSs in X, then

$$s(A, B) = \frac{1}{n}\sum_{j=1}^{n} s(G_j^A, G_j^B) = \frac{1}{n}\sum_{j=1}^{n} \frac{d(G_j^A, (G_j^B)^c)}{d(G_j^A, G_j^B) + d(G_j^A, (G_j^B)^c)} \tag{10}$$

is called the degree of similarity between A and B , where G_j^A and G_j^B are j-th IVIFNs of A and B , respectively.

Definition 5.[6, 27] Let G_i $(i = 1, 2, \ldots, n)$ be a collection of the IVIFNs, where $G_i = ([a_i, b_i], [c_i, d_i])$. If

$$Y_\omega(G_1, G_2, \cdots, G_n) = \frac{\sum_{j=1}^{n} \omega_j G_j}{\sum_{j=1}^{n} \omega_j}, \tag{11}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector, then the function Y_ω is called the weighted average operator for the IVIFNs. Particularly, if ω_j $(j = 1, 2, \ldots, n)$ are crisp values, then the weighted average operator Y_ω is calculated as follows:

$$Y_\omega(G_1, G_2, \cdots, G_n) = \frac{\sum_{j=1}^{n} \omega_j G_j}{\sum_{j=1}^{n} \omega_j} = \left(\left[\frac{\sum_{j=1}^{n} \omega_j a_j}{\sum_{j=1}^{n} \omega_j}, \frac{\sum_{j=1}^{n} \omega_j b_j}{\sum_{j=1}^{n} \omega_j}\right], \left[\frac{\sum_{j=1}^{n} \omega_j c_j}{\sum_{j=1}^{n} \omega_j}, \frac{\sum_{j=1}^{n} \omega_j d_j}{\sum_{j=1}^{n} \omega_j}\right]\right). \tag{12}$$

5 MAGDM problems and ELECTRE method with IVIFSs

5.1 Problems description for MAGDM with IVIFSs

Assume that there are m alternatives $\{A_1, A_2, \ldots, A_m\}$ and k experts $\{p_1, p_2, \ldots, p_k\}$, each alternative A_i has n attributes $\{a_1, a_2, \ldots, a_n\}$. For each alternative A_i, each expert gives evaluation on different attribute.

The multi-attribute group decision making (MAGDM) is choose the best one from all alternatives according to these evaluations. Assume that $G^t_{Mij} = [a^t_{ij}, b^t_{ij}]$ and $G^t_{Nij} = [c^t_{ij}, d^t_{ij}]$ are respectively the membership degree and non-membership degree of alternative $A_i \in A$ on an attribute a_j given by DM p_t to the fuzzy concept "excellent". In other words, the evaluation of A_i on a_j given by p_t is an IVIFN as follows:

$$G^t_{ij} = (G^t_{Mij}, G^t_{Nij}), \tag{13}$$

where $[a^t_{ij}, b^t_{ij}] \subseteq [0,1], [c^t_{ij}, d^t_{ij}] \subseteq [0,1]$ and $b^t_{ij} + d^t_{ij} \leq 1$ $(1 \leq i \leq m, 1 \leq j \leq n, 1 \leq t \leq k)$.

5.2 Determination of the weights of DMs

Since the different DMs play different roles during the process of decision making, thus the importance of DMs should be taken into consideration. The weight vector of DMs is denoted by $z = (z_1, z_2, \ldots, z_k)^T$. In the following, an approach determined the weights of DMs is given.

Suppose that the evaluation of alternative A_i given by DM p_t on each attribute are respectively the IVIFNs $G^t_{i1}, G^t_{i2}, ..., G^t_{in}$. By Eq.(12), the individual overall attribute value of A_i given by p_t is obtained as follows:

$$E^t_i = ([a^t_i, b^t_i], [c^t_i, d^t_i]) = Y_\omega(G^t_{i1}, G^t_{i2}, \cdots G^t_{in}), \tag{14}$$

where $\omega = (\omega_1, \omega_2, ..., \omega_n)^T$ is the weight vector of attributes.

Let $E^t = (E^t_1, E^t_2, \ldots, E^t_m)$ and $E^u = (E^u_1, E^u_2, \ldots, E^u_m)$ are evaluation vectors of all alternatives given by DMs p_t and p_u, respectively. Using Eqs.(8-10), the similarity degree s_{tu} between E^t and E^u is obtained, and the similarity matrix S is constructed as follows:

$$S = (s_{tu})_{k\times k}. \tag{15}$$

Obviously, S is a non-negative symmetric matrix, by the Perron-Frobenius theorem [12], there exists the maximum module eigenvalue $\lambda > 0$, and the corresponding eigenvector $x = (x_1, x_2, \ldots, x_k)^T$ satisfies that $x_t > 0$ $(t = 1, 2, \ldots, k)$ and $\lambda x = Sx$.

Let $z = \lambda x = Sx$, then each component of z is the weight of corresponding expert. The normalized vector z, the weight z_t $(t = 1, 2, \ldots, k)$ of DM p_t is obtained as follows:

$$z_t = \frac{x_t}{(x_1 + x_2 + \cdots + x_k)} \quad (t = 1, 2, \cdots, k). \tag{16}$$

5.3 ELECTRE methods based on IVIFS

Based on the idea of ELECTRE method, a new approach, named as IVIF ELECTRE, is formulated to solve a MCDM problem under interval-valued intuitionistic fuzzy environment. For each pair of alternatives k and l $(k, l = 1, 2, \ldots, m$ and $k \neq l)$, each attribute in the different alternatives can be divided into two distinct subsets. The concordance set E_{kl} of A_k and A_l is composed of all attribute for which A_k is preferred to A_l. In other words, $E_{kl} = \{j|x_{kj} \geq x_{lj}\}$, where $J = \{j|j = 1, 2, \ldots, n\}$, x_{kj} and x_{lj} denoted the evaluation of DM in the jth attribute to alternative A_k and A_l, respectively. The complementary subset, which is the discordance set, is $F_{kl} = \{j|x_{kj} < x_{lj}\}$. In the proposed IVIF ELECTRE method, we can classify different types of concordance and discordance sets using the concepts of score function, accuracy function and hesitation degree, and use concordance and discordance sets to construct concordance and discordance matrices, respectively. The decision makers can choose the best alternative using the concepts of positive and negative ideal points.

Xu [31] and Xu and Chen [36] defined the score function $S(G)$ and accuracy function $H(G)$ for an IVIFN G=([a,b],[c,d]) as follows:

$$S(G) = \frac{1}{2}(a + b - c - d), \tag{17}$$

$$H(G) = \frac{1}{2}(a + b + c + d). \tag{18}$$

Here, we define the hesitation degree for an IVIFN G=([a,b],[c,d]) as follows:

$$\pi(G) = 1 - \frac{1}{2}(a + b + c + d). \tag{19}$$

From (18) and (19), easy to see that a higher accuracy degree $H(G)$ correlates with a lower hesitancy degree $\pi(G)$.

Considering the better alternative has the higher score degree or higher accuracy degree in cases where alternatives have the same score degree. A higher score degree refers to a larger membership degree or smaller non-membership degree, and a higher accuracy degree refers to a smaller hesitation degree. Based on this, using the above three functions to compare IVIF values of different alternatives. The concordance set can be classified as concordance set, midrange concordance set and weak concordance set. Similarly, The discordance sets can also be classified as the discordance set, midrange discordance set, and weak discordance set.

Next, the concordance set, midrange concordance set, weak concordance set, discordance set, midrange discordance set, weak discordance set are defined respectively as follows.

Let $G_{kj} = ([a_{kj}, b_{kj}], [c_{kj}, d_{kj}])$ and $G_{lj} = ([a_{lj}, b_{lj}], [c_{lj}, d_{lj}])$ denote the jth attribute value of alternative A_k and A_l, respectively. The concordance set C_{kl} is composed of all attribute for which A_k is preferred to A_l,i.e.,

$$C_{kl} = \{j | [a_{kj}, b_{kj}] \geq [a_{lj}, b_{lj}], [c_{kj}, d_{kj}] < [c_{lj}, d_{lj}] \ \ and \ [\pi'_{kj}, \pi''_{kj}] < [\pi'_{lj}, \pi''_{lj}]\}, \tag{20}$$

where $J = \{j | j = 1, 2, \ldots, n\}$.

The midrange concordance set C'_{kl} is defined as

$$C'_{kl} = \{j | [a_{kj}, b_{kj}] \geq [a_{lj}, b_{lj}], [c_{kj}, d_{kj}] < [c_{lj}, d_{lj}] \ \ and \ [\pi'_{kj}, \pi''_{kj}] \geq [\pi'_{lj}, \pi''_{lj}]\}. \tag{21}$$

The major difference between (20) and (21) is the hesitancy degree; the hesitancy degree at the kth alternative with respect to the jth attribute is higher than the lth alternative with respect to the jth attribute in the midrange concordance set. Thus, Eq. (20) is more concordant than (21).

The weak concordance set C''_{kl} is defined as

$$C''_{kl} = \{j | [a_{kj}, b_{kj}] \geq [a_{lj}, b_{lj}] \ \ and \ \ [c_{kj}, d_{kj}] \geq [c_{lj}, d_{lj}]\}. \tag{22}$$

The degree of non-membership at the kth alternative with respect to the jth attribute is higher than the lth alternative with respect to the jth attribute in the weak concordance set; thus, Eq. (21) is more concordant than (22).

The discordance set is composed of all attribute for which A_k is not preferred to A_l. The discordance set D_{kl} is formulated as follows:

$$D_{kl} = \{j | [a_{kj}, b_{kj}] < [a_{lj}, b_{lj}], [c_{kj}, d_{kj}] \geq [c_{lj}, d_{lj}] \ \ and \ [\pi'_{kj}, \pi''_{kj}] \geq [\pi'_{lj}, \pi''_{lj}]\}, \tag{23}$$

The midrange discordance set D'_{kl} is defined as

$$D'_{kl} = \{j | [a_{kj}, b_{kj}] < [a_{lj}, b_{lj}], [c_{kj}, d_{kj}] \geq [c_{lj}, d_{lj}] \ \ and \ [\pi'_{kj}, \pi''_{kj}] < [\pi'_{lj}, \pi''_{lj}]\}. \tag{24}$$

The weak discordance set D''_{kl} is defined as

$$D''_{kl} = \{j|[a_{kj}, b_{kj}] < [a_{lj}, b_{lj}] \ \ and \ \ [c_{kj}, d_{kj}] < [c_{lj}, d_{lj}]\}. \tag{25}$$

The IVIF ELECTRE method is an integrated IVIFS and ELECTRE method. The relative value of the concordance set of the IVIF ELECTRE method is measured through the concordance index. The concordance index e_{kl} between A_k and A_l is defined as:

$$e_{kl} = \min_{j \in C^*}\{w_{C^*} \times d(G_{kj}, G_{lj})\}, \tag{26}$$

where $d(G_{kj}, G_{lj})$ is defined in (8), denoted the distance between jth attribute of alternatives A_k and A_l, and w_{C^*} is equal to w_C, $w_{C'}$ or $w_{C''}$, which denoted the weight of the concordance, midrange concordance, and weak concordance sets, respectively.

The concordance matrix E is defined as follows:

$$E = \begin{bmatrix} - & e_{12} & \cdots & \cdots & e_{1m} \\ e_{21} & - & e_{23} & \cdots & e_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ e_{(m-1)1} & \cdots & \cdots & - & e_{(m-1)m} \\ e_{m1} & e_{m2} & \cdots & e_{m(m-1)} & - \end{bmatrix}, \tag{27}$$

where the maximum value of e_{kl} is denoted by e^*, which is the positive ideal point, and a higher value of e_{kl} indicates that A_k is preferred to A_l.

the discordance index is defined as follows:

$$h_{\text{kl}} = \max_{j \in D^*}\{w_{D^*} \times d(G_{kj}, G_{lj})\}, \tag{28}$$

where $d(G_{kj}, G_{lj})$ is defined in (8), denoted the distance between jth attribute of alternatives A_k and A_l, and w_{D^*} is equal to w_D, $w_{D'}$ or $w_{D''}$, which denoted the weight of the discordance, midrange discordance, and weak discordance sets, respectively.

The discordance matrix H is defined as follows:

$$H = \begin{bmatrix} - & h_{12} & \cdots & \cdots & h_{1m} \\ h_{21} & - & h_{23} & \cdots & h_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ h_{(m-1)1} & \cdots & \cdots & - & h_{(m-1)m} \\ h_{m1} & h_{m2} & \cdots & h_{m(m-1)} & - \end{bmatrix}, \tag{29}$$

where the maximum value of h_{kl} is denoted by h^*, which is the negative ideal point, and a higher value of H_{kl} indicates that A_k is less favorable than A_l.

The concordance dominance matrix calculation process is based on the concept that the chosen alternative should have the shortest distance from the positive ideal solution, thus, the concordance dominance matrix K is defined as follows:

$$K = \begin{bmatrix} - & k_{12} & \cdots & \cdots & k_{1m} \\ k_{21} & - & k_{23} & \cdots & k_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ k_{(m-1)1} & \cdots & \cdots & - & k_{(m-1)m} \\ k_{m1} & k_{m2} & \cdots & k_{m(m-1)} & - \end{bmatrix}, \tag{30}$$

where $k_{kl} = e^* - e_{kl}$, which refers to the separation of each alternative from the positive ideal solution. A higher value of k_{kl} indicates that A_k is less favorable than A_l.

The discordance dominance matrix calculation process is based on the concept that the chosen alternative should have the farthest distance from the negative ideal solution, thus, the discordance dominance matrix L is defined as follows:

$$L = \begin{bmatrix} - & l_{12} & \cdots & \cdots & l_{1m} \\ l_{21} & - & l_{23} & \cdots & l_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ l_{(m-1)1} & \cdots & \cdots & - & l_{(m-1)m} \\ l_{m1} & l_{m2} & \cdots & l_{m(m-1)} & - \end{bmatrix}, \tag{31}$$

where $l_{kl} = h^* - h_{kl}$, which refers to the separation of each alternative from the negative ideal solution. A higher value of l_{kl} indicates that A_k is preferred to A_l.

In the aggregate dominance matrix determining process, the distance of each alternative to both positive and negative ideal points can be calculated to determine the ranking order of all alternatives. The aggregate dominance matrix R is defined as follows:

$$R = \begin{bmatrix} - & r_{12} & \cdots & \cdots & r_{1m} \\ r_{21} & - & r_{23} & \cdots & r_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ r_{(m-1)1} & \cdots & \cdots & - & r_{(m-1)m} \\ r_{m1} & r_{m2} & \cdots & r_{m(m-1)} & - \end{bmatrix}, \tag{32}$$

where

$$r_{kl} = \frac{l_{kl}}{k_{kl} + l_{kl}},$$

r_{kl} refers to the relative closeness to the ideal solution, with a range from 0 to 1. A higher value of r_{kl} indicates that the alternative A_k is simultaneously closer to the positive ideal point and farther from the negative ideal point than the alternative A_l, thus, it is a better alternative.

Let $\overline{T}_k = \frac{1}{m-1} \sum\limits_{l=1, l\neq k}^{m} r_{kl},\ k = 1, 2, \cdots, m,$ (33)

and $\overline{T}_k$ is the final value of evaluation. All alternatives can be ranked according to $\overline{T}_k$. The best alternative T^*, which is simultaneously the shortest distance to the positive ideal point and the farthest distance from the negative ideal point, can be generated and defined as follows:

$$T^* = \max_{1 \leq k \leq m} \{\overline{T}_k\}, \tag{34}$$

where A^* is the best alternative.

5.4 Group decision making method

In the following we shall utilize the AHP and interval-valued intuitionistic fuzzy weighted average operator Y (i.e. Eq. (12)) to propose a new MAGDM method with IVIFN information. The detailed steps are summarized as follows:

Step 1. DMs use IVIFSs to represent the evaluation information in the each attribute of alternatives;
Step 2. Use AHP to calculate the weight of attribute;
Step 3. Calculate the individual overall attribute value of each alternative by Eq.(14);
Step 4. Obtain the similarity matrix of the DMs according to Eq.(10);

Step 5. Derive the weight value of each DM from Eq.(16);
Step 6. Using the weight of DM to integrate the same attribute value of different DMs of each alternative in terms of Eq.(14);
Step 7. By the possibility degree ranking method for intervals in Section 3, calculate the ranking vector of the membership degree interval, the non-membership degree interval and the hesitancy degree interval of between the difference alternatives on each attribute, respectively.
Step 8. Obtain the concordance, midrange concordance, weak concordance, discordance, midrange discordance and weak discordance set according to Eqs.(20)-(25), respectively;
Step 9. Compute the concordance matrix, discordance matrix, concordance dominance matrix, discordance dominance matrix and aggregate dominance matrix in terms of Eqs.(26)-(32);
Step 10. Obtain the ranking order of all alternatives and the best alternative according to Eqs.(33)-(34).

6 Numerical example

In this section, we use the air-condition system selection problem given by [27] to verify the feasibility of the proposed method. The problem is described as follows:

Suppose there exist three air-condition systems $\{A_1, A_2, A_3\}$, four attributes a_1 (economical), a_2(function), a_3(being operative) and a_4(longevity) are taken into consideration in the selection problem. Three experts (DMs) $\{p_1, p_2, p_3\}$ participate in the decision making. The membership degrees and non-membership degrees for the alternative A_i on the attribute a_j given by expert p_t were listed in Tables $2-4$.

Table 2: IVIFNs given by the expert p_1.

Attribute	A_1	A_2	A_3
a_1	([0.4, 0.8], [0.0, 0.1])	([0.5, 0.7],[0.1, 0.2])	([0.5, 0.7],[0.2, 0.3])
a_2	([0.3, 0.6], [0.0, 0.2])	([0.3, 0.5],[0.2, 0.4])	([0.6, 0.8],[0.1, 0.2])
a_3	([0.2, 0.7], [0.2, 0.3])	([0.4, 0.7],[0.0, 0.2])	([0.4, 0.7],[0.1, 0.2])
a_4	([0.3, 0.4], [0.4, 0.5])	([0.1, 0.2],[0.7, 0.8])	([0.6, 0.8],[0.0, 0.2])

Table 3: IVIFNs given by the expert p_2.

Attribute	A_1	A_2	A_3
a_1	([0.5, 0.9], [0.0, 0.1])	([0.7, 0.8], [0.1, 0.2])	([0.5, 0.6], [0.1, 0.4])
a_2	([0.4, 0.5], [0.3, 0.5])	([0.5, 0.6], [0.2, 0.3])	([0.6, 0.7], [0.1, 0.2])
a_3	([0.5, 0.8], [0.0, 0.1])	([0.5, 0.8], [0.0, 0.2])	([0.4, 0.8], [0.1, 0.2])
a_4	([0.4, 0.7], [0.1, 0.2])	([0.5, 0.6], [0.3, 0.4])	([0.2, 0.6], [0.2, 0.3])

Table 4: IVIFNs given by the expert p_3.

Attribute	A_1	A_2	A_3
a_1	([0.3, 0.9], [0.0, 0.1])	([0.3, 0.8], [0.1, 0.2])	([0.2, 0.6], [0.1, 0.2])
a_2	([0.2, 0.5], [0.1, 0.4])	([0.5, 0.6], [0.1, 0.3])	([0.2, 0.6], [0.2, 0.3])
a_3	([0.4, 0.7], [0.1, 0.2])	([0.2, 0.8], [0.0, 0.2])	([0.3, 0.6], [0.1, 0.3])
a_4	([0.3, 0.6], [0.3, 0.4])	([0.3, 0.5], [0.2, 0.3])	([0.4, 0.7], [0.1, 0.2])

In the following, we will illustrate the decision making process.
(1) Calculation of weights of attributes

In order to find the weights of attributes, A commission, which is organized by sampling method, determined the importance of attribute by using AHP. A 4×4 size matrix is formed because 4 attribute are considered in this study. All the diagonal elements of the matrix will be 1, the elements of symmetrical position with respect to the diagonal are reciprocal, in other words, if a_{ij} is ith row and jth column element of matrix, then its symmetrical position is filled using $a_{ji} = 1/a_{ij}$ formula.

The comparison matrix W is obtained as follows:

$$W = \begin{pmatrix} 1 & 2 & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{3} & \frac{1}{6} \\ 3 & 3 & 1 & \frac{1}{3} \\ 4 & 6 & 3 & 1 \end{pmatrix}.$$

By computing the eigenvalues and the eigenvectors of W, we obtained that the maximum eigenvalue of W was 4.0875, the corresponding eigenvector was $\omega = (0.1905, 0.1230, 0.4046, 0.8849)^T$, consistency index CI=0.0292 and consistency ratio $CR = 0.0324 < 0.1$.

Normalized eigenvectors, the four attributes weights are obtained as follows:
$\omega_1 = 0.1213$, $\omega_2 = 0.0765$, $\omega_3 = 0.2517$, $\omega_4 = 0.5505$.

(2) Calculate the individual overall attribute value of each alternative

By Eq.(14), the individual overall attribute value of each alternative can be obtained as in Table 5.

Table 5: The individual overall attribute values of the alternatives for weight vector of attributes.

E_i^t	A_1	A_2	A_3
p_1	([0.2870,0.5393],[0.2705,0.3782])	([0.2393,0.4095],[0.4128,0.5456])	([0.5375,0.7627],[0.0571,0.2121])
p_2	([0.4373,0.7341],[0.0780,0.1857])	([0.5242,0.6746],[0.1926,0.3178])	([0.3173,0.6580],[0.1551,0.2793])
p_3	([0.3175,0.6539],[0.1980,0.3133])	([0.2901,0.6196],[0.1299,0.2627])	([0.3353,0.6551],[0.1077,0.2328])

(3) Calculation of the similarity matrix and the weight vector of DMs

The similarity matrix for the DMs is constructed by Eq.(10) as follows:

$$S = \begin{pmatrix} 1 & 0.5415 & 0.6059 \\ 0.5415 & 1 & 0.7577 \\ 0.6059 & 0.7577 & 1 \end{pmatrix}.$$

Because the maximum eigenvalue of S is 2.2746, the corresponding eigenvector is $x = (0.5373, 0.5878, 0.6048)^T$, the expert′s weights are obtained from Eq.(16) as follows: $z_1 = 0.3106$, $z_2 = 0.3398$, $z_3 = 0.3496$.

(4) Integrate the attribute value of different DMs

By Eq.(14), the attribute value of different DMs are respectively integrated as in Table 6.

Table 6: The attribute value of different DMs in the different alternatives and different attributes.

	A_1	A_2	A_3
a_1	([0.3990,0.8689],[0,0.1])	([0.4980,0.7689],[0.1,0.2])	([0.3951,0.6311],[0.1311,0.2990])
a_2	([0.2990,0.5311],[0.1369,0.3719])	([0.4379,0.5689],[0.1650,0.3311])	([0.4602,0.6961],[0.1350,0.2350])
a_3	([0.3719,0.7340],[0.0971,0.1971])	([0.3641,0.7689],[0,0.2])	([0.3650,0.6990],[0.1,0.2350])
a_4	([0.3340,0.5719],[0.2631,0.3631])	([0.3058,0.4408],[0.3893,0.4893])	([0.3942,0.6971],[0.1029,0.2340])

(5) Calculate the ranking vector

The ranking vector of the membership degree interval, the non-membership degree interval and the hesitancy degree interval of between the difference alternatives on each attribute is calculated by Eqs.(1-7), respectively, as in Table 7.

Table 7: The attribute value of different DMs in the different alternatives and different attributes.

	membership degree interval		non-membership degree interval		hesitancy degree interval	
a_1	A_1	A_2	A_1	A_2	A_1	A_2
	0.5006	0.4994	0.25	0.75	0.5873	0.4127
	A_1	A_3	A_1	A_3	A_1	A_3
	0.6286	0.3714	0.25	0.75	0.5388	0.4612
	A_2	A_3	A_2	A_3	A_2	A_3
	0.6808	0.3192	0.3207	0.6793	0.4341	0.5659
a_2	A_1	A_2	A_1	A_2	A_1	A_2
	0.3214	0.6786	0.5135	0.4865	0.5878	0.4122
	A_1	A_3	A_1	A_3	A_1	A_3
	0.27295	0.72705	0.6477	0.3523	0.59905	0.40095
	A_2	A_3	A_2	A_3	A_2	A_3
	0.34565	0.65435	0.6764	0.3236	0.5173	0.4827
a_3	A_1	A_2	A_1	A_2	A_1	A_2
	0.48325	0.51675	0.6177	0.3823	0.4723	0.5277
	A_1	A_3	A_1	A_3	A_1	A_3
	0.52875	0.47125	0.4246	0.5754	0.4995	0.5005
	A_2	A_3	A_2	A_3	A_2	A_3
	0.54255	0.45745	0.3426	0.6574	0.5273	0.4727
a_4	A_1	A_2	A_1	A_2	A_1	A_2
	0.66115	0.33885	0.25	0.75	0.56895	0.43105
	A_1	A_3	A_1	A_3	A_1	A_3
	0.35955	0.64045	0.75	0.25	0.44015	0.55985
	A_2	A_3	A_2	A_3	A_2	A_3
	0.2633	0.7367	0.75	0.25	0.365	0.635

(6) Determine the concordance, midrange concordance, weak concordance, discordance, midrange discordance and weak discordance set

Applying Eqs.(20-25) and Table 7, the concordance, midrange concordance, weak concordance, discordance, midrange discordance and weak discordance set is calculated, respectively, as follows:

$$C=\begin{pmatrix} - & - & 3 \\ 2 & - & 1 \\ 2 & 2 & - \end{pmatrix},\quad C'=\begin{pmatrix} - & 1,4 & 1 \\ 3 & - & 3 \\ 4 & 4 & - \end{pmatrix},\quad C''=\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix},$$

$$D=\begin{pmatrix} - & 2 & 2 \\ - & - & 2 \\ 3 & 1 & - \end{pmatrix},\quad D'=\begin{pmatrix} - & 3 & 4 \\ 1,4 & - & 4 \\ 1 & 3 & - \end{pmatrix},\quad D''=\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}.$$

For example, $c_{13}=\{3\}$, which is in the 1st (horizontal) row and the 3rd (vertical) column of the concordance set, is "3." $c_{12}=\{-\}$, which is in the 1st row and 2nd column of the concordance set, is "empty," and so forth.

(7) Compute the concordance matrix, discordance matrix, concordance dominance matrix, discordance dominance matrix and aggregate dominance matrix

We give the relative weights as: $[\omega_C, \omega_{C'}, \omega_{C''}, \omega_D, \omega_{D'}, \omega_{D''}] = [1, \frac{2}{3}, \frac{1}{3}, 1, \frac{2}{3}, \frac{1}{3}]$. By Eqs.(26)-(32), the concordance matrix, discordance matrix, concordance dominance matrix, discordance dominance matrix and aggregate dominance matrix are obtained, respectively, as follows:

$$E=\begin{pmatrix} - & 0.08575 & 0.02235 \\ 0.04759 & - & 0.05697 \\ 0.09643 & 0.07862 & - \end{pmatrix},\quad H=\begin{pmatrix} - & 0.1039 & 0.16309 \\ 0.09967 & - & 0.18088 \\ 0.12298 & 0.1204 & - \end{pmatrix},$$

$$K=\begin{pmatrix} - & 0.01068 & 0.07408 \\ 0.04884 & - & 0.03946 \\ 0 & 0.01781 & - \end{pmatrix},\quad L=\begin{pmatrix} - & 0.07698 & 0.01779 \\ 0.08121 & - & 0 \\ 0.0579 & 0.06048 & - \end{pmatrix}.$$

$$R = \begin{pmatrix} - & 0.8782 & 0.1936 \\ 0.6246 & - & 0 \\ 1 & 0.7725 & - \end{pmatrix}$$

(8) Compute the ranking order of all alternatives and obtain the best alternative
Applying Eq.(33),

$$\overline{T} = \begin{pmatrix} 0.5359 \\ 0.3123 \\ 0.88625 \end{pmatrix}$$

The optimal ranking order of alternatives is given by $A_3 \succ A_1 \succ A_2$. The best alternative is A_3. The ranking order given by [27] is identical. The best air-condition system is A_3.

This example shows the effectiveness of the ranking method proposed in this paper.

7 Conclusion

Regarding the MAGDM problem, the IVIF theory provides a useful and convenient way to reflect the ambiguous nature of subjective judgments and assessments. In this paper, firstly, using the normalized Hamming distance between IVIFS to construct similarity matrix and obtain the wights of DMs. Then, using possibility degree of IVIF to calculate the ranking vector. Based on this, the concordance and discordance sets, concordance and discordance matrices etc. are obtained. Finally, by computing the ranking order of all alternatives, decision makers can choose the best alternative, the example verify the correctness of the method.

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The interior and closure of fuzzy topologies induced by the generalized fuzzy approximation spaces

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Abstract

With respect to the Alexandrov fuzzy topologies induced by the generalized fuzzy approximation spaces, Wang defined interior of fuzzy set. In this paper, we give the closure of fuzzy set and discuss some properties of the interior and closure.

Keywords: Alexandrov fuzzy topology; the generalized fuzzy approximation spaces; interior; closure; properties

1 Introduction

In his classical paper [36], Zadeh introduced the notation of fuzzy sets and fuzzy set operation. Subsequently, Chang [2] applied some basic concepts from general topology to fuzzy sets and developed a theory of fuzzy topological spaces. Pu etc.[18] defined a fuzzy point which took a crisp singleton, equivalently, an ordinary point, as a special case and gave the concepts of interior and closure operator w.r.t. fuzzy topology. Later, Lai and Zhang [11] modified the second axiom in Chang's definition of fuzzy topology to define an Alexandrov fuzzy topology.

The concept of Rough sets were introduced by Z. Pawlak [19] in 1982 as an powerful mathematical tool for uncertain data while modeling the problems in many fields [17,20,27]. Because the rough sets defined with equivalence relations limited the application of it. Thus many authors changed the equivalence relations into different binary relations to expand the application of it [23,35,37,38]. In recent years, the rough sets has been combined with some mathematical theories such as algebra and topology [1,5,6,8,10, 14, 16, 21, 25, 26, 28, 29]. With respect to different binary relations, the topological properties of rough sets were further investigated in [7,14,33,34].

In 1990, Dubois and Prade [3] combining fuzzy sets and rough sets proposed rough fuzzy sets and fuzzy rough sets. Afterward Morsi and Yakout [15] investigated fuzzy rough sets defined with left-continuous t-norms and R-implicators with respect to fuzzy similarity relations. Radzikowska and Kerre [24] defined a broad family of fuzzy rough sets based on t-norms and fuzzy implicators, which are called generalized fuzzy rough sets here. In recent years, the topological properties of fuzzy rough sets were further studied in many literatures [4,9,12,13,22]. Recently, with respect to the lower fuzzy rough approximation operator determined by a fuzzy implicator, Wang [30] studied various fuzzy topologies induced by different fuzzy relations and proved that I-lower fuzzy rough approximation operators were the interior operator w.r.t. some Alexandrov fuzzy topology.

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In this paper, we give closure operator w.r.t. some Alexandrov fuzzy topology given by Wang in [30]. Combined with the definition of Wang's interior, discuss some properties of the interior and closure of fuzzy set.

2 Preliminary

Definition 2.1.[36] A fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, A(x)) : x \in X\}$$

where $A(x) : X \to [0, 1]$ is a mapping and $A(x)$ states the grade of belongness of x in A. The family of all fuzzy sets in X is denoted by $\mathscr{F}(X)$.

Let $\alpha \in [0, 1]$, then a fuzzy set $A \in \mathscr{F}(X)$ is a constant, while $A(x) = \alpha$ for all $x \in X$, denoted as α_X.

Deflnition 2.2.[36] Let A, B be two fuzzy sets of $\mathscr{F}(X)$
(1) A is contained in B if and only if $A(x) \leq B(x)$ for every $x \in X$.
(2) The union of A and B is a fuzzy set C, denoted by $A \cup B = C$, whose membership function $C(x) = A(x) \vee B(x)$ for every $x \in X$.
(3) The intersection of A and B is a fuzzy set C, denoted by $A \cap B = C$, whose membership function $C(x) = A(x) \wedge B(x)$ for every $x \in X$.
(4) The complement of A is a fuzzy set, denoted by A^c, whose membership function $A^c(x) = 1 - A(x)$ for every $x \in X$.

A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ (resp. $S : [0, 1] \times [0, 1] \to [0, 1]$) is called a t-norm (resp. t-conorm) on [0, 1] if it is commutative, associative, increasing in each argument and has a unit element 1 (resp. 0).

A mapping $I : [0, 1] \times [0, 1] \to [0, 1]$ is called a fuzzy implicator on [0, 1] if it satisfies the boundary conditions according to the Boolean implicator, and is decreasing in the first argument and increasing in the second argument.

Definition 2.3.[30] A fuzzy implicator I is said to satisfy
(1) the left neutrality property ((NP), for short), if $I(1, b) = b$ for all $b \in [0, 1]$;
(2) the confinement principle ((CP), for short), if $I(a, b) = 1 \Leftrightarrow a \leq b$, for all $a, b \in [0, 1]$;
(3) the regular property ((RP), for short), if N_I is an involutive negation, where N_I is defined as $N_I(a) = I(a, 0)$ for all $a \in [0, 1]$.

Definition 2.4. [11] A subset $\tau \subseteq \mathscr{F}(X)$ is called an Alexandrov fuzzy topology if it satisfies:
(1) $\alpha_X \in \tau$ for all $\alpha \in [0, 1]$,
(2) $\cap_{i\in\Lambda} A_i \in \tau$ for all $\{A_i\}_{i\in\Lambda} \subseteq \tau$,
(3) $\cup_{i\in\Lambda} A_i \in \tau$ for all $\{A_i\}_{i\in\Lambda} \subseteq \tau$.

Every member of τ is called a τ-open fuzzy set. A fuzzy set is τ-closed if and only if its complement is τ-open. In the sequel, when no confusion is likely to arise, we shall call a τ-open (τ-closed) fuzzy set simply an open (closed) set.

Definition 2.5. [18,31]. Let $\tau \subseteq \mathscr{F}(X)$ be a fuzzy topology. Then the interior of $A \in \mathscr{F}(X)$ w.r.t. fuzzy topology τ denoted as A^o is defined as follows:

$$A^o = \cup\{B \in \tau | B \subseteq A\}.$$

The operator A^o is called an interior operator w.r.t. fuzzy topology τ.

According to definition of the fuzzy topology, obviously A^o is an open set.

Definition 2.6. [18]. Let $\tau \subseteq \mathscr{F}(X)$ be a fuzzy topology. Then the closure of $A \in \mathscr{F}(X)$ w.r.t. fuzzy topology τ denoted as $\overline{A}$ is defined as follows:

$$\overline{A} = \cap\{B|B \supseteq A, B^c \in \tau\}$$

The operator $\overline{A}$ is called a closure operator w.r.t. fuzzy topology τ.

According to De Morgan's Law and definition of the fuzzy topology, $\overline{A}$ is a closed set.

3 Fuzzy topologies induced by the generalized fuzzy approximation spaces

A fuzzy set $R \in \mathscr{F}(X \times Y)$ is called a fuzzy relation from X to Y. If $X = Y$, then R is a fuzzy relation on X. For every fuzzy relation R on X, a fuzzy relation R^{-1} is defined as $R^{-1}(x, y) = R(y, x)$ for all $x, y \in X$. Let R be a fuzzy relation from X to Y . Then the triple (X, Y, R) is called a fuzzy approximation space. When $X = Y$ and R is a fuzzy relation on X, we also call (X, R) a fuzzy approximation space.

Definition 3.1.[30]. Let R be a fuzzy relation on X. Then R is said to be
(1) reflexive if $R(x, x) = 1$ for all $x \in X$;
(2) symmetric if $R(x, y) = R(y, x)$ for all $x, y \in X$;
(3) T-transitive if $T(R(x, y), R(y, z)) \leq R(x, z)$ for all $x, y, z \in X$.

If $T = \wedge$, then T-transitive is said to be transitive for short. A fuzzy relation R is called a fuzzy tolerance if it is reflexive and symmetric, and a fuzzy T-preorder if it is reflexive and T-transitive. Similarly, a fuzzy relation R is called a fuzzy preorder if it is reflexive and transitive.

Definition 3.2.[24,30,32]. Let (X, Y, R) be a fuzzy approximation space. Then the following mappings $\underline{R}, \overline{R} : \mathscr{F}(Y) \to \mathscr{F}(X)$ are defined as follows: for all $A \in \mathscr{F}(Y)$ and $x \in X$,

$$\underline{R}(A)(x) = \underset{y\in Y}{\wedge} I(R(x, y), A(y)) \text{ and } \overline{R}(A)(x) = \underset{y\in Y}{\vee} T(R(x, y), A(y)).$$

The mappings $\underline{R}$ and $\overline{R}$ are called $I-lower$ and $T-upper$ fuzzy rough approximation operators, respectively. The pair $(\underline{R}(A), \overline{R}(A))$ is called a generalized fuzzy rough set of A w.r.t. (X, Y, R). Also known as generalized fuzzy approximation spaces.

Let R be a fuzzy relation on X. Then a fuzzy set $A \in \mathscr{F}(X)$ is said to be
(1) I-lower definable w.r.t. fuzzy relation R if $\underline{R}(A) = A$; the family of all $I - lower$ definable fuzzy sets w.r.t. R is denoted as $\mathscr{D}_I(R)$.
(2) T-upper definable w.r.t. fuzzy relation R if $\overline{R}(A) = A$; the family of all $T - upper$ definable fuzzy sets w.r.t. R is denoted as $\mathscr{D}_T(R)$.

Proposition 3.3.[30]. Let (X, R) be a fuzzy approximation space and R be reflexive. Then
(1)$\mathscr{D}_I(R)$ is an Alexandrov fuzzy topology, if I satisfies (NP).
(2)$\mathscr{D}_T(R)$ is an Alexandrov fuzzy topology.

Let (X, R) be a fuzzy approximation space. In [30] Wang defined

$$\mathscr{R}_I(R) = \{\underline{R}(A)|A \in \mathscr{F}(X)\} \text{ and } \mathscr{R}_T(R) = \{\overline{R}(A)|A \in \mathscr{F}(X)\}.$$

To discuss the properties of generalized fuzzy rough sets, Radzikowska and Kerre [19] introduced the following auxiliary conditions: for a fuzzy implicator I and a t-norm T,
(C1) $I(a, I(b, c)) = I(T(a, b), c)$ for all $a, b, c \in [0, 1]$,
(C2) $I(a, I(b, c)) \geq I(T(a, b), c)$ for all $a, b, c \in [0, 1]$,
(C3) $I(a, I(b, c)) \leq I(T(a, b), c)$ for all $a, b, c \in [0, 1]$.

If (C1) (resp. (C2), (C3)) holds for I and T , then we say that I satisfies (C1) (resp. (C2), (C3)) for T.

Proposition 3.4.[30]. Let (X, R) be a fuzzy approximation space and R be a fuzzy T-preorder. Then
(1) $\mathscr{R}_I(R)$ is an Alexandrov fuzzy topology and $\mathscr{R}_I(R) = \mathscr{D}_I(R)$, if I satisfies (NP) and (C2) for T.
(2) $\mathscr{R}_T(R)$ is an Alexandrov fuzzy topology and $\mathscr{R}_T(R) = \mathscr{D}_T(R)$.

The above $\mathscr{D}_I(R)$, $\mathscr{D}_T(R)$, $\mathscr{R}_I(R)$ and $\mathscr{R}_T(R)$ are called fuzzy topologies induced by the generalized fuzzy approximation spaces.

4 The interior and closure of fuzzy set

Proposition 4.1.[30]. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then $\underline{R}$ is the interior operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.

Proposition 4.2. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then A is an open set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ iff $\underline{R}(A) = A^o = A$.

Proof. Suppose A is an open set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$, again $A \subseteq A$, due to definition of A^o, $A \subseteq A^o$. On the other hand,

$$\forall x \in X, \ \ \underline{R}(A)(x) = \mathop{\wedge}_{y \in X} I(R(x, y), A(y)) \leq I(R(x, x), A(x)) = I(1, A(x)) = A(x).$$

This means $\underline{R}(A) = A^o \subseteq A$. Thus $\underline{R}(A) = A^o = A$.

Conversely, suppose $\underline{R}(A) = A^o = A$, A^o is an open set, thus A is an open set.

Proposition 4.3. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then for any $A \in \mathscr{F}_{(X)}$, $[\underline{R}(A^c)]^c$ is the closure operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
Proof. For any $A \in \mathscr{F}_{(X)}$, since $\underline{R}(A^c)$ is an open set, thus $(\underline{R}(A^c))^c$ is a closed set. Again

$$\forall x \in X, \ \ \underline{R}(A^c)(x) = \mathop{\wedge}_{y \in X} I(R(x, y), A^c(y)) \leq I(R(x, x), A^c(x)) = I(1, A^c(x)) = A^c(x),$$

this means $(\underline{R}(A^c))^c \supseteq A$.

On the other hand, for any $A \subseteq B \in \mathscr{F}_{(X)}$ and $B^c \in \mathscr{D}_I(R)$. By Proposition 4.2, $\underline{R}(B^c) = B^c$, and

$$\forall x \in X, \ \ \underline{R}(A^c)(x) = \mathop{\wedge}_{y \in X} I(R(x, y), A^c(y)) \geq \mathop{\wedge}_{y \in X} I(R(x, x), B^c(x)) = \underline{R}(B^c)(x).$$

We obtain $\underline{R}(A^c) \supseteq \underline{R}(B^c) = B^c$. This means $(\underline{R}(A^c))^c \subseteq B$. By Definition of the closure, for any $A \in \mathscr{F}_{(X)}$, $[\underline{R}(A^c)]^c$ is the closure operator w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ i.e. $[\underline{R}(A^c)]^c = \overline{A}$.

Proposition 4.4. Let R be a fuzzy T-preorder on X, and I satisfy (NP) and (C2) for T. Then A is a closed set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$ iff $(\underline{R}(A^c))^c = \overline{A} = A$.

Proof. Suppose A is a closed set w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$, then A^c is an open set. Therefore $\underline{R}(A^c) = A^c$, and then $\overline{A} = (\underline{R}(A^c))^c = A$.

Conversely, suppose $\overline{A} = (\underline{R}(A^c))^c = A$, $\overline{A}$ is a closed set, thus A is a closed set.

Proposition 4.5. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A, B \in \mathscr{F}_{(X)}$ the following formula hold w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $A \subseteq \overline{A}$;
(2) $\overline{\overline{A}} = \overline{A}$;
(3) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$;
(4) $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Proof. (1) For all $x \in X$,

$$\begin{aligned}\overline{A}(x) &= (\underline{R}(A^c))^c(x) = 1 - \underline{R}(A^c)(x) = 1 - \bigwedge_{y\in X} I(R(x,y), A^c(y))\\ &\geq 1 - I(R(x,x), A^c(x)) = 1 - I(1, A^c(x)) = 1 - A^c(x) = A(x),\end{aligned}$$

thus $A \subseteq \overline{A}$.
(2) Since $\overline{A}$ is a closed set, By Proposition 4.4, $\overline{\overline{A}} = \overline{A}$.
(3) By $A \subseteq B$, we obtain $A^c \supseteq B^c$. According to Definition 3.2, obviously $\underline{R}(A^c) \supseteq \underline{R}(B^c)$, and then $\overline{A} = (\underline{R}(A^c))^c \subseteq (\underline{R}(B^c))^c = \overline{B}$.
(4) Since $A \subseteq A \cup B$, $B \subseteq A \cup B$, by (2)
$\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

On the other hand, by (1) $A \subseteq \overline{A}$, $B \subseteq \overline{B}$. Thus $A \cup B \subseteq \overline{A} \cup \overline{B}$. And then $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$. Again $\overline{A} \cup \overline{B}$ is a closed set, according to Proposition 4.4 $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Thereby $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Proposition 4.6. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A \in \mathscr{F}_{(X)}$, the following formula hold w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $\overline{A} = [(A^c)^o]^c$; (2) $A^o = [\overline{A^c}]^c$;
(3) $[\overline{A}]^c = [A^c]^o$; (4) $\overline{A^c} = [A^o]^c$.
Proof. (1) By Proposition 4.2, $(A^c)^o = \underline{R}(A^c)$, thus $[(A^c)^o]^c = [\underline{R}(A^c)]^c = \overline{A}$.

(2),(3),(4) can be proven in a similar way as for item (1).

Proposition 4.7. Let R be a fuzzy T-preorder on X, I satisfy (NP) and (C2) for T. Then for any $A, B \in \mathscr{F}_{(X)}$ and $A \subseteq B$, the following holds w.r.t. Alexandrov fuzzy topology $\mathscr{D}_I(R)$.
(1) $A^o \subseteq B^o$; (2) $A^{oo} = A^o$; (3) $(A \cap B)^o = A^o \cap B^o$.

Proof. (1) $\forall x \in X$, $\underline{R}(A)(x) = \bigwedge_{y\in X} I(R(x,y), A(y)) \leq \bigwedge_{y\in X} I(R(x,y), B(y)) = \underline{R}(B)(x)$. Thus $A^o \subseteq B^o$.
(2) Since A^o is a open set, by Proposition 4.2, $A^{oo} = A^o$.
(3) By Proposition 4.6 (2) and Proposition 4.5 (4),
$(A \cap B)^o = (\overline{(A \cap B)^c})^c = (\overline{A^c \cup B^c})^c = (\overline{A^c} \cup \overline{B^c})^c = (\overline{A^c})^c \cap (\overline{B^c})^c = A^o \cap B^o$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.2, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Weighted Lim's Geometric Mean of Positive Invertible Operators on a Hilbert Space

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Abstract

We generalize the weighted Lim's geometric mean of positive definite matrices to positive invertible operators on a Hilbert space. This mean is defined via a certain bijection map and parametrized over Hermitian unitary operators. We derive an explicit formula of the weighted Lim's geometric mean in terms of weighted metric/spectral geometric means. This kind of operator mean turns out to be a symmetric Lim-Pálfia weighted mean and satisfies the idempotency, the permutation invariance, the joint homogeneity, the self-duality, and the unitary invariance. Moreover, we obtain relations between weighted Lim geometric means and Tracy-Singh products via operator identities.

Keywords: positive invertible operator, metric geometric mean, spectral geometric mean, Lim's geometric mean, Tracy-Singh product
Mathematics Subject Classifications 2010: 47A64, 47A80.

1 Introduction

Recall that the Riccati equation for positive definite matrices A and B:

$$XA^{-1}X = B \tag{1}$$

has a unique positive solution

$$X = A\sharp B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}A^{\frac{1}{2}}, \tag{2}$$

known as the metric geometric mean of A and B. This kind of mean was introduced by Ando [2] as the maximum (with respect to the Löwner partial order) of positive semidefinite matrices X satisfying

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$$\begin{pmatrix} A & X \\ X & B \end{pmatrix} \geqslant 0.$$

The above two definitions of the metric geometric mean are equivalent. See a nice discussion about the Riccati equation and the metric geometric mean of matrices in [4].

Fiedler and Pták [3] modified the notion of the metric geometric mean to the spectral geometric mean:

$$A \diamondsuit B \;=\; (A^{-1} \sharp B)^{\frac{1}{2}} A (A^{-1} \sharp B)^{\frac{1}{2}}. \tag{3}$$

One of the most important properties of the spectral geometric mean is the positive similarity between $(A \diamondsuit B)^2$ and AB. This shows that the eigenvalues of $A \diamondsuit B$ coincide with the positive square roots of the eigenvalues of AB.

Lee and Lim [5] introduced the notion of metric geometric means and spectral geometric means on symmetric cones of positive definite matrices and developed various properties of these means. Lim [6] provided a new geometric mean of positive definite matrices varying over Hermitian unitary matrices, including the metric geometric mean as a special case. The new mean has an explicit formula in terms of metric and spectral geometric means. He established basic properties of this mean including the idempotency, joint homogeneity, permutation invariance, non-expansiveness, self-duality, and a determinantal identity. He also gave this new geometric mean for the weighted case. Lim and Pálfia [7] presented a unified framework for weighted inductive means on the cone of positive definite matrices. The metric geometric mean, spectral geometric mean, and Lim geometric mean [6] are basic examples of the two-variable weighted mean.

In this paper, we extend the notion of weighted Lim's geometric mean [6] to the case of Hilbert-space operators via a certain bijection map (see Section 2). This operator mean is parametrized over Hermitian unitary operators. An explicit formula of the weighted Lim's geometric mean is given in term of weighted metric geometric means and spectral geometric means. This kind of operator mean serves the idempotency, the permutation invariance, the joint homogeneity, the self-duality, and the unitary invariance. Moreover, we establish certain operator identities involving Lim weighted geometric means and Tracy-Singh products (see Section 3). Our results include certain literature results involving weighted metric geometric means.

2 Lim's geometric mean of operators

In this section, we discuss the notion of Lim's geometric mean of positive invertible operators on any complex Hilbert space.

Throughout, let $\mathbb{H}$ be a complex Hilbert space. Denoted by $\mathfrak{B}(\mathbb{H})$ the Banach space of bounded linear operators on $\mathbb{H}$. The set of all positive invertible operators on $\mathbb{H}$ is denoted by $\mathbb{P}$.

First of all, we recall the notions of metric/spectral geometric means of operators. Recall that for any $t \in [0,1]$, the t-weighted metric geometric mean of $A, B \in \mathbb{P}$ is defined by

$$A \sharp_t B \;=\; A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}} \tag{4}$$

For briefly, we write $A \sharp B$ for $A \sharp_{1/2} B$. The spectral geometric mean of $A, B \in \mathbb{P}$ is defined by

$$A \diamondsuit B \;=\; (A^{-1} \sharp B)^{\frac{1}{2}} A (A^{-1} \sharp B)^{\frac{1}{2}}. \tag{5}$$

We list some basic properties of metric and spectral geometric means.

Lemma 1 (e.g. [1, 3, 4])**.** *Let $A, B \in \mathbb{P}$ and $t \in [0,1]$. Then*

(i) $A \sharp_t A = A$,

(ii) $(\alpha A) \sharp_t (\beta B) = \alpha^{1-t} \beta^t (A \sharp_t B)$,

(iii) $A \sharp_t B = B \sharp_{1-t} A$,

(iv) $(A \sharp_t B)^{-1} = A^{-1} \sharp_t B^{-1}$,

(v) (Riccati Lemma) $A \sharp B$ is the unique positive invertible solution of $X A^{-1} X = B$,

(vi) $(T^* A T) \sharp_t (T^* B T) = T^* (A \sharp_t B) T$ for any invertible operator $T \in \mathfrak{B}(\mathbb{H})$,

(vii) $(T^* A T) \diamondsuit (T^* B T) = T^* (A \diamondsuit B) T$ for any unitary operator $T \in \mathfrak{B}(\mathbb{H})$.

For a Hermitian unitary operator $U \in \mathfrak{B}(\mathbb{H})$, we set

$$\mathbb{P}_U^+ := \{X \in \mathbb{P} : UXU = X\}, \quad \mathbb{P}_U^- := \{X \in \mathbb{P} : UXU = X^{-1}\}$$

Lemma 2. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator. Then the map*

$$\Phi_U \;:\; \mathbb{P}_U^+ \times \mathbb{P}_U^- \;\to\; \mathbb{P}, \quad (A, B) \;\mapsto\; A^{\frac{1}{2}} B A^{\frac{1}{2}} \tag{6}$$

is bijective with the inverse map given by

$$X \;\mapsto\; (X \sharp (UXU), X \diamondsuit (UX^{-1}U)). \tag{7}$$

Proof. The proof is quite similar to [6, Theorem 2.6]. Let $A_1, A_2 \in \mathbb{P}_U^+$ and $B_1, B_2 \in \mathbb{P}_U^-$ such that $\Phi_U(A_1, B_1) = \Phi_U(A_2, B_2)$, i.e. $A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}} = A_2^{\frac{1}{2}} B_2 A_2^{\frac{1}{2}}$. Since $A_i \in \mathbb{P}_U^+$, we have

$$\begin{aligned} U A_i^{-1} U \;&=\; (U A_i U)^{-1} \;=\; A_i^{-1}, \\ U A_i^{\frac{1}{2}} U \;&=\; (U A_i U)^{\frac{1}{2}} \;=\; A_i^{\frac{1}{2}}. \end{aligned}$$

and thus $A_i^{-1}, A_i^{\frac{1}{2}} \in \mathbb{P}_U^+$ for $i = 1, 2$. It follows that

$$\begin{aligned}
B_1^{-1} &= UB_1U \\
&= U\Big(A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}B_2A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}}\Big)U \\
&= \big(UA_1^{-\frac{1}{2}}U\big)\big(UA_2^{\frac{1}{2}}U\big)\big(UB_2U\big)\big(UA_2^{\frac{1}{2}}U\big)\big(UA_1^{-\frac{1}{2}}U\big) \\
&= A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}B_2^{-1}A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}} \\
&= A_1^{-\frac{1}{2}}A_2^{\frac{1}{2}}\Big(A_2^{-\frac{1}{2}}A_1^{\frac{1}{2}}B_1A_1^{\frac{1}{2}}A_2^{-\frac{1}{2}}\Big)^{-1}A_2^{\frac{1}{2}}A_1^{-\frac{1}{2}} \\
&= \big(A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}\big)B_1^{-1}\big(A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}\big),
\end{aligned}$$

i.e. $A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}}$ is a solution of $XB_1^{-1}X = B_1^{-1}$. Since $B_1^{-1}\sharp B_1 = I$ is the unique solution of $XB_1^{-1}X = B_1^{-1}$ (Lemma 1 (v)), we conclude $A_1^{-\frac{1}{2}}A_2A_1^{-\frac{1}{2}} = I$. This implies that $A_1 = A_2$ and then $B_1 = B_2$. Hence, Φ_U is injective. Next, let $X \in \mathbb{P}$. and consider $A = X\sharp(UXU)$ and $B = X\diamondsuit(UX^{-1}U) = A^{-\frac{1}{2}}XA^{-\frac{1}{2}}$. Consider

$$\begin{aligned}
UAU &= U\big(X\sharp(UXU)\big)U \;=\; (UXU)\sharp(U^2XU^2) \\
&= (UXU)\sharp X \;=\; X\sharp(UXU) \;=\; A
\end{aligned}$$

and

$$\begin{aligned}
UBU &= U\big(A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\big)U \;=\; \big(UA^{-\frac{1}{2}}U\big)(UXU)\big(UA^{-\frac{1}{2}}U\big) \\
&= A^{\frac{1}{2}}X^{-1}A^{\frac{1}{2}} \;=\; B^{-1}.
\end{aligned}$$

This implies that $A \in \mathbb{P}_U^+$ and $B \in \mathbb{P}_U^-$. We have that there exist $A \in \mathbb{P}_U^+$ and $B \in \mathbb{P}_U^-$ such that $\Phi_U(A, B) = A^{\frac{1}{2}}BA^{\frac{1}{2}} = X$. Thus, Φ_U is surjective. Therefore Φ_U is bijective. □

By the bijectivity of Φ_U, we can define the t-weighted Lim geometric mean of operators as follows:

Definition 3. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0, 1]$. Let $X = \Phi_U(A_1, B_1), Y = \Phi_U(A_2, B_2) \in \mathbb{P}$. The t-weighted Lim geometric mean of X and Y is defined by*

$$\mathcal{G}_U(t; X, Y) \;=\; \Phi_U(A_1\sharp_t A_2, B_1\sharp_t B_2). \tag{8}$$

We denote $\mathcal{G}_U(X, Y) = \mathcal{G}_U(1/2; X, Y)$ the Lim geometric mean.

The next theorem gives an explicit formula of $\mathcal{G}_U(X, Y)$.

Theorem 4. *Let U be a Hermitian unitary operator and $t \in [0, 1]$. Let $X, Y \in \mathbb{P}$. We have*

$$\mathcal{G}_U(t; X, Y) \;=\; (A_1\sharp_t A_2)^{\frac{1}{2}}(B_1\sharp_t B_2)(A_1\sharp_t A_2)^{\frac{1}{2}}, \tag{9}$$

where $A_1 = X\sharp(UXU), A_2 = Y\sharp(UYU), B_1 = X\diamondsuit(UX^{-1}U)$ and $B_2 = Y\diamondsuit(UY^{-1}U)$. In particular, $\mathcal{G}_I(X, Y) = X\sharp_t Y$.

Proof. Since f_U is surjective, there exist $A_1, A_2 \in \mathbb{P}_U^+$ and $B_1, B_2 \in \mathbb{P}_U^-$ such that $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. By using the inverse map (7), we have

$$\begin{aligned} (A_1, B_1) &= \Phi_U^{-1}(X) = (X\sharp(UXU), X\diamondsuit(UX^{-1}U)) \\ (A_2, B_2) &= \Phi_U^{-1}(Y) = (Y\sharp(UYU), Y\diamondsuit(UY^{-1}U)). \end{aligned}$$

For the case $U = I$, we have $\mathbb{P}_I^+ = \mathbb{P}$ and $\mathbb{P}_I^- = \{I\}$. It follows that $B_1 = B_2 = I$. By Lemma 1, we have $A_1 = X\sharp X = X$ and $A_2 = Y\sharp Y = Y$. Hence,

$$\mathcal{G}_I(t; X, Y) = (X\sharp_t Y)^{\frac{1}{2}}(I\sharp_t I)(X\sharp_t Y)^{\frac{1}{2}} = X\sharp_t Y. \qquad \square$$

Now, we give the definition of the Lim-Pálfia weighted mean [7] in the case of operators.

Definition 5. *The (two-variable) Lim-Pálfia weighted mean of positive invertible operators is the map* $\mathbb{M} : [0,1] \times \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ *satisfying*

(i) $\mathbb{M}(0, X, Y) = X$,

(ii) $\mathbb{M}(1, X, Y) = Y$,

(iii) (Idempotency) $\mathbb{M}(t, X, X) = X$ *for all* $t \in [0, 1]$.

We say that $\mathbb{M}$ *is symmetric if*

(iv) (Permutation invariancy) $\mathbb{M}(t, X, Y) = \mathbb{M}(1 - t, Y, X)$ *for all* $t \in [0, 1]$.

Theorem 6. *The* t*-weighted Lim geometric mean of operators is a symmetric Lim-Pálfia weighted mean.*

Proof. Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0, 1]$. Let $X, Y \in \mathbb{P}$. Write $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. We have by Lemma 1 that

$$\begin{aligned} \mathcal{G}_U(0; X, Y) &= \Phi_U(A_1\sharp_0 A_2, B_1\sharp_0 B_2) = \Phi_U(A_1, B_1) = X, \\ \mathcal{G}_U(1; X, Y) &= \Phi_U(A_1\sharp_1 A_2, B_1\sharp_1 B_2) = \Phi_U(A_2, B_2) = Y, \\ \mathcal{G}_U(t; X, X) &= \Phi_U(A_1\sharp_t A_1, B_1\sharp_t B_1) = \Phi_U(A_1, B_1) = X. \end{aligned}$$

This implies that $\mathcal{G}_U$ is a Lim-Pálfia weighted mean. Using Lemma 1 again, we get

$$\begin{aligned} \mathcal{G}_U(t; X, Y) &= \Phi_U(A_1\sharp_t A_2, B_1\sharp_t B_2) \\ &= \Phi_U(A_2\sharp_{1-t} A_1, B_2\sharp_{1-t} B_1) \\ &= \mathcal{G}_U(1 - t; Y, X). \end{aligned}$$

Thus, $\mathcal{G}_U$ is symmetric. $\square$

Corollary 7. *The* t*-weighted metric geometric mean of operators is a symmetric Lim-Pálfia weighted mean.*

Theorem 8. *Let $U \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $t \in [0,1]$. Let $X = \Phi_U(A_1, B_1)$ and $Y = \Phi_U(A_2, B_2)$. We have*

(i) $\mathcal{G}_U(t; X, I) = \Phi_U(A_1^{1-t}, B_1^{1-t})$ *and* $\mathcal{G}_U(t; I, Y) = \Phi_U(A_2^t, B_2^t)$,

(ii) *(Joint Homogeneity)* $\mathcal{G}_U(t; \alpha X, \beta Y) = \alpha^{1-t}\beta^t \mathcal{G}_U(t; X, Y)$ *for any* $\alpha, \beta > 0$,

(iii) *(Self-duality)* $\mathcal{G}_U(t; X, Y)^{-1} = \mathcal{G}_U(t; X^{-1}, Y^{-1})$,

(iv) *(Unitary invariance)* $\mathcal{G}_U(t; T^*XT, T^*YT) = T^*\mathcal{G}_U(t; X, Y)T$ *where* $T \in \mathfrak{B}(\mathbb{H})$ *is a unitary operator such that* $UT = TU$,

(v) $\mathcal{G}_U(t; UXU, UYU) = U\mathcal{G}_U(t; X, Y)U$,

(vi) $\mathcal{G}_U(X, X^{-1}) = I$.

Proof. The first assertion is immediate from Definition 3. For the joint homogeneity, note that

$$\alpha X \;=\; \alpha\Phi_U(A_1, B_1) \;=\; \alpha\big(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\big) \;=\; A_1^{\frac{1}{2}}(\alpha B_1)A_1^{\frac{1}{2}} \;=\; \Phi_U(A_1, \alpha B_1).$$

Similarly, $\beta Y = \Phi_U(A_2, \beta B_2)$. Using Lemma 1, we obtain

$$\begin{aligned}\mathcal{G}_U(t; \alpha X, \beta Y) &= \Phi_U(A_1 \sharp_t A_2, (\alpha B_1)\sharp_t(\beta B_2)) \;=\; \Phi_U(A_1 \sharp_t A_2, \alpha^{1-t}\beta^t(B_1 \sharp_t B_2)) \\ &= \alpha^{1-t}\beta^t \Phi_U(A_1 \sharp_t A_2, B_1 \sharp_t B_2) \;=\; \alpha^{1-t}\beta^t \mathcal{G}_U(t; X, Y).\end{aligned}$$

For the self-duality, consider

$$X^{-1} = \Phi_U(A_1, B_1)^{-1} = \left(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\right)^{-1} = A_1^{-\frac{1}{2}} B_1^{-1} A_1^{-\frac{1}{2}} = \Phi_U(A_1^{-1}, B_1^{-1}).$$

Similarly, $Y^{-1} = \Phi_U(A_2^{-1}, B_2^{-1})$. Applying Lemma 1, we get

$$\begin{aligned}\mathcal{G}_U(t; X, Y)^{-1} &= \Phi_U(A_1 \sharp_t A_2, B_1 \sharp_t B_2)^{-1} \;=\; \Phi_U((A_1 \sharp_t A_2)^{-1}, (B_1 \sharp_t B_2)^{-1}) \\ &= \Phi_U(A_1^{-1} \sharp_t A_2^{-1}, B_1^{-1} \sharp_t B_2^{-1}) \;=\; \mathcal{G}_U(t; X^{-1}, Y^{-1})\end{aligned}$$

Now, let us prove the assertion (iv). Since T is unitary, we have by Lemma 1 that

$$\begin{aligned}(T^*XT)\sharp[U(T^*XT)U] &= (T^*XT)\sharp[T^*(UXU)T] \\ &= T^*[X\sharp(UXU)]T \\ &= T^*A_1T, \\ (T^*XT)\diamondsuit[U(T^*XT)^{-1}U] &= (T^*XT)\diamondsuit[UTX^{-1}T^*U] \\ &= (T^*XT)\diamondsuit[T^*(UX^{-1}U)T] \\ &= T^*[X\diamondsuit(UX^{-1}U)]T \\ &= T^*B_1T.\end{aligned}$$

Similarly,

$$(T^*YT)\sharp[U(T^*YT)U] \;=\; T^*A_2T \quad\text{and}\quad (T^*YT)\diamondsuit[U(T^*YT)^{-1}U] \;=\; T^*B_2T$$

Then $T^*XT = \Phi_U(T^*A_1T, T^*B_1T)$ and $T^*YT = \Phi_U(T^*A_2T, T^*B_2T)$. Thus

$$\begin{aligned}
&\mathcal{G}_U(t; T^*XT, T^*YT)\\
&\quad= [(T^*A_1T)\sharp_t(T^*A_2T)]^{\frac{1}{2}}[(T^*B_1T)\sharp_t(T^*B_2T)][(T^*A_1T)\sharp_t(T^*A_2T)]^{\frac{1}{2}}\\
&\quad= [T^*(A_1\sharp_tA_2)T]^{\frac{1}{2}}[T^*(B_1\sharp_tB_2)T][T^*(A_1\sharp_tA_2)T]^{\frac{1}{2}}\\
&\quad= [T^*(A_1\sharp_tA_2)^{\frac{1}{2}}T][T^*(B_1\sharp_tB_2)T][T^*(A_1\sharp_tA_2)^{\frac{1}{2}}T]\\
&\quad= T^*(A_1\sharp_tA_2)^{\frac{1}{2}}(B_1\sharp_tB_2)(A_1\sharp_tA_2)^{\frac{1}{2}}T\\
&\quad= T^*\mathcal{G}_U(t; X, Y)T.
\end{aligned}$$

Setting $T = U$, we get the result in the assertion (v). For the last assertion, since $X^{-1} = \Phi_U(A_1^{-1}, B_1^{-1})$, we have

$$\mathcal{G}_U(X, X^{-1}) \;=\; \Phi_U(A_1\sharp A_1^{-1}, B_1\sharp B_1^{-1}) \;=\; \Phi_U(I, I) \;=\; I. \qquad \square$$

3 Weighted Lim geometric means and Tracy-Singh products

In this section, we investigate relations between Weighted Lim geometric means and Tracy-Singh products of operators. Let us recalling the notion of Tracy-Singh product.

3.1 Preliminaries on the Tracy-Singh product of operators

The projection theorem for Hilbert space allows us to decompose

$$\mathbb{H} \;=\; \bigoplus_{i=1}^{n} \mathbb{H}_i \tag{10}$$

where all $\mathbb{H}_i$ are Hilbert spaces. For each $i = 1, \dots, n$, let P_i be the natural projection map from $\mathbb{H}$ onto $\mathbb{H}_i$. Each operator $A \in \mathfrak{B}(\mathbb{H})$ can be uniquely determined by an operator matrix

$$A \;=\; [A_{ij}]_{i,j=1}^{n,n}$$

where $A_{ij} : \mathbb{H}_j \to \mathbb{H}_i$ is defined by $A_{ij} = P_iAP_j^*$ for each $i, j = 1 \dots, n$.

Recall that the tensor product of $A, B \in \mathfrak{B}(\mathbb{H})$ is the operator $A \otimes B \in \mathfrak{B}(\mathbb{H} \otimes \mathbb{H})$ such that for all $x, y \in \mathbb{H}$,

$$(A \otimes B)(x \otimes y) \;=\; (Ax) \otimes (By).$$

Definition 9. *Let $A = [A_{ij}]_{i,j=1}^{n,n}$ and $B = [B_{ij}]_{i,j=1}^{n,n}$ be operators in $\mathfrak{B}(\mathbb{H})$. The Tracy-Singh product of A and B is defined to be*

$$A \boxtimes B \;=\; \big[[A_{ij} \otimes B_{kl}]_{kl}\big]_{ij} \tag{11}$$

which is a bounded linear operator from $\bigoplus_{i,j=1}^{n,n} \mathbb{H}_i \otimes \mathbb{H}_j$ into itself.

Lemma 10 ([9, 10, 11])**.** *Let $A, B, C, D \in \mathfrak{B}(\mathbb{H})$.*

(i) $(A \boxtimes B)(C \boxtimes D) = (AC) \boxtimes (BD)$.

(ii) If $A, B \in \mathbb{P}$, then $A \boxtimes B \in \mathbb{P}$.

(iii) If $A, B \in \mathbb{P}$, then $(A \boxtimes B)^{\alpha} = A^{\alpha} \boxtimes B^{\alpha}$ for any $\alpha \in \mathbb{R}$.

(iv) If A and B are Hermitian, then $A \boxtimes B$ is also.

(v) If A and B are unitary, then $A \boxtimes B$ is also.

Lemma 11 ([8])**.** *Let $A, B, C, D \in \mathbb{P}$ and $t \in [0, 1]$. Then*

$$(A \boxtimes B)\sharp_t(C \boxtimes D) \;=\; (A\sharp_t C) \boxtimes (B\sharp_t D).$$

For each $i = 1, \dots, k$, let $\mathbb{H}_i$ be a Hilbert space and decompose

$$\mathbb{H}_i = \bigoplus_{r=1}^{n_i} \mathbb{H}_{i,r}$$

where all $\mathbb{H}_{i,r}$ are Hilbert spaces. We set $\boxtimes_{i=1}^{1} A_i = A_1$. For $k \in \mathbb{N} - \{1\}$ and $A_i \in \mathfrak{B}(\mathbb{H}_i)$ $(i = 1, \dots, k)$, we use the notation

$$\mathop{\boxtimes}_{i=1}^{k} A_i \;=\; ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{k-1}) \boxtimes A_k.$$

3.2 The compatibility between weighted Lim geometric means and Tracy-Singh products

The following theorem provides an operator identity involving t-weighted Lim geometric means and Tracy-Singh products.

Theorem 12. *Let U, V be Hermitian unitary operators, $X_1, X_2, Y_1, Y_2 \in \mathbb{P}$ and $t \in [0, 1]$.*

$$\mathcal{G}_U(t; X_1, Y_1) \boxtimes \mathcal{G}_V(t; X_2, Y_2) \;=\; \mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2). \tag{12}$$

Proof. Write

$$X_1 = \Phi_U(A_1, B_1), \quad Y_1 = \Phi_U(C_1, D_1), \quad X_2 = \Phi_V(A_2, B_2), \quad Y_2 = \Phi_V(C_2, D_2),$$

where $A_1, C_1 \in \mathbb{P}_U^+$, $B_1, D_1 \in \mathbb{P}_U^-$, $A_2, C_2 \in \mathbb{P}_V^+$, $B_2, D_2 \in \mathbb{P}_V^-$. Since U and V are Hermitian unitary operators, we have by Lemma 10 that $U \boxtimes V$ is also a Hermitian unitary operator. Thus $\mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2)$ is well-defined. By Lemma 10, we get

$$(U \boxtimes V)(A_1 \boxtimes A_2)(U \boxtimes V) \;=\; (UA_1U) \boxtimes (VA_2V) \;=\; A_1 \boxtimes A_2$$

and

$$\begin{aligned}(U \boxtimes V)(B_1 \boxtimes B_2)(U \boxtimes V) \;&=\; (UB_1U) \boxtimes (VB_2V) \;=\; B_1^{-1} \boxtimes B_2^{-1} \\ &=\; (B_1 \boxtimes B_2)^{-1}.\end{aligned}$$

Thus $A_1 \boxtimes A_2 \in \mathbb{P}_{U\boxtimes V}^+$ and $B_1 \boxtimes B_2 \in \mathbb{P}_{U\boxtimes V}^-$. Similarly, we have $C_1 \boxtimes C_2 \in \mathbb{P}_{U\boxtimes V}^+$ and $D_1 \boxtimes D_2 \in \mathbb{P}_{U\boxtimes V}^-$. Using Lemma 10, we get

$$\begin{aligned} X_1 \boxtimes X_2 \;&=\; \Phi_U(A_1, B_1) \boxtimes \Phi_V(A_2, B_2) \\ &=\; \big(A_1^{\frac{1}{2}} B_1 A_1^{\frac{1}{2}}\big) \boxtimes \big(A_2^{\frac{1}{2}} B_2 A_2^{\frac{1}{2}}\big) \\ &=\; (A_1^{\frac{1}{2}} \boxtimes A_2^{\frac{1}{2}})(B_1 \boxtimes B - 2)(A_1^{\frac{1}{2}} \boxtimes A_2^{\frac{1}{2}}) \\ &=\; (A_1 \boxtimes A_2)^{\frac{1}{2}}(B_1 \boxtimes B_2)(A_1 \boxtimes A_2)^{\frac{1}{2}} \\ &=\; \Phi_{U\boxtimes V}(A_1 \boxtimes A_2, B_1 \boxtimes B_2). \end{aligned}$$

Similarly, $Y_1 \boxtimes Y_2 = \Phi_{U\boxtimes V}(C_1 \boxtimes C_2, D_1 \boxtimes D_2)$. Then

$$\mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2) \;=\; \Phi_{U\boxtimes V}\big((A_1 \boxtimes A_2)\sharp_t(C_1 \boxtimes C_2), (B_1 \boxtimes B_2)\sharp_t(D_1 \boxtimes D_2)\big).$$

We have by applying Lemmas 10 and 11 that

$$\begin{aligned} &\mathcal{G}_U(t; X_1, Y_1) \boxtimes \mathcal{G}_V(t; X_2, Y_2) \\ &\quad=\; \Phi_U(A_1\sharp_t C_1, B_1\sharp_t D_1) \boxtimes \Phi_V(A_1\sharp_t C_2, B_2\sharp_t D_2) \\ &\quad=\; \big[(A_1\sharp_t C_1)^{\frac{1}{2}}(B_1\sharp_t D_1)(A_1\sharp_t C_1)^{\frac{1}{2}}\big] \boxtimes \big[(A_2\sharp_t C_2)^{\frac{1}{2}}(B_2\sharp_t D_2)(A_2\sharp_t C_2)^{\frac{1}{2}}\big] \\ &\quad=\; \big[(A_1\sharp_t C_1)^{\frac{1}{2}} \boxtimes (A_2\sharp_t C_2)^{\frac{1}{2}}\big]\big[(B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big]\big[(A_1\sharp_t C_1)^{\frac{1}{2}} \boxtimes (A_2\sharp_t C_2)^{\frac{1}{2}}\big] \\ &\quad=\; \big[(A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2)\big]^{\frac{1}{2}}\big[(B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big]\big[(A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2)\big]^{\frac{1}{2}} \\ &\quad=\; \Phi_{U\boxtimes V}\big((A_1\sharp_t C_1) \boxtimes (A_2\sharp_t C_2), (B_1\sharp_t D_1) \boxtimes (B_2\sharp_t D_2)\big) \\ &\quad=\; \Phi_{U\boxtimes V}\big((A_1 \boxtimes A_2)\sharp_t(C_1 \boxtimes C_2), (B_1 \boxtimes B_2)\sharp_t(D_1 \boxtimes D_2)\big) \\ &\quad=\; \mathcal{G}_{U\boxtimes V}(t; X_1 \boxtimes X_2, Y_1 \boxtimes Y_2). \end{aligned}$$ □

Corollary 13. *Let $k \in \mathbb{N}$ and $t \in [0,1]$. For each $1 \leqslant i \leqslant k$, let $U_i \in \mathfrak{B}(\mathbb{H})$ be a Hermitian unitary operator and $X_i, Y_i \in \mathbb{P}$. Then*

$$\mathop{\boxtimes}_{i=1}^{k} \mathcal{G}_{U_i}(t; X_i, Y_i) \;=\; \mathcal{G}_U\left(t; \mathop{\boxtimes}_{i=1}^{k} X_i, \mathop{\boxtimes}_{i=1}^{k} Y_i\right), \tag{13}$$

where $U = \boxtimes_{i=1}^{k} U_i$.

Proof. Since U_i is a Hermitian unitary operator for all $i = 1, \dots, k$, we have by Lemma 10 that $\boxtimes_{i=1}^{k} U_i$ is also. Using the positivity of the Tracy-Singh product, we get $\boxtimes_{i=1}^{k} X_i, \boxtimes_{i=1}^{k} Y_i \in \mathbb{P}$. Hence, the right hand side of (13) is well-defined. We reach the result by applying Thoerem 12 and induction on k. □

From Corollary 13, setting $U_i = I$ for all $i = 1, \dots, k$, we have

$$\boxtimes_{i=1}^{k} (X_i \sharp_t Y_i) \ = \ \left(\boxtimes_{i=1}^{k} X_i \right) \sharp_t \left(\boxtimes_{i=1}^{k} Y_i \right).$$

This equality were proved already in [8, Corollary 1].

Acknowledgement

The first author would like to thank the Royal Golden Jubilee Ph.D. Scholarship, grant no. PHD60K0225, from Thailand Research Fund.

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Volume 29, Number 3 May 2021
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS, LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

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Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

Editor in Chief: George Anastassiou

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

A Numerical Technique for Solving Fuzzy Fractional Optimal Control Problems[†]

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Abstract In this paper, the fuzzy fractional optimal control problem with both fixed and free final state conditions has been considered. Our problem is defined in the sense of Riemann-Liouville fractional derivative based on Hukuhara difference, and the dynamic constraint is described by a fractional differential equation of order less than 1. Using fuzzy variational approach, a necessary conditions of our problem has been derived. A numerical technique based on Grünwald-Letnikov definition of fractional derivative and the relation between right Riemann-Liouville fractional derivative and right Caputo fractional derivative is proposed. Finally, some numerical examples are given to illustrate our main results.
Keywords: Fuzzy fractional calculus;Grünwald-Letnikov fractional derivative;Fuzzy fractional optimal control problem;Fixed final state problem;Free final state problem;Fuzzy variational approach;Necessary conditions.

1. Introduction

Optimal control is the standard method for solving dynamic optimization problems, which deal with finding a control law for a given system such that a certain optimality criterion is achieved. It's playing an increasingly important role in modern system design, and considered to be a powerful mathematical tool that can be used to make decisions in real life. On the other hand, accurate modeling of some real problems in scientific fields and engineering, sometimes lead to a set of fractional differential and integral equations. Fractional optimal control problem is an optimal control problem whose dynamic system is described by fractional differential equations. We can define the fractional optimal control problem in sense of different definitions of fractional derivative, for example Riemann-Liouville fractional derivative, Caputo fractional derivative and so on.

Due to, uncertainty in the input, output and manner of many dynamical systems, meanwhile, fuzziness is a way to express an uncertain phenomena in real world. Thus, importing fuzziness in the optimal control theory, give a better display of the problems with control parameters in real world such as physical models and dynamical systems.

In the last decade, fuzzy fractional optimal control problems have attracted a great deal of attention and the interest in the filed of fuzzy fractional optimal control problems has increased. In [1], Fard and Soolaki, prove the necessary optimality conditions of pontryagin type for a class of fuzzy fractional optimal control problems with the fuzzy fractional derivative described in the Caputo sense. In [2], Fard and Salehi studied the constrained and unconstrained fuzzy fractional variational problems containing the Caputo-type fractional derivatives using the approach of the generalized differentiability. In [3], Karimyar and Fakharzadeh introduced the solution of fuzzy fractional optimal control problems by using Mittag-Leffler function.

In this paper, we will study a fixed and free final state fuzzy fractional optimal control problems with the fuzzy fractional derivative described in Riemann-Liouville type in sense of Hukuhara difference.

[†] This work is supported by National Natural Science Foundation of China(61763044).

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Then, we derive the necessary conditions of that problems based on fuzzy variational approach. A numerical algorithm is proposed to solve the necessary conditions to find the optimal fuzzy control and optimal fuzzy state as a solutions of our problems. The definitions of a strong and weak solutions of our problems are given, to guarantee the optimal solutions are a fuzzy functions.

This paper is organized as follows. In Section 2 we introduce and generalize some basic concepts and notations that are key to our discussion. In Section 3 we present basic elements of fuzzy fractional calculus and fuzzy calculus of variations. In Section 4 we establish our main results, Theorem(4.1), that provides the necessary conditions of fuzzy fractional optimal control problems with both fixed and free final state conditions. In Section 5 we propose a numerical technique to solve the necessary conditions. Finally, we discuss the applicability of the main theorem and the numerical algorithm through an examples.

2. Definitions and preliminaries

Here, we start with basic definitions and lemmas needed in the other sections for a better understanding of this work. The details of this concepts are clearly found in [7, 9, 10, 11, 12, 17].

Definition 2.1 A fuzzy set $\tilde{A} : R \to [0,1]$ is called a fuzzy number if $\tilde{A}$ is normal, convex fuzzy set, upper semi-continuous and $\text{supp}A = \overline{\{x \in R | \tilde{A}(x) > 0\}}$ is compact, where $\overline{M}$ denotes the closure of M. In the rest of this paper we use E^1 to denote the fuzzy number space.

Where it is $\alpha-$level set $\tilde{a}[\alpha] = \{x \in R : \tilde{a}(x) \geq \alpha\} = [a^l(\alpha), a^r(\alpha)], \forall \alpha \in (0,1]$, and $0-$level set $\tilde{a}[0]$ is defined as $\overline{\{x \in R | \tilde{a}(x) > 0\}}$. Obviously, the α-level set $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ is bounded closed interval in R for all $\alpha \in [0,1]$, where $a^l(\alpha)$ and $a^r(\alpha)$ denote the left-hand and right-hand end points of $\tilde{a}[\alpha]$, respectively. $\tilde{a}$ is a crisp number with value k if its membership function is defined by,

$$\tilde{a}(x) = \begin{cases} 1 & , x = k \\ 0 & , x \neq k \end{cases}$$

Thus,

$$\tilde{0}(x) = \begin{cases} 1 & , x = 0 \\ 0 & , x \neq 0. \end{cases}$$

Let $\tilde{u}, \tilde{v} \in E^1, k \in R$, we can define the addition and scalar multiplication by using α-level set respectively as

$$(\tilde{a} + \tilde{b})[\alpha] = \tilde{a}[\alpha] + \tilde{b}[\alpha],\ (k\tilde{a})[\alpha] = k\tilde{a}[\alpha],$$

where $\tilde{a}[\alpha] + \tilde{b}[\alpha]$ means the usual addition of two intervals of R, and $k\tilde{a}[\alpha]$ means the usual product between a scalar and interval of R. Furthermore, the opposite of the fuzzy number $\tilde{a}$ is $-\tilde{a}$, i.e., $-\tilde{a}(x) = \tilde{a}(-x)$, it means, $-\tilde{a}[\alpha] = [-a^r(\alpha), -a^l(\alpha)]$.

The binary operation "." in R can be extended to the binary operation "$\odot$" of two fuzzy numbers by using the extension principle. Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers, then

$$(\tilde{a} \odot \tilde{b})(z) = \sup_{x \cdot y = z} \min\{\tilde{a}(x), \tilde{b}(x)\}.$$

Using α-level set the product $(\tilde{a} \odot \tilde{b})$ is defined by

$$\begin{aligned} (\tilde{a} \odot \tilde{b})[\alpha] \quad = \quad & \Big[\min\{a^l(\alpha)b^l(\alpha), a^l(\alpha)b^r(\alpha), a^r(\alpha)b^l(\alpha), a^r(\alpha)b^r(\alpha)\}, \\ & \max\{a^l(\alpha)b^l(\alpha), a^l(\alpha)b^r(\alpha), a^r(\alpha)b^l(\alpha), a^r(\alpha)b^r(\alpha)\}\Big]. \end{aligned}$$

The metric structure is given by the Hausdorff distance $\mathbb{D} : E^1 \times E^1 \times R \to R_+ \cup \{0\}$,

$$\mathbb{D}(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max\{| a^l(\alpha) - b^l(\alpha) |, | a^r(\alpha) - b^r(\alpha) |\}.$$

A special class of fuzzy numbers is the class of triangular fuzzy numbers. For $a_1 < a_2 < a_3$ and $a_1, a_2, a_3 \in R$, the triangular fuzzy number $\tilde{a}$ is generally denoted by $\tilde{a} = (a_1, a_2, a_3)$ is determined by a_1, a_2, a_3 such that $a^l(\alpha) = a_1 + (a_2 - a_1)\alpha$ and $a^r(\alpha) = a_3 - (a_3 - a_2)\alpha$, when $\alpha = 0$ then $\tilde{a}[0] = [a_1, a_3]$ and when $\alpha = 1$ then $\tilde{a}[1] = [a_2, a_2] = a_2$.

We know that, we can identify a fuzzy number $\tilde{a} \in E^1$ by the left and right hand functions of its $\alpha-$level set, the following lemma introduce the properties of this functions.

Lemma 2.1 Suppose that $a^l : [0,1] \to R$ and $a^r : [0,1] \to R$ satisfy the conditions:

C1: a^l is bounded increasing function,

C2: a^r is bounded decreasing function,

C3: $a^l(1) \leq a^r(1)$,

C4: $\lim\limits_{\alpha \to k^-} a^l(\alpha) = a^l(k)$ and $\lim\limits_{\alpha \to k^-} a^r(\alpha) = a^r(k)$, for all $0 < k \leq 1$,

C5: $\lim\limits_{\alpha \to 0^+} a^l(\alpha) = a^l(0)$ and $\lim\limits_{\alpha \to 0^+} a^r(\alpha) = a^r(0)$.

Then $\tilde{a} : R \to [0,1]$ defined by $\tilde{a}(x) = \sup\{\alpha | a^l(\alpha) \leq x \leq a^r(\alpha)\}$ is a fuzzy number with $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$. Moreover, if $\tilde{a} : R \to [0,1]$ is a fuzzy number with $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$, then the functions $a^l(\alpha)$ and $a^r(\alpha)$ satisfy conditions **C1- C5**.

Definition 2.2 (H-difference). Let $\tilde{a}, \tilde{b} \in E^1$, where $\tilde{a}[\alpha] = [a^l(\alpha), a^r(\alpha)]$ and $\tilde{b}[\alpha] = [b^l(\alpha), b^r(\alpha)]$ for all $\alpha \in [0,1]$, the H-difference is defined by

$$\tilde{a} \ominus \tilde{b} = \tilde{c} \iff \tilde{a} = \tilde{b} + \tilde{c}.$$

Obviously, $\tilde{a} \ominus \tilde{a} = \tilde{0}$, and the α-level set of H-difference is

$$(\tilde{a} \ominus \tilde{b})[\alpha] = [a^l(\alpha) - b^l(\alpha), a^r(\alpha) - b^r(\alpha)], \forall \alpha \in [0,1].$$

Definition 2.3 (Partial ordering). Let $\tilde{a}, \tilde{b} \in E^1$, we write $\tilde{a} \preceq \tilde{b}$, if $a^l(\alpha) \leq b^l(\alpha)$ and $a^r(\alpha) \leq b^r(\alpha)$ for all $\alpha \in [0,1]$. We also write $\tilde{a} \prec \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and there exists $\alpha_0 \in [0,1]$ such that $a^l(\alpha_0) < b^l(\alpha_0)$ or $a^r(\alpha_0) < b^r(\alpha_0)$. Furthermore, $\tilde{a} = \tilde{b}$, if $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \succeq \tilde{b}$. In other words, $\tilde{a} = \tilde{b}$, if $\tilde{a}[\alpha] = \tilde{b}[\alpha]$ for all $\alpha \in [0,1]$.

In the sequel, we say that $\tilde{a}, \tilde{b} \in E^1$ are *comparable* if either $\tilde{a} \preceq \tilde{b}$ or $\tilde{a} \succeq \tilde{b}$, and *non-comparable* otherwise.

From now we consider S as a subset of R.

Definition 2.4 (Fuzzy valued function). The function $\tilde{f} : S \to E^1$ is called a fuzzy-valued function if $\tilde{f}(t)$ is assign a fuzzy number for any $e \in S$. We also denote $\tilde{f}(t)[\alpha] = [f^l(t,\alpha), f^r(t,\alpha)]$, where $f^l(t,\alpha) = (\tilde{f}(t))^l(\alpha) = \min\{\tilde{f}(t)[\alpha]\}$ and $f^r(t,\alpha) = (\tilde{f}(t))^r(\alpha) = \max\{\tilde{f}(t)[\alpha]\}$. Therefore any fuzzy-valued function $\tilde{f}$ may be understood by $f^l(t,\alpha)$ and $f^r(t,\alpha)$ being respectively a bounded increasing function of α and a bounded decreasing function of α for $\alpha \in [0,1]$. And also it holds $f^l(t,\alpha) \leq f^r(t,\alpha)$ for any $\alpha \in [0,1]$.

Definition 2.5 (Continuity of a fuzzy valued function). We say that $\tilde{f} : S \to E^1$ is continuous at $t \in S$, if both $f^l(t,\alpha)$ and $f^r(t,\alpha)$ are continuous functions at $t \in S$ for all $\alpha \in [0,1]$.

If $\tilde{f}(t)$ is continuous in the metric $\mathbb{D}$, then its definite integral exists and defined by

$$\int_a^b \tilde{f}(t)[\alpha]dt = \left[\int_a^b f^l(t,\alpha)dt, \int_a^b f^r(t,\alpha)dt\right].$$

Definition 2.6 (Distance measure between fuzzy valued functions). Suppose that $\tilde{f}, \tilde{g} : S \to E^1$ are two fuzzy functions. We define the distance measure between $\tilde{f}$ and $\tilde{g}$ by

$$\begin{aligned}\mathbb{D}_{E^1}(\tilde{f}(x), \tilde{g}(x)) &= \sup_{0 \leq \alpha \leq 1} \mathbb{H}(\tilde{f}(x)[\alpha], \tilde{g}(x)[\alpha]) \\ &= \max\{\sup_{z \in \tilde{f}(x)[\alpha]} d(z, \tilde{g}(x)[\alpha]), \sup_{y \in \tilde{g}(x)[\alpha]} d(\tilde{f}(x)[\alpha], y)\}, \ \forall x \in S.\end{aligned}$$

Where $\mathbb{H}$ is the Hausdorff metric on the family of all nonempty compact subsets of R, and

$$d(a,B)=\inf_{b\in B} d(a,b).$$

Moreover, we can define

$$\| \tilde{f}(x) \|_{E^1}^2=\mathbb{D}_{E^1}(\tilde{f}(x),\tilde{f}(x)),\ \ \forall x\in S,$$

for any $\tilde{f}:S\to E^1$.

3. Elements of fuzzy fractional calculus and fuzzy calculus of variations

Several definitions of a fractional derivative have been studied, such as Riemann-Liouville, Grünwald-Letnikov, Caputo and so on. In this paper, we deal with the problems defined by Riemann-Liouville fractional derivative. In this section, we first introduce the definition of fuzzy Riemann-Liouville integrals and derivatives in sense of Hukuhara difference.

Definition 3.1(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]\in R$ and $0<\beta\leq 1$, then the fuzzy Riemann-Liouville integral of $\tilde{f}(x)$ of order β is defined by

$${}_aI_x^{\beta}\tilde{f}(x)=\frac{1}{\Gamma(\beta)}\int_a^x \tilde{f}(t)(x-t)^{\beta-1}dt,$$

where $\Gamma(\beta)$ is the Gamma function and $x>a$.

Theorem 3.1(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]\in R$. The fuzzy Riemann-Liouville integral of $\tilde{f}(x)$ can be expressed as follows

$${}_aI_x^{\beta}\tilde{f}(x)\,[\alpha]=\left[{}_aI_x^{\beta}f^l(x,\alpha),{}_aI_x^{\beta}f^r(x,\alpha)\right],\ \ 0\leq\alpha\leq 1,$$

where

$$\begin{aligned}{}_aI_x^{\beta}f^l(x,\alpha)&=\frac{1}{\Gamma(\beta)}\int_a^x f^l(t,\alpha)(x-t)^{\beta-1}dt,\\ {}_aI_x^{\beta}f^r(x,\alpha)&=\frac{1}{\Gamma(\beta)}\int_a^x f^r(t,\alpha)(x-t)^{\beta-1}dt.\end{aligned}$$

In the next definition, we define the fuzzy Riemann-Liouville fractional derivative of order $0<\beta<1$ of a fuzzy valued function $\tilde{f}(x)$.

Definition 3.2(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]\in R$. $x_0\in(a,b)$ and then: $G(x)=\frac{1}{\Gamma(1-\beta)}\int_a^x\frac{\tilde{f}(t)dt}{(x-t)^{\beta}}$. We say that $\tilde{f}$ is Riemann-Liouville H-differentiable of order $0<\beta<1$ at x_0, if there exist an element ${}_aD_x^{\beta}\tilde{f}(x_0)\in E^1$ such that for $h>0$ sufficiently small

(1) ${}_aD_x^{\beta}\tilde{f}(x_0)=\lim\limits_{h\to 0^+}\frac{G(x_0+h)\ominus G(x_0)}{h}=\lim\limits_{h\to 0^+}\frac{G(x_0)\ominus G(x_0-h)}{h}$,
or

(2) ${}_aD_x^{\beta}\tilde{f}(x_0)=\lim\limits_{h\to 0^+}\frac{G(x_0)\ominus G(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\frac{G(x_0-h)\ominus G(x_0)}{-h}$,
or

(3) ${}_aD_x^{\beta}\tilde{f}(x_0)=\lim\limits_{h\to 0^+}\frac{G(x_0+h)\ominus G(x_0)}{h}=\lim\limits_{h\to 0^+}\frac{G(x_0-h)\ominus G(x_0)}{-h}$,
or

(4) ${}_aD_x^{\beta}\tilde{f}(x_0)=\lim\limits_{h\to 0^+}\frac{G(x_0)\ominus G(x_0+h)}{-h}=\lim\limits_{h\to 0^+}\frac{G(x_0)\ominus G(x_0-h)}{h}$.

For sake of simplicity, we say that the fuzzy valued function $\tilde{f}(x)$ is Riemann-Liouville $[(i)-\beta]-$differentiable if it is differentiable as in the Definition(3.2) case(i), $i=1,2,3,4$ respectively.

Theorem 3.2(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]\in R$ and $\tilde{f}(x)[\alpha]=[f^l(x,\alpha),f^r(x,\alpha)]$,then for $\alpha\in[0,1]$, $x\in(a,b)$ and $\beta\in(0,1)$

(i) Let us consider $\tilde{f}$ is Riemann-Liouville $[(1)-\beta]-$differentiable fuzzy-valued function, then:

$$ {}_aD_x^\beta\tilde{f}(x_0)[\alpha]=\left[{}_aD_x^\beta f^l(x_0,\alpha),{}_aD_x^\beta f^r(x_0,\alpha)\right]. $$

(ii) Let us consider $\tilde{f}$ is Riemann-Liouville $[(2)-\beta]-$differentiable fuzzy-valued function, then:

$$ {}_aD_x^\beta\tilde{f}(x_0)[\alpha]=\left[{}_aD_x^\beta f^r(x_0,\alpha),{}_aD_x^\beta f^l(x_0,\alpha)\right]. $$

Where

$$ \begin{aligned} {}_aD_x^\beta f^l(x_0,\alpha) &= \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x\frac{f^l(t,\alpha)dt}{(x-t)^\beta}\right]\Bigg|_{x=x_0}, \\ {}_aD_x^\beta f^r(x_0,\alpha) &= \left[\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_a^x\frac{f^r(t,\alpha)dt}{(x-t)^\beta}\right]\Bigg|_{x=x_0}. \end{aligned} $$

Theorem 3.3(see [6]) Let $\tilde{f}(x)$ be continuous and Lebesgue integrable fuzzy valued function in $[a,b]$ is a Riemann-Liouville H-differentiable of order $0<\beta<1$ on each point $x\in(a,b)$ in the sense of Definition(3.2) case(3) or case(4), then ${}_aD_x^\beta\tilde{f}(x)\in R$ for all $x\in(a,b)$.

Now we state some elements of fuzzy calculus of variations.

Definition 3.3(Fuzzy increment[10]). Suppose that $\tilde{x}(.)$ and $\tilde{x}(.)+\delta\tilde{x}(.)$ are fuzzy functions for which the fuzzy functional $\tilde{J}$ is defined. The increment of $\tilde{J}$, denoted by $\Delta\tilde{J}$, is

$$ \Delta\tilde{J}:=\tilde{J}(\tilde{x}+\delta\tilde{x})\ominus\tilde{J}(x), \tag{3.1} $$

Where $\delta\tilde{x}(.)$ is the variation of $\tilde{x}(.)$.

Because the increment $\Delta\tilde{J}$ depends on the fuzzy functions $\tilde{x}$ and $\delta\tilde{x}$, we denote $\Delta\tilde{J}$ by $\Delta\tilde{J}(\tilde{x},\delta\tilde{x})$.

Definition 3.4(Differentiability of a fuzzy functional[10, 15]). Suppose that $\Delta\tilde{J}$ can be written as

$$ \Delta\tilde{J}(\tilde{x},\delta\tilde{x}):=\delta\tilde{J}(\tilde{x},\delta\tilde{x})+\tilde{j}(\tilde{x},\delta\tilde{x})\cdot\parallel\delta\tilde{x}\parallel_{E^1}, \tag{3.2} $$

Where $\delta\tilde{J}$ is linear in $\delta\tilde{x}$. We say that $\tilde{J}$ is differentiable with respect to $\tilde{x}$ if for any $\epsilon>0$,

$$ D_{E^1}(\tilde{j}(\tilde{x},\delta\tilde{x}),0)<\epsilon,\ \ as\parallel\delta\tilde{x}(.)\parallel_{E^1}\to 0. $$

From now $\tilde{C}[t_0,t_1]$ represent the class of all fuzzy continuous functions on $[t_0,t_1]$.

Definition 3.5(Fuzzy relative minimum[10]) A fuzzy functional $\tilde{J}$ with domain $\tilde{C}[t_0,t_1]$, has a fuzzy relative minimizer $\tilde{x}^*=\tilde{x}^*(t)$, if

$$ \tilde{J}(\tilde{x})\succeq\tilde{J}(\tilde{x}^*), \tag{3.3} $$

for all fuzzy functions $\tilde{x}\in\tilde{C}[t_0,t_1]$.

It is clear that the inequality (3.3) holds iff

$$ J^l(\tilde{x},\alpha)\geq J^l(\tilde{x}^*,\alpha),\ \text{ and }\ J^r(\tilde{x},\alpha)\geq J^r(\tilde{x}^*,\alpha), \tag{3.4} $$

for all $\alpha\in[0,1]$ and all $\tilde{x}\in\tilde{C}[t_0,t_1]$.

The following theorem is the fundamental theorem of the calculus of variations in fuzzy environment.

Theorem 3.4 Let $\tilde{x},\delta\tilde{x}\in\tilde{C}[t_0,t_1]$ be two fuzzy functions of $t\in[t_0,t_1]$, and $\tilde{J}(\tilde{x})$ differentiable fuzzy functional of $\tilde{x}$. If $\tilde{x}^*$ is a fuzzy minimizer of $\tilde{J}$, then the variation of $\tilde{J}$ regardless of any boundary conditions must vanish on $\tilde{x}^*$, that is,

$$ \delta\tilde{J}(\tilde{x}^*,\delta\tilde{x})=0, \tag{3.5} $$

for all admissible $\delta\tilde{x}$ having the property $\tilde{x}+\delta\tilde{x}\in\tilde{C}[t_0,t_1]$.

It is obviously that the equality (3.5) holds if and only if

$$\delta J^l(\tilde{x}^*(t)[\alpha],\delta\tilde{x}(t)[\alpha],t,\alpha)=0, \tag{3.6}$$

$$\delta J^r(\tilde{x}^*(t)[\alpha],\delta\tilde{x}(t)[\alpha],t,\alpha)=0, \tag{3.7}$$

for all $\alpha\in[0,1], t\in[t_0,t_1]$ and all admissible $\delta\tilde{x}$ where,

$$\delta\tilde{x}(t)[\alpha]=[\delta x^l(t,\alpha),\delta x^r(t,\alpha)].$$

Proof. See [10]

4. Fuzzy fractional optimal control problem

In this section, we first define fuzzy fractional optimal control problem with fixed and free final state conditions, and then we derive necessary conditions for optimality by applying fuzzy variational approaches to our problem.

We define fuzzy fractional optimal control problem as:

$$\begin{aligned} &\min_{\tilde{u}}\tilde{J}(\tilde{u})=\tilde{\phi}(\tilde{x}(t_1),t_1)+\int_{t_0}^{t_1}\tilde{f}(\tilde{x}(t),\tilde{u}(t),t)dt,\\ \text{subject to:}\quad &{}_{t_0}D_t^{\beta}\tilde{x}=\tilde{g}(\tilde{x}(t),\tilde{u}(t),t)\\ &\tilde{x}(t_0)=\tilde{x}_0. \end{aligned} \tag{4.1}$$

For fixed final state problem we have additional condition $\tilde{x}(t_1)=\tilde{x}_1$. Where $\tilde{f},\tilde{g}:E^1\times E^1\times R\to E^1$ are assumed to be continuous first and second partial derivatives on $t\in I=[t_0,t_1]\subseteq R$ with respect to all their arguments and Riemann integrable, the fuzzy state $\tilde{x}(t)$ and the fuzzy control $\tilde{u}(t)$ are functions of $t\in I$, and the fuzzy state function $\tilde{x}(t)$ is Riemann-Liouville $[(1)-\beta]-$differentiable fuzzy-valued function and satisfies appropriate boundary conditions, and $\beta\in(0,1)$.

Definition 4.1 We say that an admissible fuzzy curve $(\tilde{x}^*,\tilde{u}^*)$ is solution of (4.1), if for all admissible fuzzy curve $(\tilde{x},\tilde{u})$ of (4.1),

$$\tilde{J}(\tilde{x}^*,\tilde{u}^*)\preceq\tilde{J}(\tilde{x},\tilde{u}).$$

Note that, we consider an admissible fuzzy control $\tilde{u}$ is not bounded.

Remark 4.1 If we choose $\beta=1$, problem (4.1) is reduced to classical fuzzy optimal control problem.

Definition 4.2(Fuzzy Hamiltonian Function). We define fuzzy Hamiltonian function as,

$$\tilde{H}(\tilde{x}(t),\tilde{u}(t),\tilde{\lambda}(t),t)=\tilde{f}(\tilde{x}(t),\tilde{u}(t),t)+\tilde{\lambda}(t)\tilde{g}(\tilde{x}(t),\tilde{u}(t),t). \tag{4.2}$$

It means that,

$$\tilde{H}(\tilde{x}(t),\tilde{u}(t),\tilde{\lambda}(t),t)[\alpha]=[H^l(x^l,u^l,\lambda^l,t,\alpha),H^r(x^r,u^r,\lambda^r,t,\alpha)]. \tag{4.3}$$

for any $\alpha\in[0,1]$, and where $H^l(x^l,u^l,\lambda^l,t,\alpha)$ and $H^r(x^r,u^r,\lambda^r,t,\alpha)$ are classical Hamiltonian functions.

Remark 4.2 In the following theorem, we assume that $J^l(\tilde{x}(t),\tilde{u}(t),\tilde{\lambda}(t),t)$ (or $J^r(\tilde{x}(t),\tilde{u}(t),\tilde{\lambda}(t),t)$) is stated in terms containing only $x^l(t,\alpha)$, $u^l(t,\alpha)$ and $\lambda^l(t,\alpha)$ (or only $x^r(t,\alpha)$, $u^r(t,\alpha)$ and $\lambda^r(t,\alpha)$) in order to simplify the result presentations.

4.1 Derivation of Necessary Conditions

Now we are in the position to state a fundamental result of this work in the following theorem. **Theorem 4.1**(Necessary Conditions) Assume that $\tilde{x}^*(t)$ be an admissible fuzzy state and $\tilde{u}^*(t)$ be an admissible fuzzy control. Then the necessary conditions for $\tilde{u}^*$ to be an optimal control for (4.1) and for all $\alpha \in [0,1]$, $t \in [t_0, t_1]$ are:

$$ {}_{t_0}D_t^{\beta} x^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial \lambda^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.4}$$

$$ {}_{t_0}D_t^{\beta} x^{*^r}(t,\alpha) = \frac{\partial H^r}{\partial \lambda^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.5}$$

$$ {}_t^C D_{t_1}^{\beta} \lambda^{*^l}(t,\alpha) = \frac{\partial H^l}{\partial x^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.6}$$

$$ {}_t^C D_{t_1}^{\beta} \lambda^{*^r}(t,\alpha) = \frac{\partial H^r}{\partial x^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha), \tag{4.7}$$

$$ \frac{\partial H^l}{\partial u^l}(x^{*^l}(t,\alpha), u^{*^l}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha) = 0, \tag{4.8}$$

$$ \frac{\partial H^r}{\partial u^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha) = 0. \tag{4.9}$$

with

$$ \lambda^l(t_1,\alpha) = \left.\frac{\partial \phi^l}{\partial x^l}\right|_{t=t_1}, \tag{4.10}$$

$$ \lambda^r(t_1,\alpha) = \left.\frac{\partial \phi^r}{\partial x^r}\right|_{t=t_1}. \tag{4.11}$$

for free final state problems.

Proof. First we adopt fuzzy lagrange multiplier to form an augmented functional incorporating the constraints, then we modify the performance index as,

$$ \tilde{J}_a(\tilde{u}) = \int_{t_0}^{t_1} \left[\tilde{f}(\tilde{x}(t), \tilde{u}(t), t) + \frac{d\tilde{\phi}}{dt} + \tilde{\lambda}\left(\tilde{g}(\tilde{x}(t), \tilde{u}(t), t) \ominus_{t_0} D_t^{\beta} \tilde{x}\right)\right] dt, \tag{4.12}$$

It means that,

$$ \left[J_a^l(u^l,\alpha), J_a^r(u^r,\alpha)\right] = \left[\int_{t_0}^{t_1} \left[f^l(x^l,u^l,t,\alpha) + \frac{d\phi^l}{dt} + \lambda^l(t,\alpha)\left(g^l(x^l,u^l,t,\alpha) - {}_{t_0}D_t^{\beta} x^l\right)\right] dt, \right.$$
$$ \left. \int_{t_0}^{t_1} \left[f^r(x^r,u^r,t,\alpha) + \frac{d\phi^r}{dt} + \lambda^r\left(g^r(x^r,u^r,t,\alpha) - {}_{t_0}D_t^{\beta} x^r\right)\right] dt \right]. $$

In the remaining of the proof we will ignore the similar arguments and only we consider the left hand of all functions of its α-level set.

$$ J_a^l(u^l,\alpha) = \int_{t_0}^{t_1} \left[f^l(x^l(t), u^l(t), t, \alpha) + \lambda^l(t,\alpha) g^l(x^l(t), u^l(t), t, \alpha) - \lambda^l(t,\alpha)_{t_0} D_t^{\beta} x^l(t,\alpha) + \frac{d\phi^l}{dt}\right] dt. \tag{4.13}$$

Using the definition of fuzzy Hamiltonian function, then we can rewrite equation (4.13) as,

$$J_a^l(u^l,\alpha)=\int_{t_0}^{t_1}\left[H^l(x^l(t),u^l(t),\lambda^l(t),t,\alpha)+\frac{d\phi^l}{dt}-\lambda^l(t,\alpha)_{t_0}D_t^\beta x^l(t,\alpha)\right]. \tag{4.14}$$

Taking variation of equation (4.14), we obtain

$$\delta J_a^l(u^l,\alpha)=\int_{t_0}^{t_1}\frac{\partial H^l}{\partial x^l}\delta x^l+\frac{\partial H^l}{\partial u^l}\delta u^l+\frac{\partial H^l}{\partial \lambda^l}\delta\lambda^l+\frac{\partial \phi^l}{\partial x^l}\delta x^l-\delta\lambda^l{}_{t_0}D_t^\beta x^l-\lambda^l\delta_{t_0}D_t^\beta x^l, \tag{4.15}$$

where δx^l, $\delta\lambda^l$ and δu^l are the variations of x^l, λ^l and u^l respectively.

Using the formula for fractional integration by parts, integrate the last term on the RHS of (4.15), then we obtain

$$\delta J_a^l(u^l,\alpha)=\int_{t_0}^{t_1}\left(\frac{\partial H^l}{\partial x^l}-{}_t^CD_{t_1}^\beta\lambda^l\right)\delta x^l+\frac{\partial H^l}{\partial u^l}\delta u^l+\left(\frac{\partial H^l}{\partial \lambda^l}-{}_{t_0}D_t^\beta x^l\right)\delta\lambda^l dt+\left.\left(\frac{\partial \phi^l}{\partial x^l}-\lambda^l\right)\right|_{t=t_1}\delta x^l(t_1). \tag{4.16}$$

where ${}_t^CD_{t_1}^\beta$ represent the classical right Caputo fractional derivative.

u^{*^l} is an extremal if the variation of J_a^l is zero, that is, for all $\alpha\in[0,1]$ we require

$$\int_{t_0}^{t_1}\left(\frac{\partial H^l}{\partial x^l}-{}_t^CD_{t_1}^\beta\lambda^l\right)\delta x^l+\frac{\partial H^l}{\partial u^l}\delta u^l+\left(\frac{\partial H^l}{\partial \lambda^l}-{}_{t_0}D_t^\beta x^l\right)\delta\lambda^l dt+\left.\left(\frac{\partial \phi^l}{\partial x^l}-\lambda^l\right)\right|_{t=t_1}\delta x^l(t_1)=0. \tag{4.17}$$

It is convenient to choose the coefficients of $\delta x^l,\delta u^l$, and $\delta\lambda^l$ in (4.17) to be zero. This leads to

$${}_{t_0}D_t^\beta x^{*^l}(t,\alpha)=\frac{\partial H^l}{\partial \lambda^l}(x^{*^l}(t,\alpha),u^{*^l}(t,\alpha),\lambda^{*^l}(t,\alpha),t,\alpha), \tag{4.18}$$

$${}_t^CD_{t_1}^\beta\lambda^{*^l}(t,\alpha)=\frac{\partial H^l}{\partial x^l}(x^{*^l}(t,\alpha),u^{*^l}(t,\alpha),\lambda^{*^l}(t,\alpha),t,\alpha), \tag{4.19}$$

$$\frac{\partial H^l}{\partial u^l}(x^{*^l}(t,\alpha),u^{*^l}(t,\alpha),\lambda^{*^l}(t,\alpha),t,\alpha)=0, \tag{4.20}$$

Finally, we have

$$\left.\left(\frac{\partial \phi^l}{\partial x^l}-\lambda^l\right)\right|_{t=t_1}\delta x^l(t_1)=0, \tag{4.21}$$

1. For the fixed final state problem

$$\delta x^l(t_1)=0, \tag{4.22}$$

2. For the free final state problem

$$\left.\left(\frac{\partial \phi^l}{\partial x^l}-\lambda^l\right)\right|_{t=t_1}=0. \tag{4.23}$$

Equations $(4.18)-(4.20)$ represents the necessary conditions for u^{*^l} to be an optimal with the condition (4.22) for the fixed final state problem and (4.23) for the free final state problem.

By following the same steps(using the right hand of all functions of its α-level set) for $\delta J_a^r(u^{*^r},\alpha)=0$, for all $\alpha\in[0,1]$ and $t\in[0,1]$, we will obtain

$${}_{t_0}D_t^\beta x^{*^l}(t,\alpha)=\frac{\partial H^r}{\partial \lambda^r}(x^{*^r}(t,\alpha),u^{*^r}(t,\alpha),\lambda^{*^r}(t,\alpha),t,\alpha), \tag{4.24}$$

$$ {}_{t}^{C}D_{t_1}^{\beta}\lambda^{*^r}(t,\alpha) = \frac{\partial H^l}{\partial x^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^l}(t,\alpha), t, \alpha), \tag{4.25} $$

$$ \frac{\partial H^r}{\partial u^r}(x^{*^r}(t,\alpha), u^{*^r}(t,\alpha), \lambda^{*^r}(t,\alpha), t, \alpha) = 0. \tag{4.26} $$

Equations (4.24) – (4.26) represents the necessary conditions for u^{*^r} to be an extremal with the conditions $\delta x^r(t_1) = 0$ for the fixed final state problem and $\left(\frac{\partial \phi^r}{\partial x^r} - \lambda^l\right)\Big|_{t=t_1} = 0$ for the free final state problem.

The above equations form a set of necessary conditions that the left and right hand functions of its $\alpha-$level set of the fuzzy optimal control $\tilde{u}^*$ and fuzzy optimal state $\tilde{x}^*$ must satisfy. $\square$

We know that, $\tilde{u}^*(t)$ and $\tilde{x}^*(t)$ are a fuzzy numbers with $\tilde{u}^*(t)[\alpha] = \left[u^{*^l}(t,\alpha), u^{*^r}(t,\alpha)\right]$ and $\tilde{x}^*(t)[\alpha] = \left[x^{*^l}(t,\alpha), x^{*^r}(t,\alpha)\right]$ if $u^{*^l}(t,\alpha), u^{*^r}(t,\alpha), x^{*^l}(t,\alpha)$ and $x^{*^r}(t,\alpha)$ satisfy are related properties in **C1-C5** of Lemma(2.1). In the following definition, based on the conditions **C1** and **C2** of Lemma(2.1), we introduce the definition of strong and weak solutions of our problem.

Definition 4.3(Strong and Weak Solutions).

1. (Strong Solution). We say that $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ are strong solutions of (4.1) if $u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)$,$x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ obtained from (4.4) – (4.11) satisfy the conditions **C1-C2** of Lemma(2.1), for all $t \in [t_0, t_1]$ and $\alpha \in [0,1]$.
2. (Weak Solution). We say that $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ are weak solutions of (4.1) if $u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)$,$x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ obtained from (4.4) – (4.11) do not satisfy the conditions **C1-C2** of Lemma(2.1), then we define $\tilde{u}^*(t)[\alpha]$ and $\tilde{x}^*(t)[\alpha]$ as:
   $\tilde{u}^*(t)[\alpha] =$
   $$ \begin{cases} [2u^{r^*}(t,1) - u^{l^*}(t,\alpha), u^{r^*}(t,\alpha)], \text{ if } u^{l^*}, u^{r^*} \text{are decreasing functions of } \alpha, \\ [u^{l^*}(t,\alpha), 2u^{l^*}(t,1) - u^{r^*}(t,\alpha)], \text{ if } u^{l^*}, u^{r^*} \text{are increasing functions of } \alpha, \\ [u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)], \text{ if } u^{l^*} \text{is decreasing and } u^{r^*} \text{is increasing of } \alpha \end{cases} $$
   and,
   $\tilde{x}^*(t)[\alpha] =$
   $$ \begin{cases} [2x^{r^*}(t,1) - x^{l^*}(t,\alpha), x^{r^*}(t,\alpha)], \text{ if } x^{l^*}, x^{r^*} \text{are decreasing functions of } \alpha, \\ [x^{l^*}(t,\alpha), 2x^{l^*}(t,1) - x^{r^*}(t,\alpha)], \text{ if } x^{l^*}, x^{r^*} \text{are increasing functions of } \alpha, \\ [x^{r^*}(t,\alpha), x^{l^*}(t,\alpha)], \text{ if } x^{l^*} \text{is decreasing and } x^{r^*} \text{is increasing of } \alpha \end{cases} $$

for all $t \in [t_0, t_1]$ and $\alpha \in [0,1]$.

Now, we consider fixed and free final state problems with a quadratic performance index.

4.2 Fixed Final State Problem

We can define fuzzy fractional optimal control problem with fixed final state as

$$ \begin{aligned} &\min_{\tilde{u}} \tilde{J}(\tilde{u}) = \frac{1}{2}\int_{t_0}^{t_1} \left[q(t)\tilde{x}^2 + r(t)\tilde{u}^2\right] dt, \\ &\text{subject to:} \quad {}_0D_t^{\beta}\tilde{x} = a(t)\tilde{x} + b(t)\tilde{u}, \\ &\qquad \tilde{x}(t_0) = \tilde{x}_0, \quad \tilde{x}(t_1) = \tilde{x}_1. \end{aligned} \tag{4.27} $$

where $q(t) \geq 0$ and $r(t) > 0$.

Theorem(4.1), give the necessary conditions for u^{*^l} to be an optimal as

$$ {}_{t_0}D_t^{\beta} x^l = a(t)x^l + b(t)u^l, \tag{4.28}$$

$$ {}_{t}^{C}D_{t_1}^{\beta} \lambda^l = q(t)x^l + a(t)\lambda^l, \tag{4.29}$$

$$ r(t)u^l + b(t)\lambda^l = 0. \tag{4.30}$$

Equations (4.28) and (4.30) gives

$$ {}_{t_0}D_t^{\beta} x^l = a(t)x^l - r^{-1}(t)b^2(t)\lambda^l. \tag{4.31}$$

We will obtain $x^l(t,\alpha)$ and $u^l(t,\alpha)$ by solving Equations (4.29) – (4.31) with the boundary conditions $x^l(t_0) = x_0^l$ and $x^l(t_1) = x_1^l$.

Similarly Theorem(4.1), give the necessary conditions for u^{*^r} to be an optimal as

$$ {}_{t_0}D_t^{\beta} x^r = a(t)x^r + b(t)u^r, \tag{4.32}$$

$$ {}_{t}^{C}D_{t_1}^{\beta} \lambda^r = q(t)x^r + a(t)\lambda^r, \tag{4.33}$$

$$ r(t)u^r + b(t)\lambda^r = 0. \tag{4.34}$$

Equations (4.32) and (4.34) gives

$$ {}_{t_0}D_t^{\beta} x^r = a(t)x^r - r^{-1}(t)b^2(t)\lambda^r. \tag{4.35}$$

We will obtain $x^r(t,\alpha)$ and $u^r(t,\alpha)$ by solving Equations (4.33) – (4.35) with the boundary conditions $x^r(t_0) = x_0^r$ and $x^r(t_1) = x_1^r$.

4.3 Free Final State Problem

We can define fuzzy fractional optimal control problem with free final state as

$$\begin{aligned} &\min_{\tilde{u}} \tilde{J}(\tilde{u}) = \tilde{\phi}(\tilde{x}(t_1), t_1) + \frac{1}{2}\int_{t_0}^{t_1} \left[q(t)\tilde{x}^2 + r(t)\tilde{u}^2\right] dt, \\ \text{subject to:}\quad &{}_{t_0}D_t^{\beta}\tilde{x} = a(t)\tilde{x} + b(t)\tilde{u}, \\ &\tilde{x}(t_0) = \tilde{x}_0. \end{aligned} \tag{4.36}$$

where $q(t) \geq 0$ and $r(t) > 0$.

Following the same steps, we will obtain $x^l(t,\alpha)$ and $u^l(t,\alpha)$ by solving Equations (4.29) – (4.31) with respect to the conditions

$$ x^l(t_0) = x_0^l \quad \text{and} \quad \lambda^l(t_1,\alpha) = \left.\left(\frac{\partial \phi^l}{\partial x^l}\right)\right|_{t=t_1}. \tag{4.37}$$

Also we will obtain $x^r(t,\alpha)$ and $u^r(t,\alpha)$ by solving Equations (4.33) – (4.35) with respect to the conditions

$$ x^r(t_0) = x_0^r \quad \text{and} \quad \lambda^r(t_1,\alpha) = \left.\left(\frac{\partial \phi^r}{\partial x^r}\right)\right|_{t=t_1}. \tag{4.38}$$

In the next section we propose an algorithm used to find the solution of both cases numerically, the details of this algorithm in [4, 5].

5. Numerical technique

Considering the both cases of fixed and free final state problems defined above, in order to find the solution of our problems, we use the Grünwald-Letnikov(GL-for short) approximation of the left Riemann-Liouville fractional derivative and using the relation between right Riemann-Liouville fractional derivative and right Caputo fractional derivative and then use GL-approximation, we can approximate (4.31) and (4.29) as

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} x_{m-j}^l = a(mh)x_m^l - r^{-1}(mh)b^2(mh)\lambda_m^l, \tag{5.1}$$

for $m = 1, 2, ..., N$, and

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} \lambda_{m+j}^l = q(mh)x_m^l + a(mh)\lambda_m^l + \frac{\lambda_N^l (t_1 - mh)^{-\beta}}{\gamma(1-\beta)}, \tag{5.2}$$

for $m = N-1, N-2, ..., 0$, respectively. Where N is the number of equal divisions of the interval $[0, t_1]$, the nodes are labeled as $0, 1, ..., N$. The size of each division is given as $h = \frac{t_1}{N}$, and $t_j = jh$ represent the time at node j. The coefficients are defined as

$$w_j^{\beta} = (-1)^j \binom{\beta}{j}. \tag{5.3}$$

Where x_i^l and λ_i^l represent the numerical approximations of $x^l(t,\alpha)$ and $\lambda^l(t,\alpha)$ at node i.

Similarly, we can approximate (4.35) and (4.33) as

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} x_{m-j}^r = a(mh)x_m^r - r^{-1}(mh)b^2(mh)\lambda_m^r, \tag{5.4}$$

for $m = 1, 2, ..., N$, and

$$\sum_{j=0}^{m} h^{-\beta} w_j^{(\beta)} \lambda_{m+j}^r = q(mh)x_m^r + a(mh)\lambda_m^l + \frac{\lambda_N^r (t_1 - mh)^{-\beta}}{\gamma(1-\beta)}, \tag{5.5}$$

for $m = N-1, N-2, ..., 0$, respectively.

Also x_i^r and λ_i^r represent the numerical approximations of $x^r(t,\alpha)$ and $\lambda^r(t,\alpha)$ at node i. In general, Equations (5.1) and (5.2) or Equations (5.4) and (5.5) give a set of $2N$ equations in terms of $2N$ variables, i.e., $Ax = b$, it means that, we can use any linear equation solver to find the solution. Regardless the left and right bounds of the fuzzy numbers $\tilde{x}$ and $\tilde{\lambda}$, the vector x is constructed as follows

- For fixed final state problem

$$x = [x_1 \;\; x_2 \;\; ... \;\; x_{N-1} \;\; \lambda_0 \;\; \lambda_1 \;\; ... \;\; \lambda_N]^T .$$

- For free final state problem

$$x = [x_1 \;\; x_2 \;\; ... \;\; x_N \;\; \lambda_0 \;\; \lambda_1 \;\; ... \;\; \lambda_{N-1}]^T .$$

In the next section, we will give four examples can serve to illustrate our main results.

6. Numerical examples

Example 6.1 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_{0}^{1} \left[\tilde{x}^2 + \tilde{u}^2\right] dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= t\tilde{x} + \tilde{u}, \\ \tilde{x}(0) &= (0,1,2), \quad \tilde{x}(1) = (-2,-1,1). \end{aligned}$$

Solution. We have,

$$q(t) = r(t) = b(t) = t_1 = 1, \ \text{ and } a(t) = t,$$

Then for the left bound of state and control Theorem(4.1) gives,

$${}_0D_t^{\beta}x^l = tx^l - \lambda^l, \tag{6.1}$$

$${}_t^CD_1^{\beta}\lambda^l = x^l + t\lambda^l, \tag{6.2}$$

$$u^l + \lambda^l = 0. \tag{6.3}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= \alpha, \\ x^l(1,\alpha) &= -2+\alpha. \end{aligned}$$

For the right bound of state and control, Theorem(4.1) gives,

$${}_0D_t^{\beta}x^r = tx^r - \lambda^r, \tag{6.4}$$

$${}_t^CD_1^{\beta}\lambda^r = x^r + t\lambda^r, \tag{6.5}$$

$$u^r + \lambda^r = 0. \tag{6.6}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 2-\alpha, \\ x^r(1,\alpha) &= 1-2\alpha. \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(1(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(1(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is an increasing function of α, $u^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1), furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a strong fuzzy solution of this problem.

Example 6.2 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_{1}^{2} \tilde{u}^2 dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= (2t-1)\tilde{x}\ominus \sin(t)\tilde{u}, \\ \tilde{x}(1) &= (0,1,2), \quad \tilde{x}(2)=(-2,-1,1). \end{aligned}$$

Solution. We have, $q(t)=0$, $r(t)=t_0=1$, $b(t)=-\sin(t)$, and $a(t)=(2t-1)$, then for the left bound of the state and control, Theorem(4.1) gives,

$$ {}_1D_t^{\beta}x^l = (2t-1)x^l - \sin^2(t)\lambda^l, \tag{6.7}$$

$$ {}_t^CD_2^{\beta}\lambda^l = (2t-1)\lambda^l, \tag{6.8}$$

$$ u^l - \sin(t)\lambda^l = 0. \tag{6.9}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= \alpha, \\ x^l(1,\alpha) &= -2+\alpha. \end{aligned}$$

For the right bound of state and control Theorem(4.1) gives,

$$ {}_1D_t^{\beta}x^r = (2t-1)x^r - \sin^2(t)\lambda^r, \tag{6.10}$$

$$ {}_t^CD_2^{\beta}\lambda^r = (2t-1)\lambda^r, \tag{6.11}$$

$$ u^r - \sin(t)\lambda^r = 0. \tag{6.12}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 2-\alpha, \\ x^r(1,\alpha) &= 1-2\alpha. \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(2(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1)=x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(2(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1)=x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1-C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha] = \left[u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

Example 6.3 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\int_0^1 \left[\tilde{x}^2+\tilde{u}^2\right]dt$$

subject to:

$$\begin{aligned} {}_0D_t^{\beta}\tilde{x} &= -(0,1,3)\tilde{x}+\tilde{u}, \\ \tilde{x}(0) &= (1,1,1), \quad \tilde{x}(1)=(0,0,0). \end{aligned}$$

Solution. We know that,

$$\left[{}_0D_t^\beta x^l,{}_0D_t^\beta x^r\right] = \left[-(3-2\alpha)x^l + u^l, -\alpha x^r + u^r\right],$$

then we have,

$$q(t) = r(t) = b(t) = x_0 = t_1 = 1,$$

$a(t) = -(3-2\alpha)$ and $a(t) = -\alpha$ for the left and right derivatives respectively, then for the left bound of the state and control Theorem(4.1) gives,

$$ {}_0D_t^\beta x^l = -(3-2\alpha)x^l - \lambda^l, \tag{6.13}$$

$$ {}_t^CD_1^\beta \lambda^l = x^l - (3-2\alpha)\lambda^l, \tag{6.14}$$

$$ u^l + \lambda^l = 0. \tag{6.15}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= 1, \\ x^l(1,\alpha) &= 0. \end{aligned}$$

For the right bound of the state and control Theorem(4.1) gives,

$$ {}_1D_t^\beta x^r = -\alpha x^r - \lambda^r, \tag{6.16}$$

$$ {}_t^CD_2^\beta \lambda^r = x^r - \alpha\lambda^r, \tag{6.17}$$

$$ u^r + \lambda^r = 0. \tag{6.18}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 1, \\ x^r(1,\alpha) &= 0. \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(3(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

Figure(3(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1-C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha] = \left[u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

Example 6.4 Find the fuzzy control that minimize

$$\tilde{J}(\tilde{u}(t)) = \frac{1}{2}\tilde{x}^2(1) + \frac{1}{2}\int_0^1 \left[\tilde{x}^2 + \tilde{u}^2\right] dt$$

subject to:

$$\begin{aligned} {}_0D_t^\beta \tilde{x} &= -(0,1,3)\tilde{x} + \tilde{u}, \\ \tilde{x}(0) &= (1,1,1). \end{aligned}$$

Solution. We have,

$$q(t) = r(t) = b(t) = x_0 = t_1 = 1,$$

$a(t) = -(3-2\alpha)$ and $a(t) = -\alpha$ for the left and right derivatives respectively, then Theorem(4.1) gives,

$$_{t_0}D_t^{\beta} x^l = -(3-2\alpha)x^l - \lambda^l, \tag{6.19}$$

$$_t^C D_{t_1}^{\beta} \lambda^l = x^l - (3-2\alpha)\lambda^l, \tag{6.20}$$

$$u^l + \lambda^l = 0. \tag{6.21}$$

and the boundary conditions

$$\begin{aligned} x^l(0,\alpha) &= 1, \\ \lambda^l(0,\alpha) &= x^l(1,\alpha). \end{aligned}$$

For the right bound of the state and control Theorem(4.1) gives,

$$_1D_t^{\beta} x^r = -\alpha x^r - \lambda^r, \tag{6.22}$$

$$_t^C D_2^{\beta} \lambda^r = x^r - \alpha\lambda^r, \tag{6.23}$$

$$u^r + \lambda^r = 0. \tag{6.24}$$

and the boundary conditions

$$\begin{aligned} x^r(0,\alpha) &= 1, \\ \lambda^r(0,\alpha) &= x^r(1,\alpha). \end{aligned}$$

Now, we use the numerical method to solve the above equations with the related boundary conditions, then we obtain the following results.

Figure(4(a)) show that the state $\tilde{x}^*(t)$ as a function of α, we observe that $x^{l^*}(t,\alpha)$ is an increasing function of α, $x^{r^*}(t,\alpha)$ is a decreasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, thus, $x^{l^*}(t,\alpha)$ and $x^{r^*}(t,\alpha)$ satisfy the conditions of Lemma(2.1).

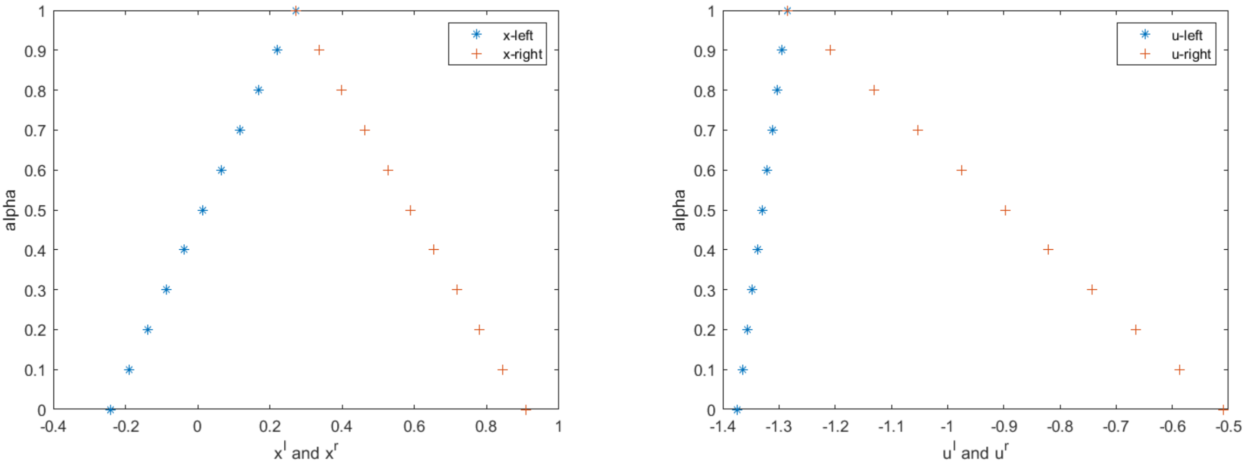
Figure(4(b)) show that the control $\tilde{u}^*(t)$ as a function of α, we find that $u^{l^*}(t,\alpha)$ is a decreasing function of α, $u^{r^*}(t,\alpha)$ is an increasing function of α and $x^{l^*}(t,1) = x^{r^*}(t,1)$, it means that $u^{l^*}(t,\alpha)$ and $u^{r^*}(t,\alpha)$ do not satisfy the conditions **C1-C2** of Lemma(2.1), then we use the definition(4.3) of weak solution, we find that

$$\tilde{u}^*(t)[\alpha] = \left[u^{r^*}(t,\alpha), u^{l^*}(t,\alpha)\right].$$

Furthermore, $\tilde{x}^*(t)$ and $\tilde{u}^*(t)$ represent a weak fuzzy solution of this problem.

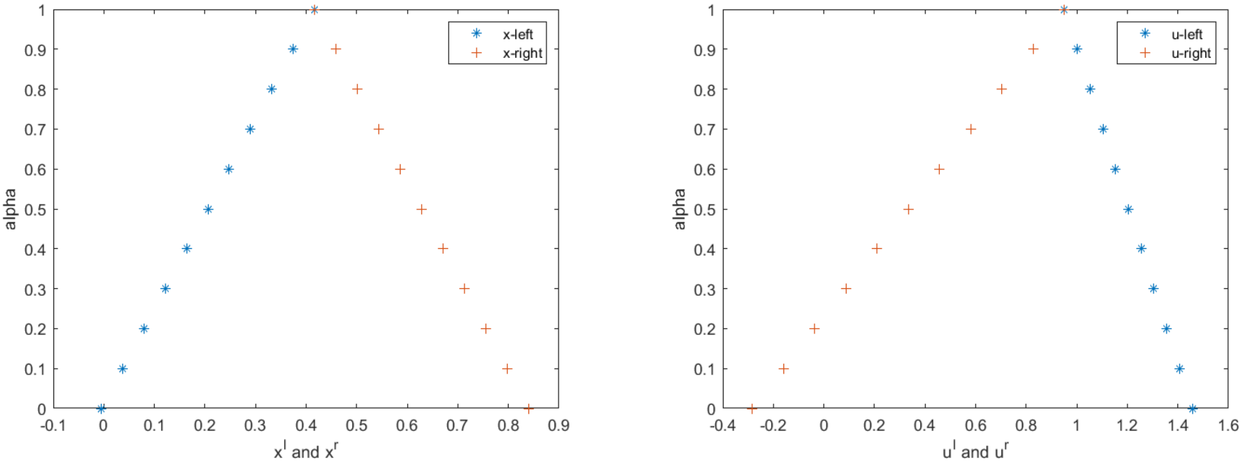
7. Conclusion

In this paper, the necessary conditions of fuzzy fractional optimal control problem with both fixed and free final state conditions at the final time has been derived using fuzzy variational approach. Our problems is defined in the sense of Riemann-Liouville fractional derivative based on Hukuhara difference. A numerical technique is proposed based on Grünwald-Letnikov definition of fractional derivative. The concepts of strong and weak solutions of our problems are given. lastly, four examples are provided to show the effectiveness of Theorem(4.1) and the numerical algorithm.



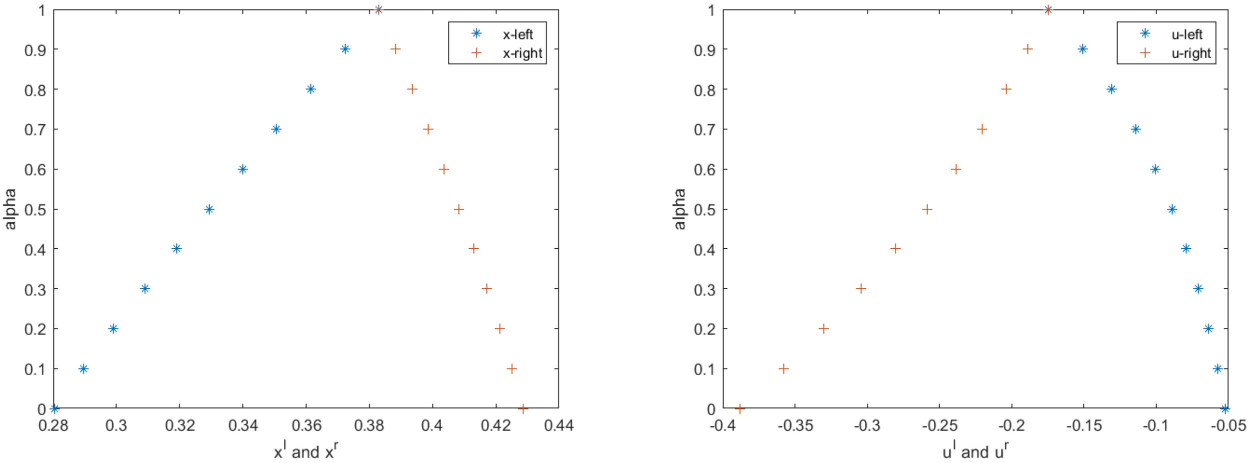
(a) **(b)**

Figure 1: Example(6.1) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.



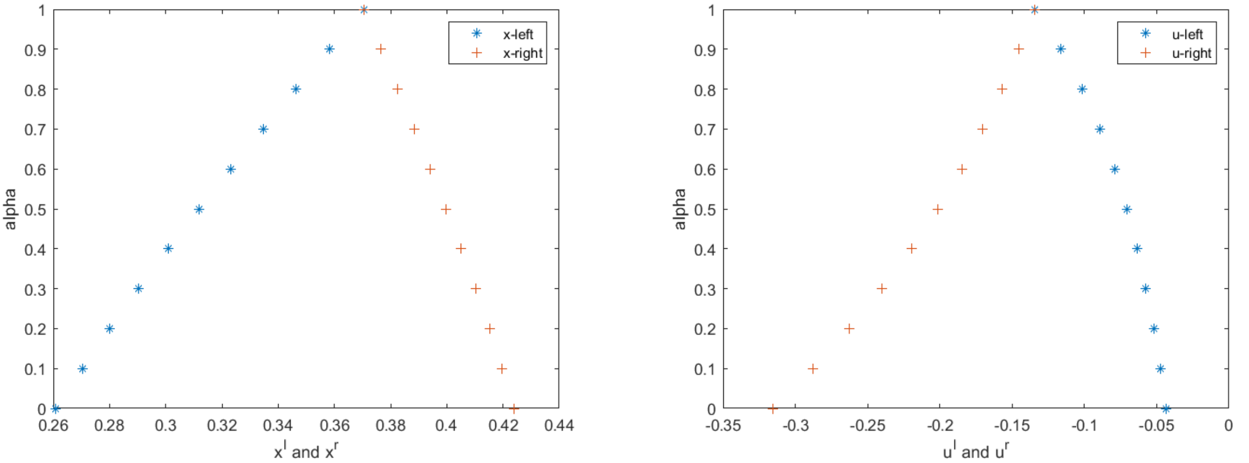
(a) **(b)**

Figure 2: Example(6.2) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.



(a) **(b)**

Figure 3: Example(6.3) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control at $t = 0.1, \beta = 0.77$.



(a) **(b)**

Figure 4: Example(6.4) (a) the state at $t = 0.1, \beta = 0.77$ (b) the control $t = 0.1, \beta = 0.77$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Differential Transform Method for Solving Fuzzy Fractional Wave Equation[†]

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Abstract: In this letter, the differential transform method (DTM) is applied to solve fuzzy fractional wave equation. The elemental properties of this method are investigated based on the two-dimensional differential transform method (DTM), generalized Taylor's formula and fuzzy Coputo's derivative. The proposed method is also illustrated by using some examples. The results reveal that DTM is a highly effective scheme for obtaining analytical solutions of the fuzzy fractional wave equation.

Mathematics Subject Classification. 65L05, 26E50

Keyword: Fuzzy numbers; Fuzzy fractional wave equation; Differential transform method; Fuzzy Caputo's derivative; Generalized Taylor formula.

1 Introduction

In 1965, the fuzzy sets were introduced for the first time by Zadeh in [28]. hundreds of examples have been supplied where the nature of uncertainty in the behavior of given system processes are fuzzy rather than stochastic nature. In the last few years, many authors have interested in the study of the theoretical framework of fuzzy initial value problems. Chang and Zadeh in [6] have introduced the concept of fuzzy derivative. Kandel and Byatt in [12] have initially presented the concept of the fuzzy differential equation. Bede and Gal in [4] have studied the concept of strongly generalized differentiable of fuzzy valued functions, which enlarged the class of differentiable fuzzy valued functions.

In 1695, the fractional calculus was first studied. The subject of fractional calculus has gained importance during the past three decades due mainly to its demonstrated applications in different area of physics and engineering in [16]. Fuzzy fractional differential equations (FFDE) play an important role in modelling of science and engineering problems. Padmapriya and Kaliyappan in [22] established analytical and numerical methods to solve fuzzy fractional differential equations. the concept of differential of fuzzy function with two variables and fuzzy wave equations studied in [26]. In the last years many authors have developed and introduced some variant methods for solving fuzzy wave equation. Kermani in [15] used finite difference method to solve the fuzzy wave equation numerically. Also, Martin and Radek in [25] used f-transforms to solve the fuzzy wave equation.

Zhou in [29] has presented the concept of the differential transform method (DTM), this method constructs an analytical solution inform of a polynomial, which is different from the tradition higher order Taylor formula method. Recently some researchers used differential transform method (DTM) to solve fuzzy fractional differential equations and fuzzy differential equations in [9, 23, 1, 19, 20].

This paper is structured as follows. In Section 2, we call some definitions on fuzzy numbers, fuzzy functions and fuzzy Caputo's derivative. In Section 3, The generalization of Taylor's formula is presented. In Section 4, the generalized two-dimensional differential transform method (DTM) for

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[†] This work is supported by National Natural Science Foundation of China (61763044).

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the solution of the fuzzy wave equation with space and time-fractional derivatives are developed and derived. Examples are shown in Section 5. Finely, conclusion is given in section 6.

2 Basic concepts

The results about fuzzy numbers space E^1, we recall that $E^1 = \{\tilde{u} : R \to [0,1] : u$ satisfies (1)(4) below $\}$ (refer to [6])

1. $\tilde{u}$ is normal, i.e., there exists $x_0 \in R$ such that $\tilde{u}(x_0) = 1$;
2. $\tilde{u}$ is convex, i.e., for all and $\lambda \in [0,1], x, y \in R$,
$$\tilde{u}(\lambda x + (1-\lambda)y) \geq \min\{\tilde{u}(x), \tilde{u}(y)\},$$
holds;
3. $\tilde{u}$ is upper semicontinuous, i.e., for any $x_0 \in R$,
$$\tilde{u}(x_0) \geq \lim_{x \longrightarrow x_0^{\pm}} \tilde{u}(x);$$
4. supp $\tilde{u} = \{x \in R | \tilde{u}(x) > 0\}$ is the support of $\tilde{u}$, and its closure cl (supp $\tilde{u}$) is compact.

For $0 < r \leq 1$, denote $[\tilde{u}]_r = \{x : \tilde{u}(x) \geq r\}$. Then from (1)-(4), follows that the r-level set $[\tilde{u}]_r$ is a closed and bounded interval for all $r \in [0,1]$.

For $\tilde{u}, \tilde{v} \in E^1,\ k \in R$, the addition and scalar multiplication are defined using the equations
$$[\tilde{u} + \tilde{v}]_r = [\tilde{u}]_r + [\tilde{v}]_r,$$
$$[k\tilde{u}]_r = k[\tilde{u}]_r,$$
respectively.

Define $D : E^1 \times E^1 \to R^+ \cup \{0\}$ using the equation
$$D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} d([\tilde{u}]_r [\tilde{v}]_r),$$
where d is Hausdorff metric space as
$$\begin{aligned} d([\tilde{u}]_r, [\tilde{v}]_r) &= \inf\{\varepsilon : [\tilde{u}]_r \subset N([\tilde{v}]_r, \varepsilon), [\tilde{v}]_r \subset N([\tilde{u}]_r, \varepsilon)\} \\ &= \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}, \end{aligned}$$
where $N([\tilde{u}]_r, \varepsilon), N([\tilde{v}]_r, \varepsilon)$ is the ε-neighborhood of $[\tilde{u}]_r, [\tilde{v}]_r$, respectively, and $\underline{u}_r, \underline{v}_r, \overline{u}_r, \overline{v}_r$ are endpoints of $[\tilde{u}]_r, [\tilde{v}]_r$, respectively.
By using the results of [13], we see that

- (E^1, D) is complete metric space,
- $D(\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}) = D(\tilde{u}, \tilde{v})$ for all $\tilde{u}, \tilde{v}, \tilde{w} \in E^1$,
- $D(k\tilde{u}, k\tilde{v}) = |k| D(\tilde{u}, \tilde{v})$.

In addition, we can introduce a partial order in E^1 by $\tilde{u} \leq \tilde{v}$ if and only if $[\tilde{u}]_r \leq [\tilde{v}]_r,\ r \in [0,1]$ if and only if $\underline{u}_r \leq \underline{v}_r, \overline{u}_r \leq \overline{v}_r, r \in [0,1]$. For applications of the partial order on E^1 (refer to [27]).

As the fuzzy number is resolved by using the interval $\tilde{u}_r = [\underline{u}_r, \overline{u}_r]$, see [8] defined another statements, parametrically, of fuzzy numbers as in following.

Definition 2.1.[31, 32] For arbitrary fuzzy numbers $\tilde{u}, \tilde{v} \in E^1, \tilde{u} = [\underline{u}_r, \overline{u}_r], \tilde{v} = [\underline{v}_r, \overline{v}_r]$, the quantity $D(\tilde{u}, \tilde{v}) = \sup_{r \in [0,1]} \max\{|\underline{u}_r - \underline{v}_r|, |\overline{u}_r - \overline{v}_r|\}$ is the distance between $\tilde{u}$ and $\overline{v}$ and also the following properties hold:

- (E^1, D) is a complete metric space,
- $D(\tilde{u} \oplus \tilde{w}, \tilde{v} \oplus \tilde{w}) = D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v}, \tilde{w} \in E^1$,
- $D(\tilde{u} \oplus \tilde{v}, \tilde{w} \oplus \tilde{e}) \leq D(\tilde{u}, \tilde{w}) + D(\tilde{v}, \tilde{e}), \forall \tilde{u}, \tilde{v}, \tilde{w}, \tilde{e} \in E^1$,
- $D(\tilde{u} \oplus \tilde{v}, \tilde{0}) \leq D(\tilde{u}, \tilde{0}) + D(\tilde{v}, \tilde{0}), \forall \tilde{u}, \tilde{v} \in E^1$,
- $D(k \odot \tilde{u}, k \odot \tilde{v}) = |k| D(\tilde{u}, \tilde{v}), \forall \tilde{u}, \tilde{v} \in E^1, k \in R$,
- $D(k_1 \odot \tilde{u}, k_2 \odot \tilde{u}) = |k_1 - k_2| D(\tilde{u}, \tilde{0}), \forall \tilde{u} \in E^1, k_1, k_2 \in R$, with $k_1 \cdot k_2 \geq 0$.

Let us recall the definition of the Hukuhara difference (H-difference) in [33]. Suppose that $\tilde{u}, \tilde{v} \in E^1$. The Hukuhara H-difference has been presented as a set $\tilde{w}$ for which $\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \tilde{u} = \tilde{v} \oplus \tilde{w}$. The H-difference is unique, but it does not always exist (a necessary condition for $\tilde{u} \ominus_{gH} \tilde{v}$ to exist is that $\tilde{u}$ contains a translate $\{c\} \oplus \tilde{v}$ of $\tilde{v}$). A generalization of the Hukuhara difference aims to overcome this situation.

Definition 2.2.[33, 31] The generalized Hukuhara difference between two fuzzy numbers $\tilde{u}, \tilde{v} \in E^1$ is defined as following:

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \begin{cases} \text{(i) } \tilde{u} = \tilde{v} \oplus \tilde{w}, \\ or \text{ (ii) } \tilde{v} = \tilde{u} \oplus (-\tilde{w}). \end{cases} \tag{2.1}$$

In terms of the $r-$levels, we get $[\tilde{u} \ominus_{gH} \tilde{v}] = [\min\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}, \max\{\underline{u}_r - \underline{v}_r, \overline{u}_r - \overline{v}_r\}]$ and if the H-difference exists, then $\tilde{u} \ominus \tilde{v} = \tilde{u} \ominus_{gH} \tilde{v}$; the conditions for existence of $\tilde{w} = \tilde{u} \ominus_{gH} \tilde{v} \in E^1$ are

$$\text{Case (i)} \quad \begin{cases} \underline{w}_r = \underline{u}_r - \underline{v}_r \text{ and } \overline{w}_r = \overline{u}_r - \overline{v}_r, \forall_r \in [0,1], \\ \text{with } \underline{w}_r \text{ increasing, } \overline{w}_r \text{ decreasing, } \underline{w}_r \leq \overline{w}_r. \end{cases} \tag{2.2}$$

$$\text{Case (ii)} \quad \begin{cases} \underline{w}_r = \overline{u}_r - \overline{v}_r \text{ and } \overline{w}_r = \underline{u}_r - \underline{v}_r, \forall_r \in [0,1], \\ \text{with } \underline{w}_r \text{ increasing, } \overline{w}_r \text{ decreasing, } \underline{w}_r \leq \overline{w}_r. \end{cases} \tag{2.3}$$

It is easy to show that (i) and (ii) are both valid if and only if $\tilde{w}$ is a crisp number. In the case, it is possible that the gH-difference of two fuzzy numbers does not exist. To address this shortcoming, a new difference between fuzzy numbers was introduced in [33].

Lemma 2.1.[10, 24] A fuzzy number $\tilde{u}$ in parametric form is a pair $[\underline{u}_r, \overline{u}_r]$ of function $\underline{u}_r$ and $\overline{u}_r$ for any $r \in [0,1]$, which satisfies the following requirements.

- $\underline{u}_r$ is a bounded non-decreasing left continuous function in (0,1];
- $\overline{u}_r$ is a bounded non-increasing left continuous function in (0,1];
- $\underline{u}_r \leq \overline{u}_r$.

Some the author of the classified fuzzy numbers into several types of fuzzy membership function. To the deepest of our study, triangular fuzzy membership function or also often referred to as triangular fuzzy numbers are the most widely used membership function.

In order to describe the fuzzy numbers and real numbers clearly, in convenience, the fuzzy numbers and fuzzy-valued functions in the whole paper are added with a tilde sign at the top, while the real-value function and interval-value functions are written directly.

A fuzzy valued function $\tilde{f}$ of two variables is a rule that assigns to each ordered pair of real numbers, (x,t), in a set D, a unique fuzzy numbers denoted by $\tilde{f}(x,t)$. The set D is the domain of $\tilde{f}$ and its range is the set of values taken by f, i.e., $\{\tilde{f}(x,t) | (x,t) \in D\}$.

The parametric representation of the fuzzy valued function $f : D \to E^1$ is expressed by $f(x,t)(r) = [\underline{f}(x,t)(r), \overline{f}(x,t)(r)]$, for all $(x,t) \in D$ and $r \in [0,1]$.

Suppose $f : D \to E^1$ be a fuzzy valued function of two variable. Then, we say that the fuzzy limit of $f(x,t)$ as (x,t) approaches to (a,b) is $L \in E^1$, and we write $\lim_{(x,t)\to(a,b)} f(x,t) = L$ if for every number $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that if $(x,t) \in D, \| (x,t) - (a,b) \| < \delta \Rightarrow D(f(x,t), L) < \varepsilon$, where $\| \cdot \|$ denotes the Euclidean norm in R^n (ref. to [3])

A fuzzy valued function $f : D \to E^1$ is said to be fuzzy continuous at $(x_0, t_0) \in D$ if $\lim_{(x,t)\to(x_0,t_0)} f(x,t) = f(x_0,t_0)$. We say that f is fuzzy continuous on D if f is fuzzy continuous at every point (x_0, t_0) in D (ref. to [3, 30]).

Definition 2.3.[11] Suppose that $\tilde{u}(x,t) : D \to E^1$ and $(x_0, t) \in D$. We say that $\tilde{u}$ is strongly generalized differentiable on (x_0, t) if there exists an element $\frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)} \in E^1$ such that

i. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t), \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)$ and the limits (in the metric D)

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t)}{h} = \lim_{h\to 0+} = \frac{(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)}{h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

ii. for all $h > 0$ sufficiently small, $\exists_{gH} \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t), \tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)}{-h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

iii. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t), \tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0 + h, t) \ominus_{gH} \tilde{u}(x_0, t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0 - h, t) \ominus_{gH} \tilde{u}(x_0, t)}{-h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)},$$

or

iv. for all $h > 0$ sufficiently small, $\exists \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t), \tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 + h, t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}(x_0, t) \ominus_{gH} \tilde{u}(x_0 - h, t)}{h} = \frac{\partial \tilde{u}}{\partial x}|_{(x_0,t)}.$$

Definition 2.4.[4] Suppose that $\tilde{u}(x,t) : D \to E^1$ and $(x_0, t) \in D$. We define the n th-order derivative of $\tilde{u}$ as follows: we say that $\tilde{u}$ is strongly generalized differentiable of the n th-order at (x_0, t) if there exists an element $\frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)} \in E^1$, $\forall s = 1, 2, \cdots, n$ such that

i. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0 + h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t), \tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 - h, t)$ and the limits (in the metric D)

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0 + h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 - h, t)}{h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

ii. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 + h, t), \tilde{u}^{(s-1)}(x_0 - h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0 + h, t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0 - h, t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0, t)}{-h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

iii. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0+h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t), \tilde{u}^{(s-1)}(x_0-h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0+h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)}{h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0-h,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0,t)}{-h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)},$$

or

iv. for all $h > 0$ sufficiently small, $\exists \tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0+h,t), \tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0-h,t)$ and the limits

$$\lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0+h,t)}{-h} = \lim_{h\to 0+} \frac{\tilde{u}^{(s-1)}(x_0,t) \ominus_{gH} \tilde{u}^{(s-1)}(x_0-h,t)}{h} = \frac{\partial^s \tilde{u}}{\partial x^s}|_{(x_0,t)}.$$

2.1 Fuzzy Coputo's derivative

We denote $C^F[a,b]$ as a space of all fuzzy valued functions which are continuous on $[a,b]$, and the space of all Kaleva integrable fuzzy-valued functions on the bounded interval $[a,b] \subset \mathbb{R}$ by $K^F[a,b]$, we denote the space of fuzzy value functions $\tilde{f}(x)$ which have continuous H-derivative up to order $n-1$ on $[a,b]$ such that $\tilde{f}^{(n-1)}(x) \in AC^F([a,b])$ by $AC^{(n)F}([a,b])$, where $AC^F([a,b])$ denote the set of all fuzzy-valued functions which are absolutely continuous (ref. to [13, 9]).

Definition 2.5.[2] Suppose $\tilde{f}(x) \in C^F[a,b] \cap K^F[a,b]$, the fuzzy Riemann Liouville integral of fuzzy valued function $\tilde{f}$ is defined as following:

$$(I^\alpha_{a+}\tilde{f})(x,r) = [(I^\alpha_{a+}\underline{f})(x,r), (I^\alpha_{a+}\overline{f})(x,r)],$$

where $0 \le r \le 1$

$$(I^\alpha_{a+}\underline{f})(x,r) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\underline{f}(t)(r)\mathrm{d}t}{(x-t)^{1-\alpha}},\ 0 \le r \le 1,$$

$$(I^\alpha_{a+}\overline{f})(x,r) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\overline{f}(t)(r)\mathrm{d}t}{(x-t)^{1-\alpha}},\ 0 \le r \le 1.$$

Suppose $\tilde{f}(x) \in C^F((0,a]) \cap K^F(0,a)$, be a given function such that $\tilde{f}(t,r) = [\underline{f}(t,r), \overline{f}(t,r)]$ for all $t \in (0,a]$ and $0 \le r \le 1$. We define $D^\alpha_{*a}\tilde{f}(t;r)$ the fuzzy fractional Riemann-Liouville derivative of order $0 < \alpha < 1$ of $\tilde{f}$ in the parametric from,

$$D^\alpha_{*a}\tilde{f}(t;r) = \frac{1}{\Gamma(1-\alpha)} \left[\frac{d}{dt} \int_0^t (t-s)^{-\alpha} \underline{f}(s,r)ds, \frac{d}{dt} \int_0^t (t-s)^{-\alpha} \overline{f}(s,r)ds \right],$$

provided that equation defines a fuzzy number $D^\alpha_{*a}\tilde{f}(t) \in E^1$. In fact,

$$D^\alpha_{*a}\tilde{f}(t,r) = [D^\alpha_{*a}\underline{f}(t,r), D^\alpha_{*a}\overline{f}(t,r)].$$

Obviously, $D^\alpha_{*a}\tilde{f}(t) = \frac{d}{dt} I^{1-\alpha}\tilde{f}(t)\ for\ t \in (0,a]$.

3 Generalized Taylor's formula

In this section, we present the generalized Taylor's formula that involves Caputo fractional derivative.

Theorem 3.1.[21] Let that $(D^\alpha_{*a})^j f(x) \in C(a,b]$ for $j = 0, 1, \cdots, n+1$, where $0 < \alpha \le 1$, that we get

$$f(x) = \sum_{i=0}^{n} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D^\alpha_{*a})^i f)(a+) + \frac{((D^\alpha_{*a})^{n+1} f)(\zeta)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha}, \tag{3.4}$$

with $a \leq \zeta \leq x$, $\forall x \in (a, b]$ and D^{α}_{*a} is the Caputo fractional derivative of order α, where $(D^{\alpha}_{*a})^j = D^{\alpha}_{*a}D^{\alpha}_{*a}\cdots D^{\alpha}_{*a}$. In case of $\alpha = 1$, the generalized Taylor's formula (3.4) reduces to the classical Taylor's formula.

Theorem 3.2.[17] Let that $(D^{\alpha}_{*a})^j f(x) \in C(a, b]$ for $j = 0, 1, \cdots\cdot, N + 1$, where $0 < \alpha \leq 1$. If $x \in [a, b]$, then

$$f(x) \simeq \sum_{i=0}^{N} \frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D^{\alpha}_{*a})^i f)(a+). \tag{3.5}$$

Furthermore, there is a value ζ with $a \leq \zeta \leq x$ so that the error term $R^{\alpha}_N(x)$ has the from

$$R^{\alpha}_N(x) = \frac{((D^{\alpha}_{*a})^{N+1} f)(\zeta)}{\Gamma((N+1)\alpha+1)}(x-a)^{(N+1)\alpha}. \tag{3.6}$$

The accuracy of $R^{\alpha}_N(x)$ increases when we choose large N and decreases as value of x moves away from the center a. Hence, we must choose N large enough so that the error does not exceed a specified bound. In the following theorem, we find precise condition under which the exponents hold for arbitrary fractional operators.

Theorem 3.3.[18] Let that $f(x) = x^{\lambda^*}g(x)$, where $\lambda^* > -1$ and $g(x)$ has the generalized power series expansion $g(x) = \sum_{n=0}^{\infty} a_n(x-a)^{n\alpha}$ with radius of convergence $R > 0, where\ \ 0 < \alpha \leq 1$. Then

$$D^{\gamma}_{*a}D^{\beta}_{*a}f(x) = D^{\gamma+\beta}_{*a}f(x) \tag{3.7}$$

for all $x \in (0, R)$ if one of the following conditions is satisfied:

1. $\beta < \lambda^* + 1$, and γ arbitrary,
2. $\beta \geq \lambda^* + 1, \gamma$ arbitrary,, and $a_j = 0$ for $j = 0, 1, \cdots\cdot, m-1$, where $m - 1 < \beta \leq m$.

4 Differential transform method and fuzzy fractional wave equation

4.1 Generalized two-dimensional differential transform method

In this section, we will derive the generalized two-dimensional differential transform method (DTM) that we get developed for the solution of the wave equation with space and time-fractional derivatives. The proposed method is based on Taylor's formula. Consider a function of two variables $u(x, t)$, and Let that it can be represented as a product of two single variable functions, $u(x, t) = f(x)g(t)$. Based on the properties of generalized two dimensional differential transform method, function $u(x, t)$ can be represented as.

$$u(x,t) = \sum_{j=0}^{\infty} F_{\alpha}(j)\cdot(x-x_0)^{j\alpha}\sum_{h=0}^{\infty} G_{\beta}(h)\cdot(t-t_0)^{h\beta} = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(j,h)(x-x_0)^{j\alpha}(t-t_0)^{h\beta}, \tag{4.8}$$

where $0 < \alpha, \beta \leq 1, U_{\alpha,\beta}(j, h) = F_{\alpha}(j)G_{\beta}(h)$ is called the spectrum of $u(x, t)$. If function $u(x, t)$ is analytical and differentiated continuously with respect to time t^* in the domain of interest, then we define the generalized two-dimensional differential transform method (DTM) of the function $u(x, t)$ as follows:

$$U_{\alpha,\beta}(j,h) = \frac{1}{\Gamma(\alpha j+1)\Gamma(\beta h+1)}[(D^{\alpha}_{x_0})^j(D^{\beta}_{t_0})^h u(x,t)]_{(x_0,t_0)}, \tag{4.9}$$

where $(D^{\alpha}_{x_0})^j = D^{\alpha}_{x_0}\cdot D^{\alpha}_{x_0}\cdots\cdot D^{\alpha}_{x_0}$. In this work, the lowercase $u(x, t)$ represents the original function while the uppercase $U_{\alpha,\beta}(j, h)$ stands for the transformed function. The generalized differential transform method (DTM) inverse of $U_{\alpha,\beta}(j, h)$ is defined as follows

$$u(x,t) = \sum_{j=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(j,h)\cdot(x-x_0)^{j\alpha}(t-t_0)^{h\beta} \tag{4.10}$$

In case of $\alpha = 1$ and $\beta = 1$. then generalized two-dimensional differential transform (DTM) (4.9) reduces to the classical two-dimensional DTM [5]. From equation (4.9) and (4.10), some basic properties of the generalized two-dimensional differential transform (DTM) are introduced below (ref. to [17]).

Theorem 4.1 If $u(x,t) = v(x,t) \pm w(x,t),$ *then* $U_{\alpha,\beta}(j,h) = V_{\alpha,\beta}(j,h) \pm W_{\alpha,\beta}(j,h).$

Theorem 4.2 If $u(x,t) = cv(x,t),$ *then* $U_{\alpha,\beta}(j,h) = cV_{\alpha,\beta}(j,h).$

Theorem 4.3 If $u(x,t) = v(x,t)w(x,t)$, then

$$U_{\alpha,\beta}(j,h) = \sum_{r=0}^{j}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(j-r,s). \tag{4.11}$$

Theorem 4.4 If $u(x,t) = D^{\alpha}_{x_0}v(x,t)$ and $0 < \alpha \le 1$, then we get

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha(j+1)+1)}{\Gamma(\alpha j+1)}V_{\alpha,\beta}(j+1,h). \tag{4.12}$$

Theorem 4.5 If $u(x,t) = D^{\alpha}_{x_0}D^{\beta}_{t_0}v(x,t)$ *and* $0 < \alpha, \beta \le 1$, then we get

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha(j+1)+1)\Gamma(\beta(h+1)+1)}{\Gamma(\alpha j+1)\Gamma(\beta h+1)}V_{\alpha,\beta}(j+1,h+1). \tag{4.13}$$

Theorem 4.6 If $u(x,t) = (x-x_0)^{n\alpha}(t-t_0)^{m\alpha},$ then $U_{\alpha,\beta}(j,h) = \delta(j-n)(h-m).$

Theorem 4.7 If $u(x,t) = D^{\gamma}_{x_0}v(x,t), m-1 < \gamma \le m$ *and* $v(x,t) = f(x)g(t)$, where $f(x)$ satisfies the conditions in Theorem 3.3, then

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha j+\gamma+1)}{\Gamma(\alpha j+1)}U_{\alpha,\beta}(j+\gamma/\alpha,h). \tag{4.14}$$

Theorem 4.8 If $u(x,t) = D^{\gamma}_{x_0}D^{\eta}_{t_0}v(x,t)$, where $m-1 < \gamma \le m, n-1 < \eta \le n$ and $v(x,t) = f(x)g(t)$, where the functions $f(x)$ and $g(x)$ satisfy the conditions given in Theorem 3.3, then

$$U_{\alpha,\beta}(j,h) = \frac{\Gamma(\alpha j+\gamma+1)}{\Gamma(\alpha j+1)}\frac{\Gamma(\beta h+\eta+1)}{\Gamma(\beta h+1)}U_{\alpha,\beta}(j+\gamma/\alpha,h+\eta/\beta). \tag{4.15}$$

4.2 Fuzzy fractional wave equation

Consider the fuzzy fractional wave equation with the indicated initial conditions and boundary conditions.

$$\frac{\partial^{\alpha}\tilde{u}}{\partial t^{\alpha}} = c^2 \odot \frac{\partial^2\tilde{u}}{\partial x^2}, \qquad 0 < \alpha \le 2, \qquad 0 < x < L, \qquad t > 0, \tag{4.16}$$

subject to the boundary conditions

$$\tilde{u}(0,t) = 0, \quad and \quad \tilde{u}(L,t) = 0, \tag{4.17}$$

and initial conditions.

$$\tilde{u}(x,0) = \tilde{f}(x), \quad and \quad \tilde{u}_t(x,0) = \tilde{g}(x). \tag{4.18}$$

We note that the case (i) of Definition 2.3 is coincident with the Hukuhara derivative [14]. We say that a function is (i) differentiable if it is differentiable as in (i) of Definition 2.3, a function is (ii)

differentiable if it is differentiable as in (ii) of Definition 2.3. In this paper we consider the two cases (i) and (ii). In Ref. [4] the authors consider four cases: the case (i) in [14] is coincident with (i); the case (iii) of Definition 2.1 is equivalent to (ii); in the other cases, the derivative is trivial because it is reduced to crisp element. For details see Theorem 7 in [4]. Thus, we only consider the cases (i) and (ii).

Lemma 4.2. [7]. Let $\tilde{u}(x,t) : D \to E^1$. Then the following statements hold.
(i) If $\tilde{u}(x,t)$ is (i)-partial differentiable for x (i.e. $\tilde{u}$ is partial differentiable for x under the meaning of Definition 2.1 (i), similarly to t), then

$$\left[\frac{\partial \tilde{u}}{\partial x}\right]_r = \left[\frac{\partial \underline{u}(x,t)(r)}{\partial x}, \frac{\partial \bar{u}(x,t)(r)}{\partial x}\right]; \tag{4.19}$$

(ii) If $\tilde{u}(x,t)$ is (ii)-partial differentiable for x (i.e. $\tilde{u}$ is partial differentiable for x under the meaning of Definition 2.1 (ii), similarly to t), then

$$\left[\frac{\partial \tilde{u}}{\partial x}\right]_r = \left[\frac{\partial \bar{u}(x,t)(r)}{\partial x}, \frac{\partial \underline{u}(x,t)(r)}{\partial x}\right]. \tag{4.20}$$

Remark 4.1. For $\tilde{u}(x,t) : D \to E^1$, the following results hold.

$$\left[\frac{\partial^2 \tilde{u}}{\partial x^2}\right]_r = \left[\frac{\partial^2 \underline{u}(x,t)(r)}{\partial x^2}, \frac{\partial^2 \bar{u}(x,t)(r)}{\partial x^2}\right], \tag{4.21}$$

in cases for that $(i,i), (ii,ii)$-$\frac{\partial^2 \tilde{u}}{\partial x^2}$ exist;

$$\left[\frac{\partial^2 \tilde{u}}{\partial x^2}\right]_r = \left[\frac{\partial^2 \bar{u}(x,t)(r)}{\partial x^2}, \frac{\partial^2 \underline{u}(x,t)(r)}{\partial x^2}\right]. \tag{4.22}$$

in cases for that $(i,ii), (ii,i)$-$\frac{\partial^2 \tilde{u}}{\partial t^2}$ exist.

Remark 4.2. In this paper, we only consider that the cases of $(i-ii)^n$-$\frac{\partial^n \tilde{u}}{\partial t^n}$ such that

$$\left[\frac{\partial^n \tilde{u}}{\partial x^n}\right]_r = \left[\frac{\partial^n \underline{u}(x,t)(r)}{\partial x^n}, \frac{\partial^n \bar{u}(x,t)(r)}{\partial x^n}\right], \tag{4.23}$$

where $(i-ii)^n$-$\frac{\partial^n \tilde{u}}{\partial t^n}$ stands for n time derivative in the cases (i) or (ii).

5 Examples

Example 5.1. Consider the following fuzzy fractional wave equation

(A)

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = 4 \odot \frac{\partial^2 \tilde{u}}{\partial x^2} \qquad 0 \le x \le 1, \quad 0 < t, \tag{5.24}$$

subject to the boundary conditions

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \qquad 0 < t, \tag{5.25}$$

and initial conditions

$$\begin{aligned} \tilde{u}(x,0) &= \tilde{f}(x) = \tilde{k}^n \odot \sin(\pi x), \quad && 0 \le x \le 1, \\ \frac{\partial \tilde{u}(x,0)}{\partial t} &= \tilde{g}(x) = 0, && 0 \le x \le 1. \end{aligned} \tag{5.26}$$

where $\tilde{k}^n \in E^1$, n=1,2,3,... fuzzy number is defined by

$$\tilde{k}(s) = \begin{cases} s, & s \in [0,1], \\ 2-s & s \in (1,2], \\ 0 & s \notin [0,2], \end{cases} \tag{5.27}$$

and $[\underline{\tilde{k}^n}](r) = r^n, [\overline{\tilde{k}^n}](r) = (2-r)^n$.
The parametric form of (5.24) is

$$\frac{\partial^2 \underline{u}}{\partial t^2} = 4\frac{\partial^2 \underline{u}}{\partial x^2} \qquad 0 \le x \le 1, \qquad 0 < t, \tag{5.28}$$

$$\frac{\partial^2 \overline{u}}{\partial t^2} = 4\frac{\partial^2 \overline{u}}{\partial x^2}, \qquad 0 \le x \le 1, \qquad 0 < t, \tag{5.29}$$

for $r \in [0,1]$, and where $\underline{u}$ stands for $\underline{u}(x,t)(r)$, similar to $\overline{u}$.
Taking the differential transform of equations (5.28) and (5.29), we get

$$(j+2)(j+1)\underline{U}(i,j+2)(r) = 4(i+2)(i+1)\underline{U}(i+2,j)(r), \tag{5.30}$$

$$(j+2)(j+1)\overline{U}(i,j+2)(r) = 4(i+2)(i+1)\overline{U}(i+2,j)(r). \tag{5.31}$$

From the initial given by equation (5.26), we get

$$\underline{u}(x,0)(r) = \sum_{i=0}^{\infty} \underline{U}(i,0)(r)x^i = \underline{k}(r)\sin(\pi x) = r^n \sum_{i=1,3,....}^{\infty} \frac{(-1)^{\frac{(i-1)}{2}}}{i!}\pi^i x^i, \tag{5.32}$$

$$\overline{u}(x,0)(r) = \sum_{i=0}^{\infty} \overline{U}(i,0)(r)x^i = \overline{k}(r)\sin(\pi x) = (2-r)^n \sum_{i=1,3,....}^{\infty} \frac{(-1)^{\frac{(i-1)}{2}}}{i!}\pi^i x^i. \tag{5.33}$$

The corresponding spectra can be obtained as follows,

$$\underline{U}(i,0)(r) = \begin{cases} 0, & \text{for i is even,} \\ \frac{(-1)^{\frac{(i-1)}{2}}}{i!} r^n \pi^i, & \text{for i is odd} \end{cases} \tag{5.34}$$

$$\overline{U}(i,0)(r) = \begin{cases} 0, & \text{for i is even,} \\ \frac{(-1)^{\frac{(i-1)}{2}}}{i!} (2-r)^n \pi^i, & \text{for i is odd} \end{cases} \tag{5.35}$$

and from equation (5.26) it can be obtained that,

$$\frac{\partial \underline{u}(x,0)(r)}{\partial t} = \sum_{i=0}^{\infty} \underline{U}(i,1)(r)x^i = 0, \tag{5.36}$$

$$\frac{\partial \overline{u}(x,0)(r)}{\partial t} = \sum_{i=0}^{\infty} \overline{U}(i,1)(r)x^i = 0. \tag{5.37}$$

Hence,

$$\underline{u}(i,1)(r) = 0, \tag{5.38}$$

$$\overline{u}(i,1)(r) = 0. \tag{5.39}$$

Substituting equations (5.34) -(5.39) to equations (5.30) and (5.31), all spectra can be found as,

$$\underline{U}(i,j)(r)=\begin{cases} 0, & \text{for } i \text{ is even } \textit{or } j \text{ is odd} \\ \dfrac{2^j(-1)^{\frac{(i+j-1)}{2}}}{i!j!}\, r^n\pi^{i+j}, & \text{for } i \text{ is odd } \textit{or } j \text{ is even} \end{cases} \tag{5.40}$$

$$\overline{U}(i,j)(r)=\begin{cases} 0, & \text{for } i \text{ is even } \textit{or } j \text{ is odd} \\ \dfrac{2^j(-1)^{\frac{(i+j-1)}{2}}}{i!j!}\, (2-r)^n\pi^{i+j}, & \text{for } i \text{ is odd } \textit{or } j \text{ is even} \end{cases} \tag{5.41}$$

So, the closed from of the solution can be easily written as

$$\begin{aligned}\underline{u}(x,t)(r)&=\underline{k}^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\underline{U}(i,j)(r)x^it^j=r^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\&=r^n\left[\left(\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\&=r^n\sin(\pi x)\cos(2\pi t),\end{aligned} \tag{5.42}$$

$$\begin{aligned}\overline{u}(x,t)(r)&=\overline{k}^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\overline{U}(i,j)(r)x^it^j=(2-r)^n\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\&=(2-r)^n\left[\left(\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\&=(2-r)^n\sin(\pi x)\cos(2\pi t).\end{aligned} \tag{5.43}$$

(B) Consider the following fuzzy fractional wave equation (5.24) with the boundary conditions:

$$\tilde{u}(0,t)=\tilde{u}(1,t)=0, \qquad 0<t, \tag{5.44}$$

and initial conditions

$$\begin{aligned}\tilde{u}(x,0)&=\tilde{f}(x)=\tilde{k}^n\oplus\sin(\pi x), && 0\le x\le 1,\\ \frac{\partial\tilde{u}(x,0)}{\partial t}&=\tilde{g}(x)=0, && 0\le x\le 1.\end{aligned} \tag{5.45}$$

By following the same steps, we will find that the solution. So, the closed from of the solution can be easily written as

$$\begin{aligned}\underline{u}(x,t)(r)&=\underline{k}^n+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\underline{U}(i,j)(r)x^it^j=r^n+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\&=r^n+\left[\left(\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\&=r^n+(\sin(\pi x)\cos(2\pi t)),\end{aligned} \tag{5.46}$$

$$\begin{aligned}\overline{u}(x,t)(r)&=\overline{k}^n+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\overline{U}(i,j)(r)x^it^j=(2-r)^n+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\&=(2-r)^n+\left[\left(\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\ldots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\&=(2-r)^n+(\sin(\pi x)\cos(2\pi t)).\end{aligned} \tag{5.47}$$

(C) Consider the following fuzzy fractional wave equation (5.24) with the boundary conditions:

$$\tilde{u}(0,t)=\tilde{u}(1,t)=0, \qquad 0<t, \tag{5.48}$$

and initial conditions

$$\begin{aligned}\tilde{u}(x,0)&=\tilde{f}(x)=\tilde{k}^n\ominus_{gH}\sin(\pi x), \quad && 0\le x\le 1,\\ \frac{\partial\tilde{u}(x,0)}{\partial t}&=\tilde{g}(x)=0, && 0\le x\le 1.\end{aligned} \tag{5.49}$$

where $\tilde{k}^n\in E^1$, n=1,2,3,... , fuzzy number is defined by

$$\tilde{k}(s)=\begin{cases}2(s-0.5), & s\in[0.5,1],\\ 2(1.5-s), & s\in(1,1.5],\\ 0 & s\notin[0.5,1.5],\end{cases} \tag{5.50}$$

and$\{\underline{\tilde{k}^n}\}(r)=(0.5+0.5r)^n$, $\{\overline{\tilde{k}^n}\}(r)=(1.5-0.5r)^n$.
By following the same steps, we will find that the solution. So, the closed from of the solution can be easily written as

$$\begin{aligned}\underline{u}(x,t)(r)&=\underline{k}^n-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\underline{U}(i,j)(r)x^it^j=(0.5+0.5r)^n-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\ &=(0.5+0.5r)^n-\left[\left(\sum_{i=1,3,\dots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\dots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\ &=(0.5+0.5r)^n-\left(\sin(\pi x)\cos(2\pi t)\right),\end{aligned} \tag{5.51}$$

$$\begin{aligned}\overline{u}(x,t)(r)&=\overline{k}^n-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\overline{U}(i,j)(r)x^it^j=(1.5-0.5r)^n-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{2^j}{j!i!}(-1)^{\frac{(i+j-1)}{2}}\pi^{i+j}x^it^j\\ &=(1.5-0.5r)^n-\left[\left(\sum_{i=1,3,\dots}^{\infty}\frac{1}{i!}(-1)^{\frac{(i-1)}{2}}(\pi x)^i\right)\left(\sum_{j=0,2,\dots}^{\infty}\frac{1}{j!}(-1)^{\frac{j}{2}}(2\pi t)^j\right)\right]\\ &=(1.5-0.5r)^n-\left(\sin(\pi x)\cos(2\pi t)\right).\end{aligned} \tag{5.52}$$

Example 5.2. Consider the following fuzzy time-fractional wave equation.

(A)

$$\frac{\partial^{1.5}\tilde{u}}{\partial t^{1.5}}=\frac{\partial^2\tilde{u}}{\partial x^2}, \qquad t>0, \tag{5.53}$$

subject to the initial conditions

$$\tilde{u}(x,0)=\tilde{f}(x)=\tilde{k}^n\odot\sin(x), \quad \frac{\partial\tilde{u}(x,0)}{\partial t}=\tilde{g}(x)=\tilde{k}^n\odot(-\sin(x)). \tag{5.54}$$

where $\tilde{k}^n\in E^1$, n=1,2,3,..., fuzzy number is defined by

$$\tilde{k}(s)=\begin{cases}2(s-0.5), & s\in[0.5,1],\\ 2(1.5-s), & s\in(1,1.5],\\ 0 & s\notin[0.5,1.5],\end{cases} \tag{5.55}$$

and$\{\underline{\tilde{k}^n}\}(r) = (0.5 + 0.5r)^n, \{\overline{\tilde{k}^n}\}(r) = (1.5 - 0.5r)^n$.
The parametric form of (5.53) is

$$\frac{\partial^{1.5}\underline{u}}{\partial t^{1.5}} = \frac{\partial^2\underline{u}}{\partial x^2}, \qquad t > 0, \tag{5.56}$$

$$\frac{\partial^{1.5}\overline{u}}{\partial t^{1.5}} = \frac{\partial^2\overline{u}}{\partial x^2}, \qquad t > 0. \tag{5.57}$$

for $r \in [0, 1]$, and where $\underline{u}$ stands for $\underline{u}(x, t)(r)$, similar to $\overline{u}$.

Let the solution $u(x, t) = f(x)g(t)$ where the function $g(t)$ satisfies the conditions given in Theorem 3.3. Then selecting $\alpha = 0.5, \beta = 1$ and applying the generalized two-dimensional differential transform method (DTM) to both sides of equations (5.56) and (5.57) by Theorem 4.7, equations (5.56) and (5.57) Transforms to

$$\underline{U}_{0.5,1}(j, h+3)(r) = \frac{(j+1)(j+2)\Gamma(\frac{h}{2}+1)}{\Gamma(\frac{h}{2}+\frac{5}{2})}\underline{U}_{0.5,1}(j+2, h)(r), \tag{5.58}$$

$$\overline{U}_{0.5,1}(j, h+3)(r) = \frac{(j+1)(j+2)\Gamma(\frac{h}{2}+1)}{\Gamma(\frac{h}{2}+\frac{5}{2})}\overline{U}_{0.5,1}(j+2, h)(r). \tag{5.59}$$

The generalized two-dimensional differential transform of the initial conditions (5.54) are given by

$$\underline{U}_{0.5,1}(j, 0)(r) = (0.5 + 0.5r)^n \frac{1}{j!}\sin(\frac{\pi j}{2}), \tag{5.60}$$

$$\underline{U}_{0.5,1}(j, 1)(r) = 0, \tag{5.61}$$

$$\underline{U}_{0.5,1}(j, 2)(r) = (0.5 + 0.5r)^n \frac{-1}{j!}\sin(\frac{\pi j}{2}), \tag{5.62}$$

$$\overline{U}_{0.5,1}(j, 0)(r) = (1.5 - 0.5r)^n \frac{1}{j!}\sin(\frac{\pi j}{2}), \tag{5.63}$$

$$\overline{U}_{0.5,1}(j, 1)(r) = 0, \tag{5.64}$$

$$\overline{U}_{0.5,1}(j, 2)(r) = (1.5 - 0.5r)^n \frac{-1}{j!}\sin(\frac{\pi j}{2}). \tag{5.65}$$

Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j, h)$ can be calculated.
So, the solution $u(x, t)$ of equations (5.56) and (5.57) is obtained

$$\begin{aligned}
\underline{u}(x, t)(r) &= (0.5 + 0.5r)^n \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right) x \\
&+ (0.5 + 0.5r)^n \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right) x^3 \\
&+ (0.5 + 0.5r)^n \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right) x^5 \\
\underline{u}(x, t)(r) &= (0.5 + 0.5r)^n \left(\sum_{j=0}^{\infty} \frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty} \frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (0.5 + 0.5r)^n \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right),
\end{aligned} \tag{5.66}$$

$$\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n\left(1-t-\frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{\Gamma(4)}t^3+.....\right)\cdot x\\
&+(1.5-0.5r)^n\left(-\frac{1}{3!}+\frac{1}{3!}t+\frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}-\frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}-\frac{1}{3!\Gamma(4)}t^3+....\right)\cdot x^3\\
&+(1.5-0.5r)^n\left(\frac{1}{5!}-\frac{1}{5!}t-\frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{5!\Gamma(4)}t^3-...\right)\cdot x^5\\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n\left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x)-\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right),\\
&= (1.5-0.5r)^n\left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x)-tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right). \qquad (5.67)
\end{aligned}$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$E_{\alpha,\beta}(z)=\tilde{k}^n\odot\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \qquad (5.68)$$

(B) Consider the following fuzzy time-fractional wave equation (5.53) with the initial conditions:

$$\tilde{u}(x,0)=\tilde{f}(x)=\tilde{k}^n\oplus\sin(x),\quad \frac{\partial\tilde{u}(x,0)}{\partial t}=\tilde{g}(x)=\tilde{k}^n\oplus(-\sin(x)). \qquad (5.69)$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j,h)$ can be calculated.

So, the solution $u(x,t)$ of equations (5.56) and (5.57) is obtained

$$\begin{aligned}
\underline{u}(x,t)(r) &= (0.5+0.5r)^n+\left(1-t-\frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{\Gamma(4)}t^3+.....\right)x\\
&+(0.5+0.5r)^n+\left(-\frac{1}{3!}+\frac{1}{3!}t+\frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}-\frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}-\frac{1}{3!\Gamma(4)}t^3+....\right)x^3\\
&+(0.5+0.5r)^n+\left(\frac{1}{5!}-\frac{1}{5!}t-\frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}}+\frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{1}{5!\Gamma(4)}t^3-...\right)x^5\\
\underline{u}(x,t)(r) &= (0.5+0.5r)^n+\left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x)-\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right),\\
&= (0.5+0.5r)^n+\left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x)-tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right), \qquad (5.70)
\end{aligned}$$

$$\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n + \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right)\cdot x\\
&+ (1.5-0.5r)^n + \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right)\cdot x^3\\
&+ (1.5-0.5r)^n + \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right)\cdot x^5\\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n + \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right),\\
&= (1.5-0.5r)^n + \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right). \qquad (5.71)
\end{aligned}$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$E_{\alpha,\beta}(z) = \tilde{k}^n \oplus \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \qquad (5.72)$$

(C) Consider the following fuzzy time fractional wave equation (5.53) with initial conditions:

$$\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \ominus_{gH} \sin(x), \quad \frac{\partial\tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \ominus_{gH} (-\sin(x)). \qquad (5.73)$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.58), (5.59) and the transformed initial conditions (5.60) -(5.65), the first few components of $U_{0.5,1}(j,h)$ can be calculated.
So, the solution $u(x,t)$ of equations (5.56) and (5.57) is obtained

$$\begin{aligned}
\underline{u}(x,t)(r) &= (0.5+0.5r)^n - \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right)x\\
&+ (0.5+0.5r)^n - \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right)x^3\\
&+ (0.5+0.5r)^n - \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right)x^5\\
\underline{u}(x,t)(r) &= (0.5+0.5r)^n - \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right),\\
&= (0.5+0.5r)^n - \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right), \qquad (5.74)
\end{aligned}$$

$$
\begin{aligned}
\overline{u}(x,t)(r) &= (1.5-0.5r)^n - \left(1 - t - \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{\Gamma(4)}t^3 + ...\right)\cdot x \\
&\quad + (1.5-0.5r)^n - \left(-\frac{1}{3!} + \frac{1}{3!}t + \frac{1}{3!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} - \frac{1}{3!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{1}{3!\Gamma(4)}t^3 + ...\right)\cdot x^3 \\
&\quad + (1.5-0.5r)^n - \left(\frac{1}{5!} - \frac{1}{5!}t - \frac{1}{5!\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \frac{1}{5!\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{1}{5!\Gamma(4)}t^3 - ...\right)\cdot x^5 \\
\overline{u}(x,t)(r) &= (1.5-0.5r)^n - \left(\sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}}}{\Gamma(\frac{3j}{2}+1)}\sin(x) - \sum_{j=0}^{\infty}\frac{(-1)^j t^{\frac{3j}{2}+1}}{\Gamma(\frac{3j}{2}+2)}\sin(x)\right), \\
&= (1.5-0.5r)^n - \left(E_{\frac{3}{2},1}(-t^{\frac{3}{2}})\sin(x) - tE_{\frac{3}{2},2}(-t^{\frac{3}{2}})\sin(x)\right). \qquad (5.75)
\end{aligned}
$$

Which is the exact solution of the fuzzy time fractional wave equations (5.56) and (5.57) where $E_{\alpha,\beta}(z)$ is the two parameters mittag-Leffer function defined by

$$
E_{\alpha,\beta}(z) = \tilde{k}^n \ominus_{gH} \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \qquad (5.76)
$$

Example 5.3. Consider the following fuzzy linear space time fractional wave equation

(A)

$$
\frac{\partial^{1.5}\tilde{u}}{\partial t^{1.5}} = \frac{1}{2}x^2 \odot \frac{\partial^{1.25}\tilde{u}}{\partial x^{1.25}} \quad x>0, \quad t>0, \qquad (5.77)
$$

subject to the initial conditions

$$
\tilde{u}(x,0) = \tilde{f}(x) = \tilde{k}^n \odot \sum_{n=0}^{\infty} a_n x^n, \qquad \frac{\partial \tilde{u}(x,0)}{\partial t} = \tilde{g}(x) = \tilde{k}^n \odot \sum_{n=0}^{\infty} b_n x^n. \qquad (5.78)
$$

where $\tilde{k}^n \in E^1$, n=1,2,3,... fuzzy number is defined by

$$
\tilde{k}(s) = \begin{cases} s, & s\in[0,1], \\ 2-s & s\in(1,2], \\ 0 & s\notin[0,2], \end{cases} \qquad (5.79)
$$

and $[\underline{\tilde{k}^n}](r) = r^n, [\overline{\tilde{k}^n}](r) = (2-r)^n$.
The parametric form of (5.77) is

$$
\frac{\partial^{1.5}\underline{u}}{\partial t^{1.5}} = \frac{1}{2}x^2\frac{\partial^{1.25}\underline{u}}{\partial x^{1.25}} \quad x>0, \quad t>0 \qquad (5.80)
$$

$$
\frac{\partial^{1.5}\overline{u}}{\partial t^{1.5}} = \frac{1}{2}x^2\frac{\partial^{1.25}\overline{u}}{\partial x^{1.25}} \quad x>0, \quad t>0 \qquad (5.81)
$$

for $r\in[0,1]$, and where $\underline{u}$ stands for $\underline{u}(x,t)(r)$, similar to $\overline{u}$.

Let the solution $u(x,t)$ can be represented as a product of single-valued functions, $u(x,t) = f(x)g(t)$ where the functions $f(x)$ and $g(t)$ satisfy the conditions given in Theorem 3.3. Selecting $\alpha = 0.5, \beta = 0.25$ and applying the generalized two-dimensional differential transform to both sides of equations (5.80) and (5.81), the fuzzy linear space-time fractional wave equations (5.80) and (5.81) transform to

$$\underline{U}_{1/2,1/4}(j,h+3)(r)=\begin{cases}\frac{1}{2}\frac{\Gamma(h/2+1)\Gamma(j/4+7/4)}{\Gamma(h/2+5/2)\Gamma(j/4+2/4)}\underline{U}_{1/2,1/4}(j+3,h)(r), & j\geq 2\\ 0, & j<2.\end{cases} \tag{5.82}$$

$$\overline{U}_{1/2,1/4}(j,h+3)(r)=\begin{cases}\frac{1}{2}\frac{\Gamma(h/2+1)\Gamma(j/4+7/4)}{\Gamma(h/2+5/2)\Gamma(j/4+2/4)}\overline{U}_{1/2,1/4}(j+3,h)(r), & j\geq 2\\ 0, & j<2.\end{cases} \tag{5.83}$$

The generalized two-dimensional transforms of the initial conditions (5.78) are given by

$$\underline{U}_{1/2,1/4}(j,0)(r)=r^n a_j, \tag{5.84}$$
$$\underline{U}_{1/2,1/4}(j,1)(r)=0, \tag{5.85}$$
$$\underline{U}_{1/2,1/4}(j,2)(r)=r^n b_j, \tag{5.86}$$
$$\overline{U}_{1/2,1/4}(j,0)(r)=(2-r)^n a_j, \tag{5.87}$$
$$\overline{U}_{1/2,1/4}(j,1)(r)=0, \tag{5.88}$$
$$\overline{U}_{1/2,1/4}(j,2)(r)=(2-r)^n b_j. \tag{5.89}$$

Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$\begin{aligned}\underline{u}(x,t)(r)&=r^n\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+r^n\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+r^n\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+r^n\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+r^n\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots,\end{aligned} \tag{5.90}$$

$$\begin{aligned}\overline{u}(x,t)(r)&=(2-r)^n\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+(2-r)^n\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+(2-r)^n\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+(2-r)^n\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+(2-r)^n\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots.\end{aligned} \tag{5.91}$$

(B) Consider the following fuzzy linear-space-time-fractional wave equation (5.77) with the initial conditions:

$$\tilde{u}(x,0)=\tilde{f}(x)=\tilde{k}^n\oplus\sum_{n=0}^{\infty}a_nx^n,\qquad \frac{\partial\tilde{u}(x,0)}{\partial t}=\tilde{g}(x)=\tilde{k}^n\oplus\sum_{n=0}^{\infty}b_nx^n. \tag{5.92}$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$\begin{aligned}\underline{u}(x,t)(r)&=r^n+\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+r^n+\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+r^n+\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+r^n+\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+r^n+\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots,\end{aligned} \tag{5.93}$$

$$\begin{aligned}\overline{u}(x,t)(r)&=(2-r)^n+\left(a_0+b_0t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_3t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_3t^{5/2}\right)\\&+(2-r)^n+\left(a_1+b_1t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_4t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_4t^{5/2}\right)\cdot x^{1/4}\\&+(2-r)^n+\left(a_2+b_2t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_5t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_5t^{5/2}\right)\cdot x^{2/4}\\&+(2-r)^n+\left(a_3+b_3t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_6t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_6t^{5/2}\right)\cdot x^{3/4}\\&+(2-r)^n+\left(a_4+b_4t+\frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)}a_7t^{3/2}+\frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)}b_7t^{5/2}\right)\cdot x+\cdots.\end{aligned} \tag{5.94}$$

(C) Consider the following fuzzy linear space-time-fractional wave equation (5.77) with the initial conditions:

$$\tilde{u}(x,0)=\tilde{f}(x)=\tilde{k}^n\ominus_{gH}\sum_{n=0}^{\infty}a_nx^n,\qquad \frac{\partial\tilde{u}(x,0)}{\partial t}=\tilde{g}(x)=\tilde{k}^n\ominus_{gH}\sum_{n=0}^{\infty}b_nx^n. \tag{5.95}$$

By following the same steps, we will find that the solution. Utilizing the recurrence relation (5.82), (5.83) and the transformed initial conditions (5.84) -(5.89), the first few components of $U_{1/2,1/4}(j,h)$ are calculated. So, from equation (4.8), the approximate solution of the fuzzy linear space-time-fractional wave equations (5.80) and (5.81) can be derived as

$$\begin{aligned}
\underline{u}(x,t)(r) = r^n - &\left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_3 t^{5/2}\right)\\
+ r^n - &\left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_4 t^{5/2}\right) \cdot x^{1/4}\\
+ r^n - &\left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_5 t^{5/2}\right) \cdot x^{2/4}\\
+ r^n - &\left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_6 t^{5/2}\right) \cdot x^{3/4}\\
+ r^n - &\left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_7 t^{5/2}\right) \cdot x + \cdots, \qquad (5.96)
\end{aligned}$$

$$\begin{aligned}
\overline{u}(x,t)(r) = (2-r)^n - &\left(a_0 + b_0 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_3 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_3 t^{5/2}\right)\\
+ (2-r)^n - &\left(a_1 + b_1 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_4 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_4 t^{5/2}\right) \cdot x^{1/4}\\
+ (2-r)^n - &\left(a_2 + b_2 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_5 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_5 t^{5/2}\right) \cdot x^{2/4}\\
+ (2-r)^n - &\left(a_3 + b_3 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_6 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_6 t^{5/2}\right) \cdot x^{3/4}\\
+ (2-r)^n - &\left(a_4 + b_4 t + \frac{\Gamma(7/4)}{\Gamma(5/2)\Gamma(2/4)} a_7 t^{3/2} + \frac{\Gamma(7/4)}{\Gamma(7/2)\Gamma(2/4)} b_7 t^{5/2}\right) \cdot x + \cdots. \qquad (5.97)
\end{aligned}$$

6 Conclusions

In this paper, the differential transform method (DTM) has been successfully applied for solving fuzzy fractional wave equation. The proposed method is also illustrated by three examples. The new method is investigated based on the two-dimensional differential transform method, generalized Taylor's formula and fuzzy Caputo,s derivative. The results reveal that DTM is a highly effective scheme for obtaining analytical solutions of the fuzzy fractional wave equation.

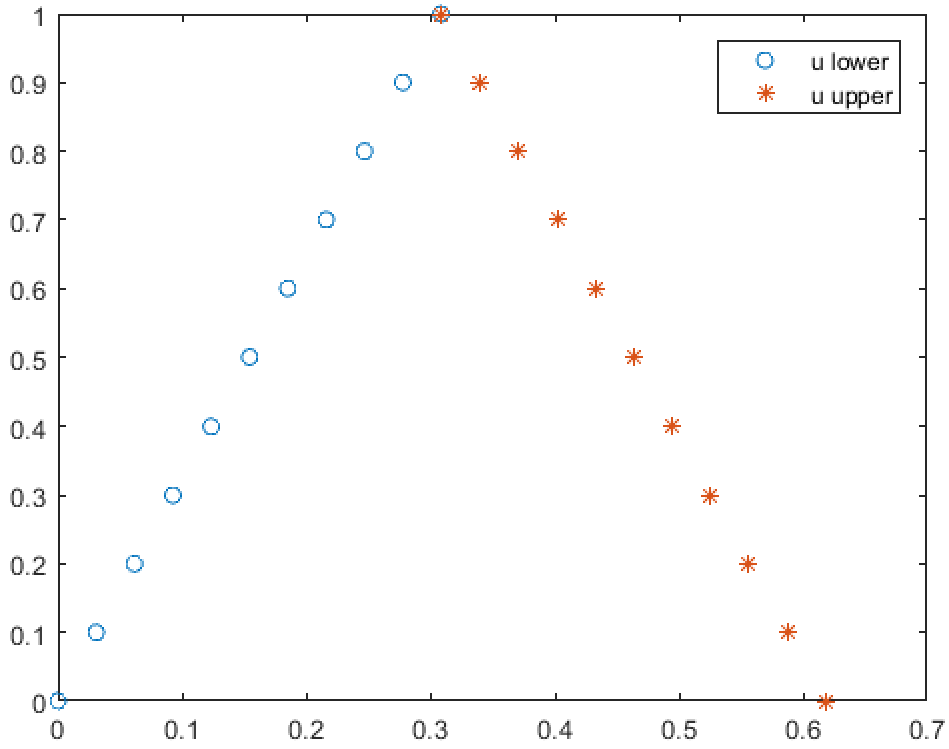

Figure 1: Example (5.1), Case (A), $t = 0.000001, x = 0.1, n = 1.$

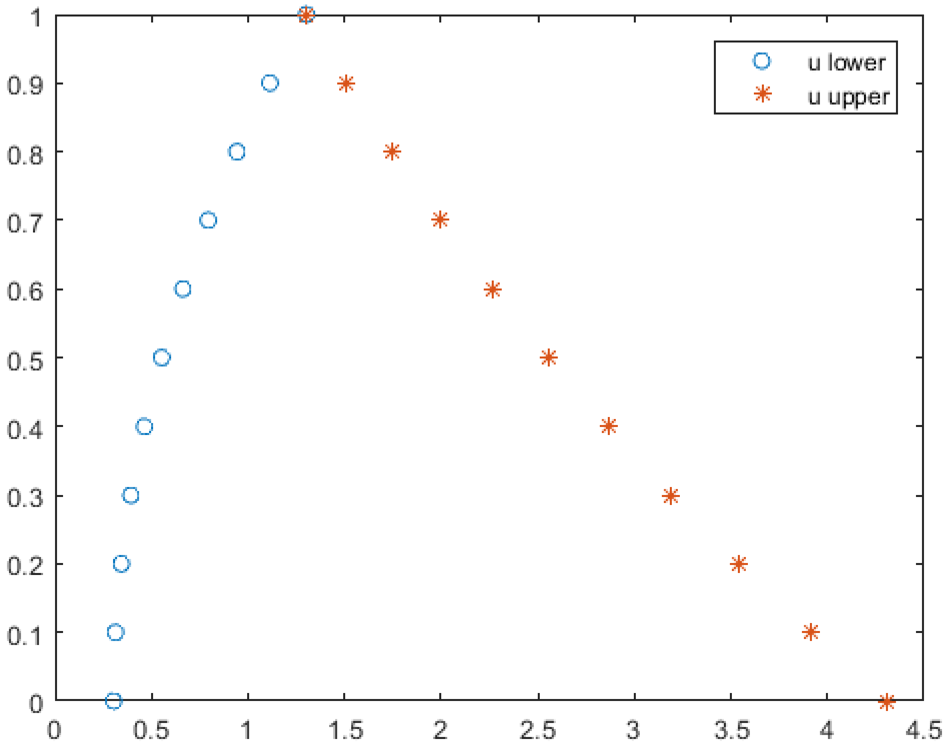

Figure 2: Example (5.1), Case (B), $t = 0.03, x = 0.1, n = 2$.

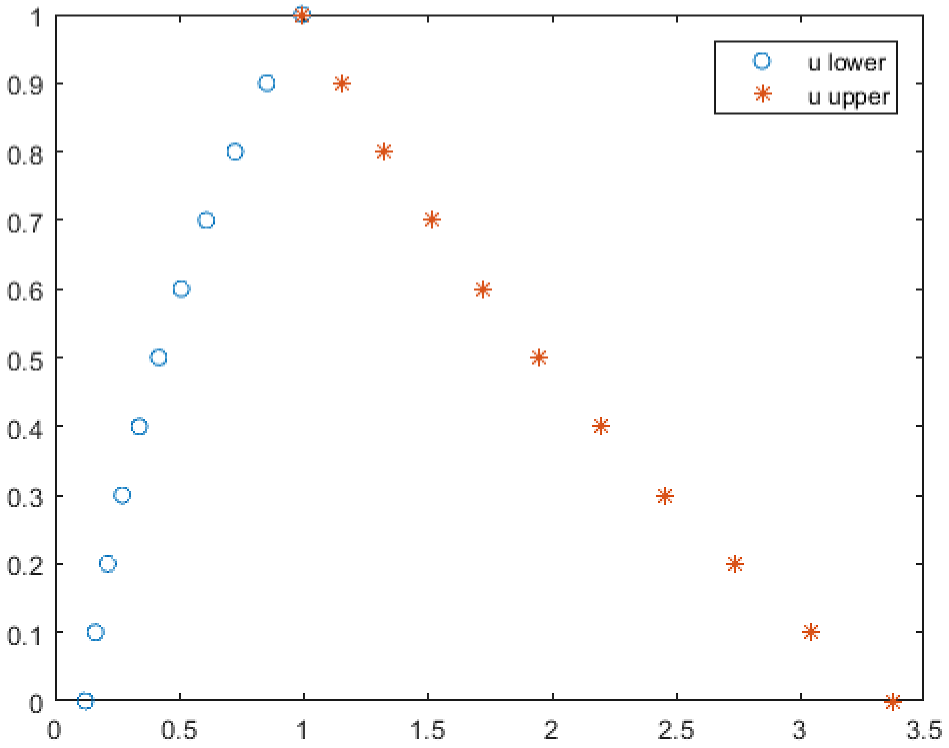

Figure 3: Example (5.1), Case (C), $t = 0.0001, x = 0.001, n = 3$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Some Generalized k-Fractional Integral Inequalities for Quasi-Convex Functions

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Abstract

Fractional integral operators generalize the concept of definite integration. Therefore these operators play a vital role in the advancement of subjects of sciences and engineering. The aim of this study is to establish the bounds of a generalized fractional integral operator via quasi-convex functions. These bounds behave as a formula in unified form, and estimations of almost all fractional integrals defined in last two decades can be obtained at once by choosing convenient parameters. Moreover, several related fractional integral inequalities are identified.

2010 Mathematics Subject Classification: 26A51, 26A33, 26D15

Key words and phrases: convex function, quasi-convex function, fractional integral operators, bounds

1 Introduction

A function $f : I \to \mathbb{R}$ is said to be *convex* if the following inequality holds:

$$f(ta + (1-t)b) \leq tf(a) + (1-t)f(b) \tag{1.1}$$

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for all $a, b \in I$ and $t \in [0, 1]$.

If inequality (1.1) is reversed, then the function f will be the concave on $[a, b]$. Convex functions are very useful in mathematical analysis. A lot of integral inequalities have been established due to convex functions in literature (for details see, [2–6, 10, 11, 18–20]. Quasi-convexity is also class of convex functions which is defined as follows:

Definition 1.1. ([10]) A function $f : I \to \mathbb{R}$ is said to be *quasi-convex* if the following inequality holds:

$$f(ta + (1-t)b) \leq \max\{f(a), f(b)\} \tag{1.2}$$

for all $a, b \in I$ and $t \in [0, 1]$.

Example 1.2. ([11, p. 83]) The function $f : [-2, 2] \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 1 & x \in [-2, -1] \\ x^2 & x \in (-1, 2] \end{cases}$$

is not a convex function on $[-2, 2]$ but it is quasi-convex function on $[-2, 2]$.

It is noted that class of quasi-convex functions contain the class of finite convex functions defined on finite closed intervals. For some recent citations and utilizations of quasi-convex functions one can see [2, 10, 11, 20] and references therein.

Fractional integral operators play an important role in generalizing the mathematical inequalities. In recent years, authors have proved various interesting mathematical inequalities due to different fractional integral operators, for example see [3–8 11, 15, 20]. The upcoming definitions and remark provide a detailed information of recent and classical fractional integral operators.

Definition 1.3. Let $f \in L_1[a, b]$ with $0 \leq a < b$. Then *Riemann-Liouville fractional integral operators* of order $\mu > 0$ are defined by

$${}^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t)dt, \quad x > a \tag{1.3}$$

and

$${}^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t)dt, \quad x < b, \tag{1.4}$$

where $\Gamma(\mu)$ is the Gamma function defined by $\Gamma(\mu) = \int_0^\infty t^{\mu-1} e^{-t} dt$.

Definition 1.4. ([16]) Let $f \in L_1[a, b]$ with $0 \leq a < b$. Then *Riemann-Liouville k-fractional integral operators* of order $\mu, k > 0$ are defined by

$${}^{\mu}I^k_{a^+}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (x-t)^{\frac{\mu}{k}-1} f(t)dt, \quad x > a \tag{1.5}$$

and

$${}^{\mu}I^k_{b^-}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (t-x)^{\frac{\mu}{k}-1} f(t)dt, \quad x < b, \tag{1.6}$$

where $\Gamma_k(\mu)$ is the k-Gamma function defined as $\Gamma_k(\mu) = \int_0^\infty t^{\mu-1} e^{-\frac{t^k}{k}} dt$.

Definition 1.5. ([14]) Let $f \in L_1[a,b]$ with $0 \leq a < b$. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). *The left-sided and right-sided fractional integrals of a function* f with respect to another function g on $[a,b]$ of order $\mu > 0$, are defined by

$$ {}_{g}^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)} \int_a^x (g(x)-g(t))^{\mu-1} g'(t) f(t)dt, \quad x > a \tag{1.7}$$

and

$$ {}_{g}^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)} \int_x^b (g(t)-g(x))^{\mu-1} g'(t) f(t)dt, \quad x < b. \tag{1.8}$$

Definition 1.6. ([15]) Let $f \in L_1[a,b]$ with $0 \leq a < b$. Also let g be an increasing and positive function on $(a,b]$, having a continuous derivative g' on (a,b). *The left-sided and right-sided fractional integrals of a function* f with respect to another function g on $[a,b]$ of order $\mu, k > 0$ are defined by

$$ {}_{g}^{\mu}I_{a^+}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1} g'(t) f(t)dt, \quad x > a \tag{1.9}$$

and

$$ {}_{g}^{\mu}I_{b^-}^{k}f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t)-g(x))^{\frac{\mu}{k}-1} g'(t) f(t)dt, \quad x < b. \tag{1.10}$$

These are compact formulas which give almost all fractional integrals by choosing suitable formations of function g. In this context the following remark is important:

Remark 1.7. Fractional integrals elaborated in (1.9) and (1.10) particularly produce several known fractional integrals corresponding to different settings of k and g.

(i) For $k = 1$ (1.9) and (1.10) fractional integrals coincide with (1.7) and (1.8).

(ii) For taking g as identity function (1.9) and (1.10) fractional integrals coincide with (1.5) and (1.6).

(iii) For $k = 1$, along with g as identity function (1.9) and (1.10) fractional integrals coincide with (1.3) and (1.4).

(iv) For $k = 1$ and $g(x) = \frac{x^\rho}{\rho}$, $\rho > 0$, (1.9) and (1.10) produce Katugampola fractional integrals defined by Chen et al. in [1].

(v) For $k = 1$ and $g(x) = \frac{x^{\tau+s}}{\tau+s}$, (1.9) and (1.10) produce generalized conformable fractional integrals defined by Khan et al. in [13].

(vi) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.9) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.10), then conformable (k,s)-fractional integrals are achieved as defined by Sidra et al. in [9].

(vii) If we take $g(x) = \frac{x^{1+s}}{1+s}$, then conformable fractional integrals are achieved as defined by Sarikaya et al. in [17].

(viii) If we take $g(x) = \frac{(x-a)^s}{s}$, $s > 0$ in (1.9) and $g(x) = -\frac{(b-x)^s}{s}$, $s > 0$ in (1.10) with $k = 1$, then conformable fractional integrals are achieved as defined by Jarad et al. in [12].

The rest of paper is organized as follows:

In Section 2, the bounds of sum of left-sided and right-sided generalized fractional integrals via quasi-convex function are established. First result provides an upper bound for generalized fractional integrals, and some particular cases are elaborated. Then bounds along with particular cases, in modulus form have been presented. Furthermore, Hadamarad type bounds are formulated. In Section 3, applications of results of Section 2 are given. Moreover concluding remarks are included at the end.

In the next sections the notation $M_a^b(f)=\max\{f(a),f(b)\}$ has been used frequently.

2 Main Results

Firstly, the following theorem set the formula for upper bounds of fractional integrals via quasi-convex functions in a unified form.

Theorem 2.1. *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be two functions such that g be differentiable and $f\in L[a,b]$ with $a<b$. Also let f be positive, quasi-convex and g be strictly increasing function with $g'\in L[a,b]$. Then for $x\in[a,b]$ and $\mu,\nu\geq k$, the following inequality holds:*

$$ {}_g^{\mu}I_{a^+}^{k}f(x)+{}_g^{\nu}I_{b^-}^{k}f(x)\leq\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}M_a^x(f)+\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}M_x^b(f). \tag{2.1}$$

Proof. As f is quasi-convex, therefore for $t\in[a,x]$, $f(t)\leq M_a^x(f)$. Under assumptions on function g, for all $x\in[a,b]$, $t\in[a,x]$ and $\mu\geq k$, the following inequality holds:

$$g'(t)(g(x)-g(t))^{\frac{\mu}{k}-1}\leq g'(t)(g(x)-g(a))^{\frac{\mu}{k}-1}. \tag{2.2}$$

From aforementioned two inequalities, the following integral inequality is yielded:

$$\int_a^x(g(x)-g(t))^{\frac{\mu}{k}-1}f(t)g'(t)dt\leq(g(x)-g(a))^{\frac{\mu}{k}-1}M_a^x(f)\int_a^x g'(t)dt. \tag{2.3}$$

By using (1.9) of Definition 1.6, the following bound of fractional integral defined in (1.9) is obtained:

$${}_g^{\mu}I_{a^+}^{k}f(x)\leq\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}M_a^x(f). \tag{2.4}$$

Again from quasi-convexity of f, for $t\in[x,b]$, $f(t)\leq M_x^b(f)$. Also for $x\in[a,b]$, $t\in[x,b]$ and $\nu\geq k$, the following inequality holds:

$$g'(t)(g(t)-g(x))^{\frac{\nu}{k}-1}\leq g'(t)(g(b)-g(x))^{\frac{\nu}{k}-1}. \tag{2.5}$$

From aforementioned two inequalities, the following integral inequality is yielded:

$$\int_x^b(g(t)-g(x))^{\frac{\nu}{k}-1}f(t)g'(t)dt\leq(g(b)-g(x))^{\frac{\nu}{k}-1}M_x^b(f)\int_x^b g'(t)dt. \tag{2.6}$$

By using (1.10) of Definition 1.6, the following bound of fractional integral defined in (1.10) is obtained:

$${}_g^{\nu}I_{b^-}^{k}f(x)\leq\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)}M_x^b(f). \tag{2.7}$$

From (2.4) and (2.7), the bound of sum of left-sided and right-sided fractional integrals is achieved. □

Special cases of Theorem 2.1, are discussed in the following corollaries.

Corollary 2.2. *If we take $\mu = \nu$ in* (2.1), *then we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) +_g^{\mu} I_{b^-}^{k} f(x) \leq \frac{1}{k\Gamma_k(\mu)} \left((g(x)-g(a))^{\frac{\mu}{k}} M_a^x(f) + (g(b)-g(x))^{\frac{\mu}{k}} M_x^b(f) \right). \quad (2.8) $$

Corollary 2.3. *If we take $k = 1$ in* (2.1), *then we get the following generalized* (*RL*) *fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+} f(x) +_g^{\nu} I_{b^-} f(x) \leq \frac{(g(x)-g(a))^{\mu}}{\Gamma(\mu)} M_a^x(f) + \frac{(g(b)-g(x))^{\nu}}{\Gamma(\nu)} M_x^b(f). \quad (2.9) $$

Corollary 2.4. *If we take $g(x) = x$ in* (2.1), *then we get the following* (*RL*) *k-fractional integral inequality:*

$$ {}^{\mu}I_{a^+}^{k} f(x) +^{\nu} I_{b^-}^{k} f(x) \leq \frac{(x-a)^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^x(f) + \frac{(b-x)^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_x^b(f). \quad (2.10) $$

Corollary 2.5. *If we take $g(x) = x$ and $k = 1$ in* (2.1), *then we get the following* (*RL*) *fractional integral inequality:*

$$ {}^{\mu}I_{a^+} f(x) +^{\nu} I_{b^-} f(x) \leq \frac{(x-a)^{\mu}}{\Gamma(\mu)} M_a^x(f) + \frac{(b-x)^{\nu}}{\Gamma(\nu)} M_x^b(f). \quad (2.11) $$

Corollary 2.6. *Under the assumptions of above theorem if f is increasing on* $[a,b]$, *then from* (2.1), *we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) +_g^{\nu} I_{b^-}^{k} f(x) \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} f(x) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} f(b). \quad (2.12) $$

Corollary 2.7. *Under the assumptions of above theorem if f is decreasing on* $[a,b]$, *then from* (2.1), *we get the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^{k} f(x) +_g^{\nu} I_{b^-}^{k} f(x) \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} f(a) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} f(x). \quad (2.13) $$

Next theorem provides the bound of generalized fractional integrals in modulus form.

Theorem 2.8. *Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be two differentiable functions with $a < b$. Also let $|f'|$ be quasi-convex and g be strictly increasing with $g' \in L[a,b]$. Then for $x \in [a,b]$ and $\mu, \nu, k > 0$, the following inequality holds:*

$$ \left| {}_g^{\mu}I_{a^+}^{k} f(x) +_g^{\nu} I_{b^-}^{k} f(x) - \left(\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)} f(a) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)} f(b) \right) \right| \quad (2.14) $$
$$ \leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)} M_a^x(|f'|) + \frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)} M_x^b(|f'|). $$

Proof. As $|f'|$ is quasi-convex, therefore for $t \in [a,x]$, we have

$$|f'(t)| \le M_a^x(|f'|). \tag{2.15}$$

From (2.15), we have

$$f'(t) \le M_a^x(|f'|). \tag{2.16}$$

Under assumptions of the function g, the following inequality holds:

$$(g(x)-g(t))^{\frac{\mu}{k}} \le (g(x)-g(a))^{\frac{\mu}{k}} \tag{2.17}$$

for all $x \in [a,b]$, $t \in [a,x]$ and $\mu, k > 0$.

From (2.16) and (2.17), we have

$$\int_a^x (g(x)-g(t))^{\frac{\mu}{k}} f'(t)dt \le (g(x)-g(a))^{\frac{\mu}{k}} M_a^x(|f'|) \int_a^x dt, \tag{2.18}$$

the left hand side calculate as follows:

$$\begin{aligned}
&\int_a^x (g(x)-g(t))^{\frac{\mu}{k}} f'(t)dt \\
&= f(t)(g(x)-g(t))^{\frac{\mu}{k}}|_a^x + \frac{\mu}{k}\int_a^x (g(x)-g(t))^{\frac{\mu}{k}-1} f(t)g'(t)dt \\
&= -f(a)(g(x)-g(a))^{\frac{\mu}{k}} + \Gamma_k(\mu+k) {}_g^{\mu}I_{a^+}^k f(x).
\end{aligned}$$

Using above calculation in (2.18), we get the following inequality:

$${}_g^{\mu}I_{a^+}^k f(x) - \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)} f(a) \le \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)} M_a^x(|f'|). \tag{2.19}$$

Also from (2.15), we can write

$$f'(t) \ge -M_a^x(|f'|). \tag{2.20}$$

Following the same procedure as we did for (2.16), we also have

$$\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)} f(a) -{}_g^{\mu} I_{a^+}^k f(x) \le \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)} M_a^x(|f'|). \tag{2.21}$$

From (2.19) and (2.21), we get the following modulus inequality:

$$\left| {}_g^{\mu}I_{a^+}^k f(x) - \frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)} f(a) \right| \le \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)} M_a^x(|f'|). \tag{2.22}$$

Again by using quasi-convexity of $|f'|$, for $t \in [x,b]$, we have

$$|f'(t)| \le M_x^b(|f'|). \tag{2.23}$$

Now for $x \in [a,b]$, $t \in [x,b]$ and $\nu, k > 0$, the following inequality holds:

$$(g(t)-g(x))^{\frac{\nu}{k}} \le (g(b)-g(x))^{\frac{\nu}{k}}. \tag{2.24}$$

By adopting the same way as we have done for (2.16), (2.17) and (2.20) one can get from (2.23) and (2.24) the following modulus inequality:

$$\left|{}_g^{\nu}I_{b^-}^{k}f(x)-\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right|\leq\frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}M_x^b(|f'|). \tag{2.25}$$

From (2.22) and (2.25) via triangular inequality, we get the modulus inequality in (2.14), which is required. □

Special cases of Theorem 2.8, are discussed in the following corollaries.

Corollary 2.9. *If we take $\mu=\nu$ in* (2.14), *then we get the following fractional integral inequality:*

$$\begin{aligned}&\left|{}_g^{\mu}I_{a^+}^{k}f(x)+{}_g^{\mu}I_{b^-}^{k}f(x)-\frac{1}{\Gamma_k(\mu+k)}\left((g(x)-g(a))^{\frac{\mu}{k}}f(a)+(g(b)-g(x))^{\frac{\mu}{k}}f(b)\right)\right| \qquad (2.26)\\&\leq\frac{1}{\Gamma_k(\mu+k)}\left((g(x)-g(a))^{\frac{\mu}{k}}(x-a)M_a^x(|f'|)+(g(b)-g(x))^{\frac{\mu}{k}}(b-x)M_x^b(|f'|)\right).\end{aligned}$$

Corollary 2.10. *If we take $k=1$ in (2.14), then we get the following generalized (RL) fractional integral inequality:*

$$\begin{aligned}&\left|{}_g^{\mu}I_{a^+}f(x)+{}_g^{\nu}I_{b^-}f(x)-\left(\frac{(g(x)-g(a))^{\mu}}{\Gamma(\mu+1)}f(a)+\frac{(g(b)-g(x))^{\nu}}{\Gamma(\nu+1)}f(b)\right)\right| \qquad (2.27)\\&\leq\frac{(g(x)-g(a))^{\mu}(x-a)}{\Gamma(\mu+1)}M_a^x(|f'|)+\frac{(g(b)-g(x))^{\nu}(b-x)}{\Gamma(\nu+1)}M_x^b(|f'|).\end{aligned}$$

Corollary 2.11. *If we take $g(x)=x$ in* (2.14), *then we get the following* (RL) *k-fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}I_{a^+}^{k}f(x)+{}^{\nu}I_{b^-}^{k}f(x)-\left(\frac{(x-a)^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(b-x)^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \qquad (2.28)\\&\leq\frac{(x-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}M_a^x(|f'|)+\frac{(b-x)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}M_x^b(|f'|).\end{aligned}$$

Corollary 2.12. *If we take $g(x)=x$ and $k=1$ in* (2.14), *then we get the following* (RL) *fractional integral inequality:*

$$\begin{aligned}&\left|{}^{\mu}I_{a^+}f(x)+{}^{\nu}I_{b^-}f(x)-\left(\frac{(x-a)^{\mu}}{\Gamma(\mu+1)}f(a)+\frac{(b-x)^{\nu}}{\Gamma(\nu+1)}f(b)\right)\right| \qquad (2.29)\\&\leq\frac{(x-a)^{\mu+1}}{\Gamma(\mu+1)}M_a^x(|f'|)+\frac{(b-x)^{\nu+1}}{\Gamma(\nu+1)}M_x^b(|f'|).\end{aligned}$$

Corollary 2.13. *Under the assumptions of above theorem if $|f'|$ is increasing on $[a,b]$, then from* (2.14), *we get the following fractional integral inequality:*

$$\begin{aligned}&\left|{}_g^{\mu}I_{a^+}^{k}f(x)+{}_g^{\nu}I_{b^-}^{k}f(x)-\left(\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \qquad (2.30)\\&\leq\frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)}|f'(x)|+\frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}|f'(b)|.\end{aligned}$$

Corollary 2.14. *Under the assumptions of above theorem if $|f'|$ is decreasing on $[a,b]$, then from (2.14), we get the following fractional integral inequality:*

$$\begin{aligned}&\left|{}_g^{\mu}I_{a^+}^{k}f(x)+{}_g^{\nu}I_{b^-}^{k}f(x)-\left(\frac{(g(x)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)+\frac{(g(b)-g(x))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \qquad (2.31)\\ &\leq \frac{(g(x)-g(a))^{\frac{\mu}{k}}(x-a)}{\Gamma_k(\mu+k)}|f'(a)|+\frac{(g(b)-g(x))^{\frac{\nu}{k}}(b-x)}{\Gamma_k(\nu+k)}|f'(x)|.\end{aligned}$$

We need the following lemma in the proof of next result.

Lemma 2.15. *Let $f:[0,\infty)\to R$ be a quasi-convex function. If $f(x)=f(a+b-x)$, then for $x\in[a,b]$, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq f(x). \qquad (2.32)$$

Proof. We have

$$\frac{a+b}{2}=\frac{1}{2}\left(\frac{x-a}{b-a}b+\frac{b-x}{b-a}a\right)+\frac{1}{2}\left(\frac{x-a}{b-a}a+\frac{b-x}{b-a}b\right). \qquad (2.33)$$

As f is quasi-convex, therefore for $x\in[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\leq \max\{f(x),f(a+b-x)\}. \qquad (2.34)$$

Using given condition $f(x)=f(a+b-x)$ in (2.34), then inequality in (2.32) is established. □

Theorem 2.16. *Let $f,g:[a,b]\longrightarrow\mathbb{R}$ be two functions such that g be differentiable and $f\in L[a,b]$ with $a<b$. Also let f be positive, quasi-convex, $f(x)=f(a+b-x)$ and g be strictly increasing with $g'\in L[a,b]$. Then for $x\in[a,b]$ and $\mu,\nu,k>0$, the following inequalities hold:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \qquad (2.35)\\ &\leq\ {}_g^{\nu+k}I_{b^-}^{k}f(a)+{}_g^{\mu+k}I_{a^+}^{k}f(b)\\ &\leq\frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]M_a^b(f).\end{aligned}$$

Proof. As f is quasi-convex, therefore for $x\in[a,b]$, we have

$$f(x)\leq M_a^b(f). \qquad (2.36)$$

Under assumptions of the function g, the following inequality holds:

$$g'(x)(g(x)-g(a))^{\frac{\nu}{k}}\leq g'(x)(g(b)-g(a))^{\frac{\nu}{k}} \qquad (2.37)$$

for all $x\in[a,b]$ and $\nu,k>0$.

From (2.36) and (2.37), we have

$$\int_a^b (g(x)-g(a))^{\frac{\nu}{k}} f(x)g'(x)dx \leq (g(b)-g(a))^{\frac{\nu}{k}} M_a^b(f)\int_a^b g'(x)dx.$$

By using (1.10) of Definition 1.6, we get

$$_g^{\nu+k}I_{b^-}^k f(a) \leq \frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{k\Gamma_k(\nu+k)} M_a^b(f). \tag{2.38}$$

Now for $x \in [a,b]$ and $\mu, k > 0$, the following inequality inequality holds:

$$g'(x)(g(b)-g(x))^{\frac{\mu}{k}} \leq g'(x)(g(b)-g(a))^{\frac{\mu}{k}}. \tag{2.39}$$

From (2.36) and (2.39), we have

$$\int_a^b (g(b)-g(x))^{\frac{\mu}{k}} f(x)g'(x)dx \leq (g(b)-g(a))^{\frac{\mu}{k}} M_a^b(f)\int_a^b g'(x)dx.$$

By using (1.9) of Definition 1.6, we get

$$_g^{\mu+k}I_{a^+}^k f(b) \leq \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{k\Gamma_k(\mu+k)} M_a^b(f). \tag{2.40}$$

Adding (2.38) and (2.40), we get the following inequality

$$_g^{\nu+k}I_{b^-}^k f(a) +_g^{\mu+k} I_{a^+}^k f(b) \leq \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)} + \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] M_a^b(f). \tag{2.41}$$

Now on the other hand multiplying (2.32) with $(g(x)-g(a))^{\frac{\nu}{k}}g'(x)$, then integrating over $[a,b]$, we have

$$f\left(\frac{a+b}{2}\right)\int_a^b (g(x)-g(a))^{\frac{\nu}{k}} g'(x)dx \leq \int_a^b (g(x)-g(a))^{\frac{\nu}{k}} g'(x)f(x)dx. \tag{2.42}$$

By using (1.10) of Definition 1.6, we get

$$\frac{k(g(b)-g(a))^{\frac{\nu}{k}+1}}{\nu+k} f\left(\frac{a+b}{2}\right) \leq k\Gamma_k(\nu+k)_g^{\nu+k}I_{b^-}^k f(a). \tag{2.43}$$

Similarly, multiplying (2.32) with $(g(b)-g(x))^{\frac{\mu}{k}}g'(x)$, then integrating over $[a,b]$, we have

$$\frac{k(g(b)-g(a))^{\frac{\mu}{k}+1}}{\mu+k} f\left(\frac{a+b}{2}\right) \leq k\Gamma_k(\mu+k)_g^{\mu+k}I_{a^+}^k f(b). \tag{2.44}$$

Adding (2.43) and (2.44), we get the following inequality

$$f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)} + \frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \leq_g^{\nu+k} I_{b^-}^k f(a) +_g^{\mu+k} I_{a^+}^k f(b). \tag{2.45}$$

From (2.41) and (2.45), we get the inequalities in (2.35), which is required. □

Special cases of Theorem 2.16, are discussed in the following corollaries.

Corollary 2.17. *If we take* $\mu=\nu$ *in* (2.35), *then we get the following fractional integral inequality:*

$$\begin{aligned}2f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] &\leq {}_g^{\mu+k}I_{b^-}^k f(a)+{}_g^{\mu+k}I_{a^+}^k f(b)\\ &\leq \frac{2}{k}\left[\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]M_a^b(f).\end{aligned}$$

Corollary 2.18. *If we take* $k=1$ *in* (2.35), *then we get the following generalized* (RL) *fractional integral inequality:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\nu+1}}{\Gamma(\nu+2)}+\frac{(g(b)-g(a))^{\mu+1}}{\Gamma(\mu+2)}\right]\\ &\leq {}_g^{\nu+1}I_{b^-}f(a)+{}_g^{\mu+1}I_{a^+}f(b)\\ &\leq \left[\frac{(g(b)-g(a))^{\nu+1}}{\Gamma(\nu+1)}+\frac{(g(b)-g(a))^{\mu+1}}{\Gamma(\mu+1)}\right]M_a^b(f).\end{aligned}\tag{2.46}$$

Corollary 2.19. *If we take* $g(x)=x$ *in* (2.35), *then we get the following* (RL) *k-fractional integral inequality:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(b-a)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(b-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right]\\ &\leq {}^{\nu+k}I_{b^-}^k f(a)+{}^{\mu+k}I_{a^+}^k f(b)\\ &\leq \frac{1}{k}\left[\frac{(b-a)^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(b-a)^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]M_a^b(f).\end{aligned}\tag{2.47}$$

Corollary 2.20. *If we take* $g(x)=x$ *and* $k=1$ *in* (2.35), *then we get the following* (RL) *fractional integral inequality:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(b-a)^{\nu+1}}{\Gamma(\nu+2)}+\frac{(b-a)^{\mu+1}}{\Gamma(\mu+2)}\right]\\ &\leq {}^{\nu+1}I_{b^-}f(a)+{}^{\mu+1}I_{a^+}f(b)\\ &\leq \left[\frac{(b-a)^{\nu+1}}{\Gamma(\nu+1)}+\frac{(b-a)^{\mu+1}}{\Gamma(\mu+1)}\right]M_a^b(f).\end{aligned}\tag{2.48}$$

Corollary 2.21. *Under the assumptions of above theorem if* f *is increasing on* $[a,b]$, *then from* (2.35), *we get the following fractional integral inequality:*

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right]\\ &\leq {}_g^{\nu+k}I_{b^-}^k f(a)+{}_g^{\mu+k}I_{a^+}^k f(b)\\ &\leq \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right]f(b).\end{aligned}\tag{2.49}$$

Corollary 2.22. *Under the assumptions of above theorem if f is decreasing on $[a,b]$, then from* (2.35), *we get the following fractional integral inequality:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+2k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+2k)}\right] \\ \le\ & {}_g^{\nu+k}I_{b^-}^k f(a) +_g^{\mu+k} I_{a^+}^k f(b) \\ \le\ & \frac{1}{k}\left[\frac{(g(b)-g(a))^{\frac{\nu}{k}+1}}{\Gamma_k(\nu+k)}+\frac{(g(b)-g(a))^{\frac{\mu}{k}+1}}{\Gamma_k(\mu+k)}\right] f(a). \end{aligned} \tag{2.50}$$

3 Applications

In this section we give applications of the results proved in the previous section. First we apply Theorem 2.1 and get the following result.

Theorem 3.1. *Under the assumptions of Theorem 2.1, we have the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^k f(b) +_g^{\nu} I_{b^-}^k f(a) \le \frac{1}{k}\left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu)}+\frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu)}\right) M_a^b(f). \tag{3.1}$$

Proof. If we put $x=a$ in (2.1), then we have

$$ {}_g^{\nu}I_{b^-}^k f(a) \le \frac{((g(b)-g(a))^{\frac{\nu}{k}}}{k\Gamma_k(\nu)} M_a^b(f). \tag{3.2}$$

If we put $x=b$ in (2.1), then we have

$$ {}_g^{\mu}I_{a^+}^k f(b) \le \frac{(g(b)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^b(f). \tag{3.3}$$

Adding inequalities (3.2) and (3.3), we get (3.1). □

Special cases of Theorem 3.1, are discussed in the following corollaries.

Corollary 3.2. *If we take $\mu=\nu$ in* (3.1), *then we have the following fractional integral inequality:*

$$ {}_g^{\mu}I_{a^+}^k f(b) +_g^{\mu} I_{b^-}^k f(a) \le \frac{2(g(b)-g(a))^{\frac{\mu}{k}}}{k\Gamma_k(\mu)} M_a^b(f). \tag{3.4}$$

Corollary 3.3. ([2]) *If we take $\mu=k=1$ and $g(x)=x$ in* (3.4), *then we get the following inequality:*

$$\frac{1}{b-a}\int_a^b f(t)dt \le M_a^b(f). \tag{3.5}$$

Next we apply Theorem 2.8 to obtain required results.

Theorem 3.4. *Under the assumptions of Theorem 2.8, we have the following fractional integral inequality:*

$$\left|{}_g^{\mu}I_{a^+}^k f(b) +{}_g^{\nu} I_{b^-}^k f(a) - \left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a) + \frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right)\right| \tag{3.6}$$
$$\leq \left(\frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)} + \frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}\right)(b-a)M_a^b(|f'|).$$

Proof. If we put $x=a$ in (2.14), then we have

$$\left|{}_g^{\nu}I_{b^-}^k f(a) - \frac{(g(b)-g(a))^{\frac{\nu}{k}}}{\Gamma_k(\nu+k)}f(b)\right| \leq \frac{(g(b)-g(a))^{\frac{\nu}{k}}(b-a)}{\Gamma_k(\nu+k)}M_a^b(|f'|). \tag{3.7}$$

If we put $x=b$ in (2.14), then we have

$$\left|{}_g^{\mu}I_{a^+}^k f(b) - \frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}f(a)\right| \leq \frac{(g(b)-g(a))^{\frac{\mu}{k}}(b-a)}{\Gamma_k(\mu+k)}M_a^b(|f'|). \tag{3.8}$$

Adding inequalities (3.7) and (3.8), we get (3.6). ☐

Special cases of Theorem 3.4, are discussed in the following corollaries.

Corollary 3.5. *If we take $\mu=\nu$ in* (3.6), *then we have the following fractional integral inequality:*

$$\left|{}_g^{\mu}I_{a^+}^k f(b) +{}_g^{\mu} I_{b^-}^k f(a) - \frac{(g(b)-g(a))^{\frac{\mu}{k}}}{\Gamma_k(\mu+k)}\left(f(a)+f(b)\right)\right| \tag{3.9}$$
$$\leq \frac{2(g(b)-g(a))^{\frac{\mu}{k}}(b-a)}{\Gamma_k(\mu+k)}M_a^b(|f'|).$$

Corollary 3.6. *If we take $\mu=k=1$ and $g(x)=x$ in* (3.9), *then we get the following inequality:*

$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \leq (b-a)M_a^b(|f'|). \tag{3.10}$$

By applying Theorem 2.16 similar relations can be established we leave it for the reader.

4 Concluding Remarks

The aim of this study is to explore bounds of fractional integrals in a compact form by using the concept of quasi-convexity. The authors are succeeded in the formulation of bounds of generalized fractional integrals (1.9) and (1.10). Theorem 2.1 provides upper bounds, Theorem 2.8 gives bounds in modulus form while Theorem 2.16 formulates bounds of Hadamard type. Section 3 consists of the applications of these bounds. Also some particular case of all these results are shown. Remark 1.7 includes all possible fractional integrals associated with generalized fractional integrals (1.9) and (1.10). The readers can obtain bounds for desired fractional integrals by putting the corresponding function g from Remark 1.7.

Acknowledgement

This research work is supported by Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Asymptotically Almost Automorphic Mild Solutions for Second Order Nonautonomous Semilinear Evolution Equations

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Abstract

The aim of this paper is to study the existence of asymptotically almost automorphic mild solution to some classes of second order semilinear evolution equation via the techniques of measure of noncompactness. The investigation is based on a new fixed point result which is a generalization of the well known Darbo's fixed point theorem. Finally examples are given to illustrate the analytical findings.

Key words: Asymptotically almost automorphic, second order nonautonomous differential equations, mild solution, evolution system, Kuratowski measures of noncompactness, fixed point.
AMS Subject Classification : 34G20

1 Introduction

This work is mainly concerned with the existence of asymptotically almost automorphic mild solution for second differential equations. More precisely, we will consider the following problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in \mathbb{R}^+ := [0, +\infty), \tag{1}$$

$$y(0) = y_0,\ y'(0) = y_1, \tag{2}$$

where $\{A(t)\}_{t\in\mathbb{R}^+}$ is a family of linear closed operators from E into E that generate an evolution system of linear bounded operators $\{\mathcal{U}(t,s)\}_{(t,s)\in\mathbb{R}^+\times\mathbb{R}^+}$ for $0 \leq s \leq t < +\infty$, $f : \mathbb{R}^+ \times E \to E$ is a Carathéodory function, and $(E, |\cdot|)$ is a real Banach space.

Evolution equations arise in many areas of applied mathematics [2, 37]. This type of equations has received much attention in recent years [1]. There are many results concerning the second-order differential equations, see for example [8, 11, 12, 20, 28, 35]. In recent years there has been an increasing interest in studying the abstract non-autonomous second order initial value problem

$$y''(t) - A(t)y(t) = f(t, y(t)),\ t \in [0, T], \tag{3}$$

$$y(0) = y_0,\ y'(0) = y_1. \tag{4}$$

The reader is referred to [10, 19, 22, 36] and the references therein. In the above mentioned works, the existence of solutions to the problem (3)-(4) is related to the existence of an evolution operator $\mathcal{U}(t; s)$ for the homogeneous equation

$$y''(t) = A(t)y(t), \quad \text{for} \quad t \geq 0.$$

For this purpose there are many techniques to show the existence of $\mathcal{U}(t, s)$ which has been developed by Kozak [25].

On the other hand, since Bochner [13] introduced the concept of almost automorphy, the automorphic functions have been applied to many areas including ordinary as well as partial differential equations, abstract differential equations, functional differential equations, integral equations, etc.; see [16, 21, 18, 27, 7]. We also refer the reader to the monographs by N'Guérékata [30, 31] for the basic theory of almost automorphic functions and applications. The concept of asymptotically almost automorphy was introduced by N'Guérékata [29]. Since then, these functions have generated lot of developments and applications, see [39, 14, 24, 17] and the references therein. In the previous works, people have established the existence of asymptotically almost automorphic mild solution of differential equations under the conditions that f satisfies or not the Lipschitz condition.

In this paper we use the technique of measures of noncompactness. It is well known that this method provides an excellent tool for obtaining existence of solutions of nonlinear differential equation. This technique works fruitfully for both integral and differential equations. More details are found in Aissani and Benchohra [3], Akhmerov *et al.* [4], Alv́ares [5], Banaś and Goebel [9], Olszowy and Wędrychowicz [33], Olszowy [34], and the references therein.

Inspired by the above works,, in this work, using the properties of the analytic semigroups, Kuratowski measure of noncompactness, fixed point theorem, we obtain an existence result without assuming that the nonlinearity f satisfies a Lipschitz type condition.

This work is organized of as follows. In Section 2, we recall some fundamental properties of asymptotically almost automorphic and facts about evolution systems. Section 3 is devoted to establishing some criteria the existence of asymptotically almost automorphic mild solutions to the problem (1)-(2). Furthermore, appropriate examples are provided in section 4 to show the feasibility of our results.

2 Preliminaries and basic results

In this section we recall certain definitions and lemmas to be used subsequently in this paper.
Throughout this paper, we denote by E a Banach space with the norm $|\cdot|$. Let $BC(\mathbb{R}^+,\mathbb{E})$ be the Banach space of all bounded and continuous functions y mapping $\mathbb{R}^+$ into E endowed with the usual supremum norm

$$\|y\|_\infty = \sup_{t\in\mathbb{R}^+} |y(t)|.$$

In what follows, let $\{A(t),\ t\in\mathbb{R}^+\}$ be a family of closed linear operators on the Banach space E with domain $D(A(t))$ which is dense in E and independent of t.

In this work the existence of solution the problem (1)-(2) is related to the existence of an evolution operator $\mathcal{U}(t,s)$ for the following homogeneous problem

$$y''(t) = A(t)y(t) \qquad t\in\mathbb{R}^+. \tag{5}$$

This concept of evolution operator has been developed by Kozak [25] and recently used by Henríquez *et al.* [22].

Definition 2.1 *A family $\mathcal{U}$ of bounded operators $\mathcal{U}(t,s) : E \to E$, $(t,s) \in \Delta := \{(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ : s \leq t\}$, is called an evolution operator of the equation (5) if de following conditions hold:*

(e_1) *For any $x \in E$ the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is continuously differentiable and*

- **(a)** *for each $t \in \mathbb{R}$, $\mathcal{U}(t,t)x = 0, \forall x \in E$,*
- **(b)** *for all $(t,s) \in \Delta$ and for any $x \in E$, $\frac{\partial}{\partial t}\mathcal{U}(t,s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathcal{U}(t,s)x|_{t=s} = -x$.*

(e_2) *For all $(t,s) \in \Delta$, if $x \in D(A(t))$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, the map $(t,s) \longmapsto \mathcal{U}(t,s)x$ is of class C^2 and*

- **(a)** $\frac{\partial^2}{\partial t^2}\mathcal{U}(t,s)x = A(t)\mathcal{U}(t,s)x$,
- **(b)** $\frac{\partial^2}{\partial s^2}\mathcal{U}(t,s)x = \mathcal{U}(t,s)A(s)x$,
- **(c)** $\frac{\partial^2}{\partial s \partial t}\mathcal{U}(t,s)x|_{t=s} = 0$.

(e_3) *For all $(t,s) \in \Delta$, then $\frac{\partial}{\partial s}\mathcal{U}(t,s)x \in D(A(t))$, there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x$ and*

- **(a)** $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{U}(t,s)x = A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$.
  Moreover, the map $(t,s) \longmapsto A(t)\frac{\partial}{\partial s}(t)\mathcal{U}(t,s)x$ is continuous,
- **(b)** $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{U}(t,s)x = \frac{\partial}{\partial t}\mathcal{U}(t,s)A(s)x$.

Throughout this paper, we will use the following definition of the concept of Kuratowski measure of noncompactness [9].

Definition 2.2 *The Kuratowski measure of noncompactness α is defined by*

$$\alpha(D) = \inf\{r > 0 : D \textit{ has a finite cover by sets of diameter} \leq r\},$$

for a bounded set D in any Banach space E.

Let us recall the basic properties of Kuratowski measure of noncompactness.

Lemma 2.3 *[9] Let E be a Banach space and $C, D \subset E$ be bounded, then the following properties hold:*

- (i_1) $\alpha(D) = 0$ *if only if D is relatively compact,*
- (i_2) $\alpha(\overline{D}) = \alpha(D)$ *; $\overline{D}$ the closure of D,*
- (i_3) $\alpha(C) \leq \alpha(D)$ *when $C \subset D$,*
- (i_4) $\alpha(C+D) \leq \alpha(C)+\alpha(D)$ *where* $C+D = \{x \mid x = y + z; y \in C; z \in D\}$,
- (i_5) $\alpha(aD) = |a|\alpha(D)$ *for any $a \in \mathbb{R}$,*
- (i_6) $\alpha(ConvD) = \alpha(D)$*, where $ConvD$ is the convex hull of D,*
- (i_7) $\mu(C \cup D) = \max(\alpha(C), \alpha(D))$*,*
- (i_8) $\alpha(C \cup \{x\}) = \alpha(C)$ *for any $x \in E$.*

Denote by $\omega^T(y, \varepsilon)$ the modulus of continuity of y on the interval $[0, T]$ i.e.

$$\omega^T(y, \varepsilon) = \sup\left\{|y(t) - y(s)|\,; t, s \in [0, T], |t - s| \leq \varepsilon\right\}.$$

Moreover, let us put

$$\omega^T(D, \varepsilon) = \sup\left\{\omega^T(y, \varepsilon); y \in D\right\},$$

$$\omega_0^T(D) = \lim_{\varepsilon\to 0} \omega^T(D, \varepsilon).$$

Lemma 2.4 *[15] Let E be a Banach space, $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$, such that*

$$\alpha(D) \leq 2\alpha(D_0).$$

Lemma 2.5 *[23] Let $D = \{y_n\}_{n=0}^{+\infty} \subset C(\mathbb{R}^+, E)$ be a bounded and countable set. Then $\alpha(D(t))$ is Lebesgue integrable on $\mathbb{R}^+$, and*

$$\alpha\left\{\int_0^t y_n(s))ds\right\}_{n=0}^{\infty} \leq 2\int_0^t \alpha(D(s))ds, \qquad t \in \mathbb{R}^+.$$

Now, we recall some basic definitions and results on almost automorphic functions and asymptotically almost automorphic functions (for more details, see [13, 31, 38]).

Definition 2.6 *A continuous function $f : \mathbb{R} \to E$ is said to be almost automorphic if for every sequence of real numbers $\{\tau_n'\}$, there exists a subsequence $\{\tau_n\}$ such that*

$$g(t) = \lim_{n\to\infty} f(t + \tau_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - \tau_n) = f(t) \quad \textit{for each } t \in \mathbb{R}.$$

Denote by $AA(\mathbb{R}, E)$ the set of all such functions.

Lemma 2.7 *[30] $AA(\mathbb{R}, E)$ is a Banach space with the supremum norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|.$$

Definition 2.8 *A continuous function $f : \mathbb{R} \times E \to E$ is said to be almost automorphic in $t \in \mathbb{R}$ for each $y \in E$ if for every sequence of real numbers $\{\tau_n'\}$, there exists a subsequence $\{\tau_n\}$ such that*

$$\lim_{n\to\infty} f(t + \tau_n, y) = g(t, y)$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - \tau_n, y) = f(t, y)$$

for each $t \in \mathbb{R}$ and each $y \in E$. The collection of those functions is denoted by $AA(\mathbb{R} \times E, E)$.

Example 2.9 *[40] The function $f : \mathbb{R} \times E \to E$ given by*

$$f(t, y) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right)\cos y$$

is almost automorphic in $t \in \mathbb{R}$ for each $y \in E$, where $E = L^2([0, 1])$.

The space of all continuous functions $h : \mathbb{R}^+ \to E$ such that $\lim_{t\to\infty} h(t) = 0$ is denoted by $C_0(\mathbb{R}^+, E)$. Moreover, we denote $C_0(\mathbb{R}^+ \times E, E)$; the space of all continuous functions from $\mathbb{R} \times E$ to E satisfying $\lim_{t\to\infty} h(t, y) = 0$ in t and uniformly in $y \in E$.

Remark 2.10 *Note that if* $\nu(t) \in C_0(\mathbb{R}^+, E)$*, then*

$$\int_0^t e^{-(t-s)}\nu(s)ds \in C_0(\mathbb{R}^+, E).$$

Definition 2.11 *A continuous function* $f : \mathbb{R}^+ \to E$ *is said to be asymptotically almost automorphic if it can be decomposed as*

$$f(t) = g(t) + h(t),$$

where

$$g(t) \in AA(\mathbb{R}, E),\ \ h(t) \in C_0(\mathbb{R}^+, E).$$

Denote by $AAA(\mathbb{R}^+, E)$ *the set of all such functions.*

Example 2.12 *The function* $f : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f(t) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right) + e^{-t}$$

is an asymptotically almost automorphic function with

$$g(t) = \sin\left(\frac{1}{2 + \cos t + \cos\sqrt{2}t}\right) \in AA(\mathbb{R}, \mathbb{R}), \quad h(t) = e^{-t} \in C_0(\mathbb{R}^+, \mathbb{R}).$$

Lemma 2.13 *[31],[32].* $AAA(\mathbb{R}^+, E)$ *is also a Banach space with the norm*

$$\|f\|_\infty = \sup_{t\in\mathbb{R}^+} |f(t)|.$$

Definition 2.14 *A continuous function* $f : \mathbb{R}^+ \times E \to E$ *is said to be asymptotically almost automorphic if it can be decomposed as*

$$f(t, y) = g(t, y) + h(t, y),$$

where

$$g(t, y) \in AA(\mathbb{R} \times E, E), \quad h(t, y) \in C_0(\mathbb{R}^+ \times E, E).$$

Denote by $AAA(\mathbb{R}^+ \times E, E)$ *the set of all such functions.*

Example 2.15 *The function $f : \mathbb{R}^+ \times E \to E$ given by*

$$f(t,x) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\cos y + e^{-t}|y|$$

is asymptotically almost automorphic in $t \in \mathbb{R}^+$ for each $y \in E$, where $E = L^2([0,1])$.

$$g(t,y) = \sin\left(\frac{1}{2+\cos t+\cos\sqrt{2}t}\right)\cos y \in AA(\mathbb{R}\times E, E),$$
$$h(t,y) = e^{-t}|y| \in C_0(\mathbb{R}^+\times E, E).$$

.

Lemma 2.16 *[26] $f : \mathbb{R}\times E \to E$ is almost automorphic, and assume that $f(t,\cdot)$ is uniformly continuous on each bounded subset $K \subset E$ uniformly for $t \in \mathbb{R}$, that is for any $\varepsilon > 0$, there exists $\varrho > 0$ such that $y, z \in K$ and $|y(t) - z(t)| < \varrho$ imply that $|f(t,y) - f(t,z)| < \varepsilon$ for all $t \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to E$ be almost automorphic. Then the function $F : \mathbb{R} \to E$ defined by $F(t) = f(t,\varphi(t))$ is almost automorphic.*

Theorem 2.17 *[6] Let Ω be a nonempty, bounded, closed and convex subset of a Banach space E, and let $\Gamma : \Omega \to \Omega$ be a continuous operator satisfying the inequality*

$$\alpha(\Gamma(D)) \le \Psi(\alpha(D))$$

for any nonempty subset D of Ω, where $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$is a nondecreasing function such that

$$\lim_{n\to+\infty} \Psi^n(t) = 0 \text{ for each } t \ge 0.$$

Then Γ has at least one fixed point in the set Ω.

3 Main results

Definition 3.1 *A function $y \in BC(\mathbb{R}^+, E)$ is said to be a mild solution to the problem (1)-(2) if y satisfies the integral equation*

$$y(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds.$$

For the proof of our main theorem, we need the following hypotheses:

(H_1) (a) There exists a constant $M \geq 1$ and $\delta > 0$, such that

$$\|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} \leq Me^{-\delta(t-s)} \quad \text{for any } (t,s) \in \Delta$$

and for any sequence of real numbers $\{\tau_n'\}$, we can extract a subsequence $\{\tau_n\}$ and for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|U(t+\tau_n, s+\tau_n) - U(t,s)\|_{\mathcal{B}(E)} \leq \varepsilon e^{-\delta(t-s)},$$

$$\|U(t-\tau_n, s-\tau_n) - U(t,s)\|_{\mathcal{B}(E)} \leq \varepsilon e^{-\delta(t-s)}$$

for each $t, s \in \mathbb{R}$. for all $n > N$, for each $t, s \in \mathbb{R}$, $t \geq s$.

(H_2) There exist a constant $\widetilde{M} \geq 0$ and $\delta > 0$, such that:

$$\left\| \frac{\partial}{\partial s} \mathcal{U}(t,s) \right\|_{\mathcal{B}(E)} \leq \widetilde{M} e^{-\delta(t-s)}, (t,s) \in \Delta.$$

(H_3) The function $f : \mathbb{R}^+ \times E \to E$ is Carathéodory and asymptotically almost automorphic i.e., $f(t,y) = g(t,y) + h(t,y)$ with

$$g(t,y) \in AA(\mathbb{R} \times E, E), \quad h(t,y) \in C_0(\mathbb{R}^+ \times E, E),$$

and $g(t,y)$ is uniformly continuous on any bounded subset $K \subset E$ uniformly for $t \in \mathbb{R}$.
Moreover,

(a) There exist $p \in L^q(\mathbb{R}, \mathbb{R}^+)$, $q \in [1, \infty)$ and a continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ such that for all $t \in \mathbb{R}^+$ and $y \in E$,

$$|g(t,y)| \leq p(t)\psi(|y|) \quad \text{and} \quad \lim_{|y| \to +\infty} \inf \frac{\psi(|y|)}{|y|} = \rho_1.$$

(b) There exist a function $\beta(t) \in C_0(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in \mathbb{R}^+$ and $y \in E$ with $|y| \leq R$,

$$|h(t,y)| \leq \beta(t)\phi(|y|) \quad \text{and} \quad \lim_{R \to +\infty} \inf \frac{\phi(R)}{R} = \rho_2.$$

(H_4) There exist a locally integrable function $\eta : \mathbb{R} \to \mathbb{R}^+$ and a continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any nonempty bounded set $D \subset E$ we have :

$$\alpha(f(t, D)) \leq \eta(t)\varphi(\alpha(D)) \text{ for a.e } t \in \mathbb{R}^+.$$

Additionally we assume that $\lim\limits_{n\to+\infty} (\psi + \phi)^n(t) = 0$ for a.e $t \in \mathbb{R}^+$. Let $\beta(t)$ be the function involved in the assumption (H_3), then

$$\int_0^t e^{-(t-s)}\beta(s)ds \in C_0(\mathbb{R}^+, \mathbb{R}^+).$$

Put

$$\rho = \sup_{t\in\mathbb{R}^+} \int_0^t e^{-(t-s)}\beta(s)ds.$$

We need the following technical lemma.

Lemma 3.2 *Assume that* (H_1) *hold. If* $\varphi(t) \in AA(\mathbb{R}, E)$*, then*

$$\Lambda(t) := \int_{-\infty}^t U(t, s)\varphi(s)\mathrm{ds}, \ \mathrm{t} \in \mathbb{R},$$

belongs to $AA(\mathbb{R}, \mathbb{E})$*.*

Proof. From (H_1) it is clear that $\Lambda(t)$ is well-defined and continuous on $\mathbb{R}$. Since $\varphi(t) \in AA(\mathbb{R}, E)$, it follows that for every sequence of real numbers $\{\tau_n'\}$, we can extract a subsequence $\{\tau_n\}$ such that

(c_1) $\lim\limits_{n\to\infty} \varphi(t + \tau_n) - \widetilde{\varphi}(t) = 0$ for each $t \in \mathbb{R}$ and,

(c_2) $\lim\limits_{n\to\infty} \widetilde{\varphi}(t - \tau_n) - \varphi(t) = 0$ for each $t \in \mathbb{R}$.

Notes that $\widetilde{\varphi}$ is also bounded on $\mathbb{R}$, and measurable. Define

$$\widetilde{\Lambda}(t) = \int_{-\infty}^t \mathcal{U}(t, s)\widetilde{\varphi}(s)\mathrm{ds}, \quad \mathrm{t} \in \mathbb{R}.$$

For $t \in \mathbb{R}$, Since $\widetilde{\varphi}$ is measurable, $\widetilde{\Lambda}$ is well-defined.
For $t \in \mathbb{R}$, we have

$$\begin{aligned}
&\left|\Lambda y)(t+\tau_n) - (\widetilde{\Lambda} y)(t)\right| \\
&= \left|\int_{-\infty}^{t+\tau_n} \mathcal{U}(t+\tau_n, s)\varphi(s)ds - \int_{-\infty}^{t} \mathcal{U}(t,s)\widetilde{\varphi}(s)ds\right| \\
&= \left|\int_{-\infty}^{t} \mathcal{U}(t+\tau_n, s+\tau_n)\varphi(s+\tau_n)ds - \int_{-\infty}^{t} \mathcal{U}(t,s)\widetilde{\varphi}(s)ds\right| \\
&\leq \int_{-\infty}^{t} \|\mathcal{U}(t+\tau_n, s+\tau_n)\|_{\mathcal{B}(E)} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| ds \\
&+ \int_{-\infty}^{t} \|\mathcal{U}(t+\tau_n, s+\tau_n) - \mathcal{U}(t,s))\|_{\mathcal{B}(E)} \widetilde{\varphi}(s) ds \\
&\leq \int_{-\infty}^{t} Me^{-\delta(t-s)} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| ds \\
&+ \int_{-\infty}^{t} \varepsilon e^{-\delta(t-s)} \left|\widetilde{\varphi}(s)\right| ds \\
&\leq M \int_{-\infty}^{t} e^{-\delta(t-s)} ds \sup_{s\in\mathbb{R}} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| \\
&+\varepsilon \int_{-\infty}^{t} e^{-\delta(t-s)} ds \sup_{s\in\mathbb{R}} \left|\widetilde{\varphi}(s)\right| \\
&\leq \frac{M}{\delta} \sup_{s\in\mathbb{R}} \left|\varphi(s+\tau_n) - \widetilde{\varphi}(s))\right| + \frac{\varepsilon}{\delta} \sup_{s\in\mathbb{R}} \left|\widetilde{\varphi}(s)\right|.
\end{aligned}$$

Using (c_1), we obtain that for $n \to \infty$,

$$\Lambda(t+\tau_n) \to \widetilde{\Lambda}(t).$$

Analogously, one can prove that,

$$\widetilde{\Lambda}(t-\tau_n) \to \Lambda(t) \text{ for each } t \in \mathbb{R} \text{ as } n \to \infty.$$

This we show that

$$\Lambda \in AA(\mathbb{R}, E).$$

Theorem 3.3 *Assume that the hypotheses $(H_1) - (H_4)$ are satisfied. If*

$$M\rho_1\|p\|_{L^q} + M\delta^{-1}\rho\rho_2 < 1, \tag{6}$$

and

$$M \max(4\|\eta\|_{L^1}, \|p\|_{L^q}\delta^{-1+\frac{1}{q}}) < 1, \tag{7}$$

then the problem (1)-(2) has a asymptotically almost automorphic mild solution.

Proof. Consider the operator $N : AAA(\mathbb{R}^+, E) \to AAA(\mathbb{R}^+, E)$ defined by

$$(Ny)(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1 + \int_0^t \mathcal{U}(t,s)f(s,y(s))ds, \tag{8}$$

where $y \in AAA(\mathbb{R}^+, E)$ with $y = \gamma + \zeta$, γ is the principal term and ζ the corrective term of y. We need to prove that N is weel- defined, that is $N(AAA(\mathbb{R}^+, E)) \subset AAA(\mathbb{R}^+, E)$. Let

$$\sigma(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1,$$

then

$$\begin{aligned} |\sigma(t)| &= |-\tfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0 + \mathcal{U}(t,0)y_1| \\ &\leq |\tfrac{\partial}{\partial s}\mathcal{U}(t,0)y_0| + |\mathcal{U}(t,0)y_1| \\ &\leq \widetilde{M}e^{-\delta t}|y_0| + Me^{-\delta t}|y_1|. \end{aligned}$$

Since $\delta > 0$, we get $\lim\limits_{t\to+\infty} |(\sigma(t)| = 0$. that is

$$\sigma \in C_0(\mathbb{R}^+, E). \tag{9}$$

By assumption $f = g + h$ where g is the principal term and h the corrective term. So we can write

$$\begin{aligned} f(t,y(t)) &= g(t,\gamma(t)) + f(t,y(t)) - f(t,\gamma(t)) + h(t,\gamma(t)) \\ &= g(t,\gamma(t)) + H(t,y(t)), \end{aligned} \tag{10}$$

In view of (10), we have

$$\begin{aligned} W(t) &= \int_0^t \mathcal{U}(t,s)f(s,y(s))ds \\ &= \int_0^t \mathcal{U}(t,s)g(s,\gamma(s))ds + \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &= \int_{-\infty}^t \mathcal{U}(t,s)g(s,\gamma(s))ds - \int_{-\infty}^0 \mathcal{U}(t,s)g(s,\gamma(s))ds \\ &+ \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &= (I_1y)(t) + (I_2y)(t), \end{aligned}$$

where

$$(I_1y)(t) = \int_{-\infty}^{t} \mathcal{U}(t,s)g(s,\gamma(s))\mathrm{ds},$$

$$\begin{aligned}(I_2y)(t) &= \int_0^t \mathcal{U}(t,s)H(s,y(s))ds \\ &- \int_{-\infty}^{0} \mathcal{U}(t,s)g(s,y(s))ds \\ &= (J_1y)(t)+(J_2y)(t),\end{aligned}$$

where

$$(J_1y)(t) = \int_0^t \mathcal{U}(t,s)H(s,y(s))ds,$$

$$(J_2y)(t) = \int_{-\infty}^{t} \mathcal{U}(t,s)g(s,\gamma(s))\mathrm{ds}.$$

Using (H_3) and Lemma 2.16 , we deduce that $s \to g(s,\gamma(s))$ is in $AA(\mathbb{R},E)$. Thus, by Lemma 3.2 we obtain

$$(I_1y)(t) \in AA(\mathbb{R},E). \tag{11}$$

Let's prove that $J_1 \in C_0(\mathbb{R}^+,E), J_2 \in C_0(\mathbb{R}^+,E)$.
Ideed by definition $H \in C_0(\mathbb{R}^+,E)$, that means given $\varepsilon > 0$, there exists $T > 0$ such that if $t \geq T$, we have $|H(t,y)| \leq \varepsilon$. Therefore if $t \geq T$, we get

$$\begin{aligned}\int_T^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)}|H(s,y(s))|ds &\leq M\varepsilon\int_T^t e^{-\delta(t-s)}ds \\ &\leq \frac{M}{\delta}\varepsilon,\end{aligned}$$

then

$$|(J_1y)(t)| \leq \frac{M}{\delta}\varepsilon \qquad \text{if } t \geq T.$$

So,

$$J_1 \in C_0(\mathbb{R}^+,E). \tag{12}$$

Next, let us show that $J_2 \in C_0(\mathbb{R}^+, E)$.

$$\begin{aligned}
|(J_2y)(t)| &\leq \int_{-\infty}^{0} \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} |g(s,y(s))| ds \\
&\leq M \sup_{t\in\mathbb{R}} |g(t,y(t))| \int_{0}^{T} e^{-\delta(t-s)} ds \\
&+ M\|g\|_\infty \frac{e^{-\delta(t}}{\delta} \qquad \to 0 \text{ as } \to \infty.
\end{aligned}$$

So,

$$J_2 \in C_0(\mathbb{R}^+, E). \tag{13}$$

Finaly combining (9),(11), (12) and (13) proves our claim that $N \in AAA(\mathbb{R}^+, E)$. Next, we will prove that the operator N satisfies all the assumptions of Theorem 2.17. We will break the proof into several steps.
Let

$$B_R = \left\{ y \in AAA(\mathbb{R}^+, E) : \|y\|_\infty \leq R \right\},$$

where R be any positive constant. Then B_R is a bounded, closed and convex subset of $AAA(\mathbb{R}^+, E)$.

Step 1: $N(y) \in B_R$ for any $y \in B_R$.
In fact, if we assume that the assertion is false, then $R < |(Ny)(t)|$. This yields that

$$\begin{aligned}
R < |(Ny)(t)| &\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)} |g(s,y(s)| ds \\
&+ \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} |h(s,y(s)| ds \\
&\leq \int_0^t \|\mathcal{U}(t,s)\|_{B(E)}\, p(s)\psi(|y(s)|) ds \\
&+ \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)} \beta(s)\phi(|y(s)|) ds \\
&\leq M\psi(R) \int_0^t e^{-\delta(t-s)} p(s) ds \\
&+ M\ \phi(R) \int_0^t e^{-\delta(t-s)} \beta(s) ds.
\end{aligned}$$

For $t \geq 0$, it follows from the Hölder inequality that

$$R < |(Ny)(t)| \quad \leq \quad M\psi(R)\|p\|_{L^q} + M\rho_2\phi(R).$$

Dividing both sides by R and taking the $\liminf$ as $R \to +\infty$, we have

$$M\rho_1\|p\|_{L^q} + M\delta^{-1}\rho\rho_2 > 1,$$

which contradicts (6). Hence, the operator N transforms the set B_R into itself.

Step 2. N is continuous.

Let $(y_n)_{n\in N}$ be a sequence in B_R such that $y_n \to y$ in B_R.

Case 1. If $t \in [0,T]$; $T > 0$, then, we have

$$|(Ny_n)(t) - (Ny)(t)| \leq M\int_0^t |f(s, y_n(s)) - f(s, y(s))|\, ds.$$

Since the functions f is Carathéodory, the Lebesgue dominated convergence theorem implies that

$$\|Ny_n - Ny\|_\infty \to 0 \quad \text{as} \;\; n \to +\infty.$$

Case 2. Since the functions f is Carathéodory, we can see that

$$|f(s, y_n(s)) - f(s, y(s))| \leq \frac{\delta\varepsilon}{M} \quad \text{for } t \geq T. \tag{14}$$

If $t \in (T, \infty)$, $T > 0$, then (14) and the hypotheses give us that

$$\begin{aligned} |Ny_n(t) - Ny(t)| \; &\leq \int_0^t \|\mathcal{U}(t,s)\|_{\mathcal{B}(E)}\Big|f(s, y_n(s)) - f(s, y(s))\Big|ds \\ &\leq M\frac{\delta\varepsilon}{M}\int_0^t e^{-\delta(t-s)}ds \\ &\leq \frac{M}{\delta}\frac{\delta\varepsilon}{M} \\ &\leq \varepsilon. \end{aligned} \tag{15}$$

Then the inequality (15) reduces to

$$\|N(y_n) - N(y)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Now, we conclude that N is continuous from B_R to B_R.

Step 3: $N(B_R)$ is equicontinuous.

Let $t_1, t_2 \in [0, T]$ with $t_2 > t_1$ and $y \in B_R$. Then, we have

$$\begin{aligned}
&|(N_1y)(t_2) - (N_1y(t_1)| \\
&= \left| \int_0^{t_1} (\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s))g(s, y(s)) \right. \\
&\left. + \int_{t_1}^{t_2} \mathcal{U}(t_2, s)g(s, y(s))ds \right| \\
&+ \left| \int_0^{t_1} (\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s))h(s, y(s)) \right. \\
&\left. + \int_{t_1}^{t_2} \mathcal{U}(t_2, s)h(s, y(s))ds \right| \\
&\leq \int_0^{t_1} \|\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)\|_{B(E)}\ p(s)\psi(|y(s)|)ds \\
&+M \int_{t_1}^{t_2} e^{-\delta(t-s)} p(s)\psi(|y(s)|)ds. \\
&+ \int_0^{t_1} \|\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)\|_{B(E)}\ \beta(s)\phi(|y(s)|)ds \\
&+M \int_{t_1}^{t_2} e^{-\delta(t-s)} \beta(s)\phi(|y(s)|)ds.
\end{aligned}$$

It follows from the Hölder inequality that

$$\begin{aligned}
&|(N_1y)(t_2) - (N_1y(t_1)| \\
&\leq \int_0^{t_1} \|\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)\|_{\mathcal{B}(E)}\ p(s)\psi(|y(s)|)ds \\
&+\frac{M\|p\|_{L^q}\psi(R)}{\delta^{1-\frac{1}{q}}} \left(e^{-\frac{q\delta}{q-1}(t-t_2)} - e^{-\frac{q\delta}{q-1}(t-t_2)}\right)^{1-\frac{1}{q}} \\
&+ \int_0^{t_1} \|\mathcal{U}(t_2, s) - \mathcal{U}(t_1, s)\|_{B(E)}\ \beta(s)\phi(|y(s)|)ds \\
&+\frac{M\phi(R)\sup\limits_{t\in\mathbb{R}}\beta(t)}{\delta}(e^{-\delta(t-t_2)} - e^{-\delta(t-t_1)}).
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $t_2 - t_1 \to 0$, which implies that $N(B_R)$ is equicontinuous.
Consider the measure of noncompacteness $\mu(B)$ defined on the family of bounded subsets of the space $AAA(\mathbb{R}^+, E)$ (see [33]) by

$$\mu(B) = \omega_0^T(B) + \sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}.$$

Step 4: $\mu(N(B)) \leq M\max(4\|\eta\|_{L^1}, \|p\|_{L^q}\delta^{-1+\frac{1}{q}})(\varphi+\psi)(\mu(B))$ for all $B\subset B_R$. For all $B\subset B_R$, $N(B)$ is bounded. Hence, by Lemma 2.4, there exists a countable set $B_1=\{y\}_{n=1}^{\infty}\subset B$, such that

$$(N(B)) \leq 2\alpha(N(B_1)). \tag{16}$$

Using the properties of α, Lemma 2.4, Lemma 2.5 and assumptions (H_1) and (H_4), we get

$$\begin{aligned}
\alpha(NB_1(t)) &\leq \alpha\left(\left\{\int_0^t \mathcal{U}(t,s)f(s,y_n(s))ds\right\}_{n=0}^{\infty}\right)\\
&\leq 2M\int_0^t \{\alpha\left(f(s,y_n(s))ds)\right)\}_{n=0}^{\infty} ds\\
&\leq 2M\int_0^t \eta(s)\varphi\left(\{(\alpha(y_n(s))\}_{n=0}^{\infty})\right)) ds\\
&\leq 2M\int_0^t \eta(s)\varphi(\alpha(B(s)))ds.
\end{aligned}$$

Form inequality (16), it follows that

$$\alpha(NB(t)) \leq 4M\int_0^t \eta(s)\varphi(\alpha(B(s)))ds,$$

then

$$\alpha(N(B(t)) \leq 4M\|\eta\|_{L^1}\varphi(\sup_{t\in\mathbb{R}^+}\alpha(B(t))).$$

Since

$$\sup_{t\in\mathbb{R}^+}\alpha(B(t)) \leq \sup_{t\in\mathbb{R}^+}\alpha(B(t)) + \lim_{t\to+\infty}\sup\{|y(t)| : t\geq T, y\in E\}),$$

then

$$\alpha(N(B(t)) \leq 4M\|\eta\|_{L^1}\varphi(\sup_{t\in\mathbb{R}^+}\alpha(B(t)) + \lim_{t\to+\infty}\sup\{|y(t)| : t\geq T, y\in E\}). \tag{17}$$

On the other hand, we have

$$\begin{array}{rcl}
|(Ny)(t)| & \leq & \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
& + & M\displaystyle\int_{-\infty}^{t} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds + |(I_2y)(t)| \\
& + & M\displaystyle\int_{-\infty}^{T} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds. \\
& + & M\displaystyle\int_{T}^{t} e^{-\delta(t-s)}p(s)\psi(|\gamma(s)|)ds + |I_2(t)|. \\
& \leq & \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
& + & M\displaystyle\int_{-\infty}^{T} e^{-\delta(t-s)}p(s)ds\psi(\sup_{s\in\mathbb{R}}|\gamma(s)|) \\
& + & M\displaystyle\int_{T}^{t} e^{-\delta(t-s)}p(s)ds\psi(\sup\{|\gamma(t)| : t\geq T, y\in E\}) \\
& + & \sup\{|(I_2y)(t)| : t\geq T, y\in E\}).
\end{array}$$

Next, applying the Hölder inequality we derive

$$\begin{array}{rcl}
|(Ny)(t)| & \leq & \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
& + & \dfrac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}e^{-\delta(t-T)}\psi(\|y\|_\infty). \\
& + & \dfrac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}(1-e^{-\frac{q\delta}{q-1}t})^{1-\frac{1}{q}}\psi(\sup\{|y(t)| : t\geq T, y\in E\}) \\
& + & \sup\{|(I_2y)(t)| : t\geq T, y\in E\}).
\end{array}$$

Then

$$\begin{array}{rcl}
|(Ny)(t)| & \leq & \widetilde{M}e^{-\delta t}\,|y_1| + Me^{-\delta t}\,|y_0| \\
& + & \dfrac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}e^{-\delta T}\psi(\|y\|_\infty). \\
& + & \dfrac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}\psi(\sup\{|y(t)| : t\geq T, y\in E\}) \\
& + & \sup\{|(I_2y)(t)| : t\geq T, y\in E\}).
\end{array}$$

Since $\delta \geq 0$, $I_2 \in C_0(\mathbb{R}^+, E)$ and

$$\lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\} \leq \sup_{t\in\mathbb{R}} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\},$$

then

$$\begin{aligned} &\lim_{T\to+\infty} \sup\{|(Ny)(t) : t \geq T, y \in E\}) \\ &\leq \tfrac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}} \psi(\sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}). \end{aligned} \tag{18}$$

Further, combining (17) and (18), we get

$$\begin{aligned} &\sup_{t\in J} \alpha((NB)(t)) + \lim_{T\to+\infty} \sup\{|(Ny)(t) : t \geq T, y \in E\}) \\ &\leq 4M\|\eta\|_{L^1} \varphi(\sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}) \\ &+ \tfrac{M\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}} \psi(\sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\} \\ &\leq M \max(4\|\eta\|_{L^1}, \tfrac{\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}})(\varphi + \psi)(\sup_{t\in J} \alpha(B(t)) + \lim_{T\to+\infty} \sup\{|y(t)| : t \geq T, y \in E\}). \end{aligned} \tag{19}$$

From **Step 3** and inequality (19), we conclude that

$$\mu(N(B)) \leq M \max\left(4\|\eta\|_{L^1}, \frac{\|p\|_{L^q}}{\delta^{1-\frac{1}{q}}}\right)(\varphi + \psi)(\mu(B)).$$

It follows from Lemma 2.17 that N has at least one fixed point $y \in B_R$, which is just a asymptotically almost automorphic mild solution of problem (1)-(2) on $\mathbb{R}^+$.

4 An Example

Consider the second order differential equation of the form;

$$\left\{\begin{array}{lll} \dfrac{\partial^2}{\partial t^2} z(t,\tau) & = \dfrac{\partial^2}{\partial \tau^2} z(t,\tau) + 2\sin\left(\dfrac{1}{2+\cos t + \cos\sqrt{2}t}\right)\dfrac{\partial}{\partial t} z(t,\tau) & \\ & +\dfrac{\sin^2 t}{12\sqrt{1+t^2}} \sin\left(\dfrac{1}{2+\cos t+\cos\sqrt{2}t}\right)(|z(t,\tau)| + \ln(1+|z(t,\tau)|)) & \\ & +\dfrac{\sin^2 t}{15\sqrt{1+t^2}} \dfrac{\sin \pi z(t,\tau)}{(1+|z(t,\tau)|)}, & t \in \mathbb{R}^+, \quad \tau \in [0,\pi], \\ z(t,0) = & z(t,\pi) = 0, & t \in \mathbb{R}^+, \\ \dfrac{\partial}{\partial t} z(0,\tau) & = \psi(\tau), & \tau \in [0,\pi]. \end{array}\right. \tag{20}$$

Let $E = L^2([0,\pi],\mathbb{R}^+)$ be the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}^+$, and let $H^2([0,\pi],\mathbb{R}^+)$ be the Sobolev space of functions $x : [0,\pi] \to \mathbb{R}^+$, such that $x'' \in L^2([0,\pi],\mathbb{R}^+)$. We consider the operator $A_1 z(\tau) = z''(\tau)$ with domain $D(A_1) = H^2(\mathbb{R}^+,\mathbb{C})$, which is the infinitesimal generator of strongly continuous cosine function $C(t)$ on E. Moreover, A_1 has discrete spectrum, the spectrum of A_1 consists of eigenvalues n^2 for $n \in \mathbb{Z}$, with associated eigenvector

$$\omega_n(\xi) = \frac{1}{\sqrt{2\pi}} e^{in\xi}, n \in \mathbb{Z},$$

the set $\{\omega_n \in \mathbb{Z}\}$ is an orthonormal basis of E. In particular,

$$A_1 x = -\sum_{n=1}^{\infty} n^2 \langle x, w_n\rangle w_n \text{ for } x \in D(A).$$

The cosine function $C(t)$ is given by

$$C(t)x = \sum_{n=1}^{\infty} \cos(nt)\langle x, w_n\rangle w_n \text{ for } x \in D(A), t \in \mathbb{R}^+,$$

form a cosine function on H, with associated sine function

$$S(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle x, w_n\rangle w_n \text{ for } x \in D(A), t \in \mathbb{R}^+.$$

From [35], for all $x \in H^2([0,\pi],\mathbb{R}^+), t \in \mathbb{R}^+$, $\|C(t)\|_{B(E)} \leq e^{-t}$ and $\|S(t)\|_{B(E)} \leq e^{-t}$.
Now, we define an operator $A(t) : D(A) \subset H \to H$ by

$$\begin{cases} D(A(t)) = D(A) \\ A(t) = A_1 + b(t,\tau). \end{cases}$$

where $b(t,\tau) = 2\sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right)$
Note that $A(t)$ generates an evolutionary process $\mathcal{U}(t,s)$ of the form

$$\mathcal{U}(t,s) = S(t-s)e^{\int_s^t b(t,s)ds}$$

Since $b(t,\tau) = 2\sin\left(\frac{1}{2+\cos t + \cos\sqrt{2}t}\right) \leq 2$, we have

$$\mathcal{U}(t,s) = S(t-s)e^{-2(t-s)} \tag{21}$$

and

$$\|\mathcal{U}\|_{\mathcal{B}(E)} \leq \|S\|_{\mathcal{B}(E)}e^{-2(t-s)} \leq e^{-3(t-s)}$$

We conclude that $\mathcal{U}(t,s)$ is a evolutionary process exponentially stable with $M = 1$ and $\delta = 3$.
It follows from the estimate (21) that $\mathcal{U}(t,s) : E \to E$ is well defined and satisfies the conditions of Definition 2.1.
Hence conditions (H_1) and (H_2) are satisfied.
Now, let

$$z(t)(\tau) = w(t)(\tau),\ t \geq 0,\ \tau \in [0,\pi],$$

$$g(t,z)(\tau) \quad = \quad \frac{\sin^2 t}{12\sqrt{1+t^2}} \sin\Big(\frac{1}{2+\cos t + \cos\sqrt{2}t}\Big)(|z(t,\tau)| + \ln(1+|z(t,\tau)|)),$$

$$h(t,z)(\tau) = \frac{\sin^2 t \sin \pi z(t,\tau)}{15\sqrt{1+t^2}(1+|z(t,\tau)|)}.$$

Then it is easy to verify that $g : \mathbb{R} \times E \times E$ is continuous and

$$g \in AA(\mathbb{R} \times E; E).$$

We can estimate for the functions g:

$$g(t,z)(\tau) \leq \frac{\sin^2 t}{12\sqrt{1+t^2}}(|z(t,\tau)| + \ln(1+|z(t,\tau)|)).$$

Hence conditions $(H_3)(a)$ is satisfied with

$$p(t) = \frac{\sin^2 t}{3\sqrt{1+t^2}}, \quad \psi(t) = \frac{1}{4}(t + \ln(1+t)).$$

Then it is easy to verify that $p \in L^2(\mathbb{R})$ and $\rho_1 = \dfrac{1}{4}$.
On the other hand, it is clear that $h : \mathbb{R}^+ \times E \times E$ is continuous and

$$h \in C_0(\mathbb{R}^+ \times E; E).$$

We can also estimate for the functions h:

$$h(t,z)(\tau) \leq \frac{\pi}{15\sqrt{1+t^2}}|z(t,\tau)|.$$

Hence conditions $(H_3)(b)$ is satisfied with

$$\beta(t) = \frac{\pi}{15\sqrt{1+t^2}}, \quad \phi(R) = R.$$

Then it is easy to verify that $\beta \in C_0(\mathbb{R}^+, \mathbb{R})$, $\rho_2 = 1$ and $\rho \leq \frac{\pi}{15}$. Furthermore:

$$f(t; z) = g(t; z) + h(t; z) \in AA(\mathbb{R}^+ \times E; E).$$

We can also estimate for the functions f:

$$f(t, z)(\tau) \leq \frac{2\sin^2 t}{\sqrt{1+t^2}} |z(t, \tau)|. \tag{22}$$

By (22), for every $t \in J$, and $B \in D \subset E$, we have

$$\alpha(f(t, D) \leq \frac{\sin^2 t}{12\sqrt{1+t^2}} \alpha(D),$$

Hence conditions $(H4)$ is satisfied with

$$\eta(t) = \frac{1}{6\sqrt{1+t^2}}, \quad \varphi(t) = \frac{\sin^2 t}{2}.$$

Moreover, we have

$$(\psi + \varphi)(t) = \frac{\sin^2 t}{2} + \frac{1}{4}(t + \ln(1+t)) \leq t.$$

We conclude that (see Lemma 2.1. [6])

$$\lim_{n \to +\infty} (\psi + \phi)^n(t) = 0 \text{ for a.e } t \in \mathbb{R}^+.$$

Consequently, can be written in the abstract form (1)-(2) with $A(t)$ and f as defined above. Thus, Theorem 3.3 yields that equation (20) has a asymptotically almost automorphic mild solution.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Applying hybrid coupled fixed point theory to the nonlinear hybrid system of second order differential equations

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Abstract

In this work, we apply hybrid fixed point theory method to prove the existence of solution of systems of second order ordinary nonlinear hybrid differential equations with periodic boundaries.

Key words and phrases. Second order differential equation; hybrid systems; boundary value problems; coupled fixed point.

AMS Mathematics subject Classification. 34A12, 34A38.

1 Introduction

Let $X \neq \phi$, and $F : X \times X \to X$ is a mapping. A point $(x, y) \in X \times X$ is said to a coupled fixed point of F in $X \times X$ if $F(x, y) = x$ and $F(y, x) = y$. The notation of coupled fixed point was introduced in 2006 by Baskar and Lshmikantham [1]. It will known that the existence of the fixed point play an important role for showing the existence of solutions of nonlinear integral [2, 3] , differential equations [4, 5] and iterative process [6].

In 1964, Krasnoselskii [7] initiated the idea of study the hybrid fixed point theory for the function which can be written as the sum of two other functions.In 2013, Dhage [2] obtained hybrid fixed point theorems for the operator which can be written as the sum of two other operators using Krasnoselskii fixed point theorem techniques and developed a Krasnoselskii fixed point technique helpful to analyze the existence of solution of nonlinear Volterra fractional integral equations under some conditions.

Recently, in 2015, Dhage and Dhage [8] proved the existence of solutions of the boundary value problems of second order ordinary nonlinear differential equations using the hybrid fixed point theorem which was obtained by themselves [9] .

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More recently, in 2017, Yang et al. [10] introduced the notation of hybrid coupled fixed point theorems and applied this ideas to prove the existence of system of fractional differential equations of order $\alpha, 0 < \alpha < 1$.

In this paper, we apply the hybrid fixed point theorem to study and prove the existence of solution of a certain system of boundary -value problems with periodic boundaries(for short BVPPB) of second order ordinary nonlinear hybrid differential equations.

To this end, the remainder of the article is organized as follows. Section 2, is given some preliminaries and basic definitions. Section3, is established the existence results of coupled nonlinear system of second order differential equations.

2 Preliminaries

Throughout this paper, let E be a nonempty set and $(E, \leq, \|.\|)$ be a partially ordered normed linear space. If $Q : E \to E$ is a mapping. Then Q is said to be monotone nondecreasing if $a \leq b$ the $Q(a) \leq Q(b)$, for all $a, b \in E$. Two elements $a, b \in E$ are said to be comparable if $a \leq b$ or $b \leq a$. If C is nonempty subset of E,C is said to be chain if each two elements $a, b \in C$ are comparable.

Definition 1 *[10] Let $Q : E \to E$ be a mapping. Q is called partially compact if for each chain C subset of E, $Q(C)$ is reltively compact subset of E.*

Definition 2 *[2, 10] Let $Q : E \to E$ be a mapping. Given an element $a \in E$. Define orbit $\Gamma(a; Q)$ as:*

$$\Gamma(a; Q) = \{a, Qa, Q^2a, Q^3a, ..., Q^na, ...\}.$$

If for any sequence $\{a_n\} \in \Gamma(a; Q)$ such that: $a_n \to a^*$ as $n \to \infty$ then $Qa_n \to Qa^*$, for each $a \in E$, then Q is called $\Gamma-$ orbitally continuous in E. Furthermore, $(E, \leq, \|.\|)$ is said to be $\Gamma-$ orbitally complete if each sequence $\{a_n\} \in \Gamma(a; Q)$ converges to an element $a^* \in E$.

Definition 3 *[2, 10] A mapping $\phi : E \to E$ is said to be $D-$ function if it is upper semi continuous and monotone nondecreasing such that: $\phi(0) = 0$.*

Definition 4 *[2, 10] A mapping $\Upsilon : E \to E$ is called partially nonlinear $D-$ contraction in E , if for each comparable elements $a, b \in E$, there exist a $D-$ function $\phi : \Re^+ \to \Re^+$ such that the following conditions are satisfied:*

(i) $\|\Upsilon a - \Upsilon b\| \leq \phi(\|a - b\|)$, *and*

(ii) $\phi(r) < r$ *for all* $r > 0$.

Definition 5 *[10] Let $(E, \leq, \|.\|)$ be a $\Gamma-$ orbitally complete linear space. The positive cone $K \in E$ is defined as: $K = \{x \in E : x \geq 0\}$*

The following theorem will be used as tool to prove the main results.

Theorem 6 *[10] Let E is a partially ordered $\Gamma-$ orbitally complete normed linear space and K is the positive cone of E. Let K is normal and C be a nonempty closed subset of E. Consider $P, Q : C \to C$ are two monotone nondecreasing mappings such that the following are satisfied:* (1) *P is $\Gamma-$ orbitally continuous and a partially nonlinear $D-$ contraction,*

(2) *Q is $\Gamma-$ orbitally continuous and a partially compact,*

(3) *there exist an element $a \in C$ such that: $a \leq Pa + Qy$ for all $y \in C$ and*

(4) *every pair of elements in C has an upper and lower bounded.*

Then $F(x, y) = Px + Qy$ has a coupled fixed point in $E \times E$.

Now, we give some notations and definitions about the problem which we will study the existence of its solution.

Definition 7 *Consider $I = [0, b] \in \Re$ where $b > 0$. The periodic boundary value problem of second-order nonlinear differential equation can be written as:*

$$\frac{d^2x(t)}{dt^2} = f(t, x(t) + h(t, x(t)), \qquad x(0) = x(b), x'(0) = x'(b) \tag{1}$$

for all $t \in I$, where $f, h : I \times \Re \to \Re$ are continuous function. The solution of the differential equation (1) is the function $x \in C^2(I, \Re)$ that satisfies equation (1), where by $x \in C^2(I, \Re)$ we mean the space of twice continuously differentiable real-valued functions on I.

Definition 8 *the space $C(I,\Re)$ is the space of all continuous real-value function defined on I. It is easy to prove that: $C(I,\Re)$ is a Banach space with the norm:*

$$\|x\| = \sup_{t\in I} |x(t)|. \tag{2}$$

Therefore, it is also clear that $C(I,\Re)$ is partially ordered with respect to the partially order relation:

$$x \leq y \ \text{ if and only if } \quad x(t) \leq y(t) \ \text{ for all } \ t \in I. \tag{3}$$

Also $C(I,\Re)$ is a partially ordered $\Gamma-$ orbitally complete linear space with normal cone $K_C = \{x \in C(I,\Re) : x(t) \geq 0, t \in I\}$ [10].

Lemma 9 *[11] Consider $\sigma, g \in L^1(I,\Re)$, then x is a solution of the differential equation:*

$$x''(t) + \sigma(t)x(t) = g(t), \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

if and only if x is a solution of the integral equation:

$$x(t) = \int_0^b G_\sigma(t,s)g(s)ds,$$

where $G_\sigma(t,s)$ is a Green's function associated with the differential equation:

$$x''(t) + \sigma(t)x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

Remark 10 *The Green function G_σ is continuous an nonnegative on $I \times I$ and there exist the number :*

$$M_\sigma = \max\{|G_\sigma(t,s)| : t,s \in [0,b]\}$$

for all $\sigma \in L^1(I,\Re^+)$.

Lemma 11 *[8] Consider the differential equations (1). Let $F(t,x) = f(t,x) + \lambda x$, such that:*

$(P1)$ The function $f : I \times \Re \to \Re$ is continuous and there exist a constant $k_1 > 0$ such that$|\ F(t,x)| \leq k_1$ for all $t \in I$,

$(P2)$ The function $h : I \times \Re \rightarrow \Re$ is continuous and exist a constant $k_2 > 0$ such that$|\ h(t, x)| \leq k_2$ for all $t \in I$.

Then a function $v \in C(I, \Re)$ is a solution of the differential equation:

$$\frac{d^2x(t)}{dt^2} + \lambda x(t) = F(t, x(t) + h(t, x(t)), \qquad x(0) = x(b), x'(0) = x'(b) \tag{4}$$

if and only if v is the solution of the integral equation:

$$x(t) = \int_0^b G(t, s)F(s, x(s))ds + \int_0^b G(t, s)h(s, x(s))ds, \tag{5}$$

for all $t \in I$, where $G(t, s)$ is a Green's function associated with the differential equation:

$$x''(t) + \lambda x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

3 Main results

Now , consider the system of nonlinear differential equations:

$$\begin{aligned}
&\tfrac{d^2x(t)}{dt^2} = f(t, x(t) + h(t, y(t)), \quad t \in I,\\
&\tfrac{d^2y(t)}{dt^2} = f(t, y(t) + h(t, x(t)), \quad t \in I,\\
&x(0) = x(b) \quad , \quad x'(0) = x'(b),\\
&y(0) = y(b) \quad , \quad y'(0) = y'(b).
\end{aligned} \tag{6}$$

Then the hybrid system (6) can be written as:

$$\begin{aligned}
&\tfrac{d^2x(t)}{dt^2} + \lambda x(t) = F(t, x(t) + h(t, y(t)), \quad t \in I,\\
&\tfrac{d^2y(t)}{dt^2} + \lambda y(t) = F(t, y(t) + h(t, x(t)), \quad t \in I,\\
&x(0) = x(b) \quad , \quad x'(0) = x'(b),\\
&y(0) = y(b) \quad , \quad y'(0) = y'(b),
\end{aligned} \tag{7}$$

where $F(t, x(t)) = f(t, x(t)) + \lambda x(t)$. By applying Lemma 11, we have the following Lemma .

Lemma 12 *Consider the hybrid system of differential equations (7), such that the conditions (P1) and (P2) hold.*

Then a function $(v_1, v_2) \in C(I, \Re) \times C(I, \Re)$ *is a solution of the differential equation the system (7) :*

if and only if (v_1, v_2) *is the solution of the system of nonlinear integral equation:*

$$\begin{aligned} x(t) &= \int_0^b G(t,s)F(s,x(s))ds + \int_0^b G(t,s)h(s,y(s))ds, \\ y(t) &= \int_0^b G(t,s)F(s,y(s))ds + \int_0^b G(t,s)h(s,x(s))ds \end{aligned} \tag{8}$$

for all $t \in I$*, where* $G(t,s)$ *is a Green's function associated with the differential equation:*

$$x''(t) + \lambda x(t) = 0, \qquad t \in I,$$

$$x(0) = x(b) \ \ , \ \ x'(0) = x'(b),$$

By using the continuity of the integrals, we can define the two mappings: $T_1 : C(I, \Re) \to C(I, \Re)$ and $T_2 : C(I, \Re) \to C(I, \Re)$ as:

$$T_1(x(t)) = \int_0^b G(t,s)F(s,x(s))ds,$$

and

$$T_2(x(t)) = \int_0^b G(t,s)h(s,x(s))ds.$$

Then we can define the following :

$$T(x(t), y(t)) = T_1 x(t) + T_2 y(t)$$

The coupled fixed point of the operator T is the solution of the system (7).

With the two conditions (P1) and (P2) consider the following set of assumptions:

$(P3)$ There exists $\lambda > 0$ and $\varepsilon > 0$ such that:

$$0 \leq [f(t,x) + \lambda x] - [f(t,y) + \lambda y] \leq \varepsilon(x - y),$$

for all $t \in I$ and $x, y \in \Re$ such that: $x \geq y$.

$(P4)$ The function $h(t,x)$ nondecreasing.

$(P5)$ There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) \leq f(t, v(t)) + h(t, y(t)),$$

$$v(0) \leq v(b) \quad , \quad v'(0) \leq v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$.

Lemma 13 *Assume that (P1) and (P3) holds. Let* $\varepsilon M b < 1$ *. Then the operator* T_1 *is* $\Gamma-$ *orbitally continuous, nondecreasing and a partially nonlinear* $D-$ *contraction.*

Proof. Let $E = C(I, \Re)$. The proof will be done in 3 steps.

Step 1: To prove that T_1 is nondecreasing.

Let $x, y \in E$, such that: $x \geq y$ Then by (P3), we have that:

$$T_1 x(t) = \int_0^b G(t,s) F(s, x(s)) ds \geq \int_0^b G(t,s)) F(s, y(s)) ds \geq T_1 y(t),$$

for all $t \in I$. Thus T_1 is nondecreasing operator.

Step 2: To prove that T_1 is $\Gamma-$ orbitally continuous.

Take a sequence $\{x_n\} \in \Gamma(x; T_1)$ for any $x \in E$ with $x_n \to x^*$ as $n \to \infty$. Since T_1 is continues by (P1) again, then we have that:

$$\begin{aligned} \lim_{n\to\infty} (T_1 x_n)(t) &= \textstyle\int_0^b G(t,s) \lim_{n\to\infty} F(s, x_n(s)) ds \\ &= \textstyle\int_0^b G(t,s) F(s, x^*(s)) ds \\ &= (T_1 x^*)(t), \end{aligned} \tag{9}$$

for all $t > 0$. Hence , T_1 is $\Gamma-$ orbitally continuous.

Step 3: To prove that T_1 is partially nonlinear $D-$ contraction.

For any comparable element $x, y \in E$ such that:$x \geq y$. For $t \in I$, we have that:

$$\begin{aligned} \mid (T_1 x)(t) - (T_1 y)(t) \mid &= \mid \textstyle\int_0^b G(t,s) [F(s, x(s)) - F(s, y(s))] ds \mid \\ &\leq \textstyle\int_0^b G(t,s) \mid [F(s, x(s)) - F(s, y(s))] \mid ds \\ &\leq M \varepsilon b \|x - y\|, \end{aligned} \tag{10}$$

Let $\tau = M\varepsilon b$. New define the mapping $\Theta(l) = \tau l$, for each $l \in \Re^+$. Thus $\Theta(l) < l$ and $\Theta(0) = 0$. Therefore, we proved that:$\|T_1 x - T_1 y\| \leq \Theta(\|x - y\|)$. Hence T_1 is partially nonlinear $D-$ contraction.

Lemma 14 *Assume that (P2) and (P4) holds. Then the operator* T_2 *is* $\Gamma-$ *orbitally continuous, nondecreasing and a partially compact in* $E = C(I, \Re$ *.*

Proof. The proof is also done in 3 steps.

Step 1: To prove that: T_2 is $\Gamma-$ orbitally continuous.

Consider for any $x \in E$ with $x_n \to x^*$ as $n \to \infty$. Since T_2 is continues by (P2), then we have that:

$$\begin{aligned} \lim_{n\to\infty}(T_2x_n)(t) &= \textstyle\int_0^b G(t,s)\lim_{n\to\infty} h(s,x_n(s))ds \\ &= \textstyle\int_0^b G(t,s)h(s,x^*(s))ds \\ &= (T_2x^*)(t), \end{aligned} \tag{11}$$

for all $t > 0$. Hence , T_2 is $\Gamma-$ orbitally continuous.

Step 2: To prove that: T_2 is nondecreasing.

Let $x, y \in E$, such that: $x \geq y$ Then by (P2), we have that:

$$T_2x(t) = \int_0^b G(t,s)h(s,x(s))ds \geq \int_0^b G(t,s))h(s,y(s))ds \geq T_2y(t),$$

for all $t \in I$. Thus T_2 is nondecreasing operator.

Step 3: To prove that T_2 is partially compact in $E = C(I, \Re)$. Let C be an arbitrary chain in E. We show $T_2(C)$ is uniformly bounded and equicontinuous set in E. Consider $x \in C$ be arbitrary, then we get that:

$$\begin{aligned} \mid T_2x(t) \mid &=\mid \textstyle\int_0^b G(t,s)h(s,x(s))ds \mid \\ &\leq \textstyle\int_0^b G(t,s) \mid h(s,x(s)) \mid ds \\ &\leq Mk_2b, \end{aligned} \tag{12}$$

Thus , we have that: $\|T_2x\| \leq \tau$, where $\tau = Mk_2b$. Hence, $T_2(C)$ is uniformly bounded subset in E.

To prove $T_2(C)$ is an equicontinuous : let $t_1, t_2 \in I$ such that: $t_1 - t_2 < 0$. We have that:

$$\begin{aligned} \mid T_2x(t_2) - T_2x(t_1) \mid &=\mid \textstyle\int_0^b [G(t_1,s) - G(t_2,s)]h(s,x(s))ds \mid \\ &\leq \textstyle\int_0^b \mid G(t_1,s) - G(t_2,s) \mid h(s,x(s))ds \\ &\leq \textstyle\int_0^b \mid G(t_1,s) - G(t_2,s) \mid k_2 ds \to 0, \end{aligned} \tag{13}$$

as $t_1 \to t_2$. Hence $T_2(C)$ is a compact subset of E . Therefore, T_2 is partially compact in $E = C(I, \Re)$.

Theorem 15 *Let the hypotheses (P1), (P2), (P3), (P4) , (P5) and* $\varepsilon Mb < 1$ *hold. Then the hybrid system (6) has a coupled solution on* I*.*

Proof. By (P5) There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) \le f(t, v(t) + h(t, y(t)),$$

$$v(0) \le v(b) \quad , \quad v'(0) \le v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$. Then , we get that: There exists an element $v \in C^2(I, \Re)$ such that:

$$v''(t) + \lambda v(t) \le F(t, v(t)) + h(t, y(t)),$$

$$v(0) \le v(b) \quad , \quad v'(0) \le v'(b),$$

for all $t \in I$ and $y \in C(I, \Re)$. Applying Theorem 6, we get that : the mapping $T(x, y) = T_1 x + T_2 y$ has a coupled fixed point. This coupled fixed point is the solution of the hybrid system (6).

Corollary 16 *Let the hypotheses (P1), (P2), (P3), (P4) and* $\varepsilon Mb < 1$ *hold. Consider there exists an element* $v \in C^2(I, \Re)$ *such that:*

$$v''(t) \ge f(t, v(t) + h(t, y(t)),$$

$$v(0) \ge v(b) \quad , \quad v'(0) \ge v'(b),$$

for all $t \in I$ *and* $y \in C(I, \Re)$*.*

Then the hybrid system (6) has a coupled solution on I*.*

Example 17 *Let* $I = [0, 1]$*. Suppose we have the following system of differential equations:*

$$\begin{aligned} & x''(t) = tan^{-1}x(t) - x(t) + h(t, y(t)), \quad t \in I, \\ & y''(t) = tan^{-1}y(t) - y(t) + h(t, x(t)), \quad t \in I, \\ & x(0) = x(1) \quad , \quad x'(0) = x'(1), \\ & y(0) = y(1) \quad , \quad y'(0) = y'(1), \end{aligned} \tag{14}$$

and $t \in [0,1]$. *Consider* $g : I \times \Re \to \Re$ *such that:*

$$g(t,x) == \begin{cases} 1, & \text{if } x \leq 1 \\ \frac{2x}{1+x}, & \text{if } x > 1. \end{cases} \tag{15}$$

It is clear that $f(t,x) = \tan^{-1} x(t) - x(t)$ *and* f *and* g *are continues functions.Define* $F(t,x) = \tan^{1} x(t) - x(t)$. *Then we get that:* $|F(t,x)| < \frac{1}{\pi} = k_1$. *Therefore ,* $f(t,x)$ *is satisfied condition (P1) . Also, since we have that:*

$$0 \leq \tan^{1} x - \tan^{1} y \leq \frac{1}{1+\kappa^2}(x-y) \quad \forall x, y \in \Re, \;\; x > y, \;\; x > \kappa > y.$$

Thus f *is satisfied condition (P3) with* $\lambda = 1$. *It is also clear that:* $\lambda > \varepsilon = \frac{1}{1+\kappa^2}$ *, where* $x > \kappa > y$. *Also,* $g(t,x)$ *is nondecreasing in* x *for all* $t \in I$ *and bounded . Thus* g *is satisfied the conditions (P2) and (P4). Finally , the function :*

$$v(t) = -2\int_0^1 G(t,s)ds + \int_0^1 G(t,s))ds,$$

is satisfied the condition (P5). Then the hybrid system has (14) a coupled solution.

4 Declaration of conflicting interests

The author declared no potential conflicts on interest with respect to the research, authorship, and/or publication of this article.

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CO-ORDINATED CONVEX FUNCTIONS OF THREE VARIABLES AND SOME ANALOGOUS INEQUALITIES WITH APPLICATIONS

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Abstract: Co-ordinated convex function of two variables and corresponding inequalities have been studied before by many researchers. This paper deals with Co-ordinated convex function of three variables. In the present paper the idea of co-ordinated convex function of two variables in a rectangle from the plane R^2 is extended to that of three variables in a rectangle from space R^3. Moreover corresponding extended right handed Hermite-Hadamard type inequalities are also incorporated. At the end some applications of resulting inequalities to special means are also given.

Key Words: Hermite-Hadamard's inequality, co-ordinated convex function, Hölder's integral inequality, power mean inequality, arithmetic mean, logarithmic mean.

1. Introduction

It is said that the notion *convex* was pioneered by Archimedes. While estimating value of π, he noticed that the perimeter of a convex figure is smaller than that of any other convex figure, encompassing it. As per J. L. Jensen, idea of convex function is as primeval as an increasing function or a positive function. A documented result spontaneously identified with convex function is Hermite-Hadamard (HH) inequality

$$g\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g(u)du \leq \frac{g(\alpha)+g(\beta)}{2}, \tag{1.1}$$

where g is a real valued convex function on the real interval I and $\alpha, \beta \in I$ with $\alpha < \beta$. The idea of co-ordinated convex (CC) function on a rectangle from the plane was coined by Dragomir and Pearce in 2000, see [6]. Co-ordinated convexity is more general than convexity. It only requires a function to be convex on each

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2010 *Mathematics Subject Classification.* 26A51, 26D15, 52A15
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coordinate individually and independently, not simultaneously as in case of convexity. Thereafter Dragomir and many other mathematicians worked in this field, see [1, 5, 8, 9, 11, 13, 14, 16–18, 20–22]. The idea of co-ordinated convexity was joined with other classes of convex functions like s-convex, m-convex, (α, m)-convex, h-convex, log-convex, $((s, m), QC)$-convex functions and many interesting results were obtained, see [2–4, 10, 12, 15, 19] and references there in.

Dragomir and Pearce [6] defined CC function of two variables as follows. The principal outcomes of the present article are actuated by this definition.

Definition 1.1. *A function $g : \triangle \longrightarrow R$ is called CC on $\triangle$ if the partial functions $g_u : [\gamma, \delta] \longrightarrow R$ and $g_v : [\alpha, \beta] \longrightarrow R$, defined as $g_u(y) = g(u, y)$ and $g_v(x) = g(x, v)$ respectively, are convex. Here $\triangle$ is bi-dimensional interval given by $\triangle = [\alpha, \beta] \times [\gamma, \delta]$.*

2. Main Results

In this section we mainly establish results on the basis of definition of CC function of three variables on a rectangle from the space R^3. We derive HH type inequality for CC functions of three variables and then establish inequalities related to the algebraic combination of middle term and right side terms of this inequality, to be called right HH type inequalities. Let us first define CC function of three variables. Motivated by Definition 1.1, CC function of three variables is defined as:

Definition 2.1. *A function $g : \Delta \longrightarrow R$ is called CC on Δ if the partial functions $g_{u,v} : [\zeta, \eta] \longrightarrow R$, $g_{u,w} : [\gamma, \delta] \longrightarrow R$ and $g_{v,w} : [\alpha, \beta] \longrightarrow R$, defined as $g_{u,v}(z) = g(u, v, z), g_{u,w}(y) = g(u, y, w)$ and $g_{v,w}(x) = g(x, v, w)$ respectively, are convex. Here and on wards Δ is tri-dimensional interval given by $\Delta = [\alpha, \beta] \times [\gamma, \delta] \times [\zeta, \eta]$.*

Lemma 2.2. *If a function $g : \Delta \longrightarrow R$ is CC function on Δ, then the subsequent inequality is induced*

$$\begin{aligned} &g\left(lc + (1-l)d, me + (1-m)h, nr + (1-n)s\right)\\ &\leq lmng(c, e, r) + lm(1-n)g(c, e, s) + l(1-m)ng(c, h, r)\\ &\quad + l(1-m)(1-n)g(c, h, s) + (1-l)mng(d, e, r) + (1-l)m(1-n)g(d, e, s)\\ &\quad + (1-l)(1-m)ng(d, h, r) + (1-l)(1-m)(1-n)g(d, h, s), \end{aligned}$$

for all $l, m, n \in [0, 1]$ and $(c, e, r), (c, e, s), (c, h, r)$ e.t.c. are in Δ.

Proof. The Ineq. (**??**) can be proved simply by applying Definition 2.1, to the given function g. ☐

Lemma 2.3. *Every convex function is CC. The converse in general is not true.*

Proof. It can be proved by using Definition 2.1 of CC function. Moreover the function $g : [0, 1]^3 \longrightarrow [0, \infty)$ given by $g(u, v, w) = uvw$, for all $(u, v, w) \in [0, 1]^3$ is CC function but is not convex on $[0, 1]^3$. ☐

Motivated by Dragomir [7], we extend HH Ineq. (1.1) for CC functions of three variables as follows:

Theorem 2.4. *Suppose that $g : \Delta \subset R^3 \longrightarrow R$ is CC function on Δ, then the subsequent inequality is induced*

$$\begin{aligned}
&g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right)\\
&\le \frac{1}{3}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta g\left(u, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right)du + \frac{1}{\delta-\gamma}\int_\gamma^\delta g\left(\frac{\alpha+\beta}{2}, v, \frac{\zeta+\eta}{2}\right)dv\right.\\
&\quad\left.+\frac{1}{\eta-\zeta}\int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, w\right)dw\right] \qquad (2.1)\\
&\le \frac{1}{3}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\gamma^\delta\int_\alpha^\beta g\left(u, v, \frac{\zeta+\eta}{2}\right)dudv + \frac{1}{(\delta-\gamma)(\eta-\zeta)}\right.\\
&\quad\left.\int_\gamma^\delta\int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, v, w\right)dvdw + \frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\alpha^\beta\int_\zeta^\eta g\left(u, \frac{\gamma+\delta}{2}, w\right)dudw\right] \qquad (2.2)\\
&\le \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_\alpha^\beta\int_\gamma^\delta\int_\zeta^\eta g(u,v,w)dudvdw \qquad (2.3)\\
&\le \frac{1}{6}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta\{g(u,v,\zeta)+g(u,v,\eta)\}dudv\right.\\
&\quad+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta\{g(\alpha,v,w)+g(\beta,v,w)\}dvdw\\
&\quad\left.+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\alpha^\beta\int_\zeta^\eta\{g(u,\gamma,w)+g(u,\delta,w)\}dudw\right] \qquad (2.4)\\
&\le \frac{1}{12}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}du\right.\\
&\quad+\frac{1}{\delta-\gamma}\int_\gamma^\delta\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}dv\\
&\quad\left.+\frac{1}{\eta-\zeta}\int_\zeta^\eta\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}dw\right] \qquad (2.5)\\
&\le \frac{1}{8}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)\right.\\
&\quad\left.+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)\right]. \qquad (2.6)
\end{aligned}$$

Proof. Since $g : \Delta \longrightarrow R$ is CC function, it follows that the function $g_{u,v} : [\zeta, \eta] \longrightarrow R$, defined by $g_{u,v}(z) = g(u, v, z)$ is convex on $[\zeta, \eta]$. Then from Inequality (1.1), we have

$$g_{\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}}\left(\frac{\zeta+\eta}{2}\right) \le \frac{1}{\eta-\zeta}\int_\zeta^\eta g_{\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}}(w)dw, \quad \text{where } w \in [\zeta, \eta]$$

$$g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, \frac{\zeta+\eta}{2}\right) \le \frac{1}{\eta-\zeta}\int_\zeta^\eta g\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}, w\right)dw.$$

Similar arguments applied to the functions $g_{u,w}$ and $g_{v,w}$ respectively, we get

$$g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\le\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv$$

and

$$g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\le\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}g\left(u,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du,$$

adding above three inequalities, we get Ineq. (2.1) as follows

$$\begin{aligned}&g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)\\&\le\frac{1}{3}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}g\left(u,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\right.\\&\quad\left.+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},w\right)dw\right].\end{aligned}$$

Now consider the following inequality

$$\begin{aligned}&g_{\frac{\alpha+\beta}{2},v}\left(\frac{\zeta+\eta}{2}\right)\le\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g_{\frac{\alpha+\beta}{2},v}(w)dw,\qquad\text{where } w\in[\zeta,\eta],\\&g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)\le\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},v,z\right)dw,\\&\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}g\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\le\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}g\left(\frac{\alpha+\beta}{2},v,w\right)dvdw,\end{aligned}$$

adding all such inequalities, we get Ineq. (2.2) as follows

$$\begin{aligned}&\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f\left(x,\frac{\gamma+\delta}{2},\frac{\zeta+\eta}{2}\right)du+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}f\left(\frac{\alpha+\beta}{2},v,\frac{\zeta+\eta}{2}\right)dv\\&+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}f\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2},w\right)dw\le\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}f\left(u,v,\frac{\zeta+\eta}{2}\right)dudv\\&+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}f\left(\frac{\alpha+\beta}{2},v,w\right)dvdw\\&+\frac{1}{(\eta-\zeta)(\beta-\alpha)}\int_{\alpha}^{\beta}\int_{\zeta}^{\eta}f\left(u,\frac{\gamma+\delta}{2},w\right)dwdu.\end{aligned}$$

Considering

$$g_{u,v}\left(\frac{\zeta+\eta}{2}\right)\le\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g_{u,v}(w)dw\le\frac{g_{u,v}(m)+g_{u,v}(n)}{2},\ \text{ where } w\in[\zeta,\eta],$$

gives

$$g\left(u,v,\frac{\zeta+\eta}{2}\right)\le\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g(u,v,w)dw\le\frac{g(u,v,\zeta)+g(u,v,\eta)}{2},\tag{2.7}$$

integrating Ineq. (2.7) w.r.t u and v over the intervals $[\alpha,\beta]$ and $[\gamma,\delta]$ respectively, then multiplying by $\frac{1}{(\beta-\alpha)(\delta-\gamma)}$, we have

$$\begin{aligned}
&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta g\left(u,v,\frac{\zeta+\eta}{2}\right)dudv\\
&\le \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_\alpha^\beta\int_\gamma^\delta\int_\zeta^\eta g(u,v,w)dudvdw\\
&\le \frac{1}{2(\beta-\alpha)(\delta-\gamma)}\left[\int_\alpha^\beta\int_\gamma^\delta g(u,v,\zeta)dudv+\int_\alpha^\beta\int_\gamma^\delta g(u,v,\eta)dudv\right],
\end{aligned}$$

by the same argument applied on the functions $g_{v,w}$ and $g_{u,w}$, we tend to get two additional such inequalities. Then by adding these inequalities, we have Ineqs. (2.3) and (2.4) as follows

$$\begin{aligned}
&\frac{1}{3}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta g\left(u,v,\frac{\zeta+\eta}{2}\right)dudv+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\right.\\
&\left.\int_\gamma^\delta\int_\zeta^\eta g\left(\frac{\alpha+\beta}{2},v,w\right)dvdw+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\zeta^\eta\int_\alpha^\beta g\left(u,\frac{\gamma+\delta}{2},w\right)dudw\right]\\
&\le \frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_\alpha^\beta\int_\gamma^\delta\int_\zeta^\eta g(u,v,w)dudvdw\\
&\le \frac{1}{6}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta g(u,v,\zeta)dudv+\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta g(u,v,\eta)dudv\right.\\
&\quad+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta g(\alpha,v,w)dvdw+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta g(\beta,v,w)dvdw\\
&\quad\left.+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\eta^\zeta\int_\alpha^\beta g(u,v,\zeta)dudw+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_\eta^\zeta\int_\alpha^\beta g(u,v,\eta)dudw\right].
\end{aligned}$$

By convexity of $g_{\alpha,v}$, we have

$$\begin{aligned}
\frac{1}{\eta-\zeta}\int_\zeta^\eta g_{\alpha,v}(w)dw &\le \frac{g_{\alpha,v}(\zeta)+g_{\alpha,v}(\eta)}{2}, \quad \text{where } w\in[\zeta,\eta]\\
\frac{1}{\eta-\zeta}\int_\zeta^\eta g(\alpha,v,w)dw &\le \frac{g(\alpha,v,\zeta)+g(\alpha,v,\eta)}{2},
\end{aligned}$$

integrating this inequality w.r.t. v over the interval $[\gamma,\delta]$, we have

$$\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta g(\alpha,v,w)dvdw\le\frac{1}{2(\delta-\gamma)}\int_\gamma^\delta\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)\}dv,$$

similarly by convexity of $g_{\beta,v}$ we have

$$\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_\gamma^\delta\int_\zeta^\eta g(\beta,v,w)dvdw\le\frac{1}{2(\delta-\gamma)}\int_\gamma^\delta\{g(\beta,v,\zeta)+g(\beta,v,\eta)\}dv,$$

after further calculation we have six such inequalities, adding these we get Ineq. (2.5) as follows

$$\begin{aligned}
&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\{g(u,v,\zeta)+g(u,v,\eta)\}\,dudv\\
&+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}\{g(\alpha,v,w)+g(\beta,v,w)\}\,dvdw\\
&+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_{\alpha}^{\beta}\int_{\zeta}^{\eta}\{g(u,v,\zeta)+g(u,v,\eta)\}\,dudw\\
&\le\frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}\,du\right.\\
&+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv\\
&\left.+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}\,dw\right].
\end{aligned}$$

Now to get last inequality we use the convexity of $g_{\alpha,\gamma}$, which implies

$$\begin{aligned}
&\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g_{\alpha,\gamma}(w)dw\le\frac{g_{\alpha,\gamma}(\zeta)+g_{\alpha,\gamma}(\eta)}{2},\ \text{where } w\in[\zeta,\eta],\\
&\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}g(\alpha,\gamma,w)dw\le\frac{g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)}{2},
\end{aligned}$$

adding such inequalities, we get Ineq. (2.6) as follows

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\left[\int_{\alpha}^{\beta}g(u,\gamma,\zeta)du+\int_{\alpha}^{\beta}g(u,\gamma,\eta)du+\int_{\alpha}^{\beta}g(u,\delta,\zeta)du+\int_{\alpha}^{\beta}g(u,\delta,\eta)du\right]\\
&+\frac{1}{\delta-\gamma}\left[\int_{\gamma}^{\delta}g(\alpha,v,\zeta)dv+\int_{\gamma}^{\delta}g(\alpha,v,\eta)dv+\int_{\gamma}^{\delta}g(\beta,v,\zeta)dv+\int_{\gamma}^{\delta}g(\beta,v,\eta)dv\right]\\
&+\frac{1}{\eta-\zeta}\left[\int_{\zeta}^{\eta}g(\alpha,\gamma,w)dw+\int_{\zeta}^{\eta}g(\alpha,\delta,w)dw+\int_{\zeta}^{\eta}g(\beta,\gamma,w)dw+\int_{\zeta}^{\eta}g(\beta,\delta,w)dw\right]\\
&\le\frac{3}{2}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)\right.\\
&\left.+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)\right].
\end{aligned}$$

Hence proved. □

Motivated by notion given in [22], we now present right handed HH type inequalities related to inequality given in Theorem 2.4, for differentiable CC functions on rectangle from the space R^3. In order to establish further results we need the following lemma.

Note: From here onwards we use '$\mathcal{A}$' to represent the following algebraic combination of middle and the right sided terms of HH type inequality given in Theorem

2.4.

$$
\begin{aligned}
&\frac{1}{8}\left[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)+g(\beta,\delta,\zeta)\right.\\
&\left.+g(\beta,\delta,\eta)\right]-\frac{1}{4}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\left\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\right\}du\right.\\
&+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\left\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\right\}dv\\
&\left.+\frac{1}{\eta-\zeta}\int_{\zeta}^{\eta}\left\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\right\}dw\right]+\frac{1}{2}\left[\frac{1}{(\beta-\alpha)(\delta-\gamma)}\right.\\
&\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\{g(u,v,\zeta)+g(u,v,\eta)\}dudv+\frac{1}{(\delta-\gamma)(\eta-\zeta)}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}\{g(\alpha,v,w)+g(\beta,v,w)\}dvdw\\
&\left.+\frac{1}{(\beta-\alpha)(\eta-\zeta)}\int_{\alpha}^{\beta}\int_{\zeta}^{\eta}\left\{g(u,\gamma,w)+g(u,\delta,w)\right\}dudw\right]\\
&-\frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\int_{\zeta}^{\eta}g(u,v,w)dudvdw.
\end{aligned}
$$

Lemma 2.5. *Let $g:\Delta\subset R^3\longrightarrow R$ be partial differentiable function on Δ. If $g_{lmn}\in L(\Delta)$, then the subsequent identity is induced*

$$
\begin{aligned}
\mathcal{A}=\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}&\int_{0}^{1}\int_{0}^{1}\int_{0}^{1}(1-2l)(1-2m)(1-2n)\\
&g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn.
\end{aligned}
$$

Proof. Considering the following triple integral and integrating it by parts w.r.t. l,m and n respectively, we get

$$
\begin{aligned}
&\int_{0}^{1}\int_{0}^{1}\int_{0}^{1}(1-2l)(1-2m)(1-2n)\\
&\quad g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\,dldmdn\\
&=\int_{0}^{1}\int_{0}^{1}(1-2m)(1-2n)\left[\frac{(1-2l)}{\alpha-\beta}g_{mn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\Big|_{0}^{1}\right.\\
&\quad\left.-\int_{0}^{1}\frac{(-2)}{\alpha-\beta}g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dl\right]dmdn\\
&=\int_{0}^{1}\int_{0}^{1}\frac{(1-2m)(1-2n)}{\beta-\alpha}[g_{mn}(\alpha,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)+g_{mn}(\beta,m\gamma+(1-m)\delta,\\
&\quad n\zeta+(1-n)\eta)-2\int_{0}^{1}g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dl]dmdn\\
&=\frac{1}{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}\{[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)\\
&\quad+g(\beta,\gamma,\eta)+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)]-2\left[\int_{0}^{1}\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+g(u,\delta,\eta)\}\,du+\int_0^1\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv\\
&+\int_0^1\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\Big\}dw\Big]\\
&+4\Big[\int_0^1\int_0^1\{g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\zeta)+g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\eta)\}dldm\\
&+\int_0^1\int_0^1\{g(\alpha,m\gamma+(1-m)\delta,\zeta n+(1-n)\eta)+g(\beta,m\gamma+(1-m)\delta,\zeta n+(1-n)\eta)\}\,dmdn\\
&+\int_0^1\int_0^1\{g(l\alpha+(1-l)\beta,\gamma,\zeta n+(1-n)\eta)+g(l\alpha+(1-l)\beta,\delta,\zeta n+(1-n)\eta)\}\,dldn\Big]\\
&-8\int_0^1\int_0^1\int_0^1 g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn\Big\}, \qquad (2.8)
\end{aligned}
$$

on multiplying both sides Eq. (2.8) by $\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}$, we get

$$
\begin{aligned}
&\frac{1}{8}\,[g(\alpha,\gamma,\zeta)+g(\alpha,\gamma,\eta)+g(\alpha,\delta,\zeta)+g(\alpha,\delta,\eta)+g(\beta,\gamma,\zeta)+g(\beta,\gamma,\eta)\\
&+g(\beta,\delta,\zeta)+g(\beta,\delta,\eta)]-\frac{1}{4}\Big[\int_0^1\{g(u,\gamma,\zeta)+g(u,\gamma,\eta)+g(u,\delta,\zeta)+g(u,\delta,\eta)\}\,du\\
&+\int_0^1\{g(\alpha,v,\zeta)+g(\alpha,v,\eta)+g(\beta,v,\zeta)+g(\beta,v,\eta)\}\,dv\\
&+\int_0^1\{g(\alpha,\gamma,w)+g(\alpha,\delta,w)+g(\beta,\gamma,w)+g(\beta,\delta,w)\}\,dw\Big]\\
&+\frac{1}{2}\Big[\int_0^1\int_0^1\{g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\zeta)+g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,\eta)\}dldm\\
&+\int_0^1\int_0^1\{g(\alpha,m\gamma+(1-m)\delta,n\zeta+(1-\zeta n)\eta)+g(\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)\}dmdn\\
&+\int_0^1\int_0^1\{g(l\alpha+(1-l)\beta,\gamma,n\zeta+(1-n)\eta)+g(l\alpha+(1-l)\beta,\delta,n\zeta+(1-n)\eta)\}\,dldn\Big]\\
&-\int_0^1\int_0^1\int_0^1 g(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)dldmdn\\
&=\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1[(1-2l)(1-2m)(1-2n)\\
&\qquad g_{lmn}(l\alpha+(1-l)\beta,m\gamma+(1-m)\delta,n\zeta+(1-n)\eta)]dldmdn. \qquad (2.9)
\end{aligned}
$$

By changing variable on left hand side of Eq. (2.9), we get the required result. □

Theorem 2.6. *Let $g:\Delta\subset R^3\longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|$ is a CC function on Δ, then the subsequent inequality is induced*

$$
\begin{aligned}
|\mathcal{A}|\leq&\ \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{512}\,[|g_{lmn}(\alpha,\gamma,\zeta)|+|g_{lmn}(\alpha,\gamma,\eta)+|g_{lmn}(\alpha,\delta,\zeta)\\
&+|g_{lmn}(\alpha,\delta,\eta)|+|g_{lmn}(\beta,\gamma,\zeta)|+|g_{lmn}(\beta,\gamma,\eta)|+|g_{lmn}(\beta,\delta,\zeta)|+|g_{lmn}(\beta,\delta,\eta)|].
\end{aligned}
$$

Proof. From Lemma 2.5, properties of modulus, co-ordinated convexity of $|g_{lmn}|$ and integrating by parts w.r.t. l, m and n respectively, we have

$$\begin{aligned}
&|\mathcal{A}| \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |1-2l||1-2m||1-2n|\,[l|g_{lmn}(\alpha, m\gamma+(1-m)\delta, \\
&\quad \zeta n+(1-n)\eta)| + (1-l)|g_{lmn}(\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|]\,dldmdn \\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8\times 4}\int_0^1\int_0^1 |1-2m||1-2n|\,[|g_{mn}(\alpha, m\gamma+(1-m)\delta, \\
&\quad n\zeta+(1-n)\eta)| + |g_{mn}(\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|]\,dmdn \\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8\times 4\times 4}\int_0^1 |1-2n|\,[|g_n(\alpha,\gamma,\zeta n+(1-n)\eta)| \\
&\quad +|g_n(\alpha,\delta,n\zeta+(1-n)\eta)|+|g_n(\beta,\gamma,n\zeta+(1-n)\eta)||g_n(\beta,\delta,n\zeta+(1-n)\eta)|]dn \\
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{512}\,[|g_{lmn}(\alpha,\gamma,\zeta)| + |g_{lmn}(\alpha,\gamma,\eta) + |g_{lmn}(\alpha,\delta,\zeta) \\
&\quad +|g_{lmn}(\alpha,\delta,\eta)| + |g_{lmn}(\beta,\gamma,\zeta)| + |g_{lmn}(\beta,\gamma,\eta)| + |g_{lmn}(\beta,\delta,\zeta)| + |g_{lmn}(\beta,\delta,\eta)|].
\end{aligned}$$

Hence proved. □

Theorem 2.7. *Let $g : \Delta \subset R^3 \longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|^q$ with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, is a CC function on Δ, then the subsequent inequality is induced*

$$\begin{aligned}
|\mathcal{A}| \leq\ & \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8(p+1)^{\frac{3}{p}}}\left(\frac{1}{8}\right)^{\frac{1}{q}} \\
& [|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q + |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q \\
& + |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q + |g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q]^{\frac{1}{q}}.
\end{aligned}$$

Proof. From Lemma 2.5, properties of modulus, Hölder's integral inequality and Lemma 2.2, we have

$$\begin{aligned}
&|\mathcal{A}| \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)| \\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|dldmdn \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|^p dldmdn\right)^{\frac{1}{p}} \\
&\quad \left(\int_0^1\int_0^1\int_0^1 |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|^q dldmdn\right)^{\frac{1}{q}} \\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |1-2l|^p\,|1-2m|^p\,|1-2n|^p dldmdn\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\Big(\int_0^1\int_0^1\int_0^1 (lmn|g_{lmn}(\alpha,\gamma,\zeta)|^q + lm(1-n)|g_{lmn}(\alpha,\gamma,\eta)|^q\\
&+ l(1-m)n|g_{lmn}(\alpha,\delta,\zeta)|^q + l(1-m)(1-n)|g_{lmn}(\alpha,\delta,\eta)|^q\\
&+ (1-l)mn|g_{lmn}(\beta,\gamma,\zeta)|^q + (1-l)m(1-n)|g_{lmn}(\beta,\gamma,\eta)|^q\\
&+(1-l)(1-m)n|g_{lmn}(\beta,\delta,\zeta)|^q + (1-l)(1-m)(1-n)|g_{lmn}(\beta,\delta,\eta)|^q)\, dldmdn\Big)^{\frac{1}{q}}\\
=&\frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{(p+1)^3}\right)^{\frac{1}{p}}\left(\frac{1}{8}\right)^{\frac{1}{q}}(|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q\\
&+ |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q + |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q\\
&+|g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q)^{\frac{1}{q}}\ .
\end{aligned}
$$

Hence proved. ☐

Theorem 2.8. *Let $g : \Delta \subset R^3 \longrightarrow R$ be partial differentiable function on Δ. If $|g_{lmn}|^q$, $q \geq 1$ is a CC function on Δ, then the subsequent inequality is induced*

$$
\begin{aligned}
|\mathcal{A}| \leq& \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{64}\left(\frac{1}{8}\right)^{\frac{1}{q}}\\
&[|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q + |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q\\
&+ |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q + |g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. From Lemma 2.5, properties of modulus, power mean inequality and Lemma 2.2, we have

$$
\begin{aligned}
&|\mathcal{A}|\\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|\\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|dldmdn\\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|dldmdn\right)^{1-\frac{1}{q}}\\
&\quad \Big(\int_0^1\int_0^1\int_0^1 |(1-2l)(1-2m)(1-2n)|\\
&\quad |g_{lmn}(l\alpha+(1-l)\beta, m\gamma+(1-m)\delta, n\zeta+(1-n)\eta)|^q dldmdn\Big)^{\frac{1}{q}}\\
&\leq \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\Big(\int_0^1\int_0^1\int_0^1 |1-2l||1-2m||1-2n|\\
&\quad (lmn|g_{lmn}(\alpha,\gamma,\zeta)|^q + lm(1-n)|g_{lmn}(\alpha,\gamma,\eta)|^q + l(1-m)n|g_{lmn}(\alpha,\delta,\zeta)|^q\\
&\quad + l(1-m)(1-n)|g_{lmn}(\alpha,\delta,\eta)|^q + (1-l)mn|g_{lmn}(\beta,\gamma,\zeta)|^q +\\
&\quad (1-l)m(1-n)|g_{lmn}(\beta,\gamma,\eta)|^q + (1-l)(1-m)n|g_{lmn}(\beta,\delta,\zeta)|^q\\
&\quad + (1-l)(1-m)(1-n)|g_{lmn}(\beta,\delta,\eta)|^q)dldmdn\Big)^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)}{8}\left(\frac{1}{8}\right)^{1-\frac{1}{q}}\left(\frac{1}{64}\right)^{\frac{1}{q}} \\
&\quad (|g_{lmn}(\alpha,\gamma,\zeta)|^q + |g_{lmn}(\alpha,\gamma,\eta)|^q + |g_{lmn}(\alpha,\delta,\zeta)|^q + |g_{lmn}(\alpha,\delta,\eta)|^q \\
&\quad + |g_{lmn}(\beta,\gamma,\zeta)|^q + |g_{lmn}(\beta,\gamma,\eta)|^q + |g_{lmn}(\beta,\delta,\zeta)|^q + |g_{lmn}(\beta,\delta,\eta)|^q)^{\frac{1}{q}},
\end{aligned}
$$

after simplification we get the required result. □

Remark. *Since $\frac{1}{8} < \frac{1}{(p+1)^{\frac{3}{p}}}$ for $p > 1$, therefore estimation given in Theorem 2.8 with an improved and simplified constant is even better than that of Theorem 2.7.*

3. Some Applications to special Means

In Section 2, we established some inequalities based on CC functions of three variables. Now we apply these inequalities to get estimates for absolute values of different patterns of some special means. For this let us first have a look at following special means of positive real numbers ϵ, κ $(\epsilon \neq \kappa)$.

Arithmetic mean

$$A(\epsilon,\kappa) = \frac{\epsilon+\kappa}{2}.$$

Harmonic Mean

$$H(\epsilon,\kappa) = \frac{2\epsilon\kappa}{\epsilon+\kappa}.$$

Logarithmic mean

$$L(\epsilon,\kappa) = \frac{\kappa-\epsilon}{ln(\kappa)-ln(\epsilon)}.$$

Genralised Log-mean

$$L_i(\epsilon,\kappa) = \left[\frac{\kappa^{i+1}-\epsilon^{i+1}}{(i+1)(\kappa-\epsilon)}\right]^{\frac{1}{i}},\ i \in Z \setminus \{-1,0\}.$$

Proposition 3.1. *Let $\alpha,\beta,\gamma,\delta,\zeta,\eta \in R^+$ such that $\alpha<\beta, \gamma<\delta, \zeta<\eta$, we have*

$$
\begin{aligned}
&\left|A_iA_jA_k - \left(L_i^iA_jA_k + A_iL_j^jA_k + A_iA_jL_k^k\right) + \left(L_i^iL_j^jA_k + L_i^iA_jL_k^k + A_iL_j^jL_k^k\right) - L_i^iL_j^jL_k^k\right| \\
&\qquad \le (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}A_{i-1}A_{j-1}A_{k-1}. \qquad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
&\left|A_iA_jA_k - \left(L_i^iA_jA_k + A_iL_j^jA_k + A_iA_jL_k^k\right) + \left(L_i^iL_j^jA_k + L_i^iA_jL_k^k + A_iL_j^jL_k^k\right) - L_i^iL_j^jL_k^k\right| \\
&\qquad \le (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{8(p+1)^{\frac{3}{p}}}\left(A_{q(i-1)}\,A_{q(j-1)}\,A_{q(k-1)}\right)^{\frac{1}{q}}. \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
&\left|A_iA_jA_k - \left(L_i^iA_jA_k + A_iL_j^jA_k + A_iA_jL_k^k\right) + \left(L_i^iL_j^jA_k + L_i^iA_jL_k^k + A_iL_j^jL_k^k\right) - L_i^iL_j^jL_k^k\right| \\
&\qquad \le (ijk)\frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64}\left(A_{q(i-1)}\,A_{q(j-1)}\,A_{q(k-1)}\right)^{\frac{1}{q}}. \qquad (3.3)
\end{aligned}
$$

Where $A_i = A\left(\alpha^i,\beta^i\right)$, $A_j = A\left(\gamma^j,\delta^j\right)$, $A_k = A\left(\zeta^k,\eta^k\right)$, $L_i^i = L_i^i(\alpha,\beta)$, $L_j^j = L_j^j(\gamma,\delta)$, $L_k^k = L_k^k(\zeta,\eta)$, $A_{i-1} = A\left(\alpha^{i-1},\beta^{i-1}\right)$, $A_{q(i-1)} = A\left(\alpha^{q(i-1)},\beta^{q(i-1)}\right)$ etc.

Proof. The assertions in Proposition 3.1 follow by taking $g(u,v,w) = u^i v^j w^k$ as the CC function in Theorem 2.6, Theorem 2.7 and Theorem 2.8 respectively. Where $u, v, w \in R^+$ and $i, j, k \in Z^+$. □

Proposition 3.2. *Let $\alpha, \beta, \gamma, \delta, \zeta, \eta \in R^+$ such that $\alpha < \beta, \gamma < \delta, \zeta < \eta$, we have*

$$\Big| H_i H_j H_k - \Big(L^i_{-i} H_j H_k + H_i L^j_{-j} H_k + H_i H_j L^k_{-k} \Big) + \Big(L^i_{-i} L^j_{-j} H_k + L^i_{-i} H_j L^k_{-k} + H_i L^j_{-j} L^k_{-k} \Big) - L^i_{-i} L^j_{-j} L^k_{-k} \Big|$$
$$\leq (ijk) \frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64} (H_i H_j H_k) \left(H^{-1}_{i+1} H^{-1}_{j+1} H^{-1}_{k+1} \right) \Big(L^i_{-i} L^j_{-j} L^k_{-k} \Big).$$

$$\Big| H_i H_j H_k - \Big(L^i_{-i} H_j H_k + H_i L^j_{-j} H_k + H_i H_j L^k_{-k} \Big) + \Big(L^i_{-i} L^j_{-j} H_k + L^i_{-i} H_j L^k_{-k} + H_i L^j_{-j} L^k_{-k} \Big) - L^i_{-i} L^j_{-j} L^k_{-k} \Big|$$
$$\leq (ijk) \frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{8(p+1)^{\frac{3}{p}}} (H_i H_j H_k) \Big(H^{-1}_{q(i+1)} H^{-1}_{q(j+1)} H^{-1}_{q(k+1)} \Big)^{\frac{1}{q}} \Big(L^i_{-i} L^j_{-j} L^k_{-k} \Big).$$

$$\Big| H_i H_j H_k - \Big(L^i_{-i} H_j H_k + H_i L^j_{-j} H_k + H_i H_j L^k_{-k} \Big) + \Big(L^i_{-i} L^j_{-j} H_k + L^i_{-i} H_j L^k_{-k} + H_i L^j_{-j} L^k_{-k} \Big) - L^i_{-i} L^j_{-j} L^k_{-k} \Big|$$
$$\leq (ijk) \frac{[(\beta-\alpha)(\delta-\gamma)(\eta-\zeta)]^2}{64} (H_i H_j H_k) \Big(H^{-1}_{q(i+1)} H^{-1}_{q(j+1)} H^{-1}_{q(k+1)} \Big)^{\frac{1}{q}} \Big(L^i_{-i} L^j_{-j} L^k_{-k} \Big).$$

Where $H_i = H\left(\alpha^i, \beta^i\right), H_j = H\left(\gamma^j, \delta^j\right), H_k = H\left(\zeta^k, \eta^k\right), L^i_i = L^i_{-i}(\alpha, \beta)$, $L^j_{-j} = L^j_j(\gamma, \delta), L^k_{-k} = L^k_k(\zeta, \eta), H^{-1}_{i+1} = H^{-1}\left(\alpha^{i+1}, \beta^{i+1}\right)$, $H^{-1}_{q(i+1)} = H^{-1}\left(\alpha^{q(i+1)}, \beta^{q(i+1)}\right)$ etc.

Proof. The assertions in Proposition 3.2 follow by taking $g(u,v,w) = \frac{1}{u^i v^j w^k}$ as CC function in Theorem 2.6, Theorem 2.7 and Theorem 2.8 respectively. Where $u, v, w \in R^+$ and $i, j, k \in Z^+$. □

4. CONCLUSION

In this paper the idea of CC function of two variables in a rectangle from the plane R^2 is extended to that of three variables in a rectangle from the space R^3. Then HH type inequality for CC function of three variables is established. Consequently by using this inequality the analogous extended right handed HH type inequalities for CC functions of three variables are obtained. Thus obtained right handed inequalities are utilized to give bounds for algebraic combinations of some special means.

Motivated by these results one can also find extensions of existing left handed HH type inequalities. Furthermore it is asserted that in the same way idea of CC function of three variables in a rectangle from the space R^3 can be further extended to CC functions of n variables in a rectangle from n-dimensional Euclidean space R^n.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Studies on the Higher Order Difference Equation

$$x_{n+1} = \beta x_{n-l} + \alpha x_{n-k} + \frac{a x_{n-t}}{b x_{n-t} + c}$$

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ABSTRACT

The main objective of this paper is to study the local and the global stability of the solutions, the periodic character and the boundedness of the difference equation

$$x_{n+1} = \beta x_{n-l} + \alpha x_{n-k} + \frac{a x_{n-t}}{b x_{n-t} + c}, \qquad n = 0,\ 1,\ ...,$$

where the parameters $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1}, ...,\ x_{-1}$, x_0 are positive real numbers where $s = max\{l,\ k,\ t\}$.

Keywords: Difference equations, Stability, Global stability, Boundedness, Periodic solutions.

Mathematics Subject Classification: 39A10

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1. INTRODUCTION

The higher-order difference equations are of paramount importance in applications. Such equations also seem naturally as discrete analogues and as numerical solutions of differential which model various diverse phenomena in biology, ecology, physiology, physics, engineering, economics and so on [1-9]. The theory of difference equations gets a central position in applicable analysis. That is, the theory of difference equations will continue to play an important role in mathematics as a whole. Hence, it is very interesting to study the behavior of solutions of a difference equations and to discuss the local and global asymptotic stability of their equilibrium points [10-15]. In recent years, the behavior of solutions of various difference equations has been one of the main topics in the theory of difference equations [16-34].

Abo-Zeid [35] obtained the global asymptotic stability of all solutions of the difference equation

$$x_{n+1} = \frac{Ax_{n-2}}{B+Cx_nx_{n-1}x_{n-2}}, \qquad n = 0, 1, ...,$$

where $A,\ B,\ C$ are positive real numbers and the initial conditions $x_{-2},\ x_{-1},\ x_0$ are real numbers.

Abu-Saris et al. [36] studied the globally asymptotically stability of the equilibrium solution of the rational difference equation

$$x_{n+1} = \frac{a+x_nx_{n-k}}{x_n+x_{n-k}}, \qquad n = 0, 1, ...,$$

where k is a nonnegative integer, $a \geq 0$, and $x_{-k},\ ...,\ x_0 > 0$.

You-Hui et al. [37] investigated the global attractivity of the nonlinear difference equation

$$y_{n+1} = \frac{p+qy_n}{1+y_n+ry_{n-k}}, \qquad n = 0, 1, ...,$$

where $p,\ q,\ r \in [0,\ \infty)$, $k \geq 1$ is a positive integer and the initial conditions $y_{-k},\ \ldots,\ y_{-1}$ are nonnegative real numbers and y_0 is a positive real number.

Zayed et al. [38] investigated the boundedness character, the periodicity character, the convergence and the global stability of positive solutions of the difference equation,

$$x_{n+1}=\frac{\alpha_0x_n+\alpha_1x_{n-l}+\alpha_2x_{n-k}}{\beta_0x_n+\beta_1x_{n-l}+\beta_2x_{n-k}},\qquad n=0,1,...,$$

where the coefficients $\alpha_i,\ \beta_i\in(0,\ \infty)$ for $i=0,\ 1,\ 2,$ and $l,\ k$ are positive integers such that $l<k.$ The initial conditions $x_{-k},\ ,\ ...,\ x_{-l},\ ...,\ x_{-2},\ x_{-1},x_0$ are arbitrary positive real numbers.

El-Dessoky [39] investigated some qualitative behavior of the solutions of the difference equation

$$x_{n+1}=ax_{n-l}+bx_{n-k}+\frac{cx_{n-s}}{dx_{n-s}-e},\qquad n=0,\ 1,\ ...,$$

where the parameters $a,\ b,\ c,\ d$ and e are positive real numbers and the initial conditions $x_{-t},\ x_{-t+1},...,\ x_{-1},$ x_0 are positive real numbers where $t=max\{l,\ k,\ s\}.$

Our goal is to obtain some qualitative behavior of the positive solutions of the difference equation

$$x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\frac{ax_{n-t}}{bx_{n-t}+c},\qquad n=0,\ 1,\ ..., \tag{1}$$

where the parameters $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers and the initial conditions $x_{-s},\ x_{-s+1},...,\ x_{-1},$ x_0 are positive real numbers where $s=max\{l,\ k,\ t\}.$

2. LOCAL STABILITY

In this section, we study the local stability of the equilibrium point of equation (1).

The equilibrium points of Eq. (1) are given by

$$\overline{x}=\beta\overline{x}+\alpha\overline{x}+\frac{a\overline{x}}{b\overline{x}+c},$$

$$b(1-\alpha-\beta)\overline{x}^2+c(1-\alpha-\beta)\overline{x}=a\overline{x}.$$

So, $\overline{x}_0=0$ is forever an equilibrium point of the difference equation (1). If $\alpha+\beta<1$, then the positive equilibrium point of the Eq. (1) is given by

$$\overline{x}_1=\frac{a}{b(1-\alpha-\beta)}-\frac{c}{b}.$$

Let $f:(0,\ \infty)^3\longrightarrow(0,\ \infty)$ be a continuous function defined by

$$f(u,\ v,\ w)=\beta u+\alpha v+\frac{aw}{bw+c}.$$

Therefore, it follows that

$$\frac{\partial f(u,\ v,\ w)}{\partial u}=\beta,\ \frac{\partial f(u,\ v,\ w)}{\partial v}=\alpha,\ \frac{\partial f(u,\ v,\ w)}{\partial w}=\frac{ac}{(bw+c)^2}. \tag{2}$$

THEOREM 2.1. *The zero equilibrium* $\overline{x}_0$ *of the difference equation (1) is locally asymptotically stable if*

$$c\,(\alpha+\beta)+a<c. \tag{3}$$

Proof: So, we can write Eq. (2) at zero equilibrium point $\overline{x}_0=0$ in the form

$$\frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial u}=\beta=p_1,\ \frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial v}=\alpha=p_2\text{ and }\frac{\partial f(\overline{x}_0,\ \overline{x}_0,\ \overline{x}_0)}{\partial w}=\frac{a}{c}=p_3.$$

Then the linearized equation of Eq. (1) about $\overline{x}_0$ is

$$y_{n+1}-p_1y_{n-k}-p_2y_{n-l}-p_3y_{n-s}=0,$$

According to Theorem 1.6 page 7 in [1], then Eq. (1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$|\beta| + |\alpha| + \left|\tfrac{a}{c}\right| < 1,$$

and so

$$c\,(\alpha + \beta) + a < c.$$

The proof is complete.

Example 1. Consider $l = 2$, $k = 1$, $t = 3$, $\beta = 0.3$, $\alpha = 0.2$, $a = 0.5$, $b = 0.7$ and $c = 6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the zero solution of the difference equation (1) is local stability (see Fig. 1).

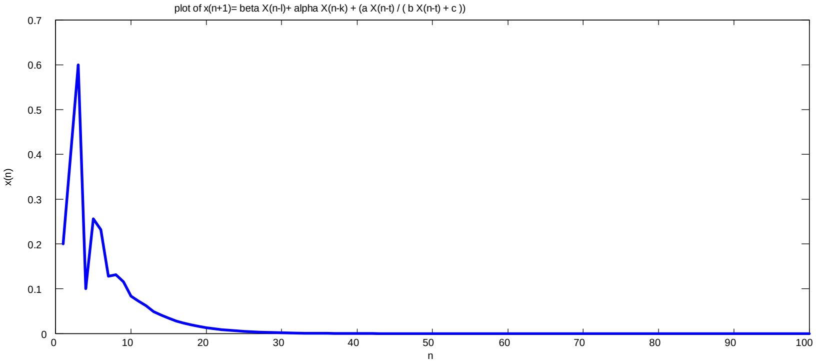

Figure 1. Plot the behavior of zero solution of Eq. (1) is local stable.

Theorem 2.2. *The positive equilibrium $\overline{x}_1$ of the difference equation (1) is locally asymptotically stable if*

$$c\,(1 - \alpha - \beta) < a. \tag{4}$$

Proof: So, we can write Eq. (2) at the positive equilibrium point $\overline{x}_1 = \frac{a}{b(1-\alpha-\beta)} - \frac{c}{b}$

$$\frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial u} = \beta = p_1,\ \ \frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial v} = \alpha = p_2 \text{ and } \frac{\partial f(\overline{x}_1,\, \overline{x}_1,\, \overline{x}_1)}{\partial w} = \frac{c(1-\alpha-\beta)^2}{a} = p_3.$$

Then the linearized equation of Eq. (1) about $\overline{x}_1$ is

$$y_{n+1} - p_1 y_{n-k} - p_2 y_{n-l} - p_3 y_{n-s} = 0,$$

According to Theorem 1.6 page 7 in [1], then Eq. (1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| < 1.$$

Thus,

$$|\beta| + |\alpha| + \left|\frac{c(1-\alpha-\beta)^2}{a}\right| < 1,$$

and so

$$\frac{c(1-\alpha-\beta)^2}{a} < 1 - a - b,$$

if $\alpha + \beta < 1$, then

$$c\,(1 - \alpha - \beta) < a.$$

The proof is complete.

Example 2. Figure (2) shows the solution of the difference equation (1) is local stability if $l = 2$, $k = 1$, $t = 3$, $\beta = 0.3$, $\alpha = 0.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$.

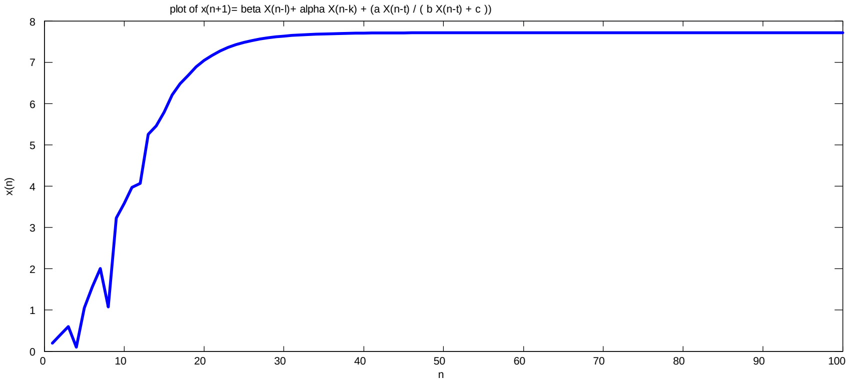

Figure 2. Draw the behavior of the positve solution of Eq. (1) is local stable.

Example 3. The solution of the difference equation (1) is unstable if $l = 2$, $k = 1$, $t = 3$, $\beta = 0.9$, $\alpha = 0.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$. (See Fig. 3).

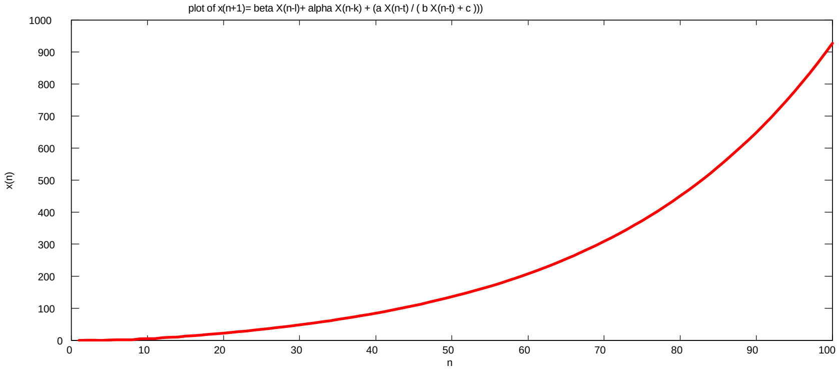

Figure 3. Sketch the behavior of the solution of Eq. (1) is unstable.

3. GLOBAL STABILITY

In this section, the global asymptotic stability of equation (1) is studied.

THEOREM 3.1. *The equilibrium point $\overline{x}_0$ is a global attractor of difference equation (1) if*

$$\alpha + \beta + \frac{a}{c} < 1. \tag{5}$$

Proof: Suppose that ζ and η are real numbers and assume that $F : [\zeta, \eta]^3 \longrightarrow [\zeta, \eta]$ is a function defined by

$$F(x,\ y,\ z) = \beta x + \alpha y + \tfrac{az}{bz+c}.$$

Then
$$\tfrac{\partial F(x,\ y,\ z)}{\partial x} = \beta,\ \tfrac{\partial F(x,\ y,\ z)}{\partial y} = \alpha \text{ and } \tfrac{\partial F(x,\ y,\ z)}{\partial z} = \tfrac{ac}{(bz+c)^2}.$$

Now, we can see that the function $F(x,\ y,\ z)$ increasing in $x,\ y$ and z. Then

$$\begin{aligned} & \left[\beta x+\alpha x+\frac{ax}{bx+c}-x\right](x-\overline{x}_0) \\ \leq\ & \left[-(1-\alpha-\beta)\,x+\frac{ax}{bx}+\frac{ax}{c}\right](x-0)\leq-\left(1-\alpha-\beta-\frac{a}{c}\right)x^2<0 \end{aligned}$$

If $\alpha+\beta+\frac{a}{c}<1$, then $F(x,\ y,\ z)$ satisfies the negative feedback property

$$[F(x,\ x,\ x)-x]\,(x-\overline{x}_0)<0,\quad \text{for } \overline{x}_0=0.$$

According to Theorem 1.10 page 15 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

Example 4. Consider $l=2$, $k=1$, $t=3$, $\beta=0.03$, $\alpha=0.02$, $a=0.5$, $b=0.7$ and $c=4$ and the initial conditions $x_{-3}=0.5$, $x_{-2}=0.7$, $x_{-1}=0.6$ and $x_0=1.1$, the zero solution of the difference equation (1) is global stability (see Fig. 4).

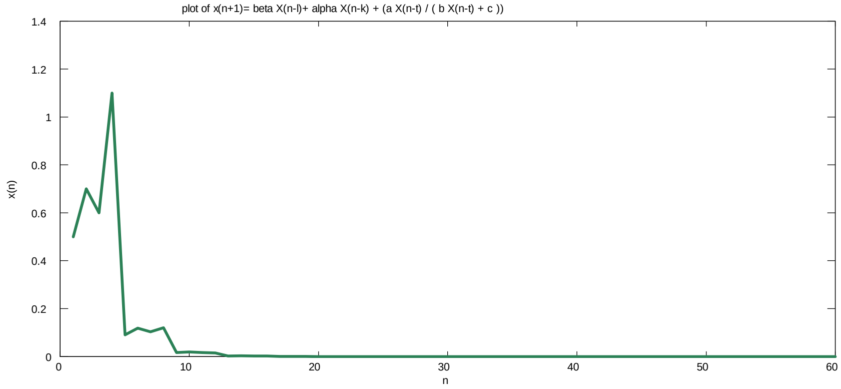

Figure 4. Plot the behavior of the zero solution of Eq. (1) is global stability.

Theorem 3.2. *The equilibrium point $\overline{x}_1$ is a global attractor of difference equation (1) if*

$$\beta+\alpha<1. \tag{6}$$

Proof: Suppose that ζ and η are real numbers and assume that $g:[\zeta,\eta]^3\longrightarrow[\zeta,\eta]$ is a function defined by

$$g(u,\ v,\ w)=\beta u+\alpha v+\tfrac{aw}{bw+c}.$$

Then

$$\tfrac{\partial g(u,\ v,\ w)}{\partial u}=\beta,\ \tfrac{\partial g(u,\ v,\ w)}{\partial v}=\alpha \text{ and } \tfrac{\partial g(u,\ v,\ w)}{\partial w}=\tfrac{ac}{(bw+c)^2}.$$

Now, we can see that the function $g(u,\ v,\ w)$ increasing in $u,\ v$ and w.

Let (m,M) be a solution of the system $M=g(M,\ M,\ M)$ and $m=g(m,\ m,\ m)$. Then from Eq. (1), we see that

$$M=\beta M+\alpha M+\frac{aM}{bM+c},\ m=\beta m+\alpha m+\frac{am}{bm+c},$$

thus

$$\begin{aligned} b(1\alpha-\beta)M^2-c(1-\alpha-\beta)M &= aM, \\ b(1\alpha-\beta)m^2-c(1-\alpha-\beta)m &= am. \end{aligned}$$

Subtracting we obtain

$$\begin{aligned} b(1-\alpha-\beta)(M^2-m^2)-(a+c(1-\alpha-\beta))(M-m) &= 0, \\ (M-m)\{b(1-\alpha-\beta)(M+m)-a-c(1-\alpha-\beta)\} &= 0 \end{aligned}$$

under the condition $0 \neq b(1-\alpha-\beta)$, we see that

$$M = m.$$

According to Theorem 1.15 page 18 in [1], then $\overline{x}_1$ is a global attractor of Eq. (1). This completes the proof.

Example 5. The solution of the difference equation (1) is global stability when $l=2$, $k=1$, $t=3$, $\beta=0.1$, $\alpha=0.2$, $a=2$, $b=1$ and $c=0.01$ and the initial conditions $x_{-3}=0.2$, $x_{-2}=0.4$, $x_{-1}=0.6$ and $x_0=0.1$. (See Fig. 5).

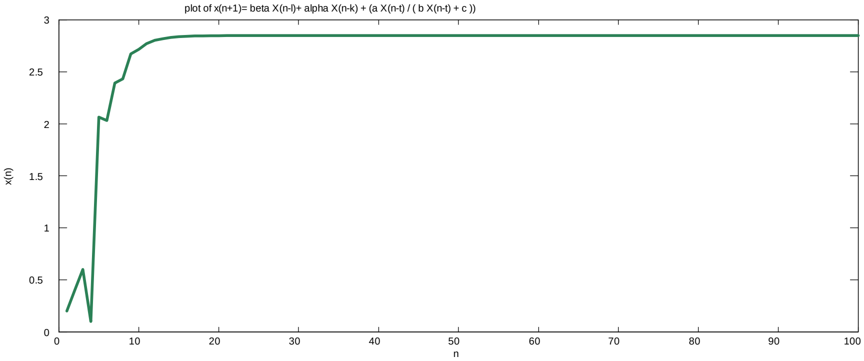

Figure 5. Plot the behavior of the solution of Eq. (1) is global stability.

4. PERIODIC SOLUTIONS

THEOREM 4.1. *Let l, k and t are both odd positive integers then for all β, α, a, b and c are positive real numbers, then Eq. (1) has a prime period two solution if*

$$\alpha+\beta<1 \text{ and } c(1-\alpha-\beta)<a. \tag{7}$$

Proof: First, suppose that there exists distinct nonnegative solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when l, k and t are both odd, then $x_{n+1}=x_{n-l}=x_{n-k}=x_{n-t}=P$. It follows Eq. (1) that

$$P=\beta P+\alpha P+\tfrac{aP}{bP+c} \text{ and } Q=\beta Q+\alpha Q+\tfrac{aQ}{bQ+c}.$$

Therefore,

$$b(1-\alpha-\beta)P^2+(c(1-\alpha-\beta)-a)P=0, \tag{8}$$

$$b(1-\alpha-\beta)Q^2+(c(1-\alpha-\beta)-a)Q=0, \tag{9}$$

Subtracting (9) from (8) gives

$$P+Q=\tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} \tag{10}$$

Again, adding (8) and (9) yields

$$PQ = 0. \tag{11}$$

where $a > c(1-\alpha-\beta)$ and $1 > \alpha+\beta$. Let P and Q are the two distinct nonnegative real roots of the quadratic

$$b(1-\alpha-\beta)t^2 - (a - c(1-\alpha-\beta))t = 0, \tag{12}$$

and so

$$a - c(1-\alpha-\beta) > 0 \text{ and } 1-\alpha-\beta > 0, \tag{13}$$

from Inequality (13), we obtain Inequality (7).

Second suppose that Inequality (7) is true. We will show that Eq. (1) has a prime period two solution.

Therefore P and Q are distinct nonnegative real numbers.

Set

$$x_{-l} = P,\ x_{-k} = P,\ x_{-t} = P,\ ,...,\ x_{-3} = P,\ x_{-2} = Q,\ x_{-1} = P,\ x_0 = Q.$$

We would like to show that

$$x_1 = x_{-1} = P = \tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} \quad \text{and} \quad x_2 = x_0 = Q = 0.$$

It follows from Eq. (1) that

$$\begin{aligned}
x_1 &= \beta P + \alpha P + \tfrac{aP}{bP+c} = (\beta+\alpha)P + \frac{a\left(\frac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right)}{b\left(\frac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right)+c},\\
&= (\beta+\alpha)\left(\tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right) + \tfrac{a(a-c(1-\alpha-\beta))}{b(a-c(1-\alpha-\beta))+cb(1-\alpha-\beta)} = (\beta+\alpha)\left(\tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)}\right) + \tfrac{a(a-c(1-\alpha-\beta))}{ab},\\
&= \tfrac{(\beta+\alpha)(a-c(1-\alpha-\beta))+(1-\alpha-\beta)(a-c(1-\alpha-\beta))}{b(1-\alpha-\beta)} = \tfrac{(a-c(1-\alpha-\beta))(\beta+\alpha+1-\alpha-\beta)}{b(1-\alpha-\beta)} = \tfrac{a-c(1-\alpha-\beta)}{b(1-\alpha-\beta)} = P.
\end{aligned}$$

and

$$x_2 = \beta Q + \alpha Q + \tfrac{aQ}{bQ+c} = 0 = Q,$$

Then by induction we get

$$x_{2n} = Q \quad \text{and} \quad x_{2n+1} = P \quad \text{for all} \quad n \geq -2.$$

Thus Eq. (1) has the prime period two solution

$$...,\ P,\ Q,\ P,\ Q,\ ...,$$

where P and Q are the distinct nonnegative roots of the quadratic Eq. (12) and the proof is complete.

THEOREM 4.2. *Let l, k and t are both even positive integers then for all β, α, a, b and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Let that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when l, k and t are both even, then $x_{n+1} = P$ and $x_{n-l} = x_{n-k} = x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha Q + \tfrac{aQ}{bQ+c} \text{ and } Q = \beta P + \alpha P + \tfrac{aP}{bP+c}.$$

Therefore,

$$bPQ + cP = b(\beta+\alpha)Q^2 + (a + c(\beta+\alpha))Q, \tag{14}$$

$$bPQ + cQ = b(\beta+\alpha)P^2 + (a + c(\beta+\alpha))P, \tag{15}$$

Subtracting (15) from (14) gives

$$P + Q = -\tfrac{a+c(1+\beta+\alpha)}{b(\beta+\alpha)} \tag{16}$$

Again, adding (14) and (15) yields

$$PQ = \tfrac{c(a+c(1+\beta+\alpha))}{b^2(\beta+\alpha)(1+\beta+\alpha)}. \tag{17}$$

From (16) and (17), we have

$$(P+Q)\,PQ = -\tfrac{c(a+c(1+\beta+\alpha)+c)^2}{b^3(\beta+\alpha)^2(1+\beta+\alpha)} < 0$$

This contradicts the hypothesis that both P and Q are positive. Thus, the proof is now completed.

Theorem 4.3. *Let $l,\ k$ are even and t is odd positive integers then for all $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Let that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $l,\ k$ are even and t is odd, then $x_{n+1} = x_{n-t} = P$ and $x_{n-l} = x_{n-k} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha Q + \tfrac{aP}{bP+c} \text{ and } Q = \beta P + \alpha P + \tfrac{aQ}{bQ+c}.$$

Therefore,

$$bP^2 + cP = b\left(\alpha+\beta\right)PQ + c\left(\alpha+\beta\right)Q + aP, \tag{18}$$

$$bQ^2 + cQ = b\left(\alpha+\beta\right)PQ + c\left(\alpha+\beta\right)P + aQ, \tag{19}$$

By subtracting (18) from (19), we deduce

$$P + Q = \tfrac{a-c(1+\alpha+\beta)}{b} \tag{20}$$

Again, by adding (18) and (19), we get

$$PQ = -\left(\tfrac{c(\alpha+\beta)(a-c(1+\alpha+\beta))}{b^2(\alpha+\beta+1)}\right). \tag{21}$$

If $a > c\left(1+\alpha+\beta\right)$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Therefor, the proof is completed.

Theorem 4.4. *If $l,\ t$ are even and k is odd positive integers then Eq. (1) has no positive prime period two solution.*

Proof: Let there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $l,\ t$ are even and k is odd, then $x_{n+1} = x_{n-k} = P$ and $x_{n-l} = x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha P + \tfrac{aQ}{bQ+c} \text{ and } Q = \beta P + \alpha Q + \tfrac{aP}{bP+c}.$$

Therefore,

$$b\left(1-\alpha\right)PQ + c\left(1-\alpha\right)P = b\beta Q^2 + \left(c\beta + a\right)Q, \tag{22}$$

$$b\left(1-\alpha\right)PQ + c\left(1-\alpha\right)Q = b\beta P^2 + \left(c\beta + a\right)P, \tag{23}$$

By subtracting (23) from (22), we get

$$P + Q = -\left(\tfrac{a+c(1-\alpha+\beta)}{b\beta}\right) \tag{24}$$

While, by adding (22) and (23), we deduce

$$PQ = \tfrac{c(1-\alpha)(a+c(1-\alpha+\beta))}{b^2\beta(1-\alpha+\beta)}. \tag{25}$$

If $\alpha < 1$ and $\alpha < 1+\beta$ then from (24) and (25), we have

$$PQ\left(P+Q\right) = -\tfrac{c(1-\alpha)(a+c(1-\alpha+\beta))^2}{b^3\beta^2(1-\alpha+\beta)} < 0$$

This contradicts the hypothesis that both P, Q are positive. Thus, the proof is now completed.

Theorem 4.5. *Suppose that $k,\ t$ are even and l is odd positive integers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when $k,\ t$ are even and l is odd, then $x_{n+1} = x_{n-l} = P$ and $x_{n-k} = x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta P + \alpha Q + \tfrac{aQ}{bQ+c} \text{ and } Q = \beta Q + \alpha P + \tfrac{aP}{bP+c}.$$

Therefore,

$$b\left(1-\beta\right)PQ + c\left(1-\beta\right)P = b\alpha Q^2 + \left(c\alpha + a\right)Q, \tag{26}$$

$$b\left(1-\beta\right)PQ + c\left(1-\beta\right)Q = b\alpha P^2 + \left(c\alpha + a\right)P, \tag{27}$$

By subtracting (27) from (26), we have

$$P + Q = -\left(\tfrac{a+c(1+\alpha-\beta)}{b\alpha}\right) \tag{28}$$

Again, by adding (26) and (27),we deduce

$$PQ = \tfrac{c(1-\beta)(a+c(1+\alpha-\beta))}{b^2\alpha(1+\alpha-\beta)}. \tag{29}$$

If $\beta < 1$ and $\beta < 1 + \alpha$ then from (28) and (29), we have

$$PQ\,(P+Q) = -\tfrac{c(1-\beta)(a+c(1+\alpha-\beta))^2}{b^2\alpha^2(1+\alpha-\beta)} < 0$$

This contradicts the hypothesis that both P, Q are positive. Thus, the proof is now completed.

THEOREM 4.6. *Let k is even and $l,\ t$ are odd positive integers then for all $\beta,\ \alpha,\ a,\ b$ and c are positive real numbers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when k is even and $l,\ t$ are odd, then $x_{n+1} = x_{n-l} = x_{n-t} = P$ and $x_{n-k} = Q$. It follows Eq. (1) that

$$P = \beta P + \alpha Q + \tfrac{aP}{bP+c} \text{ and } Q = \beta Q + \alpha P + \tfrac{aQ}{bQ+c}.$$

Therefore,

$$b\,(1-\beta)\,P^2 + c\,(1-\beta)\,P = b\alpha PQ + c\alpha Q + aP, \tag{30}$$

$$b\,(1-\beta)\,Q^2 + c\,(1-\beta)\,Q = b\alpha PQ + c\alpha P + aQ, \tag{31}$$

By subtracting (31) from (30), we get

$$P + Q = \tfrac{a-c(1+\alpha-\beta)}{b(1-\beta)} \tag{32}$$

Again, by adding (30) and (31), we have

$$PQ = -\tfrac{c\alpha(a-c(1+\alpha-\beta))}{b^2(1-\beta)(1+\alpha-\beta)}. \tag{33}$$

where $\beta < 1$, $\beta < 1 + \alpha$ and $c\,(1 + \alpha - \beta) < a$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Therefor, the proof is completed.

THEOREM 4.7. *If l is even and $k,\ t$ are odd positive integers, then Eq. (1) has no positive prime period two solution.*

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when l is even and $k,\ t$ are odd, then $x_{n+1} = x_{n-k} = x_{n-t} = P$ and $x_{n-l} = Q$. It follows Eq. (1) that

$$P = \beta Q + \alpha P + \tfrac{aP}{bP+c} \text{ and } Q = \beta P + \alpha Q + \tfrac{aQ}{bQ+c}.$$

Therefore,

$$b\,(1-\alpha)\,P^2 + c\,(1-\alpha)\,P = b\beta PQ + c\beta Q + aP, \tag{31}$$

$$b\,(1-\alpha)\,Q^2 + c\,(1-\alpha)\,Q = b\beta PQ + c\beta P + aQ, \tag{32}$$

Subtracting (32) from (31) gives

$$P + Q = \tfrac{a-c(1+\beta-\alpha)}{b(1-\alpha)} \tag{33}$$

Again, adding (31) and (32) yields

$$PQ = -\frac{c\beta(a-c(1+\beta-\alpha))}{b^2(1-\alpha)(1+\beta-\alpha)}. \tag{34}$$

If $\alpha < 1$, $\alpha < 1+\beta$ and $c(1+\beta-\alpha) < a$, then PQ is negative. But P, Q are both positive, and we have a contradiction. Thus, the proof is completed.

THEOREM 4.8. *Let t is even and l, k are odd positive integers. If*

$$c(1-\alpha-\beta)+a \neq 0,$$

then Eq. (1) has no prime period two solution.

Proof: Assume that there exists distinct positive solution P and Q, such that

$$...P,\ Q,\ P,\ Q, ...,$$

is a prime period two solution of Eq.(1).

We see from Eq. (1) when t is even and l, k are odd, then $x_{n+1} = x_{n-k} = x_{n-l} = P$ and $x_{n-t} = Q$. It follows Eq. (1) that

$$P = \beta P + \alpha P + \frac{aQ}{bQ+c} \text{ and } Q = \beta Q + \alpha Q + \frac{aP}{bP+c}.$$

Therefore,

$$b(1-\alpha-\beta)PQ + c(1-\alpha-\beta)P = aQ, \tag{35}$$

$$b(1-\alpha-\beta)PQ + c(1-\alpha-\beta)Q = aP, \tag{36}$$

Subtracting (47) from (46) gives

$$(c(1-\alpha-\beta)+a)(P-Q) = 0$$

Since $c(1-\alpha-\beta)+a \neq 0$, then $P = Q$. This is a contradiction. Thus, the proof is completed.

Example 6. Figure (6) shows the Eq. (1) has a period two solution when $l = 1$, $k = 3$, $t = 5$, $\beta = 0.1$, $\alpha = 0.2$, $a = 0.5$, $b = 0.07$ and $c = 0.05$ and the initial conditions $x_{-5} = P$, $x_{-4} = Q$, $x_{-3} = P$, $x_{-2} = Q$, $x_{-1} = P$ and $x_0 = Q$ where $P = \frac{a-c(1+\beta-\alpha)}{b(1-\alpha)}$ and $Q = 0$.

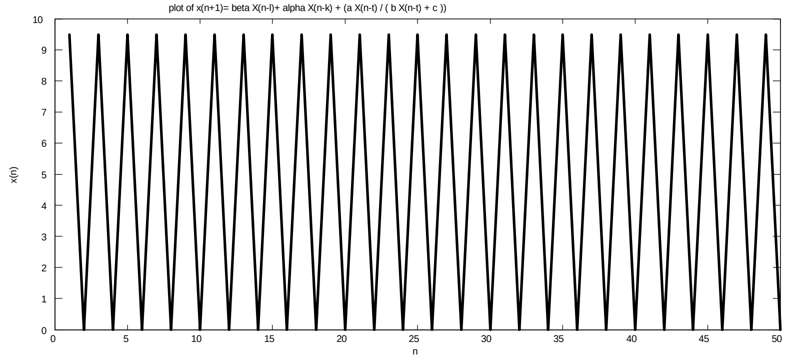

Figure 6. Sketch the solution of Eq. (1) has a period two solution.

Example 7. Consider $l = 5$, $k = 2$, $t = 4$, $\beta = 0.6$, $\alpha = 0.2$, $a = 0.4$, $b = 0.7$ and $c = 0.5$ and the initial conditions $x_{-5} = 1.2$, $x_{-4} = 1.4$, $x_{-3} = 0.6$, $x_{-2} = 1.1$, $x_{-1} = 0.3$ and $x_0 = 0.8$ the solution of Eq. (1) has no period two solution (See Fig. 7).

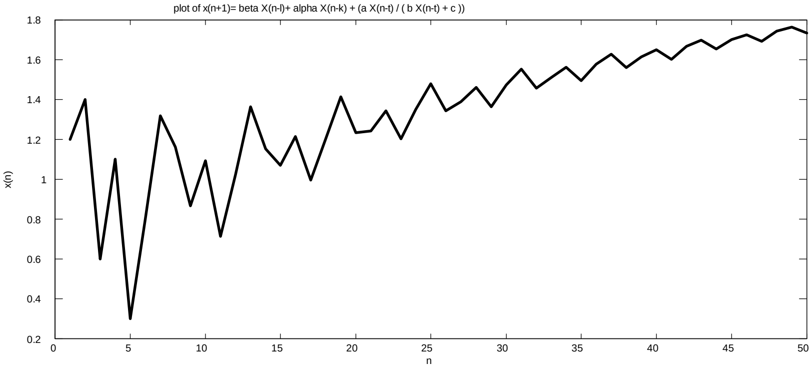

Figure 7. Draw the solution of Eq. (1) has no periodic.

5. BOUNDEDNESS OF THE SOLUTIONS

In this section, we investigate the boundedness nature of the positive solutions of equation (1).

Theorem 5.1. *Every solution of difference equation (1) is bounded if* $\beta+\alpha<1$.

Proof: Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned} x_{n+1} &= \beta x_{n-l}+\alpha x_{n-k}+\tfrac{a x_{n-t}}{b x_{n-t}+c}, \\ &\leqslant \beta x_{n-l}+\alpha x_{n-k}+\tfrac{a x_{n-t}}{b x_{n-t}}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{a}{b} \quad \text{for all } n\geq 1. \end{aligned}$$

By using a comparison, we can right hand side as follows

$$t_{n+1}=\beta t_{n-l}+\alpha t_{n-k}+\tfrac{a}{b}.$$

and this equation is locally asymptotically stable if $\beta+\alpha<1$, and converges to the equilibrium point $\overline{t}=\frac{a}{b(1-\beta-\alpha)}$.Therefore

$$\lim_{n\to\infty}\sup x_n \leqslant \tfrac{a}{b(1-\beta-\alpha)}.$$

Thus the solution is bounded.

Theorem 5.2. *Every solution of difference equation (1) is unbounded if* $\beta>1 or\ \alpha>1$.

Proof: Let $\{x_n\}_{n=-s}^{\infty}$ be a solution of Equation (1).Then from Equation (1) we see that

$$x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{a x_{n-t}}{b x_{n-t}+c}>\beta x_{n-l} \quad f\text{or all} \quad n\geq 1.$$

We see that the right hand side can be written as follows

$$t_{n+1}=\beta t_{n-l}.$$

then

$$t_{ln+i}=\beta^n t_{l+i}+const., \qquad i=0,1,...,l,$$

and this equation is unstable because $\beta>1$, and $\lim\limits_{n\to\infty} t_n=\infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded from above.

Similarly from Equation (1) we see that

$$x_{n+1}=\beta x_{n-l}+\alpha x_{n-k}+\tfrac{a x_{n-t}}{b x_{n-t}+c}>\alpha x_{n-k} \ for\ all\ n\geq 1.$$

We see that the right hand side can be written as follows

$$t_{n+1}=\alpha t_{n-k}.$$

then

$$t_{kn+i} = \alpha^n t_{k+i} + const., \qquad i = 0, 1, ..., k,$$

and this equation is unstable because $\alpha > 1$, and $\lim_{n\to\infty} t_n = \infty$.Then by using ratio test $\{x_n\}_{n=-s}^{\infty}$ is unbounded from above. Thus, the proof is now completed.

Example 8. We assume $l = 2$, $k = 1$, $t = 3$, $\beta = 0.4$, $\alpha = 1.2$, $a = 3$, $b = 0.7$ and $c = 0.6$ and the initial conditions $x_{-3} = 0.2$, $x_{-2} = 0.4$, $x_{-1} = 0.6$ and $x_0 = 0.1$, the solution of the difference equation (1) is unbounded (see Fig. 8).

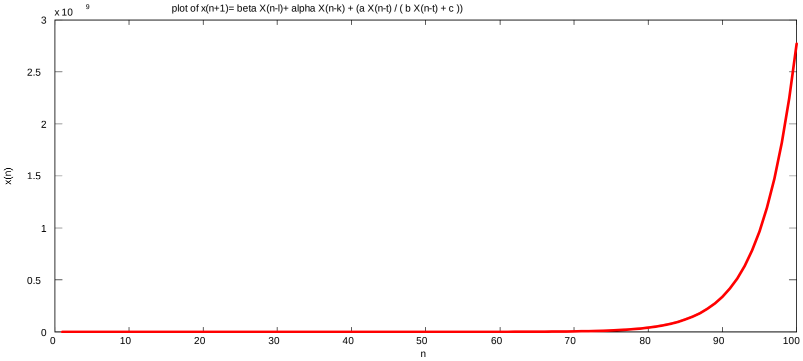

Figure 8. Plot the behavior of the solution of Eq. (1) is unbounded.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Hermite-Hadamard inequality for Sugeno integral based on harmonically convex functions

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Abstract. For the classical Hermite-Hadamard inequality of harmonically convex functions, i.e.,

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

an upper bound is proved in the framework of the Sugeno integral.

Keywords: the Sugeno integral; the Hermite-Hadamard inequality; harmonically convex function.

2010 Mathematics Subject Classification: 26D15, 28A25, 28E15, 39B62.

1. Introduction

One of the most important integral inequalities which is related to harmonically convex functions is classical Hermite-Hadamard integral inequality. Double inequality

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}.$$

is known as Hermite-Hadamard integral inequality for harmonically convex functions, where $f \in L([a,b])$ [7, 5]. When we are trying to obtain these inequalities in the spirit of monotone measures and non-additive integrals, we get different results than the classic form.

The concept of the fuzzy integral was introduced and initially examined by Sugeno [17]. Further theoretical investigations of the integral and its generalizations have been pursued by many researchers [14, 15, 12, 2, 8, 1]. The study of inequalities for the Sugeno integral was initiated by Román-Flores and Chalco-Cano [13]. In this article, at the first we prove some Hermite-Hadamard type inequalities for harmonically convex functions in the case of non-additive integrals. Consequently, upper bound for these functions are established. In fact, the main purpose of this article is to obtain an approximation for non-solvable integral of this type.

This paper is organized as follows. Some necessary preliminaries are presented in Section 2. We address the essential problems in Sections 3 and upper bound for the Sugeno integral based on a harmonically convex function is presented. Finally, a conclusion is drawn and a problem for further investigations is given in Section 4.

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2. **Preliminaries**

In this section, we are going to review some well known results from the theory of non-additive measures.

Definition 2.1. [8, 18] Let Σ be a Σ-algebra of subsets of X and let $\mu : \Sigma \to [0, \infty)$ be a non-negative, extended real-valued set function, we say that μ is a monotone measure (or fuzzy measure) iff:

- **(FM1):** $\mu(\emptyset) = 0$;
- **(FM2):** $E, F \in \Sigma$ and $E \subseteq F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);
- **(FM3):** $(E_n) \subseteq \Sigma, \quad E_1 \subseteq E_2 \subseteq \ldots$ imply $\lim_{n\to+\infty} \mu(E_n) = \mu(\bigcup_{i=1}^{\infty} E_i)$ (continuity from below);
- **(FM4):** $(E_n) \subseteq \Sigma, \quad E_1 \supseteq E_2 \supseteq \ldots, \mu(E_1) < \infty$ imply $\lim_{n\to+\infty} \mu(E_n) = \mu(\bigcap_{i=1}^{\infty} E_i)$ (continuity from above).

Let (X, Σ, μ) be a monotone measure space and f is a non-negative real-valued function on X. We denote the set of all non-negative measurable functions f by $\mathcal{F}_+$ and F_α denote the set $\{x \in X \mid f(x) \geq \alpha\}$, the α-level of f, for $\alpha \geq 0$. $F_0 = \{x \in X \mid f(x) > 0\} = supp(f)$ is the support of f. We know that: $\alpha \leq \beta \Rightarrow \{f \geq \beta\} \subseteq \{f \geq \alpha\}$.

Definition 2.2. [17, 8, 18] Let μ be a monotone measure (or fuzzy measure) on (X, Σ). If $f \in \mathcal{F}_+$ and $A \in \Sigma$, then the Sugeno integral (or fuzzy integral) of f on A, with respect to the monotone measure μ is defined by

$$⨍_A f d\mu := \bigvee_{\alpha\geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$, $\wedge$ denotes the operation sup and inf on $[0, \infty)$ respectively. In particular if $A = X$, then

$$⨍_X f d\mu := ⨍ f d\mu = \bigvee_{\alpha\geq 0} (\alpha \wedge \mu(F_\alpha)).$$

The following properties of the Sugeno integral are well known and can be found in [18, 19].

Proposition 2.3. Let (X, Σ, μ) be a fuzzy measure space, with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. We have

1. $⨍_A f d\mu \leq \mu(A)$;
2. $⨍_A k d\mu \leq k \wedge \mu(A)$, for k non-negative constant;
3. if $f \leq g$ on A, then $⨍_A f d\mu \leq ⨍_A g d\mu$;
4. if $A \subset B$, then $⨍_A f d\mu \leq ⨍_B f d\mu$;
5. if $\mu(A) < \infty$, then $⨍_A f d\mu \geq \alpha \Leftrightarrow \mu(A \cap \{f \geq \alpha\}) \geq \alpha$;
6. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow ⨍_A f d\mu \leq \alpha$;
7. $⨍_A f d\mu < \alpha \Leftrightarrow$ there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) < \alpha$;
8. $⨍_A f d\mu > \alpha \Leftrightarrow$ there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.4. Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap F_\alpha)$. Then, due to (5) and (6) of Proposition 2.3, we have that

$$F(\alpha) = \alpha \Rightarrow ⨍_A f d\mu = \alpha.$$

Thus, from a numerical point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$.

The following proposition shows how to transform a Sugeno integral $⨍_A f \mathrm{d}\mu$, which is defined on a monotone measure space (X, Σ, μ), into another Sugeno integral $⨍ g \mathrm{d}m$ defined on the Lebesgue measure space $([0, \infty), \overline{B_+}, m)$, where $\overline{B_+}$ is the class of all Borel sets in $[0, \infty)$ and m is the Lebesgue measure.

Proposition 2.5. [18] For any $A \in \Sigma$

$$⨍_A f \mathrm{d}\mu = ⨍ \mu(A \cap F_\alpha) \mathrm{d}m,$$

where $F_\alpha = \{x \in X \mid f(x) \geq \alpha\}$ and m is the Lebesgue measure.

Definition 2.6. [16] A t-norm is a function $T : [0, 1] \times [0, 1] \to [0, 1]$ satisfying the following conditions:

(T_1): $T(x, 1) = T(1, x) = x$ for any $x \in [0, 1]$;

(T_2): For any $x_1, x_2, y_1, y_2 \in [0, 1]$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, $T(x_1, y_1) \leq T(x_2, y_2)$;

(T_3): $T(x, y) = T(y, x)$ for any $x, y \in [0, 1]$;

(T_4): $T(T(x, y), z) = T(x, T(y, z))$ for any $x, y, z \in [0, 1]$.

A function $S : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-conorm [9] if there is a t-norm T such that $S(x, y) = 1 - T(1 - x, 1 - y)$.

Example 2.7. The following functions are t-norms:

1: $T_M(x, y) = x \wedge y$.

2: $T_P(x, y) = x.y$.

3: $T_L(x, y) = (x + y - 1) \vee 0$.

Hereafter, we assume that (X, Σ, μ) is a monotone measure space. To simplify the calculation of the Sugeno integral, for a given $f \in \mathcal{F}_+(X)$ and $A \in \Sigma$, we write

$$\Gamma = \{\alpha : \alpha \geq 0, \ \ \mu(A \cap F_\alpha) > \mu(A \cap F_\beta) \ \ for \ \ any \ \ \beta > \alpha\}.$$

It is easy to see that

$$⨍_A f \mathrm{d}\mu = \bigvee_{\alpha \in \Gamma} (\alpha \wedge \mu(A \cap F_\alpha)).$$

Remark 2.8. A binary operator T on $[0, 1]$ is called a t-seminorm[16] if it satisfies the above condition (T_1) and (T_2). Notice that if T is a t-seminorm, for any $x, y \in [0, 1]$, we have $T(x, y) \leq T(x, 1) = x$ and $T(x, y) \leq T(1, y) = y$, and consequently, $T(x, y) \leq T_M(x, y)$.

By using the concept of t-seminorm, García and Álvarez [16] proposed the following family of fuzzy integral.

Definition 2.9. Let T be a t-seminorm. Then the seminormed Sugeno's fuzzy integral of a function $f \in \mathcal{F}_+$ over $A \in \Sigma$ with respect to T and the fuzzy measure μ is defined by

$$\int_{T,A} f \mathrm{d}\mu = \bigvee_{\alpha \in [0,1]} T(\alpha, \mu(A \cap F_\alpha)).$$

Notice that the Sugeno integral of $f \in \mathcal{F}_+$ over $A \in \Sigma$ is the seminormed Sugeno's fuzzy integral of f over $A \in \Sigma$ with respect to the t-seminorm T_M.

Proposition 2.10. (García and Álvarez [16])Let (X, Σ, μ) be a monotone measure space and T be a t-seminorm. Then,

1: For any $A \in \Sigma$ and $f, g \in \mathcal{F}_+$ with $f \le g$, we have
$$\int_{T,A} f \mathrm{d}\mu \le \int_{T,A} g \mathrm{d}\mu.$$

2: For $A, B \in \Sigma$ with $A \subset B$ and any $f \in \mathcal{F}_+$,
$$\int_{T,A} f \mathrm{d}\mu \le \int_{T,B} f \mathrm{d}\mu.$$

Definition 2.11. [7] Let $I \subset \mathbb{R} - \{0\}$ is a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex on I if the inequality
$$f\left(\frac{ab}{ta + (1-t)b}\right) \le tf(b) + (1-t)f(a) \tag{2.1}$$
holds, for all $a, b \in I$ and $t \in [0, 1]$. If the inequality (2.1) is reversed, then f is said to be harmonically concave. We note that for $t = \frac{1}{2}$, we have the definition of Jensen type of harmonic convex functions, that is
$$f\left(\frac{2ab}{a+b}\right) \le \frac{f(a) + f(b)}{2}, \ \forall a, b \in I.$$

Proposition 2.12. [7] Let $I \subset \mathbb{R} - \{0\}$ be a real interval and $f : I \to \mathbb{R}$ is function, then:

1: if $I \subset (0, +\infty)$ and f is convex and nondecreasing, then f is harmonically convex.
2: if $I \subset (0, +\infty)$ and f is harmonically convex and nonincreasing, then f is convex.
3: if $I \subset (-\infty, 0)$ and f is harmonically convex and nondecreasing, then f is convex.
4: if $I \subset (-\infty, 0)$ and f is convex and nonincreasing, then f is harmonically convex.

Proposition 2.13. [4] If $[a, b] \subset I \subseteq (0, \infty)$ and we consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right] \to \mathbb{R}$ defined by $g(t) = f(\frac{1}{t})$, then f is harmonically convex on $[a, b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b}, \frac{1}{a}\right]$.

Proposition 2.14. [6] A function $f : (0, \infty) \to \mathbb{R}$ is harmonically convex if and only if $xf(x)$ is convex.

Theorem 2.15. Let $f : [a, b] \subseteq (0, \infty) \to [0, +\infty)$ be a convex function with $f(a) \ne f(b)$.Then
$$⨍_a^b f \mathrm{d}\mu \le \bigvee_{\alpha \in \Gamma} \left(\alpha \wedge \mu\left([a, b] \cap \left\{x \ge \frac{\alpha(b-a) + af(b) - bf(a)}{f(b) - f(a)}\right\}\right)\right)$$
where $\Gamma = [f(a), f(b))$ for $f(b) > f(a)$ and $\Gamma = [f(b), f(a))$ for $f(a) > f(b)$.

Proof. As f is convex function, for $x \in [a, b]$ we have,
$$f(x) = f\left((1 - \frac{x-a}{b-a})a + \frac{x-a}{b-a}b\right) \le (1 - \frac{x-a}{b-a})f(a) + \frac{x-a}{b-a}f(b)$$

and so by (3) of Proposition 2.3

$$\fint_a^b f\mathrm{d}\mu \leq \fint_a^b \left((1-\frac{x-a}{b-a})f(a)+\frac{x-a}{b-a}f(b)\right)\mathrm{d}\mu = \fint_a^b g(x)\mathrm{d}\mu.$$

In order to calculate the integral in the right hard part of the last inequality, we consider the distribution function $F(\alpha)$ given by

$$F(\alpha)=\mu([a,b]\cap\{g\geq\alpha\})=\mu\left([a,b]\cap\left\{\frac{b-x}{b-a}f(a)+\frac{x-a}{b-a}f(b)\geq\alpha\right\}\right).$$

If $f(a)<f(b)$, then

$$F(\alpha)=\mu\left([a,b]\cap\left\{x\geq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)=\mu\left([\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)},b]\right).$$

Thus $\Gamma=[f(a),f(b))$ and we only consider $\alpha\in[f(a),f(b))$.

If $f(a)>f(b)$, then

$$F(\alpha)=\mu\left([a,b]\cap\left\{x\leq\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}\right\}\right)=\mu\left([a,\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\right).$$

Thus $\Gamma=[f(b),f(a))$ and only need $\alpha\in[f(b),f(a))$.

This completes the proof. □

Remark 2.16. In the case $f(a)=f(b)$ in Theorem 2.15, we have $g(x)=f(x)$ and so

$$\fint_a^b f\mathrm{d}\mu\leq\fint_a^b g\mathrm{d}\mu=\fint_a^b f(a)\mathrm{d}\mu=f(a)\wedge\mu([a,b]).$$

Corollary 2.17. Let $f:[a,b]\subseteq(0,\infty)\to(0,\infty)$ be a convex function and Σ be the Borel field and μ be the Lebesgue measure on $X=\mathbb{R}$, then

$$\fint_a^b f\mathrm{d}\mu\leq\begin{cases}\bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge(b-\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)})\right) & ,f(a)<f(b)\\ f(a)\wedge(b-a) & ,f(a)=f(b)\\ \bigvee_{\alpha\in[f(b),f(a))}\left(\alpha\wedge(\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}-a)\right) & ,f(a)>f(b)\end{cases}$$

So

$$\fint_a^b f\mathrm{d}\mu\leq\begin{cases}\frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}\wedge(b-a) & ,f(a)<f(b)\\ f(a)\wedge(b-a) & ,f(a)=f(b)\\ \frac{(b-a)f(a)}{f(a)-f(b)+(b-a)}\wedge(b-a) & ,f(a)>f(b).\end{cases}$$

Proof. In the case where $f(a)<f(b)$, we have

$$\bigvee_{\alpha\in[f(a),f(b))}\left(\alpha\wedge(b-\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)})\right)=\frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}.$$

In fact, $\alpha = \frac{(b-a)f(b)}{f(b)-f(a)+(b-a)}$ is as the solution of the equation $F(\alpha) = \alpha$, where F is the distribution function. So taking into account (1) of Proposition 2.3 ($\fint_a^b f\mathrm{d}\mu \leq \mu([a,b]) = b-a$) and Remark 2.4 we have

$$\fint_a^b f\mathrm{d}\mu \leq \frac{(b-a)f(b)}{f(b)-f(a)+(b-a)} \wedge (b-a).$$

Proofs the other cases is analogous. ☐

Note that Corollary 2.17 is the same as the Sadarangani Theorem [3].

3. **Main Results**

Let $I \subset \mathbb{R} - \{0\}$ be a harmonically convex function and $a, b \in I$ with $a < b$ and $f \in L([a,b])$. The following inequalities

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{3.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for harmonically convex functions.

Unfortunately, as we will see in the following example, in general, the Hermite-Hadamard inequality is not valid in the fuzzy context.

Example 3.1. Let μ be the usual Lebesgue measure on $\mathbb{R}$ and the function $f(x) = \frac{3}{7}x^2$ on $X = [\frac{1}{2}, 1]$. Obviously, this function is convex and nondecreasing as a result f is harmonically convex function on $[\frac{1}{2}, 1]$. With the above inequality we have

$$\fint_{\frac{1}{2}}^1 \frac{f(x)}{x^2}dx = \fint_{\frac{1}{2}}^1 \frac{3}{7}dx = \frac{3}{7} \wedge \mu([\frac{1}{2}, 1]) = \frac{3}{7} \simeq 0.42.$$

on the other hand, $\frac{f(\frac{1}{2})+f(1)}{2} = \frac{15}{56} \simeq 0.26$.

This proves that the right-hand side of inequality (3.1) is not satisfied for the Sugeno integrals.

The aim of this work is to show a the Hermite-Hadamard type inequality for the Sugeno integral in the case where f is a harmonically convex function.

Lemma 3.2. Let $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave, then

$$\fint_a^b f\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[f(a),f(b))} \left(\alpha \wedge \mu[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}, b]\right) & , f(a) < f(b) \\ f(a) \wedge \mu([a,b]) & , f(a) = f(b) \\ \bigvee_{\alpha\in[f(b),f(a))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\right) & , f(a) > f(b). \end{cases}$$

Proof. Since $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ is harmonically convex function on the interval $[a,b]$, then by Proposition 2.13 the function $g : [\frac{1}{b}, \frac{1}{a}] \to \mathbb{R}$, $g(s) = f(\frac{1}{s})$ is convex on $[\frac{1}{b}, \frac{1}{a}]$. Obviously for any $x \in [a,b]$, $f(x) = g(\frac{1}{x})$, and therefor applying Theorem 2.15 to g, we have

$$\fint_a^b f(x)\mathrm{d}\mu = \fint_a^b g(\frac{1}{x})\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[g(\frac{1}{a}),g(\frac{1}{b}))}\Big(\alpha\wedge\mu[\frac{\alpha(b-a)+ag(\frac{1}{b})-bg(\frac{1}{a})}{g(\frac{1}{b})-g(\frac{1}{a})},b]\Big) & ,g(\frac{1}{a})<g(\frac{1}{b}) \\ g(\frac{1}{a})\wedge\mu([a,b]) & ,g(\frac{1}{a})=g(\frac{1}{b}) \\ \bigvee_{\alpha\in[g(\frac{1}{b}),g(\frac{1}{a}))}\Big(\alpha\wedge\mu[a,\frac{\alpha(b-a)+ag(\frac{1}{b})-bg(\frac{1}{a})}{g(\frac{1}{b})-g(\frac{1}{a})}]\Big) & ,g(\frac{1}{a})>g(\frac{1}{b}) \end{cases}$$

$$= \begin{cases} \bigvee_{\alpha\in[f(a),f(b))}\Big(\alpha\wedge\mu[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)},b]\Big) & ,f(a)<f(b) \\ f(a)\wedge\mu([a,b]) & ,f(a)=f(b) \\ \bigvee_{\alpha\in[f(b),f(a))}\Big(\alpha\wedge\mu[a,\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}]\Big) & ,f(a)>f(b). \end{cases}$$

□

Corollary 3.3. Let $f:[a,b]\subseteq(0,\infty)\to(0,\infty)$ be a harmonically convex function which is not concave, Σ be the Borel field and μ be the Lebesgue measure on $X=\mathbb{R}$, then

$$\fint_a^b f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(b)}{f(b)-f(a)+b-a}\wedge(b-a) & ,f(a)<f(b) \\ f(a)\wedge(b-a) & ,f(a)=f(b) \\ \frac{(b-a)f(a)}{f(a)-f(b)+b-a}\wedge(b-a) & ,f(a)>f(b). \end{cases}$$

Remark 3.4. If $[a,b]\subseteq(0,\infty)$ and f is harmonically convex and nonincreasing, then taking into account (2) of Proposition 2.12 the function f is convex and hance the upper bound for the Sugeno integral of f mentioned in article "Hermite-Hadamard inequality for fuzzy integral", were written by K. sadarangani is established.

Remark 3.5. If $[a,b]\subseteq(-\infty,0)$ and f is harmonically convex and nondecreasing, then taking into account (3) of Proposition 2.12 the function f is convex and hance the upper bound for the Sugeno integral of f is established.

Example 3.6. Let μ be a Lebesgue measure and consider function $f(x) = e^{-\frac{1}{x}}$ on $[\frac{1}{3}, \frac{3}{4}]$. Obviously, this function is non-negative and harmonically convex but neither convex, nor concave. we have,

$$\begin{aligned}
\fint_{\frac{1}{3}}^{\frac{3}{4}} f d\mu &= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu \left([\frac{1}{3}, \frac{3}{4}] \cap \left\{ e^{-\frac{1}{x}} \geq \alpha \right\} \right) \right) \\
&= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu \left([\frac{1}{3}, \frac{3}{4}] \cap \left\{ -\frac{1}{x} \geq \ln \alpha \right\} \right) \right) \\
&= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu \left([\frac{1}{3}, \frac{3}{4}] \cap \{ -1 \geq x ln\alpha \} \right) \right) \\
&= \bigvee_{\alpha \geq 0} \left(\alpha \wedge \mu \left([\frac{1}{3}, \frac{3}{4}] \cap \left\{ x \geq \frac{-1}{ln\alpha} \right\} \right) \right).
\end{aligned}$$

As result with the solution of the equation

$$\frac{1}{ln\alpha} + \frac{3}{4} = \alpha$$

we conclude that $\alpha \simeq 0/175$. Then $\fint_{\frac{1}{3}}^{\frac{3}{4}} f d\mu \simeq 0/175$.

On the other hand, since $f(\frac{3}{4}) = \frac{1}{e^{\frac{4}{3}}}$ and $f(\frac{1}{3}) = \frac{1}{e^3}$. By Corollary 3.3, we have

$$\fint_{\frac{1}{3}}^{\frac{3}{4}} f d\mu \leq \frac{f(\frac{3}{4})(\frac{3}{4} - \frac{1}{3})}{f(\frac{3}{4}) - f(\frac{1}{3}) + (\frac{3}{4} - \frac{1}{3})} \wedge (\frac{3}{4} - \frac{1}{3})$$

$$\simeq 0/234 \wedge \frac{5}{12} = 0/234 \wedge 0/416 = 0/234$$

that is a logical inequality.

Example 3.7. The function $f(x) = x - \ln(x+1)$ is nondecreasing and harmonic convex function on $[\frac{1}{2}, 1]$. $f(1) = 1 - \ln 2$ and $f(\frac{1}{2}) = \frac{1}{2} - \ln(\frac{3}{2})$. As $f(1) > f(\frac{1}{2})$, Corollary 3.3 gives us,

$$\fint_{\frac{1}{2}}^{1} f d\mu \leq \frac{(1 - \frac{1}{2}) f(1)}{f(1) - f(\frac{1}{2}) + \frac{1}{2}} \wedge (\frac{1}{2}) \simeq 0.718 \wedge \frac{1}{2} = \frac{1}{2}.$$

Thus, we find an upper bound for the Sugeno integral of this function on $[\frac{1}{2}, 1]$.

Example 3.8. The function $f(x) = e^{x^2+x}$ is nondecreasing and harmonic convex function on $[1, 2]$ and $f(1) = e^2$ and $f(2) = e^5$. As follows we find an upper bound for the Sugeno integral of this function,

$$\fint_{1}^{2} e^{x^2+x} d\mu \leq \frac{e^5}{e^5 - e^2 + 1} \wedge (1) \simeq 1.0449 \wedge 1 = 1.$$

Remark 3.9. $f(x) = log(x)$ is a harmonically convex function but not convex, that is why in the Corollary 3.3, does not apply because it is concave. For concave function, we use the Sadarangani paper.

Corollary 3.10. Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a harmonically convex function which is not concave and $g : \mathbb{R} \to \mathbb{R}$ is a linear function, then $f \circ g$ is harmonically convex[10] and so,

$$\fint_a^b (f \circ g)\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha \in [f(g(a)),f(g(b)))} \left(\alpha \wedge \mu[\frac{\alpha(b-a)+af(g(b))-bf(g(a))}{f(g(b))-f(g(a))}, b]\right) & , f(g(a)) < f(g(b)) \\ f(g(a)) \wedge \mu([a,b]) & , f(g(a)) = f(g(b)) \\ \bigvee_{\alpha \in [f(g(b)),f(g(a)))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(g(b))-bf(g(a))}{af(g(b))-bf(g(a))}]\right) & , f(g(a)) > f(g(b)). \end{cases}$$

Corollary 3.11. Let $f : [a,b] \subseteq (0,\infty) \to \mathbb{R}$ be a harmonically convex function which is not concave and $g : \mathbb{R} \to \mathbb{R}$ is a linear function, Σ be the Borel field and μ be the Lebesgue measure on $X = \mathbb{R}$, then $f \circ g$ is harmonic convex function[10] and so,

$$\fint_a^b (f \circ g)\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(g(b))}{f(g(b))-f(g(a))+b-a} \wedge (b-a) & , f(g(a)) < f(g(b)) \\ f(g(a)) \wedge (b-a) & , f(g(a)) = f(g(b)) \\ \frac{(b-a)f(g(a))}{f(g(a))-f(g(b))+b-a} \wedge (b-a) & , f(g(a)) > f(g(b)). \end{cases}$$

Remark 3.12. In the case g be harmonic convex function and f be relative convex function, we know that $f \circ g$ is harmonically convex function [11]. Thus similar results of Corollary 3.10 and Corollary 3.11 hold.

Corollary 3.13. Let $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ be a harmonically convex function which is not concave function, Σ be the Borel field and μ be the Lebesgue measure on $X = \mathbb{R}$, then

$$\int_{T_P,[a,b]} f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)^2 f(b)}{f(b)-f(a)+b-a} & , f(a) < f(b) \\ (b-a)f(a) & , f(a) = f(b) \\ \frac{(b-a)^2 f(a)}{f(a)-f(b)+b-a} & , f(a) > f(b). \end{cases}$$

Proof. For harmonically convex function $f : [a,b] \subseteq (0,\infty) \to (0,\infty)$ with $f(a) \neq f(b)$ according to Proposition 2.10 and Corollary 3.3 with t-norm T_p, we have

$$\int_{T_P,[a,b]} f\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)f(b)}{f(b)-f(a)+b-a}.(b-a) & , f(a) < f(b) \\ f(a).(b-a) & , f(a) = f(b) \\ \frac{(b-a)f(a)}{f(a)-f(b)+b-a}.(b-a) & , f(a) > f(b) \end{cases}$$

$$= \begin{cases} \frac{(b-a)^2 f(b)}{f(b)-f(a)+b-a} & , f(a) < f(b) \\ (b-a)f(a) & , f(a) = f(b) \\ \frac{(b-a)^2 f(a)}{f(a)-f(b)+b-a} & , f(a) > f(b). \end{cases}$$

□

Example 3.14. Let μ be the Lebesgue measure on $\mathbb{R}$. Consider the function $f(x) = \frac{1}{x^2}$ on $X = [1, 3]$. Obviously, this function is harmonically convex and positive on $X = [1, 3]$. As $f(1) = 1$ and $f(3) = \frac{1}{9}$, using Corollary 3.13, we can get the following estimate:

$$\int_{T_P,[1,3]} \frac{1}{x^2} \mathrm{d}\mu \leq \frac{(3-1)^2 f(1)}{f(1) - f(3) + (3-1)} = \frac{18}{13}.$$

Now, let's introduce the most important theorem of this article. With the help of it, an upper bound in the framework of the Sugeno integral for Hermite-Hadamard inequality of harmonically convex functions can be established.

Theorem 3.15. Let $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ be a harmonically convex function which is not concave, then

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu \leq \fint_a^b f \mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha \in [f(a), f(b))} \left(\alpha \wedge \mu[\frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}, b] \right) & , f(a) < f(b) \\ f(a) \wedge \mu([a, b]) & , f(a) = f(b) \\ \bigvee_{\alpha \in [f(b), f(a))} \left(\alpha \wedge \mu[a, \frac{\alpha(b-a)+af(b)-bf(a)}{f(b)-f(a)}] \right) & , f(a) > f(b) \end{cases}$$

where $m_0 = min\{a^2, b^2\}$.

Proof. Let f be a harmonically convex function which is not concave and $m_0 = min\{a^2, b^2\}$. By Proposition 2.5 we have,

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu = \fint_a^b \mu([a, b] \cap F_\alpha) \mathrm{d}m \tag{3.2}$$

where m is the Lebesgue measure and

$$F_\alpha = \{x \in X : m_0 \frac{f(x)}{x^2} \geq \alpha\}.$$

Obviously,

$$\left([a, b] \cap \left\{ f(x) \geq \frac{x^2}{m_0} \alpha \right\} \right) \subseteq ([a, b] \cap \{f(x) \geq \alpha\}).$$

By monotonicity μ, we deduce

$$\mu \left([a, b] \cap \left\{ f(x) \geq \frac{x^2}{m_0} \alpha \right\} \right) \leq \mu ([a, b] \cap \{f(x) \geq \alpha\}).$$

Now, by Proposition 2.3 and Proposition 2.5, we obtain

$$\fint_a^b \mu \left([a, b] \cap \{f \geq \frac{x^2}{m_0} \alpha\} \right) \mathrm{d}m \leq \fint_a^b \mu ([a, b] \cap \{f \geq \alpha\}) \mathrm{d}m = \fint_a^b f \mathrm{d}\mu. \tag{3.3}$$

Combining (3.2 , 3.3), we have

$$\fint_a^b m_0 \frac{f(x)}{x^2} \mathrm{d}\mu \leq \fint_a^b f \mathrm{d}\mu.$$

The last inequality follows from Lemma 3.2. □

Corollary 3.16. If $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ be a harmonically convex function which is not concave then,

$$\fint_a^b xf(x)\mathrm{d}\mu \leq \begin{cases} \bigvee_{\alpha\in[af(a),bf(b))} \Big(\alpha \wedge \mu[\frac{\alpha(b-a)+abf(b)-baf(a)}{bf(b)-af(a)}, b]\Big) & ,af(a) < bf(b) \\ af(a) \wedge \mu([a, b]) & ,af(a) = bf(b) \\ \bigvee_{\alpha\in[bf(b),af(a))} \Big(\alpha \wedge \mu[a, \frac{\alpha(b-a)+abf(b)-baf(a)}{bf(b)-af(a)}]\Big) & ,af(a) > bf(b). \end{cases}$$

Proof. f is harmonically convex function.Therefore, according to the Proposition 2.14 $xf(x)$ is convex. Finally, the proof is complete by using Theorem 2.15. □

Corollary 3.17. If $f : [a, b] \subseteq (0, \infty) \to (0, \infty)$ be a harmonically convex function which is not concave, Σ be Borel field and μ be a Lebesgue measure on $X = \mathbb{R}$, then

$$\fint_a^b xf(x)\mathrm{d}\mu \leq \begin{cases} \frac{(b-a)bf(b)}{bf(b)-af(a)+b-a} \wedge (b-a) & ,af(a) < bf(b) \\ af(a) \wedge (b-a) & ,af(a) = bf(b) \\ \frac{(b-a)af(a)}{af(a)-bf(b)+b-a} \wedge (b-a) & ,af(a) > bf(b). \end{cases}$$

Example 3.18. Let μ be the usual Lebesgue measure on X and the function $f(x) = \frac{3}{5}x^2$ on $X = [1, 2]$. Obviously, this function is convex and nondecreasing. So by (1) of Proposition 2.12 f is harmonically convex on $[1, 2]$. With use the Corollary 3.17 we have

$$\fint_1^2 xf(x)dx \leq \frac{(2-1)2f(2)}{2f(2)-f(1)+(2-1)} \wedge (2-1) \simeq 0.923.$$

On the other hand, $\fint_1^2 xf(x)dx \simeq 0.87$. This show that the Corollary 3.17 is valid.

4. **Conclusion**

In this paper, we have researched the Hermite-Hadamard inequality for the Sugeno integral based on harmonically convex functions. For further investigations we propose to consider the Hermite-Hadamard inequality for the Choquet integral, and also for some other non-additive integrals. In the future research, we will continue to explore other integral inequalities for non-additive measures and integrals based on harmonically convex function.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On the Order, Type and Zeros of Meromorphic Functions and Analytic Functions of $[p,q]$-Order in the Unit Disc

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Abstract

In this paper, the authors investigate the $[p,q]$-order and $[p,q]$-type of f_1+f_2, f_1f_2, f_1/f_2, where f_1, f_2 are meromorphic functions or analytic functions with the same $[p,q]$-order and different $[p,q]$-type in the unit disc, and the authors also study the $[p,q]$-order and $[p,q]$-type of f and its derivative. At the end, the authors investigate the relationship between two different $[p,q]$-convergence exponents of f. The obtained results are the improvements and supplements to many previous results.

Key words:meromorphic function; analytic function; unit disc; $[p,q]$-order; $[p,q]$-type
AMS Subject Classification(2000): 30D35, 30D15

1. Notations and Results

We use $\mathbb{C}$ to denote the complex plane and $\Delta=\{z:|z|<1\}$ to denote the unit disc. By a meromorphic function f, we mean a meromorphic function in the complex plane or a meromorphic function in the unit disc. We shall assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory in the complex plane or in the unit disc (*see* $[4,10,14-17,19,20]$). Firstly for $r\in(0,+\infty)$, we define $\exp_1 r=e^r$ and $\exp_{i+1} r=\exp(\exp_i r), i\in\mathbb{N}$, for all r sufficiently large in $(0,+\infty)$, we define $\log_1 r=\log r$ and $\log_{i+1} r=\log(\log_i r), i\in\mathbb{N}$, we also denote $\exp_0 r=r=\log_0 r$ and $\exp_{-1} r=\log_1 r$. Moreover, we denote the logarithmic measure of a set $E\subset[0,1)$ by $m_lE=\int_E\frac{dt}{1-t}$. Throughout this paper, we use p,q to denote positive integers satisfying $1\leq q\leq p$. Secondly, we recall some notations about meromorphic functions and analytic functions.

Definition 1.1 (*see* $[4,17,19,20]$). The order $\sigma(f)$ and lower order $\mu(f)$ of a meromorphic function f in the complex plane are respectively defined by

$$\sigma(f)=\varlimsup_{r\to\infty}\frac{\log T(r,f)}{\log r},\qquad \mu(f)=\varliminf_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

where $T(r,f)$ is the characteristic function of a meromorphic function f in the complex plane or in the unit disc.

Definition 1.2 (*see* [4, 19, 20]). Let f be a meromorphic function in the complex plane or an entire function satisfying $0<\sigma(f)<\infty$, then the type of f is respectively defined by

$$\tau(f)=\varlimsup_{r\to\infty}\frac{T(r,f)}{r^{\sigma(f)}},\qquad \tau_M(f)=\varlimsup_{r\to\infty}\frac{\log M(r,f)}{r^{\sigma(f)}}.$$

Definition 1.3 (*see* [8, 9, 11, 13]). The $[p,q]$-order of a meromorphic function f in the complex plane is defined by

$$\sigma_{[p,q]}(f)=\varlimsup_{r\to\infty}\frac{\log_p T(r,f)}{\log_q r}.$$

If f is a transcendental entire function, the $[p,q]$-order of f is defined by (*see* [11, 13])

$$\sigma_{[p,q]}(f)=\varlimsup_{r\to\infty}\frac{\log_p T(r,f)}{\log_q r}=\varlimsup_{r\to\infty}\frac{\log_{p+1} M(r,f)}{\log_q r}.$$

If f is a polynomial, then $\sigma_{[p,q]}(f)=0$ for any $p\geq q\geq 1$. From Definition 1.3, if $q=1$, we denote $\sigma_{[1,1]}=\sigma_1(f)=\sigma(f)$, and $\sigma_{[p,1]}=\sigma_p(f)$. Similar with Definition 1.2, we can also give the definitions of $\tau_p(f)$ and $\tau_{M,p}(f)$ when $p>1$. In order to keep accordance with Definition 1.1, we give Definition 1.3 by making a small change to the original definition of entire functions of $[p,q]$-order (*see* [8, 9]).

Definition 1.4 (*see* [3, 7]). The iterated p-order of a meromorphic function f in Δ is defined by

$$\sigma_p(f)=\varlimsup_{r\to 1^-}\frac{\log_p T(r,f)}{-\log(1-r)}\qquad (p\in\mathbb{N}).$$

For an analytic function f in Δ, we also define

$$\sigma_{M,p}(f)=\varlimsup_{r\to 1^-}\frac{\log_{p+1} M(r,f)}{-\log(1-r)}.$$

Remark 1.1. If $p=1$, then we denote $\sigma_1(f)=\sigma(f)$ and $\sigma_{M,1}(f)=\sigma_M(f)$, and we have $\sigma(f)\leq\sigma_M(f)\leq\sigma(f)+1$ (*see* [6, 12, 16, 17]) and $\sigma_{M,p}(f)=\sigma_p(f)$ $(p\geq 2)$ (*see* [3, 7]).

Definition 1.5 (*see* [2]). Let f be a meromorphic function in Δ, then the $[p,q]$-order and lower $[p,q]$-order of f are respectively defined by

$$\sigma_{[p,q]}(f)=\varlimsup_{r\to 1^-}\frac{\log_p T(r,f)}{\log_q\left(\frac{1}{1-r}\right)},\qquad \mu_{[p,q]}(f)=\varliminf_{r\to 1^-}\frac{\log_p T(r,f)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Definition 1.6 (*see* [2]). Let f be an analytic function in Δ, then the $[p, q]$-order and lower $[p, q]$-order about maximum modulus of f are respectively defined by

$$\sigma_{M,[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1} M(r, f)}{\log_q \left(\frac{1}{1-r}\right)}, \qquad \mu_{M,[p,q]}(f) = \underline{\lim_{r\to 1^-}} \frac{\log_{p+1} M(r, f)}{\log_q \left(\frac{1}{1-r}\right)}.$$

Definition 1.7 (*see* [2]). The $[p, q]$-type of a meromorphic function f of $[p, q]$-order in Δ with $0 < \sigma_{[p,q]}(f) = \sigma_1 < \infty$ is defined by

$$\tau_{[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p-1} T(r, f)}{\left[\log_{q-1} \left(\frac{1}{1-r}\right)\right]^{\sigma_1}}.$$

For an analytic function f in Δ, and the $[p, q]$-type about maximum modulus of f of $[p, q]$-order with $0 < \sigma_{M,[p,q]}(f) = \sigma_2 < \infty$ is defined by

$$\tau_{M,[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p M(r, f)}{\left[\log_{q-1} \left(\frac{1}{1-r}\right)\right]^{\sigma_2}}.$$

Definition 1.8 The lower $[p, q]$-type of a meromorphic function f of lower $[p, q]$-order in Δ with $0 < \mu_{[p,q]}(f) = \mu_1 < \infty$ is defined by

$$\underline{\tau}_{[p,q]}(f) = \underline{\lim_{r\to \infty}} \frac{\log_{p-1} T(r, f)}{\left[\log_{q-1} \left(\frac{1}{1-r}\right)\right]^{\mu_1}}.$$

Similarly for an analytic function f in Δ, and the lower $[p, q]$-type about maximum modulus of f of lower $[p, q]$-order with $0 < \mu_{M,[p,q]}(f) = \mu_2 < \infty$ is defined by

$$\underline{\tau}_{M,[p,q]}(f) = \underline{\lim_{r\to \infty}} \frac{\log_p M(r, f)}{\left[\log_{q-1} \left(\frac{1}{1-r}\right)\right]^{\mu_2}}.$$

Remark 1.2. From Definitions 1.7 and 1.8, it is easy to see that $\tau_{[p,q]}(f) \le \tau_{M,[p,q]}(f)$ and $\underline{\tau}_{[p,q]}(f) \le \underline{\tau}_{M,[p,q]}(f)$.

Definition 1.9. For any $a \in \mathbb{C} \cup \{\infty\}$, we use $n\left(r, \frac{1}{f-a}\right)$ to denote the unintegrated counting function for the sequence of a-point of a meromorphic function f in Δ. Then the $[p, q]$-exponents of convergence of a-point of f about $n\left(r, \frac{1}{f-a}\right)$ is defined by

$$\lambda^n_{[p,q]}(f, a) = \overline{\lim_{r\to 1^-}} \frac{\log_p n\left(r, \frac{1}{f-a}\right)}{\log_q \left(\frac{1}{1-r}\right)}.$$

Definition 1.10. Let $N\left(r, \frac{1}{f-a}\right)$ be the integrated counting function for the sequence of a-point of a meromorphic function f in Δ. Then the $[p,q]$-exponents of convergence of a-point of f about $N\left(r, \frac{1}{f-a}\right)$ is defined by

$$\lambda^N_{[p,q]}(f,a) = \overline{\lim_{r\to 1^-}} \frac{\log_p N\left(r, \frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)}.$$

Remark 1.3. Similar with Definitions 1.9 and 1.10, we can also give the definitions of the $[p,q]$-exponents of convergence of distinct a-point of f about $n\left(r, \frac{1}{f-a}\right)$ and $N\left(r, \frac{1}{f-a}\right)$, i.e., $\overline{\lambda}^n_{[p,q]}(f,a)$ and $\overline{\lambda}^N_{[p,q]}(f,a)$.

The order and type are two important indicators in revealing the growth of the entire functions or meromorphic functions, many authors have investigated the growth of entire functions or meromorphic functions in the complex plane or in the unit disc ($e.g.$, $see[4, 8-10, 14-20]$) since the first half of the twentieth century. In the following, we list some classic results in the complex plane.

Theorem A(*see* [4, 10, 19, 20]). If f_1 and f_2 are meromorphic functions of finite order with $\sigma(f_1) = \sigma_3$ and $\sigma(f_2) = \sigma_4$, then $\sigma(f_1+f_2) \le \max\{\sigma_3, \sigma_4\}, \sigma(f_1f_2) \le \max\{\sigma_3,\sigma_4\}, \sigma(f_1/f_2) \le \max\{\sigma_3, \sigma_4\}$; if $\sigma_3 < \sigma_4$, then $\sigma(f_1+f_2) = \sigma(f_1f_2) = \sigma(f_1/f_2) = \sigma_4$.

Theorem B (*see* [20]). If f_1 and f_2 are meromorphic functions of finite order, then $\mu(f_1+f_2) \le \min\{\max\{\sigma(f_1), \mu(f_2)\}, \max\{\mu(f_1), \sigma(f_2)\}\}$, $\mu(f_1f_2) \le \min\{\max\{\sigma(f_1), \mu(f_2)\}, \max\{\mu(f_1), \sigma(f_2)\}\}$. Furthermore, if $\sigma(f_1) < \mu(f_2)$, then $\mu(f_1+f_2) = \mu(f_1f_2) = \mu(f_2)$; or if $\sigma(f_2) < \mu(f_1)$, then $\mu(f_1+f_2) = \mu(f_1f_2) = \mu(f_1)$.

Theorem C (*see* [10]). If f_1 and f_2 are entire functions of finite order satisfying $\sigma(f_1) = \sigma(f_2) = \sigma_5$, then the following two statements hold:
(i) If $\tau_M(f_1) = 0$ and $0 < \tau_M(f_2) < \infty$, then $\sigma(f_1f_2) = \sigma_5, \tau_M(f_1f_2) = \tau_M(f_2)$.
(ii) If $\tau_M(f_1) < \infty$ and $\tau_M(f_2) = \infty$, then $\sigma(f_1f_2) = \sigma_5, \tau_M(f_1f_2) = \infty$.

Theorem D (*see* [18]). Let $f_1(z)$ and $f_2(z)$ be entire functions satisfying $0 < \sigma_p(f_1) = \sigma_p(f_2) = \sigma_6 < \infty,\ 0 \le \tau_{M,p}(f_1) < \tau_{M,p}(f_2) \le \infty$. Then the following statements hold:
(i) If $p \ge 1$, then $\sigma_p(f_1+f_2) = \sigma_6,\ \tau_{M,p}(f_1+f_2) = \tau_{M,p}(f_2)$;
(ii) If $p > 1$, then $\sigma_p(f_1f_2) = \sigma_6,\ \tau_{M,p}(f_1f_2) = \tau_{M,p}(f_2)$.

Theorem E (*see* [18]). Let $p \ge 1,\ f(z)$ be an entire function or a meromorphic function in the complex plane satisfying $0 < \sigma_p(f) < \infty$. If $p \ge 1$, then $\sigma_p(f) = \sigma_p(f'), \tau_{M,p}(f') = \tau_{M,p}(f)$; if $p > 1$, then $\sigma_p(f) = \sigma_p(f'), \tau_p(f') = \tau_p(f)$.

From Theorems A-E, we can easily obtain the following similar propositions in the unit disc.

Propositions (i) If f_1 and f_2 are meromorphic functions satisfying $\sigma_{[p,q]}(f_1) = \sigma_6$ and $\sigma_{[p,q]}(f_2) = \sigma_7$ in Δ, then $\sigma_{[p,q]}(f_1 \pm f_2) \leq \max\{\sigma_6,\ \sigma_7\}, \sigma_{[p,q]}(f_1 f_2) \leq \max\{\sigma_6, \sigma_7\}$ and $\sigma_{[p,q]}(f_1/f_2) \leq \max\{\sigma_6, \sigma_7\}$.

(ii) If $\sigma_6 \neq \sigma_7$ in Proposition (i), then $\sigma_{[p,q]}(f_1 \pm f_2) = \sigma_{[p,q]}(f_1 f_2) = \sigma_{[p,q]}(f_1/f_2) = \max\{\sigma_6, \sigma_7\}$.

(iii) If f_1 and f_2 are meromorphic functions in Δ, then $\mu_{[p,q]}(f_1 + f_2) \leq \max\{\sigma_{[p,q]}(f_1), \mu_{[p,q]}(f_2)\}$ or $\mu_{[p,q]}(f_1+f_2) \leq \max\{\mu_{[p,q]}(f_1), \sigma_{[p,q]}(f_2)\}$ and $\mu_{[p,q]}(f_1 f_2) \leq \max\{\sigma_{[p,q]}(f_1), \mu_{[p,q]}(f_2)\}$ or $\mu_{[p,q]}(f_1 f_2) \leq \max\{\mu_{[p,q]}(f_1), \sigma_{[p,q]}(f_2)\}$.

(iv) If f_1 and f_2 are meromorphic functions in Δ satisfying $\sigma_{[p,q]}(f_1) < \mu_{[p,q]}(f_2) \leq \infty$, then $\mu_{[p,q]}(f_1 + f_2) = \mu_{[p,q]}(f_1 f_2) = \mu_{[p,q]}(f_1/f_2) = \mu_{[p,q]}(f_2)$.

(v) If f_1 and f_2 are analytic functions in Δ satisfying $\sigma_{M,[p,q]}(f_1) = \sigma_8$ and $\sigma_{M,[p,q]}(f_2) = \sigma_9$, then $\sigma_{M,[p,q]}(f_1 \pm f_2) \leq \max\{\sigma_8, \sigma_9\}$ and $\sigma_{M,[p,q]}(f_1 f_2) \leq \max\{\sigma_8, \sigma_9\}$. If $\sigma_8 \neq \sigma_9$, then $\sigma_{M,[p,q]}(f_1 \pm f_2) = \max\{\sigma_8, \sigma_9\}$.

(vi) If f_1 and f_2 are analytic functions in Δ, then $\max\{\mu_{M,[p,q]}(f_1 \pm f_2), \mu_{M,[p,q]}(f_1 f_2)\} \leq \max\{\sigma_{M,[p,q]}(f_1), \mu_{M,[p,q]}(f_2)\}$ or $\max\{\mu_{M,[p,q]}(f_1 \pm f_2), \mu_{M,[p,q]}(f_1 f_2)\} \leq \max\{\mu_{M,[p,q]}(f_1), \sigma_{M,[p,q]}(f_2)\}$.

(vii) If f_1 and f_2 are analytic functions of $[p, q]$-order in Δ, for any $r \in [0, 1)$, by the inequality $T(r, f) \leq \log^+ M(r, f) \leq \frac{4}{1-r} T\left(\frac{1+r}{2}, f\right)$ (*see* [4, 17]), we easily obtain that if $p = q \geq 2$ and $\sigma_{[p,q]}(f) > 1$, or $p > q \geq 1$, then $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f)$ and $\tau_{[p,q]}(f) = \tau_{M,[p,q]}(f)$. Similarly, we have $\mu_{[p,q]}(f) = \mu_{M,[p,q]}(f)$ and $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f)$ if $p = q \geq 2$ and $\mu_{[p,q]}(f) > 1$, or $p > q \geq 1$.

Combining Theorems D and E, a natural question is: Can we get the similar results with Theorems D, E for meromorphic functions or analytic functions of $[p, q]$ order in Δ? In fact, we obtain the following results:

Theorem 1.1. Let f_1 and f_2 be meromorphic functions in Δ satisfying $0 < \sigma_{[p,q]}(f_1) = \sigma_{[p,q]}(f_2) = \sigma_{10} < \infty$ and $0 \leq \tau_1 = \tau_{[p,q]}(f_1) < \tau_{[p,q]}(f_2) = \tau_2 \leq \infty$. Then $\sigma_{[p,q]}(f_1 + f_2) = \sigma_{[p,q]}(f_1 f_2) = \sigma_{[p,q]}(f_1/f_2) = \sigma_{10}$, and the following two statements hold:

(i) If $p > 1$ and $p \geq q \geq 1$, then $\tau_{[p,q]}(f_1 + f_2) = \tau_{[p,q]}(f_1 f_2) = \tau_{[p,q]}(f_1/f_2) = \tau_{[p,q]}(f_2)$.
(ii) If $p = q = 1$, then $\tau_2 - \tau_1 \leq \max\{\tau(f_1 + f_2), \tau(f_1 f_2), \tau(f_1/f_2)\} \leq \tau_2 + \tau_1$.

Theorem 1.2. Let f_1 and f_2 be meromorphic functions in Δ satisfying $0 < \sigma_{[p,q]}(f_1) = \mu_{[p,q]}(f_2) < \infty$ and $0 \leq \tau_{[p,q]}(f_1) < \underline{\tau}_{[p,q]}(f_2) \leq \infty$, then $\mu_{[p,q]}(f_1+f_2) = \mu_{[p,q]}(f_1 f_2) = \mu_{[p,q]}(f_1/f_2) = \mu_{[p,q]}(f_2)$. And if $p > 1$ and $p \geq q \geq 1$, we have $\underline{\tau}_{[p,q]}(f_1 + f_2) = \underline{\tau}_{[p,q]}(f_1 f_2) = \underline{\tau}_{[p,q]}(f_1/f_2) = \underline{\tau}_{[p,q]}(f_2)$.

In the following, when f_1 and f_2 are analytic functions of $[p, q]$-order in the unit disc we have the similar results.

Theorem 1.3. Let f_1 and f_2 be analytic functions in Δ satisfying $0 < \sigma_{M,[p,q]}(f_1) = \sigma_{M,[p,q]}(f_2) = \sigma_{11} < \infty$ and $0 \leq \tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) \leq \infty$, then $\sigma_{M,[p,q]}(f_1+f_2) = \sigma_{11}$ and $\tau_{M,[p,q]}(f_1+f_2) = \tau_{M,[p,q]}(f_2)$.

Remark 1.4. By Proposition (vii), we know that Theorem 1.3 is of the same with Theorem 1.1 for $p > q \geq 1$ and $p = q \geq 2, \sigma_{[p,q]}(f) > 1$. For the case $p = q = 1$, the result of Theorem 1.3 is better than that of Theorem 1.1.

Corollary 1.1. Let f_1 and f_2 be analytic functions in Δ satisfying $0 < \sigma_{M,[p,q]}(f_1) = \mu_{M,[p,q]}(f_2) < \infty$ and $0 \leq \tau_{M,[p,q]}(f_1) < \underline{\tau}_{M,[p,q]}(f_2) \leq \infty$, then $\mu_{M,[p,q]}(f_1 + f_2) = \mu_{M,[p,q]}(f_2)$ and $\underline{\tau}_{M,[p,q]}(f_1 + f_2) = \underline{\tau}_{M,[p,q]}(f_2)$.

Theorem 1.4. Let f be an analytic function of $[p,q]$-order in Δ, then $\sigma_{M,[p,q]}(f) = \sigma_{M,[p,q]}(f')$, $\mu_{M,[p,q]}(f) = \mu_{M,[p,q]}(f')$. If $0 < \sigma_{M,[p,q]}(f) < \infty$ or $0 < \mu_{M,[p,q]}(f) < \infty$, then $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$, $\underline{\tau}_{M,[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f')$.

Theorem 1.5. Let f be a meromorphic function of $[p,q]$-order in Δ, then
(i) If $p \geq q \geq 2$ and $p > q = 1$, then $\sigma_{[p,q]}(f) = \sigma_{[p,q]}(f'), \mu_{[p,q]}(f) = \mu_{[p,q]}(f')$ and $\tau_{[p,q]}(f) = \tau_{[p,q]}(f')$, $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{[p,q]}(f')$ for $0 < \sigma_{[p,q]}(f) < \infty$ or $0 < \mu_{[p,q]}(f) < \infty$.
(ii) If $p = q = 1$, then $\sigma(f) = \sigma(f'), \mu(f) = \mu(f')$ and $\tau_{[1,1]}(f') \leq 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \leq 2\underline{\tau}_{[1,1]}(f)$.

Theorem 1.6. Let f be a meromorphic function of $[p,q]$-order in Δ, $a \in \mathbb{C} \cup \{\infty\}$. Then the following statements hold:

(i) If $p > q \geq 1$, then $\lambda^{N}_{[p,q]}(f,a) = \lambda^{n}_{[p,q]}(f,a)$.
(ii) If $p = q = 1$, then $\lambda^{N}(f,a) \leq \lambda^{n}(f,a) \leq \lambda^{N}(f,a) + 1$ $(see$ [12]).
(iii) If $p = q \geq 2$, then $\lambda^{N}_{[p,p]}(f,a) \leq \lambda^{n}_{[p,p]}(f,a) \leq \max\left\{\lambda^{N}_{[p,p]}(f,a), 1\right\}$. Furthermore, we have $\lambda^{N}_{[p,p]}(f,a) = \lambda^{n}_{[p,p]}(f,a)$ if $\lambda^{N}_{[p,p]}(f,a) \geq 1$, and if $\lambda^{N}_{[p,p]}(f,a) < 1$ then $\lambda^{N}_{[p,p]}(f,a) \leq \lambda^{n}_{[p,p]}(f,a) \leq 1$.

Remark 1.4. The conclusions of Theorem 1.6 also hold between $\overline{\lambda}^{n}_{[p,q]}(f,a)$ and $\overline{\lambda}^{N}_{[p,q]}(f,a)$.

2. Preliminary Lemmas

Lemma 2.1 (*see* [4, 19, 20]). Let f_1, f_2, $\cdots$, $f_m(z)$ be meromorphic functions in Δ, where $m \geq 2$ is a positive integer. Then

(i) $T(r, f_1 f_2 \cdots f_m) \leq \sum\limits_{i=1}^{m} T(r, f_i)$,

(ii) $T(r, f_1 + f_2 + \cdots + f_m) \leq \sum\limits_{i=1}^{m} T(r, f_i) + \log m$.

Lemma 2.2 (*see* [6, 17]). Let f be a meromorphic function in Δ, and let $k \geq 1$ be an integer.

Then

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\left\{\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right\}$, possibly outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dt}{1-t} < \infty$.

Lemma 2.3 (*see* [1]). Let $g : (0,1) \to R$ and $h : (0,1) \to R$ be monotone increasing functions such that $g(r) \leq h(r)$ holds outside of an exceptional set $E_2 \subset [0,1)$ for which $\int_{E_2} \frac{dt}{1-t} < \infty$. Then there exisits a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.

Lemma 2.4 (*see* [5, 15]) Suppose that f is meromorphic in Δ with $f(0) = 0$. Then

$$m(r,f) \leq \left[1 + \varphi(\frac{r}{R})\right] T(R,f') + N(R,f'), \tag{3.13}$$

where $0 < r < R < 1, \varphi(t) = \frac{1}{\pi}\log\frac{1+t}{1-t}$.

3. Proofs of Theorems 1.1 -1.6

Proof of Theorem 1.1. Assume that $0 \leq \tau_1 = \tau_{[p,q]}(f_1) < \tau_{[p,q]}(f_2) = \tau_2 < \infty$, by Definition 1.7, it is easy to see that for any given $\varepsilon > 0$ and $r \to 1^-$, we have

$$T(r,f_1) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}, \tag{3.1}$$

$$T(r,f_2) \leq \exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}. \tag{3.2}$$

By using (3.1)-(3.2) and Lemma 2.1, we have

$$T(r, f_1 + f_2) \leq T(r,f_1) + T(r,f_2) + \log 2$$

$$\leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\} + \exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\} + \log 2$$

$$\leq 2\exp_{p-1}\left\{(\tau_2 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{10}}\right\}.$$

Hence $\sigma_{[p,q]}(f_1 + f_2) \leq \sigma_{10}$. In addition, if $p = q = 1$, we can get $\tau_{[p,q]}(f_1 + f_2) \leq \tau_1 + \tau_2$, if $p > 1$, then $\tau_{[p,q]}(f_1 + f_2) \leq \tau_2$ for any $p \geq q \geq 1$. On the other hand, for any given $\varepsilon > 0$, there exists a sequence $\{r_n\}_{n=1}^{\infty} \to 1^-$ satisfying

$$T(r_n,f_1) \leq \exp_{p-1}\left\{(\tau_1 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}, \tag{3.3}$$

$$T(r_n,f_2)\geq \exp_{p-1}\left\{(\tau_2-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}. \tag{3.4}$$

By (3.3)-(3.4) and Lemma 2.1, we obtain

$$T(r_n,f_1+f_2)\geq T(r_n,f_2)-T(r_n,f_1)-\log 2$$

$$\geq \exp_{p-1}\left\{(\tau_2-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}-\exp_{p-1}\left\{(\tau_1+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{10}}\right\}-\log 2. \tag{3.5}$$

By (3.5) we have $\sigma_{[p,q]}(f_1+f_2)\geq\sigma_{10}$. Furthermore, if $p=q=1$, then $\tau_{[p,q]}(f_1+f_2)\geq\tau_2-\tau_1$ and $\tau_{[p,q]}(f_1+f_2)\geq\tau_2$ for $p>1$ and $p\geq q\geq 1$.

Therefore, we have $\sigma_{[p,q]}(f_1+f_2)=\sigma_{10}$, and if $p>1$, then $\tau_{[p,q]}(f_1+f_2)=\tau_{[p,q]}(f_2)$, if $p=q=1$, then $\tau_2-\tau_1\leq\tau_{[p,q]}(f_1+f_2)\leq\tau_2+\tau_1$. Since $T(r,f_1f_2)\leq T(r,f_1)+T(r,f_2)$, $T(r,f_1f_2)\geq T(r,f_2)-T(r,f_1)-O(1)$ and $T\left(r,\frac{1}{f_2}\right)=T(r,f_2)+O(1)$, by the above proof, $\sigma_{[p,q]}(f_1f_2)=\sigma_{[p,q]}(f_1/f_2)=\sigma_{10}$, $\tau_{[p,q]}(f_1f_2)=\tau_{[p,q]}(f_1/f_2)=\tau_{[p,q]}(f_2)$ for $p>1$ and $\tau_2-\tau_1\leq\max\{\tau(f_1f_2),\tau(f_1/f_2)\}\leq\tau_2+\tau_1$ if $p=q=1$ also can hold. Moreover, Theorem 1.1 also holds for $\tau_{[p,q]}(f_2)=\tau_2=\infty$.

Proof of Theorem 1.2. Without loss of generality, we suppose that $0\leq\tau_3=\tau_{[p,q]}(f_1)<\underline{\tau}_{[p,q]}(f_2)=\tau_4<\infty$. Assume that $\sigma_{[p,q]}(f_1)=\mu_{[p,q]}(f_2)=\mu_3$, and by Definition 1.8, it is easy to see that for any given $\varepsilon>0$, there exists a sequence $\{r_n\}_{n=1}^{\infty}\to 1^-$ satisfying

$$T(r_n,f_1)<\exp_{p-1}\left\{(\tau_3+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}, \tag{3.6}$$

$$T(r_n,f_2)<\exp_{p-1}\left\{(\tau_4+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}. \tag{3.7}$$

By (3.6)-(3.7) and Lemma 2.1, we have

$$T(r_n,f_1+f_2)\leq T(r_n,f_1)+T(r_n,f_2)+\log 2$$

$$\leq \exp_{p-1}\left\{(\tau_3+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}+\exp_{p-1}\left\{(\tau_4+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}+\log 2$$

$$\leq 2\exp_{p-1}\left\{(\tau_4+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\mu_3}\right\}.$$

Hence $\mu_{[p,q]}(f_1+f_2)\leq\mu_3$. In addition, if $p>1$, then $\underline{\tau}_{[p,q]}(f_1+f_2)\leq\tau_4$ for $p\geq q\geq 1$. On the other hand, for any given $\varepsilon>0$ and $r\to 1^-$, we have

$$T(r,f_1)\leq \exp_{p-1}\left\{(\tau_3+\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\}, \tag{3.8}$$

$$T(r,f_2)\geq \exp_{p-1}\left\{(\tau_4-\varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\}. \tag{3.9}$$

By (3.8)-(3.9) and Lemma 2.1, we obtain

$$T(r, f_1 + f_2) \geq T(r, f_2) - T(r, f_1) - \log 2$$

$$\geq \exp_{p-1}\left\{(\tau_4 - \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\} - \exp_{p-1}\left\{(\tau_3 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\mu_3}\right\} - \log 2. \quad (3.10)$$

By (3.10) we have $\mu_{[p,q]}(f_1 + f_2) \geq \mu_3$ and $\underline{\tau}_{[p,q]}(f_1 + f_2) \geq \tau_4$ for $p > 1$ and $p \geq q \geq 1$. Thus we have $\mu_{[p,q]}(f_1 + f_2) = \mu(f_2)$ and if $p > 1$ and $p \geq q \geq 1$, then $\underline{\tau}_{[p,q]}(f_1 + f_2) = \underline{\tau}_{[p,q]}(f_2)$. Since $T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2)$, $T(r, f_1 f_2) \geq T(r, f_2) - T(r, f_1) - O(1)$ and $T\left(r, \frac{1}{f_2}\right) = T(r, f_2) + O(1)$, by the above proof, $\mu_{[p,q]}(f_1 f_2) = \mu_{[p,q]}(f_1/f_2) = \mu_{[p,q]}(f_2)$ and $\underline{\tau}_{[p,q]}(f_1 f_2) = \underline{\tau}_{[p,q]}(f_1/f_2) = \underline{\tau}_{[p,q]}(f_2)$ also hold if $p > 1$ and $p \geq q \geq 1$.

The conclusions of Theorem 1.2 also hold for $\tau_3 = \tau_{[p,q]}(f_1) < \underline{\tau}_{[p,q]}(f_2) = \tau_4 = \infty$.

Proof of Theorem 1.3. Set $0 \leq \tau_5 = \tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) = \tau_6 < \infty$, by Definition 1.7, for any given ε $(0 < 2\varepsilon < \tau_6 - \tau_5)$, there exists a sequence $\{r_n\}_{n=1}^{\infty} \to 1^-$ satisfying

$$M(r_n, f_1) \leq \exp_p\left\{(\tau_5 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}, \quad (3.11)$$

$$M(r_n, f_2) > \exp_p\left\{(\tau_6 - \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}. \quad (3.12)$$

We can choose a sequence $\{z_n\}_{n=1}^{\infty}$ satisfying $|z_n| = r_n (n = 1, 2, \cdots)$ and $|f_2(z_n)| = M(r_n, f_2)$, by (3.11)-(3.12) we have

$$M(r_n, f_1 + f_2) \geq |f_1(z_n) + f_2(z_n)| \geq |f_2(z_n)| - |f_1(z_n)| \geq M(r_n, f_2) - M(r_n, f_1)$$

$$\geq \exp_p\left\{(\tau_6 - \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\} - \exp_p\left\{(\tau_5 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\}$$

$$\geq \frac{1}{2}\exp_p\left\{(\tau_6 - \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r_n}\right)\right]^{\sigma_{11}}\right\} \quad (r_n \to 1^-).$$

Hence $\sigma_{M,[p,q]}(f_1 + f_2) \geq \sigma_{11}$ and $\tau_{M,[p,q]}(f_1 + f_2) \geq \tau_6$. On the other hand, we have

$$M(r, f_1 + f_2) \leq M(r, f_1) + M(r, f_2)$$

$$\leq \exp_p\left\{(\tau_5 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\} + \exp_p\left\{(\tau_6 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\}$$

$$\leq 2\exp_p\left\{(\tau_6 + \varepsilon)\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{11}}\right\},$$

therefore $\sigma_{M,[p,q]}(f_1 + f_2) \leq \sigma_{11}$ and $\tau_{M,[p,q]}(f_1 + f_2) \leq \tau_6$. Thus we can get $\sigma_{M,[p,q]}(f_1 + f_2) = \sigma_{11}$ and $\tau_{M,[p,q]}(f_1 + f_2) = \tau_{M,[p,q]}(f_2)$. Moreover, Theorem 1.3 also holds for $\tau_{M,[p,q]}(f_1) < \tau_{M,[p,q]}(f_2) = \tau_6 = \infty$.

Proof of Theorem 1.4. Since f is an analytic function in the unit disc, from the formula

$$f(z)=f(0)+\int_0^z f'(\zeta)d\zeta \qquad (|z|=r<1)$$

where the integral route is a line from 0 to z in the unit disc. We obtain that

$$M(r,f)\le |f(0)|+|\int_0^z f'(\zeta)d\zeta|\le |f(0)|+rM(r,f')\le |f(0)|+M(r,f'),$$

i.e.

$$M(r,f')\ge M(r,f)-|f(0)|. \tag{3.13}$$

By (3.13), we have

$$\sigma_{M,[p,q]}(f')\ge \sigma_{M,[p,q]}(f),\ \ \mu_{M,[p,q]}(f')\ge \mu_{M,[p,q]}(f).$$

On the other hand, in the circle $|z|=r\in(0,1)$, we take a point z_0 satisfying $|f'(z_0)|=M(r,f')$. By the Cauchy inequality

$$f'(z_0)=\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z_0)^2}d\zeta,$$

where $C=\{\zeta:|\zeta-z_0|=s(r)-r\}$ and $s(r)=1-d(1-r)$, $d\in(0,1)$. We deduce that

$$M(r,f')=|f'(z_0)|\le \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(\zeta)}{(\zeta-z_0)^2}(s(r)-r)\right|d\theta\le \frac{M(s(r),f)}{s(r)-r},$$

i.e.

$$M(r,f')\le \frac{M(s(r),f)}{(1-d)(1-r)}. \tag{3.14}$$

By (3.14), then $\sigma_{M,[p,q]}(f')\le \sigma_{M,[p,q]}(f)$, $\mu_{M,[p,q]}(f')\le \mu_{M,[p,q]}(f)$. Hence

$$\sigma_{M,[p,q]}(f)=\sigma_{M,[p,q]}(f'),\ \ \mu_{M,[p,q]}(f')=\mu_{M,[p,q]}(f). \tag{3.15}$$

If $0<\sigma_{M,[p,q]}(f)<\infty$ and by (3.13), (3.15), we can get $\tau_{M,[p,q]}(f')\ge \tau_{M,[p,q]}(f)$. Then by (3.14)-(3.15), if $p\ge q=1$ we can obtain

$$\frac{\log_p M(r,f')}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f')}}\le \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}},\frac{\log_p M(s(r),f)}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}}\right\}$$

$$\le \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left(\frac{1}{1-r}\right)^{\sigma_{M,[p,1]}(f)}},\frac{\log_p M(s(r),f)}{\left[\frac{1}{1-s(r)}\right]^{\sigma_{M,[p,1]}(f)}}\cdot\left(\frac{1}{d}\right)^{\sigma_{M,[p,1]}(f)}\right\},$$

let $d \to 1$, therefore $\tau_{M,[p,q]}(f') \leq \tau_{M,[p,q]}(f)$, then $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$. If $p \geq q \geq 2$, then

$$\frac{\log_p M(r,f')}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f')}} \leq \max\left\{\frac{\log_p\left[\frac{1}{(1-r)(1-d)}\right]}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f)}}, \frac{\log_p M(s(r),f)}{\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_{M,[p,q]}(f)}}\right\},$$

thus we have $\tau_{M,[p,q]}(f') \leq \tau_{M,[p,q]}(f)$ and $\tau_{M,[p,q]}(f) = \tau_{M,[p,q]}(f')$. If $0 < \mu_{M,[p,q]}(f) < \infty$, we can similarly obtain $\underline{\tau}_{M,[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f')$.

Proof of Theorem 1.5. By Lemma 2.2, we have

$$\begin{aligned} T(r,f') &= m(r,f') + N(r,f') \leq m(r,f) + m\left(r,\frac{f'}{f}\right) + 2N(r,f) \\ &\leq 2T(r,f) + m\left(r,\frac{f'}{f}\right) \leq (2+\varepsilon)T(r,f) + O\left\{\log\frac{1}{1-r}\right\} \quad (r \notin E_1). \quad (3.16) \end{aligned}$$

From (3.16) and Lemma 2.3, we have $\sigma_{[p,q]}(f') \leq \sigma_{[p,q]}(f), \mu_{[p,q]}(f') \leq \mu_{[p,q]}(f)$ for $p \geq q \geq 1$, $\tau_{[p,q]}(f') \leq \tau_{[p,q]}(f)$, $\underline{\tau}_{[p,q]}(f') \leq \underline{\tau}_{[p,q]}(f)$, for $p > 1$ and $\tau_{[1,1]}(f') \leq 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \leq 2\underline{\tau}_{[1,1]}(f)$ for $p = q = 1$. On the other hand, set $R = s(r) = 1 - d(1-r), d \in (0,1)$ in Lemma 2.4, we have

$$T(r,f) < \left(2 + \frac{1}{\pi}\log\frac{3}{(1-d)(1-r)}\right)T(s(r),f'). \quad (3.17)$$

By (3.17) and by the similar proof in Theorem 1.4, we have $\sigma_{[p,q]}(f) \leq \sigma_{[p,q]}(f')$, $\mu_{[p,q]}(f) \leq \mu_{[p,q]}(f')$ for $p \geq q \geq 1$, $\tau_{[p,q]}(f) \leq \tau_{[p,q]}(f')$, $\underline{\tau}_{[p,q]}(f) \leq \underline{\tau}_{[p,q]}(f')$ for $p \geq q \geq 2$ and $\tau_{[p,q]}(f) \leq (\frac{1}{d})^{\sigma_{[p,q]}(f)}\tau_{[p,q]}(f')$ for $p > q = 1$, letting $d \to 1$, therefore the following statements hold:

If $p \geq q \geq 2$ and $p > q = 1$, then $\sigma_{[p,q]}(f) = \sigma_{[p,q]}(f'), \mu_{[p,q]}(f) = \mu_{[p,q]}(f')$ and $\tau_{[p,q]}(f) = \tau_{[p,q]}(f')$ for $0 < \sigma_{[p,q]}(f) < \infty$, $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{[p,q]}(f')$ for $0 < \mu_{[p,q]}(f) < \infty$.

If $p = q = 1$, then $\sigma(f) = \sigma(f'), \mu(f) = \mu(f')$ and $\tau_{[1,1]}(f') \leq 2\tau_{[1,1]}(f)$, $\underline{\tau}_{[1,1]}(f') \leq 2\underline{\tau}_{[1,1]}(f)$.

Proof of Theorem 1.6. Without loss of generality, assume that $f(a) \neq 0$, by

$$N\left(r,\frac{1}{f-a}\right) = \int_0^r \frac{n\left(t,\frac{1}{f-a}\right) - n\left(0,\frac{1}{f-a}\right)}{t}dt \quad (0 < r < 1),$$

we have

$$n\left(r,\frac{1}{f-a}\right) \leq \frac{1}{\log\left(1+\frac{1-r}{2r}\right)}\int_r^{r+\frac{1-r}{2}} \frac{n\left(t,\frac{1}{f-a}\right)}{t}dt \leq \frac{1}{\log\left(1+\frac{1-r}{2r}\right)}N\left(\frac{1+r}{2},\frac{1}{f-a}\right), \quad (3.18)$$

where $0 < r < 1, \log(1+\frac{1-r}{2r}) \sim \frac{1-r}{2r}, r \to 1^-$. By (3.18), we have

$$\overline{\lim_{r\to 1^-}}\frac{\log_p n\left(r,\frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)} \leq \max\left\{\overline{\lim_{r\to 1^-}}\frac{\log_p N\left(\frac{1+r}{2},\frac{1}{f-a}\right)}{\log_q\left(\frac{1}{1-r}\right)}, \overline{\lim_{r\to 1^-}}\frac{\log_p\left(\frac{2r}{1-r}\right)}{\log_q\left(\frac{1}{1-r}\right)}\right\}. \quad (3.19)$$

By (3.19), we can obtain
(i) if $p > q \geq 1$, then $\lambda^n_{[p,q]}(f,a) \leq \lambda^N_{[p,q]}(f,a)$;
(ii) if $p = q = 1$, then $\lambda^n(f,a) \leq \lambda^N(f,a) + 1$;
(iii) if $p = q \geq 2$, then $\lambda^n_{[p,p]}(f,a) \leq \max\left\{\lambda^N_{[p,p]}(f,a), 1\right\}$.
On the other hand, by

$$N\left(r, \frac{1}{f-a}\right) = \int_{r_0}^{r} \frac{n\left(t, \frac{1}{f-a}\right)}{t}dt + N\left(r_0, \frac{1}{f-a}\right) \leq n\left(r, \frac{1}{f-a}\right)\log\left(\frac{r}{r_0}\right) + O(1), \qquad (3.20)$$

where $0 < r_0 < r < 1$. By (3.20), we can get
(i) if $p > q \geq 1$, then $\lambda^N_{[p,q]}(f,a) \leq \lambda^n_{[p,q]}(f,a)$;
(ii) if $p = q = 1$, then $\lambda^N(f,a) \leq \lambda^n(f,a)$;
(iii) if $p = q \geq 2$, then $\lambda^N_{[p,p]}(f,a) \leq \lambda^n_{[p,p]}(f,a)$.

Therefore, the conclusions of Theorem 1.6 hold.

Author Contributions: Conceptualization and providing original idea, J. Tu; Formal analysis and manuscript editing, K.Q.Hu and H. Zhang.

Funding: This project is supported by the National Natural Science Foundation of China(11561031, 11861005, 41472130), the Natural Science Foundation of China of Jiangxi Province in China(20161BAB201020).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Reachable sets for semilinear integrodifferential control systems

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Abstract

In this paper, we consider a control system for semilinear integrodifferential equations in Hilbert spaces with Lipschitz continuous nonlinear term. Our method is to find the equivalence of approximate controllability for the given semilinear system and the linear system excluded the nonlinear term, which is based on results on regularity for the mild solution. Finally, we give a simple example to which our main result can be applied.

Keywords: approximate controllability, semilinear control system, lipschtiz continuity, approximate controllability, reachable set

AMS Classification Primary 35B37; Secondary 93C20

1 Introduction

Let H and V be real Hilbert spaces such that V is a dense subspace in H. In this paper, we are concerned with the control results for the following retarded semilinear control system in Hilbert space H:

$$\begin{cases} x^{'}(t) = Ax(t) + g(t, x(t), \int_0^t k(t,s,x(s))ds)) + Bu(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

This research was supported by Basic Science Research Program through the National research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2019R1F1A1048077)

where $t > 0$, B is a bounded linear controller, and $u(t)$ is an appropriate control functions. Let A be the operator associated with a bounded sesquilinear form defined in $V \times V$ satisfying Gårding inequality. Then it is well known that $S(t)$ generated by A is an analytic semigroup in both H and V^*, where V^* is the dual space of V, and so the system (1.1) may be considered as an system in both H and V^*. g is is a nonlinear mapping as detailed in Section 2.

Whether the reachable set associated with control space in dense subset of H. This is called an approximate controllability problem. As for linear evolution systems in general Banach, there are many papers and monographs, see [1, 2], Triggiani [3], Curtain and Zwart [4] and references and therein.

The controllability for nonlinear control systems has been studied by many authors, for example, control of nonlinear infinite dimensional systems in [5], controllability for parabolic equations with uniformly bounded nonlinear terms in [6], local controllability of neutral functional differential systems in [7].

Recently, the approximate controllability for semilinear control systems can be founded in [8, 9, 10], their results give sufficient condition on strict assumptions on the control action operator B. Similar considerations of semilinear systems have been dealt with in many references [11, 12, 13, 14].

We investigate the equivalence of approximate controllability for (1.1) such that excluded the nonlinear term and the controller. The solution mapping from the initial space to the solution space is Lipschitz continuous in $[0, T]$. We no longer require the strict range condition on B, and the uniform boundedness in [6] but instead we need the regularity and a variation of solutions of the given equations. For the basis of our study we construct the fundamental solution and establish variations of constant formula of solutions for the linear systems, see [15, 16].

Based on L^2-regularity properties of semilinear integrodifferential equations in Hilbert space and the regularity of solutions discussed in Section 2. We will obtain the relations between the reachable set of the semilinear system and that of its corresponding linear system in Section 3. Finally, a simple example to which our main result can be applied is given.

2 Regularity for retarded semilinear equations

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norms on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$.

For $l \in V^*$ we denote (l, v) by the value $l(v)$ of l at $v \in V$. The norm of l as element of V^* is given by

$$||l||_* = \sup_{v \in V} \frac{|(l, v)|}{||v||}.$$

Therefore, we assume that V has a stronger topology than H and, for brevity, we may regard that

$$||u||_* \le |u| \le ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\mathrm{Re}\ a(u, u) \ge \omega_1 ||u||^2 - \omega_2 |u|^2, \tag{2.2}$$

where $\omega_1 > 0$ and ω_2 is a real number. Let A be the operator associated with this sesquilinear form:

$$(Au, v) = -a(u, v), \quad u,\ v \in V. \tag{2.3}$$

Then A is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to

$$D(A) = \{u \in V : Au \in H\}$$

is also denoted by A. It is well known that A generates an analytic semigroup in both of H and V^*(see [17]).

From the following inequalities

$$\omega_1 ||u||^2 \le \mathrm{Re}\, a(u, u) + \omega_2 |u|^2 \le |Au|\,|u| + \omega_2 |u|^2 \le \max\{1, \omega_2\} ||u||_{D(A)} |u|,$$

where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$, it follows that there exists a constant $C > 0$ such that

$$||u|| \le C ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.4}$$

Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.5}$$

where each space is dense in the next one, which is continuous injection.

Lemma 2.1. *With the notations (2.1), (2.4), and (2.5), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^**(Section 1.3.3 of [18]).*

Assumption (K). Let $k : \mathbb{R}^+ \times [\,-h, 0] \times V \to H$ be a nonlinear mapping satisfying the following:

(K1) For any $x \in V$ the mapping $k(\cdot, \cdot, x)$ is measurable;

(K2) There exist positive constants K_0, K_1 such that

$$\begin{aligned}&|k(t,s,x) - k(t,s,y)| \le K_1||x-y||,\\ &|k(t,s,0)| \le K_0\end{aligned}$$

for all $(t,s) \in \mathbb{R}^+ \times [-h, 0]$ and $x, y \in V$.

Assumption (G). Let $g : \mathbb{R}^+ \times V \times H \to H$ be a nonlinear mapping satisfying the following:

(G1) For any $x \in V$, $y \in H$ the mapping $g(\cdot, x, y)$ is measurable;

(G2) There exist positive constants L_0, L_1, L_2 such that

$$\begin{aligned}&|g(t,x,y) - g(t,\hat{x},\hat{y})| \le L_1||x-\hat{x}|| + L_2|y-\hat{y}|,\\ &|g(t,0,0)| \le L_0\end{aligned}$$

for all $t \in \mathbb{R}^+$, $x, \hat{x} \in V$, and $y, \hat{y} \in H$.

For $x \in L^2(-h, T; V),\ T > 0$ we set

$$G(t,x) = g(t, x(t), \int_0^t k(t,s,x(s))ds).$$

The above operator g is the semilinear case of the nonlinear part of quasilinear equations considered by Yong and Pan [19]. The mild solution of (1.1) is represented by

$$x(t) = S(t)x_0 + \int_0^t \big\{G(s, x(s)0 + Bu(s)\big\}ds, \quad t \ge 0.$$

Lemma 2.2. *Let $x \in L^2(0,T;V)$, $T > 0$. Then $G(\cdot, x) \in L^2(0,T;H)$ and*

$$||G(\cdot,x)||_{L^2(0,T;H)} \le (L_0 + K_0L_2)\sqrt{T} + (L_1 + L_2K_1T)||x||_{L^2(0,T;V)}.$$

Moreover if $x_1,\ x_2 \in L^2(0,T;V)$, then

$$||G(\cdot,x_1) - G(\cdot,x_2)||_{L^2(0,T;H)} \le (L_1 + L_2K_1T)||x_1 - x_2||_{L^2(0,T;V)}. \tag{2.6}$$

Proof. Hence, from (K2), (G2) and the above inequality it is easily seen that

$$\begin{aligned}
&||G(\cdot,x)||_{L^2(0,T;H)} \le ||G(\cdot,0)|| + ||G(\cdot,x)-G(\cdot,0)||\\
&\le L_0\sqrt{T} + L_1||x||_{L^2(0,T;V)} + L_2||\int_0^{\cdot} k(\cdot,s,x(s))ds||_{L^2(0,T;H)}\\
&\le L_0\sqrt{T} + L_1||x||_{L^2(0,T;V)} + L_2K_1T||x||_{L^2(0,T;V)} + K_0L_2\sqrt{T}\\
&\le (L_0+K_0L_2)\sqrt{T} + (L_1+L_2K_1T)||x||_{L^2(0,T;V)}
\end{aligned}$$

Similarly, we can prove (2.6). □

In view of Lemma 2.2, we can apply the regularity results of Theorem 3.1 of [10] to (1.1), and so we obtain the following results.

Proposition 2.1. *1) Let $x_0 \in H$ and $k \in L^2(0,T;V^*)$, $T>0$. Then there exists a unique solution x of (2.7) belonging to*

$$L^2(0,T;V)\cap W^{1,2}(0,T;V^*) \subset C([0,T];H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|x_0| + ||k||_{L^2(0,T;V^*)}), \tag{2.7}$$

where C_1 is a constant depending on T.

2) If $x_0 \in H$ and $k \in L^2(0,T;V^*)$, then the mapping

$$H\times L^2(0,T;V^*) \ni (x_0,k) \mapsto x \in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$$

is Lipschitz continuous.

Here, we note that by using interpolation theory, we have that for $z \in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$, there exists a constant $C_2>0$ such that

$$||z||_{C([0,T];H)} \le C_2||z||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}. \tag{2.8}$$

3 Approximately reachable sets

Let U be a Banach space and the controller operator B is bounded linear operator from another Banach space U to X.

Let $S(t)$ be an analytic semigroup generation by A. Then we may assume that there exists a positive constant C_0 such that

$$||S(t)|| \leq C_0, \quad ||AS(t)|| \leq C_0/t(t > 0). \tag{3.1}$$

The solution $x(t) = x(t; x_0, G, u)$ of initial value problem (1,1) is the following form:

$$x(t; x_0, G, u) = S(t)x_0 + \int_0^t S(t-s)\{G(t, x(s)) + Bu(s)\}ds, \quad t > 0,$$

For $T > 0$, $x_0 \in H$ and $u \in L^2(0, T; U)$ we define reachable sets as follows.

$$\begin{aligned} &L_T(x_0) = \{x(T; x_0, 0, u) : u \in L^2(0, T; U)\}, \\ &R_T(x_0) = \{x(T; x_0, G, u) : u \in L^2(0, T; U)\}, \\ &L(x_0) = \bigcup_{T>0} L_T(x_0), \quad R(x_0) = \bigcup_{T>0} R_T(x_0). \end{aligned}$$

Definition 3.1. *(1) System (1.1) is said to be H-approximately controllable for initial value x_0 (resp. in time T) if $\overline{R(x_0)} = H$ (resp. $\overline{R_T(x_0)} = H$).*
(2) The linear system corresponding (1.1) is said to be H-approximately controllable for initial value x_0 (resp. in time T) if $\overline{L(x_0)} = H$ (resp. $\overline{L_T(x_0)} = H$).

Remark 3.1. *Since A generate an analytic semigroup, the following (1)-(4) are equivalent for the linear system (see [2, Theorem 3.10]).*

(1) $\overline{L(x_0)} = H \quad \forall x_0 \in H.$

(2) $\overline{L(0)} = H.$

(3) $\overline{L_T(x_0)} = H \quad \forall x_0 \in H.$

(4) $\overline{L_T(0)} = H.$

Theorem 3.1. *For any $T > 0$ we have*

$$\overline{R_T(0)} \subset \overline{L_T(0)}.$$

Proof. Let $z_0 \notin \overline{L_T(0)}$. Since $\overline{L_T(0)}$ is a balanced closed convex subspace, we have $\alpha z_0 \notin \overline{L_T(0)}$ for every $\alpha \in \mathbb{R}$, and

$$\inf\{||z_0 - z|| : z \in \overline{L_T(0)}\} = d.$$

By the formula (2.7) we have

$$||x(\cdot; 0, G, u)||_{L^2(0,T;V)} \leq C_1||B||||u||_{L^2(0,T;U)}, \tag{3.2}$$

where C_1 is the constant in Proposition 2.1. For every $u \in L^2(0,T;U)$, we choose a constant $\alpha > 0$ such that

$$C_0\{(L_0 + K_0L_2)\sqrt{T} + (L_1 + L_2K_1T)C_1||B||||u||_{L^2(0,T;U)}\} < \alpha d. \tag{3.3}$$

Hence form (3.2), (3.3) and by using Hölder inequality, it follows that

$$\begin{aligned}
&|x(T;0,G,u) - \alpha z_0| \\
&\geq |\int_0^T S(T-s)Bu(s)ds - \alpha z_0| - |\int_0^T S(T-s)G(s,x(s))ds| \\
&\geq \alpha d - C_0\{(L_0 + K_0L_2)\sqrt{T} + (L_1 + L_2K_1T)||x||_{L^2(0,T;V)}\} \\
&\geq \alpha d - C_0\{(L_0 + K_0L_2)\sqrt{T} + (L_1 + L_2K_1T)C_1||B||||u||_{L^2(0,T;U)}\} > 0.
\end{aligned}$$

Thus, we have $\alpha z_0 \notin \overline{R_T(0)}$. □

Lemma 3.1. *Suppose that* $k \in L^2(0,T;H)$ *and* $x(t) = \int_0^t S(t-s)k(s)ds$ *for* $0 \leq t \leq T$. *Then there exists a constant* C_3 *such that*

$$||x||_{L^2(0,T;D(A))} \leq C_1||k||_{L^2(0,T;H)}, \tag{3.4}$$

$$||x||_{L^2(0,T;H)} \leq C_3T||k||_{L^2(0,T;H)}, \tag{3.5}$$

and

$$||x||_{L^2(0,T;V)} \leq C_3\sqrt{T}||k||_{L^2(0,T;H)}. \tag{3.6}$$

Proof. The assertion (3.4) is immediately obtained by (2.7). Since

$$\begin{aligned}
||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2dt \leq C_0\int_0^T(\int_0^t |k(s)|ds)^2dt \\
&\leq C_0\int_0^T t\int_0^t |k(s)|^2dsdt \leq C_0\tfrac{T^2}{2}\int_0^T |k(s)|^2ds
\end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \leq T\sqrt{C_0/2}||k||_{L^2(0,T;H)}.$$

From (2.4), (3.4), and (3.5) it holds that

$$||x||_{L^2(0,T;V)} \leq C\sqrt{C_1T}(M/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_3 > 0$ such that

$$C_3 = \max\{\sqrt{C_0/2}, C\sqrt{C_1}(C_0/2)^{1/4}\},$$

the proof is complete. □

Theorem 3.2. *Under Assumptions (K) and (G), for any $x_0 \in H$ we have*

$$\overline{L_T(x_0)} \subset \overline{R_T(x_0)}.$$

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Proof. Let $u \in L^2(0,T;U)$ be arbitrary fixed. Then by (2.7) we have

$$||x_u||_{L^2(0,T;V)} \leq C_1(|x_0| + ||B||||u||_{L^2(0,T;U)}),$$

where x_u is the solution of (1.1) corresponding to the control u. For any $\epsilon > 0$, we can choose a constant $\delta > 0$ satisfying

$$\begin{aligned} \min\{\sqrt{\delta}, \delta\} < \min\big[&\big\{7C_3(L_1 + L_2K_1T))\big\}^{-1}, \\ &\epsilon\big\{C_3(L_0 + K_0L_2\sqrt{T})\big\}^{-1}, \\ &\epsilon\big\{C_3(L_1 + L_2K_1T)(C_1C_2||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} + \epsilon)\big\}^{-1}, \\ &\epsilon\big\{C_3(C_0||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} + \epsilon)(L_1 + L_2K_1T)\big\}^{-1}, \\ &\epsilon\big\{(C_3^2(L_0 + K_0L_2)\sqrt{T} + \epsilon)(L_1 + L_2K_1T)\big\}^{-1}\big]/6. \end{aligned} \tag{3.7}$$

Set

$$\begin{aligned} x_1 := x(T-\delta; x_0, G, u) = S(T-\delta)x_0 + \\ + \int_0^{T-\delta} S(T-\delta-s)G(s, x_u(s))ds + \int_0^{T-\delta} S(T-\delta-s)Bu(s)ds, \end{aligned}$$

where $x_u(t) = x(t; x_0, G, u)$ for $0 < t \leq T$. Consider the following problem:

$$\begin{cases} y^{'}(t) = Ay(t) + Bu(t), \quad \delta < t \leq T, \\ y(T-\delta) = x_1, \quad y(s) = 0 \quad -h \leq s \leq 0. \end{cases} \tag{3.8}$$

The solution of (3.8) with respect to the control $w \in L^2(T-\delta, T; U)$ is denoted by

$$\begin{aligned} y_w(T) &= S(\delta)x_1 + \int_{T-\delta}^{T} S(T-s)Bw(s)ds \\ &= S(T)x_0 + S(\delta)\int_0^{T-\delta} S(T-\delta-s)G(s, x_u(s))ds \\ &+ S(\delta)\int_0^{T-\delta} S(T-\delta-s)Bu(s)ds + \int_{T-\delta}^{T} S(T-s)Bw(s)ds. \end{aligned} \tag{3.9}$$

Then since $z \in \overline{L_T(x_0)}$, and $\overline{L_T(x_0)} = \overline{L(0)}$ is independent of the time T and initial data x_0(see Remark 2.1), there exists $w_1 \in L^2(T-\delta, T; U)$ such that

$$\sup_{T-\delta \le t \le T} |y_{w_1}(t) - z| < \frac{\epsilon}{6}, \tag{3.10}$$

and hence, by (3.9),

$$|\int_{T-\delta}^{t} S(T-s)Bw_1(s)ds| \le C_0||x_u||_{L^2(0,T-\delta;V)} + \frac{\epsilon}{6}, \quad t-\delta \le t \le T. \tag{3.11}$$

Now, we set

$$v(s) = \begin{cases} u & \text{if } \ 0 \le s \le T-\delta, \\ w_1(s) & \text{if } \ T-\delta < s < T. \end{cases}$$

Then $v \in L^2(0,T;U)$. Observing that

$$x_v(t; G, v) = S(t)x_0 + \int_0^t S(t-\tau)\{G(\tau, x_v(\tau)) + Bv(\tau)\}d\tau,$$

from (3.9) and (3.10) we obtain that

$$\begin{aligned} &|x(T; x_0, G, v) - z| \le |y_{w_1}(T) - z| + |x(T; x_0, G, v) - y_{w_1}(T)| \\ &\le |y_{w_1}(T) - z| \\ &\quad + |\int_0^T S(T-s)G(s, x_v(s))ds - S(\delta)\int_0^{T-\delta} S(T-\delta-s)G(s, x_u(s))ds| \\ &\quad + |\int_0^T S(T-s)Bv(s)ds - S(\delta)\int_0^{T-\delta} S(T-\delta-s)Bu(s)ds \\ &\quad - \int_{T-\delta}^T S(T-s)Bw_1(s)ds| \\ &\le \frac{\epsilon}{6} + |\int_{T-\delta}^T S(T-s)G(s, x_{w_1}(s))ds| \\ &\le \frac{\epsilon}{6} + II. \end{aligned} \tag{3.12}$$

Here, we remind that the x_{w_1} is represented by

$$\begin{aligned} x_{w_1}(t) =& S(t)x(T-\delta; x_0, G, u) \\ &+ \int_{T-\delta}^t S(T-s)G(s, x_{w_1}(s))ds + \int_{T-\delta}^t S(T-s)Bw_1(s))ds \end{aligned}$$

for $T-\delta<t\le T$. Here, by (2.7) we have

$$\begin{aligned} ||S(\cdot)x(T-\delta;x_0,G,u)||_{L^2(0,T;V)} &\le C_1|x(T-\delta;x_0,G,u)| \\ &\le C_1C_2||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)}. \end{aligned} \tag{3.13}$$

Put

$$p(t)=\int_{T-\delta}^{t}S(t-s)G(s,x_{w_1}(s))ds,\quad T-\delta<t\le T,$$

and

$$q(t):=\int_{t-\delta}^{T}S(t-s)Bw_1(s)ds\quad T-\delta<t\le T.$$

Then with aid of (3.6) of Lemma 3.1 and Lemma 2.2, we have

$$\begin{aligned} &||p||_{L^2(T-\delta,T;V)}\le C_3\sqrt{\delta}||G(\cdot,x_{w_1})||_{L^2(T-\delta,T;V)} \\ &\le C_3\sqrt{\delta}\{(L_0+K_0L_2)\sqrt{T}+(L_1+L_2K_1T)||x_{w_1}||_{L^2(T-\delta,T;V)}\}, \end{aligned} \tag{3.14}$$

and by (3.11),

$$||q||_{L^2(T-\delta,T;V)}\le\sqrt{\delta}(C_0||x_u||_{L^2(0,T-\delta;V)}+\frac{\epsilon}{6}). \tag{3.15}$$

Since $C_3\sqrt{\delta}(L_1+L_2K_1T))<1$ by virtue of (3.7), by (3.13)-(3.15), we get

$$\begin{aligned} ||x_{w_1}||_{L^2(T-\delta,T;V)}\le&\{C_1C_2||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \\ &+\sqrt{\delta}(C_0||x_u||_{L^2(0,T-\delta;V)}+\frac{\epsilon}{6}) \\ &+C_3\sqrt{\delta T}(L_0+K_0L_2)\}\{1-C_3\sqrt{\delta}(L_1+L_2K_1T))\}^{-1}. \end{aligned} \tag{3.16}$$

Hence, with aid of (3.6), (3.7), (3.16), and by using the Hölder inequality, we have

$$\begin{aligned} II=&|\int_{T-\delta}^{T}S(T-s)G(s,x_{w_1}(s))ds| \\ &\le C_3\sqrt{\delta T}\{(L_0+K_0L_2)+(L_1+L_2K_1T)||x_{w_1}||_{L^2(T-\delta,T;V)}\} \\ &\le C_3\sqrt{\delta T}(L_0+K_0L_2)+C_3\sqrt{\delta}(L_1+L_2K_1T)\big\{C_1C_2||x_u||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \\ &+\sqrt{\delta}(C_0||x_u||_{L^2(0,T-\delta;V)}+\frac{\epsilon}{6}) \\ &+C_3\sqrt{\delta T}(L_0+K_0L_2)\big\}\big\{1-C_3\sqrt{\delta}(L_1+L_2K_1T))\big\}^{-1}<\frac{5\epsilon}{6}. \end{aligned} \tag{3.17}$$

Therefore, by (3.12) and (3.17), we have

$$||x(T;x_0,G,v)-z||_H<\epsilon,$$

that is, $z\in\overline{R_T(x_0)}$ and the proof is complete. □

Remark 3.2. *Noting that $H([0,T];U)$ is dense in $L^2(0,T;U)$, we can obtain the same results of Theorem 3.2 corresponding to (1.1) with control space*

$$H([0,T];U) = \{w : [0,T] \to U : |w(t) - w(s)| \le H_0|t-s|^\theta,\ 0 < \theta < 1,\ H_0 > 0\}$$

instead of $L^2(0,T;U)$

From Theorems 3.1-2, we obtain the following control results of (1.1).

Corollary 3.1. *Under Assumptions (K) and (G), for $T > 0$ we have*

$$\overline{L_T(x_0)} = H \iff \overline{R_T(x_0)} = H.$$

Therefore, the approximate controllability of linear system (1.1) with $g = 0$ is equivalent to the condition for the approximate controllability of the nonlinear system (1.1).

Acknowledgement This research was supported by Basic Science Research Program through the National research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2015R1D1A1A09059030).

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On modified degenerate poly-tangent numbers and polynomials

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Abstract : In this paper we introduce the modified degenerate degenerate poly-tangent polynomials and numbers. We also give some properties, explicit formulas, several identities, a connection with modified degenerate poly-tangent numbers and polynomials, and some integral formulas. Finally, we investigate the zeros of the modified degenerate poly-tangent polynomials by using computer.

Key words : Tangent numbers and polynomials, degenerate poly-tangent numbers and polynomials, Cauchy numbers, Stirling numbers, modified degenerate poly-tangent polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials, poly-Bernoulli numbers and polynomials, poly-Euler numbers and polynomials(see [1-11]). In this paper, we define modified degenerate poly-tangent polynomials and numbers and study some properties of the modified degenerate poly-tangent polynomials and numbers. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Carlitz [1] has defined the degenerate Stirling numbers of the first kind and second kind, $S_1(n, k, \lambda)$ and $S_2(n, k, \lambda)$ by means of

$$\left(\frac{1-(1-t)^{\lambda}}{\lambda}\right)^{k} = k! \sum_{n=k}^{\infty} S_1(n,k,\lambda)\frac{t^n}{n!}, \tag{1.1}$$

$$\left((1+\lambda t)^{1/\lambda}-1\right)^{k} = k! \sum_{n=k}^{\infty} S_2(n,k,\lambda)\frac{t^n}{n!}. \tag{1.2}$$

Howard [12] has defined the degenerate weighted Stirling numbers of the first kind and second kind, $S_1(n, k, x, \lambda)$ and $S_2(n, k, x, \lambda)$ by means of

$$(1-t)^{\lambda-x}\left(\frac{1-(1-t)^{\lambda}}{\lambda}\right)^{k} = k! \sum_{n=k}^{\infty} S_1(n,k,x,\lambda)\frac{t^n}{n!}, \tag{1.3}$$

$$(1+\lambda t)^{x/\lambda}\left((1+\lambda t)^{1/\lambda}-1\right)^{k} = k! \sum_{n=k}^{\infty} S_2(n,k,x,\lambda)\frac{t^n}{n!}. \tag{1.4}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k).$$

The generalized raising factorial $< x|\lambda >_n$ with increment λ is defined by

$$< x|\lambda >_n = \prod_{k=0}^{n-1}(x+\lambda k).$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}.$$

The degenerate poly-Bernoulli numbers $\mathcal{B}_n^{(k)}(\lambda)$ were introduced by Kaneko [5] by using the following generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-(1+\lambda t)^{-1/\lambda}} = \sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}(\lambda)\frac{t^n}{n!}, \quad (k\in\mathbb{Z}), \tag{1.5}$$

where

$$\mathrm{Li}_k(t) = \sum_{n=1}^{\infty}\frac{t^n}{n^k} \tag{1.6}$$

is the kth polylogarithm function.

The degenerate poly-Euler polynomials $\mathcal{E}_n^{(k)}(x,\lambda)$ are defined by generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(x,\lambda)\frac{t^n}{n!}, \quad (k\in\mathbb{Z}). \tag{1.7}$$

The familiar degenerate tangent polynomials $\mathbf{T}_n(x,\lambda)$ are defined by the generating function([7]):

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathbf{T}_n(x,\lambda)\frac{t^n}{n!}, \quad (|2t|<\pi). \tag{1.8}$$

When $x=0$, $\mathbf{T}_n(0,\lambda)=\mathbf{T}_n(\lambda)$ are called the degenerate tangent numbers. The degenerate tangent polynomials $\mathbf{T}_n^{(r)}(x,\lambda)$ of order r are defined by

$$\left(\frac{2}{(1+\lambda t)^{2/\lambda}+1}\right)^r(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathbf{T}_n^{(r)}(x,\lambda)\frac{t^n}{n!}, \quad (|2t|<\pi). \tag{1.9}$$

It is clear that $r=1$ we recover the degenerate tangent polynomials $\mathbf{T}_n(x,\lambda)$.

The degenerate Bernoulli polynomials $\mathbf{B}_n^{(r)}(x,\lambda)$ of order r are defined by the following generating function

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x,\lambda)\frac{t^n}{n!}, \quad (|t|<2\pi). \tag{1.10}$$

The degenerate Frobenius-Euler polynomials of order r, denoted by $\mathbf{H}_n^{(r)}(u,x,\lambda)$, are defined as

$$\left(\frac{1-u}{(1+\lambda t)^{1/\lambda}-u}\right)^r(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}. \tag{1.11}$$

The values at $x=0$ are called degenerate Frobenius-Euler numbers of order r; when $r=1$, the polynomials or numbers are called ordinary degenerate Frobenius-Euler polynomials or numbers.

The degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are defined by the generating function:

$$\frac{2\mathrm{Li}_k\left(1-e^{-t}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}, \quad (k\in\mathbb{Z}). \tag{1.12}$$

When $x=0$, $\mathcal{T}_n^{(k)}(0,\lambda)=\mathcal{T}_n^{(k)}(x,\lambda)$ are called the degenerate poly-tangent numbers. Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [1-12]). In the following section, we introduce the modified degenerate poly-tangent polynomials and numbers. After that we will investigate some their properties. We also give some relationships both between these polynomials and modified degenerate poly-tangent polynomials and between these polynomials and cauchy numbers. Finally, we investigate the zeros of the modified degenerate poly-tangent polynomials by using computer.

2. Modified degenerate poly-tangent polynomials

In this section, we define modified degenerate poly-tangent numbers and polynomials and provide some of their relevant properties.

The modified degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are defined by the generating function:

$$\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!},\quad (k\in\mathbb{Z}). \tag{2.1}$$

When $x=0$, $\mathcal{T}_n^{(k)}(0,\lambda)=\mathcal{T}_n^{(k)}(x,\lambda)$ are called the degenerate poly-tangent numbers. Upon setting $k=1$ in (2.1), we have

$$\mathcal{T}_n^{(1)}(x,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\lambda^{n-1}S_1(l,1)\mathbf{T}_{n-l}(x,\lambda)\text{ for }n\geq 1.$$

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda}\\&=\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

By comparing the coefficients on both sides of (2.2), we have the following theorem.

Theorem 2.1. For $n\in\mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(\lambda)(x|\lambda)_{n-l}.$$

The following elementary properties of the degenerate poly-tangent numbers $\mathcal{T}_n^{(k)}(\lambda)$ and polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ are readily derived form (2.1). We, therefore, choose to omit details involved.

Theorem 2.2. For $k\in\mathbb{Z}$, we have

$$\begin{aligned}&(1)\quad \mathcal{T}_n^{(k)}(x+y,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)(y|\lambda)_{n-l}.\\&(2)\quad \mathcal{T}_n^{(k)}(2-x,\lambda)=\sum_{l=0}^{n}(-1)^l\binom{n}{l}\mathcal{T}_{n-l}^{(k)}(2,\lambda)<x|\lambda>_l.\end{aligned}$$

Theorem 2.3 For any positive integer n, we have

$$\begin{aligned}&(1)\quad \mathcal{T}_n^{(k)}(mx,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)((m-1)x|\lambda)_{n-l}.\\&(2)\quad \mathcal{T}_n^{(k)}(x+1,\lambda)-\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{n-1}\binom{n}{l}\mathcal{T}_l^{(k)}(x,\lambda)(1|\lambda)_{n-l}.\end{aligned} \tag{2.3}$$

From (1.6), (1.8), and (2.1), we get

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}=\left(2\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda}\\
&=\sum_{l=0}^{\infty}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{l+1}}{(l+1)^k}\frac{2(1+\lambda t)^{x/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\\
&=\sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i\frac{2(1+\lambda t)^{x/\lambda}\left(1+\lambda t\right)^{-i/\lambda}}{(1+\lambda t)^{2/\lambda}+1} \qquad (2.4)\\
&=\sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\mathbf{T}_j(x,\lambda)(-1)^{n-j}<i|\lambda>_{(n-j)}\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n}\frac{1}{(l+1)^k}\binom{l+1}{i}(-i)^{n+i-j}\binom{n}{j}\mathbf{T}_j(x,\lambda)<i|\lambda>_{(n-j)}\right)\frac{t^n}{n!}.
\end{aligned}
$$

By comparing the coefficients on both sides of (2.4), we have the following theorem.

Theorem 2.4 For $n\in\mathbb{Z}_+$, we have

$$
\begin{aligned}
\mathcal{T}_n^{(k)}(x,\lambda)&=\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n}\frac{(-i)^{n+i-j}}{(l+1)^k}\binom{l+1}{i}\binom{n}{j}\mathbf{T}_j(x,\lambda)<i|\lambda>_{(n-j)}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\frac{(-i)^i}{(l+1)^k}\binom{l+1}{i}\mathbf{T}_n(x-i,\lambda).
\end{aligned}
$$

By (2.1), we note that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}=2\sum_{l=0}^{\infty}(-1)^l(1+\lambda t)^{2l/\lambda}\sum_{l=0}^{\infty}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{l+1}}{(l+1)^k}(1+\lambda t)^{x/\lambda}\\
&=2\sum_{l=0}^{\infty}\sum_{i=0}^{l}\frac{\left(1-(1+\lambda t)^{-1/\lambda}\right)^{i+1}}{(i+1)^k}(-1)^{l-i}(1+\lambda t)^{(2l-2i)/\lambda}(1+\lambda t)^{x/\lambda}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{2(-1)^{l+j-i}\binom{i+1}{j}}{(i+1)^k}(1+\lambda t)^{(2l-2i+x)/\lambda}\left(1+\lambda t\right)^{-j/\lambda}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\sum_{m=0}^{n}\frac{2(-1)^{l+j-i}\binom{i+1}{j}\binom{n}{m}(2l-2i+x|\lambda)_m<j|\lambda>_{(n-m)}}{(i+1)^k}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients on both sides, we have the following theorem.

Theorem 2.5 For $n\in\mathbb{Z}_+$, we have

$$
\begin{aligned}
\mathcal{T}_n^{(k)}(x,\lambda)&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\sum_{m=0}^{n}\frac{2(-1)^{l+j-i}\binom{i+1}{j}\binom{n}{m}(2l-2i+x|\lambda)_m<j|\lambda>_{(n-m)}}{(i+1)^k}\\
&=\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1}\frac{2(-1)^{l+j-i}\binom{i+1}{j}(2l-2i-j+x|\lambda)_m}{(i+1)^k}.
\end{aligned}
$$

3. Some identities involving degenerate poly-tangent numbers and polynomials

In this section, we give several combinatorics identities involving degenerate poly-tangent numbers and polynomials in terms of degenerate Stirling numbers, generalized falling factorial functions, generalized raising factorial functions, Beta functions, degenerate Bernoulli polynomials of higher order, and degenerate Frobenius-Euler functions of higher order.

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(1-(1-(1+\lambda t)^{-1/\lambda})\right)^{-x}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-(1+\lambda t)^{-1/\lambda})^l\\
&=\sum_{l=0}^{\infty}<x>_l\frac{((1+\lambda t)^{1/\lambda}-1)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{-l/\lambda}\right)\\
&=\sum_{l=0}^{\infty}<x>_l\sum_{n=0}^{\infty}S_2(n,l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(-l,\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}(-l,\lambda)<x>_l\right)\frac{t^n}{n!},
\end{aligned}\tag{3.1}$$

where $<x>_l=x(x+1)\cdots(x+l-1)(l\geq 1)$ with $<x>_0=1$.

By comparing the coefficients on both sides of (3.1), we have the following theorem.

Theorem 3.1 For $n\in\mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}(-l,\lambda)<x>_l.$$

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(1-(1-(1+\lambda t)^{-1/\lambda})\right)^{-x}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-(1+\lambda t)^{-1/\lambda})^l\\
&=\sum_{l=0}^{\infty}<x>_l\frac{(1+\lambda t)^{-l/\lambda}((1+\lambda t)^{1/\lambda}-1)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\sum_{l=0}^{\infty}<x>_l\sum_{n=0}^{\infty}S_2(n,l,-l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l,-l,\lambda)\mathcal{T}_{n-i}^{(k)}(\lambda)<x>_l\right)\frac{t^n}{n!},
\end{aligned}\tag{3.2}$$

where $<x>_l=x(x+1)\cdots(x+l-1)(l\geq 1)$ with $<x>_0=1$.

By comparing the coefficients on both sides of (3.2), we have the following theorem.

Theorem 3.2 For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i} S_2(i,l,-l,\lambda)\mathcal{T}_{n-i}^{(k)}(\lambda) < x >_l .$$

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!} &= \left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)\left(((1+\lambda t)^{1/\lambda}-1)+1\right)^x \\
&= \frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\sum_{l=0}^{\infty}\binom{x}{l}\left((1+\lambda t)^{1/\lambda}-1\right)^l \\
&= \sum_{l=0}^{\infty}(x)_l\frac{\left((1+\lambda t)^{1/\lambda}-1\right)^l}{l!}\left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right) \qquad (3.3) \\
&= \sum_{l=0}^{\infty}(x)_l\sum_{n=0}^{\infty}S_2(n,l,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides of (3.3), we have the following theorem.

Theorem 3.3 For $n \in \mathbb{Z}_+$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l,\lambda)\mathcal{T}_{n-i}^{(k)}.$$

By (1.2), (1.10), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!} \\
&= \left(\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\right)(1+\lambda t)^{x/\lambda} \\
&= \frac{((1+\lambda t)^{1/\lambda}-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^r(1+\lambda t)^{x/\lambda}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!} \\
&= \frac{((1+\lambda t)^{1/\lambda}-1)^r}{r!}\left(\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(\lambda)\frac{t^n}{n!}\right)\frac{r!}{t^r} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}}S_2(l+r,r,\lambda)\sum_{i=0}^{n-l}\binom{n-l}{i}\mathbf{B}_i^{(r)}(x,\lambda)\mathcal{T}_{n-l-i}^{(k)}(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.4 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{n}\sum_{i=0}^{n-l}\frac{\binom{n}{l}\binom{n-l}{i}}{\binom{l+r}{r}}S_2(l+r,r)T_{n-l-i}^{(k)}\mathbf{B}_i^{(r)}(x,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&=\frac{((1+\lambda t)^{1/\lambda}-u)^r}{(1-u)^r}\left(\frac{1-u}{(1+\lambda t)^{1/\lambda}-u}\right)^r(1+\lambda t)^{x/\lambda}\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\\
&=\sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}\sum_{i=0}^{r}\binom{r}{i}(1+\lambda t)^{i/\lambda}(-u)^{r-i}\frac{1}{(1-u)^r}\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}\\
&=\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{n=0}^{\infty}\mathbf{H}_n^{(r)}(u,x,\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(i,\lambda)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{(1-u)^r}\sum_{i=0}^{r}\binom{r}{i}(-u)^{r-i}\sum_{l=0}^{n}\binom{n}{l}\mathbf{H}_l^{(r)}(u,x,\lambda)\mathcal{T}_{n-l}^{(k)}(i,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.5 For $n\in\mathbb{Z}_+$ and $r\in\mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\frac{1}{(1-u)^r}\sum_{i=0}^{r}\sum_{l=0}^{n}\binom{r}{i}\binom{n}{l}(-u)^{r-i}\mathbf{H}_l^{(r)}(u,x,\lambda)\mathcal{T}_{n-l}^{(k)}(i,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\frac{(1+\lambda t)^{1/\lambda}+1}{(1+\lambda t)^{1/\lambda}+1}\\
&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{1/\lambda}+1}(1+\lambda t)^{x/\lambda}\left(\frac{(1+\lambda t)^{1/\lambda}}{(1+\lambda t)^{2/\lambda}+1}+\frac{1}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\left(\sum_{n=0}^{\infty}\mathcal{E}_n^{(k)}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{1}{2}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\mathcal{E}_{n-l}^{(k)}(x,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.6 For $n\in\mathbb{Z}_+$ and $r\in\mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda)=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(1,\lambda)+\mathbf{T}_n(\lambda)\right)\mathcal{E}_{n-l}^{(k)}(x,\lambda).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{T}_n^{(k)}(x,\lambda)\frac{t^n}{n!}&=\frac{2\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\frac{1-(1+\lambda t)^{-1/\lambda}}{1-(1+\lambda t)^{-1/\lambda}}\\
&=\frac{\mathrm{Li}_k\left(1-(1+\lambda t)^{-1/\lambda}\right)}{1-(1+\lambda t)^{-1/\lambda}}\left(\frac{2(1+\lambda t)^{x/\lambda}}{(1+\lambda t)^{2/\lambda}+1}-\frac{2(1+\lambda t)^{(x-1)/\lambda}}{(1+\lambda t)^{2/\lambda}+1}\right)\\
&=\left(\sum_{n=0}^{\infty}\mathcal{B}_n^{(k)}(\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\mathbf{T}_n(x,\lambda)-\mathbf{T}_n(x-1,\lambda)\right)\frac{t^n}{n!}\right)\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\left(\mathbf{T}_n(x,\lambda)-\mathbf{T}_n(x-1,\lambda)\right)\mathcal{B}_{n-l}^{(k)}(x,\lambda)\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

Theorem 3.7 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\mathcal{T}_n^{(k)}(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(x,\lambda) - \mathbf{T}_n(x-1,\lambda)\right) \mathcal{B}_{n-l}^{(k)}(\lambda).$$

By Theorem 3.6 and Theorem 3.7, we have the following corollary.

Corollary 3.8 For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}&\sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(1,\lambda) + \mathbf{T}_n(\lambda)\right) \mathcal{E}_{n-l}^{(k)}(x,\lambda)\\ &= 2\sum_{l=0}^{n} \binom{n}{l} \left(\mathbf{T}_n(x,\lambda) - \mathbf{T}_n(x-1,\lambda)\right) \mathcal{B}_{n-l}^{(k)}(\lambda).\end{aligned}$$

3. Distribution of zeros of the degenerate poly-tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$. The degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ can be determined explicitly. A few of them are

$$\begin{aligned}
&\mathcal{T}_0^{(k)}(x,\lambda) = 0,\\
&\mathcal{T}_1^{(k)}(x,\lambda) = 1,\\
&\mathcal{T}_2^{(k)}(x,\lambda) = -3 + 2^{1-k} - \lambda + 2x\\
&\mathcal{T}_3^{(k)}(x,\lambda) = 4 - 3\cdot 2^{2-k} + 2\cdot 3^{1-k} + 9\lambda - 3\cdot 2^{1-k}\lambda + 2\lambda^2 - 9x\\
&\qquad + 3\cdot 2^{1-k}x - 6\lambda x + 3x^2,\\
&\mathcal{T}_4^{(k)}(x,\lambda) = 3 + 3^{3-2k} + 7\cdot 2^{1-k} + 3\cdot 2^{3-k} - 8\cdot 3^{1-k} - 4\cdot 3^{2-k} - 24\lambda\\
&\qquad + 3\cdot 2^{3-k}\lambda + 3\cdot 2^{4-k}\lambda - 4\cdot 3^{2-k}\lambda - 33\lambda^2 + 11\cdot 2^{1-k}\lambda^2 - 6\lambda^3\\
&\qquad + 16x - 3\cdot 2^{4-k}x + 8\cdot 3^{1-k}x + 54\lambda x - 3\cdot 2^{2-k}\lambda x - 3\cdot 2^{3-k}\lambda x\\
&\qquad + 22\lambda^2 x - 18x^2 + 3\cdot 2^{2-k}x^2 - 18\lambda x^2 + 4x^3.
\end{aligned}$$

We investigate the beautiful zeros of thedegenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ by using a computer. We plot the zeros of the poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ for $n = 30, k = -5, -1, 1, 5, \lambda = 1/2$, and $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 30$ and $k = -5$. In Figure 1(top-right), we choose $n = 30$ and $k = -1$. In Figure 1(bottom-left), we choose $n = 30$ and $k = 1$. In Figure 1(bottom-right), we choose $n = 30$ and $k = 5$. Stacks of zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$ from a 3-D structure are presented(Figure 2). In Figure 2(left), we choose $k = -5$. In Figure 2(middle), we choose $k = 1$. In Figure 2(right), we choose $k = 5$. Our numerical results for approximate solutions of real zeros of $\mathcal{T}_n^{(k)}(x,\lambda), \lambda = 1/2$ are displayed(Tables 1, 2).

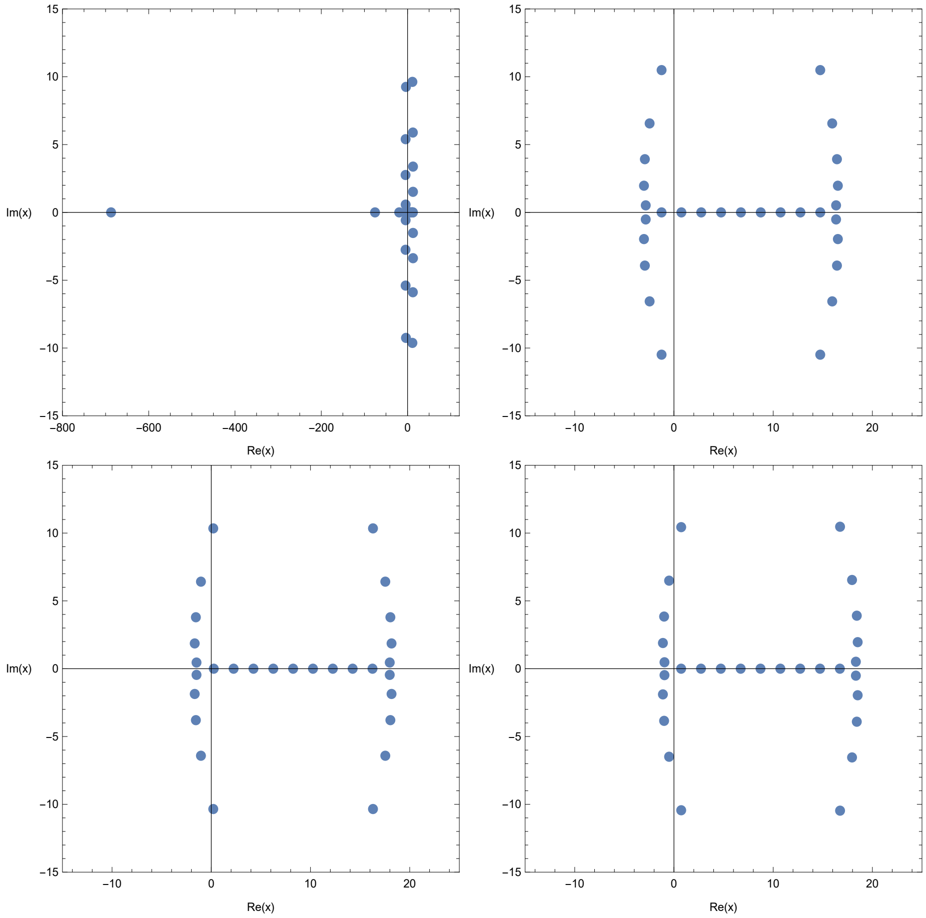

Figure 1: Zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$

Table 1. Numbers of real and complex zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$

	$k=-10$		$k=1$		$k=10$	
degree n	real	complex zeros	real	complex zeros	real	complex zeros
2	1	0	1	0	1	0
3	2	0	2	0	2	0
4	3	0	3	0	3	0
5	4	0	4	0	4	0
6	5	0	5	0	5	0
7	6	0	2	4	2	4
8	5	2	3	4	3	4
9	6	2	4	4	4	4
10	5	4	5	4	5	4
11	6	4	6	4	6	4
12	7	4	7	4	5	6

The plot of real zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \leq n \leq 30$ structure are presented(Figure 3). In Figure 3(left), we choose $k=-5$ and $\lambda=1/2$. In Figure 3(middle),we choose $k=1$ and $\lambda=1/2$. In Figure 3(right), we choose $k=5$ and $\lambda=1/2$.

We observe a remarkable regular structure of the complex roots of the degenerate poly-tangent

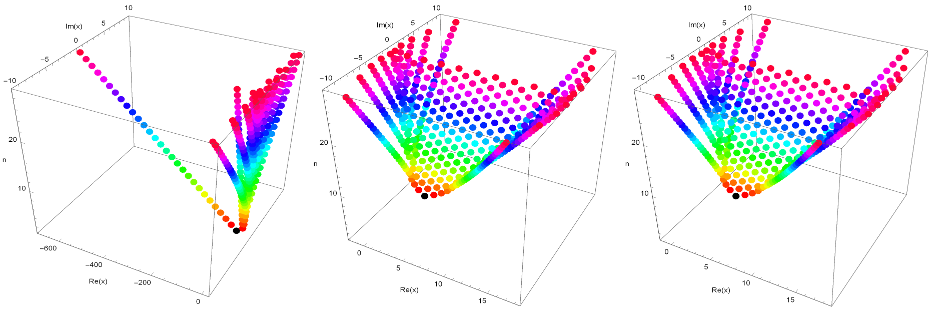

Figure 2: Stacks of zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$

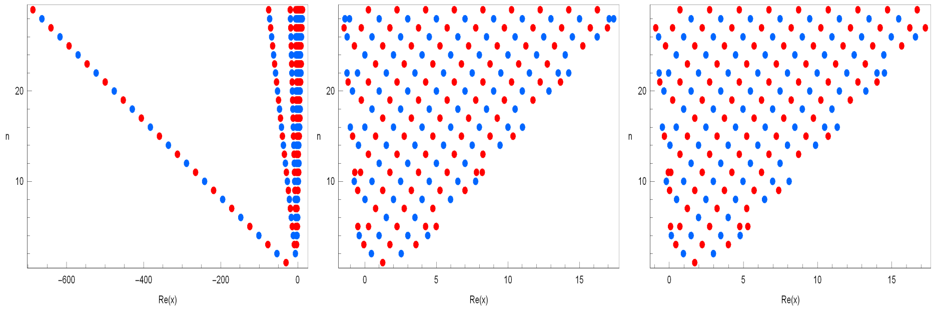

Figure 3: Real zeros of $\mathcal{T}_n^{(k)}(x,\lambda)$ for $1 \le n \le 30$

polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$. We also hope to verify a remarkable regular structure of the complex roots of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$(Table 1).

Next, we calculated an approximate solution satisfying poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda) = 0$ for $x \in \mathbb{R}$. The results are given in Table 2 and Table 3.

Table 2. Approximate solutions of $\mathcal{T}_n^{(k)}(x,\lambda) = 0, \lambda = 1/2, k = -5$

degree n	x
2	30.250
3	−53.896, −6.1044
4	−77.421, −8.8591, −2.9699
5	−100.91, −11.489, −3.9628, −1.6365
6	−124.39, −14.080, −4.7720, −2.3421, −0.66874
7	−147.85, −16.655, −5.4611, −3.0181, −1.0879, 0.076439

Table 3. Approximate solutions of $\mathcal{T}_n^{(k)}(x,\lambda)=0, \lambda=1/2, k=5$

degree n	x
2	1.7188
3	0.95682, 2.9807
4	0.44597, 2.2234, 3.9869
5	0.13979, 1.4750, 3.4758, 4.7844
6	0.090663, 0.71964, 2.7246, 4.7571, 5.3017
7	1.9752, 3.9751

By numerical computations, we will make a series of the following conjectures:

Conjecture 4.1. Prove that $\mathcal{T}_n^{(k)}(x,\lambda), x\in\mathbb{C}$, has $Im(x,\lambda)=0$ reflection symmetry analytic complex functions. However, $T_n^{(k)}(x,\lambda), k\neq 1$, has not $Re(x,\lambda)=a$ reflection symmetry for $a\in\mathbb{R}$.

Using computers, many more values of n have been checked. It still remains unknown if the conjecture fails or holds for any value n(see Figures 1, 2, 3). We are able to decide if $\mathcal{T}_n^{(k)}(x,\lambda))=0$ has $n-1$ distinct solutions(see Tables 1, 2, 3).

Conjecture 4.2. Prove that $\mathcal{T}_n^{(k)}(x,\lambda))=0$ has $n-1$ distinct solutions.

Since $n-1$ is the degree of the polynomial $\mathcal{T}_n^{(k)}(x,\lambda)$, the number of real zeros $R_{\mathcal{T}_n^{(k)}(x,\lambda)}$ lying on the real plane $Im(x,\lambda)=0$ is then $R_{\mathcal{T}_n^{(k)}(x,\lambda)}=n-1-C_{\mathcal{T}_n^{(k)}(x,\lambda)}$, where $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{\mathcal{T}_n^{(k)}(x,\lambda)}$ and $C_{\mathcal{T}_n^{(k)}(x,\lambda)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the degenerate poly-tangent polynomials $\mathcal{T}_n^{(k)}(x,\lambda)$ which appear in mathematics and physics.

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

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On the Carlitz's type twisted (p,q)-Euler polynomials and twisted (p,q)-Euler zeta function

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Abstract : In this paper we construct Carlitz's type twisted (p,q)-Euler zeta function. In order to define Carlitz's type twisted (p,q)-Euler zeta function, we introduce the Carlitz's type twisted (p,q)-Euler numbers and polynomials by generalizing the Euler numbers and polynomials, Carlitz's type q-Euler numbers and polynomials. We also give some interesting properties, explicit formulas, a connection with Carlitz's type twisted (p,q)-Euler numbers and polynomials. Finally, we investigate the zeros of the Carlitz's type twisted (p,q)-Euler polynomials by using computer.

Key words : Euler numbers and polynomials, q-Euler numbers and polynomials, (h,q)-Euler numbers and polynomials, Carlitz's type twisted (p,q)-Euler numbers and polynomials, (p,q)-Euler zeta function, twisted (p,q)-Euler zeta function.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many mathematicians have studied in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials(see [1-10]). In this paper, we define Carlitz's type twisted (p,q)-Euler numbers and polynomials and study some properties of the Carlitz's type twisted (p,q)-Euler numbers and polynomials.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0,-1,-2,-3,\ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

We remember that the classical Euler numbers E_n and Euler polynomials $E_n(x)$ are defined by the following generating functions(see [1, 2, 3, 4, 5])

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.1}$$

and

$$\left(\frac{2}{e^t+1}\right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (|t| < \pi). \tag{1.2}$$

respectively.

The (p,q)-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p-q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2q^{n-3} + pq^{n-2} + q^{n-1}.$$

It is clear that (p,q)-number contains symmetric property, and this number is q-number when $p=1$. In particular, we can see $\lim_{q\to 1}[n]_{p,q} = n$ with $p=1$.

By using (p,q)-number, we define the (p,q)-analogue of Euler polynomials and numbers, which generalized the previously known numbers and polynomials, including the Carlitz's type q-Euler numbers and polynomials. We begin by recalling here the Carlitz's type q-Euler numbers and polynomials(see 1, 2, 3, 4, 5]).

Definition 1. The Carlitz's type q-Euler polynomials $E_{n,q}(x)$ are defined by means of the generating function

$$F_q(t,x)=\sum_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_q t}. \tag{1.3}$$

and their values at $x=0$ are called the Carlitz's type q-Euler numbers and denoted $E_{n,q}$.

Many kinds of of generalizations of these polynomials and numbers have been presented in the literature(see [1-10]). Based on this idea, we generalize the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$. It follows that we define the following (p,q)-analogues of the the Carlitz's type q-Euler number $E_{n,q}$ and q-Euler polynomials $E_{n,q}(x)$ (see [6, 7, 9, 10]).

Definition 2. For $0<q<p\le 1$, the Carlitz's type (p,q)-Euler numbers $E_{n,p,q}$ and polynomials $E_{n,p,q}(x)$ are defined by means of the generating functions

$$F_{p,q}(t)=\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m]_{p,q}t}. \tag{1.4}$$

and

$$F_{p,q}(t,x)=\sum_{n=0}^{\infty}E_{n,p,q}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_{p,q}t}, \tag{1.5}$$

respectively.

In the following section, we define Carlitz's type twisted (p,q)-Euler zeta function. We introduce the Carlitz's type twisted (p,q)-Euler polynomials and numbers. After that we will investigate some their properties. Finally, we investigate the zeros of the Carlitz's type twisted (p,q)-Euler polynomials by using computer.

2. Twisted (p,q)-Euler numbers and polynomials

In this section, we define twisted (p,q)-Euler numbers and polynomials and provide some of their relevant properties. Let r be a positive integer, and let ω be rth root of 1.

Definition 2. For $0<q<p\le 1$, the Carlitz's type twisted (p,q)-Euler numbers $E_{n,p,q,\omega}$ and polynomials $E_{n,p,q,\omega}(x)$ are defined by means of the generating functions

$$F_{p,q,\omega}(t)=\sum_{n=0}^{\infty}E_{n,p,q,\omega}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m \omega^m e^{[m]_{p,q}t}. \tag{2.1}$$

and

$$F_{p,q,\omega}(t,x)=\sum_{n=0}^{\infty}E_{n,p,q,\omega}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m \omega^m e^{[m+x]_{p,q}t}, \tag{2.2}$$

respectively.

Setting $p=1$ in (2.1) and (2.2), we can obtain the corresponding definitions for the Carlitz's type twisted q-Euler number $E_{n,q,\omega}$ and q-Euler polynomials $E_{n,q,\omega}(x)$ respectively. Obviously, if we put $\omega=1$, then we have

$$E_{n,p,q,\omega}(x)=E_{n,p,q}(x),\quad E_{n,p,q,\omega}=E_{n,p,q}.$$

Putting $p=1$, we have

$$\lim_{q\to 1}E_{n,p,q,\omega}(x)=E_{n,\omega}(x),\quad \lim_{q\to 1}E_{n,p,q,\omega}=E_{n,\omega}.$$

By using above equation (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,p,q,\omega}\frac{t^n}{n!} &= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m\omega^m e^{[m]_{p,q}t}\\ &= \sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+\omega q^{l+1}p^{n-l}}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.3}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 3. For $n\in\mathbb{Z}_+$, we have

$$E_{n,p,q,\omega} = [2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+\omega q^{l+1}p^{n-l}}.$$

If we put $p=1$ in the above theorem we obtain

$$E_{n,p,q,\omega} = [2]_q\left(\frac{1}{1-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+\omega q^{l+1}}.$$

By (2.2), we obtain

$$E_{n,p,q,\omega}(x) = [2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+\omega q^{l+1}p^{n-l}}. \tag{2.4}$$

By using (2.2) and (2.4), we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,p,q,\omega}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+\omega q^{l+1}p^{n-l}}\right)\frac{t^n}{n!}\\ &= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m\omega^m e^{[m+x]_{p,q}t}.\end{aligned} \tag{2.5}$$

Since $[x+y]_{p,q} = p^y[x]_{p,q}+q^x[y]_{p,q}$, we see that

$$E_{n,p,q,\omega}(x) = [2]_q\sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left(\frac{1}{p-q}\right)^l\frac{1}{1+\omega q^{k+1}p^{n-k}}. \tag{2.6}$$

Next, we introduce Carlitz's type twisted (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q,\omega}(x)$.

Definition 4. The Carlitz's type twisted (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q,\omega}(x)$ are defined by

$$E^{(h)}_{n,p,q}(x) = [2]_q\sum_{m=0}^{\infty}(-1)^m q^m p^{hm}\omega^m[m+x]_{p,q}^n. \tag{2.7}$$

By using (2.7) and (p,q)-number, we have the following theorem.

Theorem 5. For $n\in\mathbb{Z}_+$, we have

$$E^{(h)}_{n,p,q,\omega}(x) = [2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl}p^{(n-l)x}\frac{1}{1+\omega q^{l+1}p^{n-l+h}}.$$

By (2.6) and Theorem 2.4, we have

$$E_{n,p,q,\omega}(x) = \sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}E^{(n-l)}_{l,p,q,\omega}$$

The following elementary properties of the (p,q)-analogue of Euler numbers $E_{n,p,q,\omega}$ and polynomials $E_{n,p,q,\omega}(x)$ are readily derived form (2.1) and (2.2). We, therefore, choose to omit details involved.

Theorem 6. (Distribution relation) For any positive integer m(=odd), we have

$$E_{n,p,q,\omega}(x) = \frac{[2]_q}{[2]_{q^m}}[m]_{p,q}^n \sum_{a=0}^{m-1}(-1)^a q^a \omega^a E_{n,p^m,q^m,\omega^m}\left(\frac{a+x}{m}\right), n \in \mathbb{Z}_+.$$

Theorem 7. (Property of complement) For $n \in \mathbb{Z}_+$, we have

$$E_{n,p^{-1},q^{-1},\omega^{-1}}(1-x) = (-1)^n \omega p^n q^n E_{n,p,q,\zeta}(x).$$

Theorem 8. For $n \in \mathbb{Z}_+$, we have

$$\omega q E_{n,p,q,\omega}(1) + E_{n,p,q,\omega} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

By (2.1) and (2.2), we get

$$-[2]_q \sum_{l=0}^{\infty}(-1)^{l+n} q^{l+n}\omega^{l+n} e^{[l+n]_{p,q}t} + [2]_q \sum_{l=0}^{\infty}(-1)^l q^l \omega^l e^{[l]_{p,q}t} = [2]_q \sum_{l=0}^{n-1}(-1)^l q^l \omega^l e^{[l]_{p,q}t}. \tag{2.8}$$

Hence we have

$$(-1)^{n+1}q^n\omega^n \sum_{m=0}^{\infty} E_{m,p,q,\omega}(n)\frac{t^m}{m!} + \sum_{m=0}^{\infty} E_{m,p,q,\omega}\frac{t^m}{m!} = \sum_{m=0}^{\infty}\left([2]_q \sum_{l=0}^{n-1}(-1)^l q^l \omega^l [l]_{p,q}^m\right)\frac{t^m}{m!}. \tag{2.9}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.9), we have the following theorem.

Theorem 9. For $n \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n-1}(-1)^l q^l \omega^l [l]_{p,q}^m = \frac{(-1)^{n+1}q^n\omega^n E_{m,p,q,\omega}(n) + E_{m,p,q,\omega}}{[2]_q}.$$

We investigate the zeros of the twisted (p,q)-Euler polynomials $E_{n,p,q,\omega}(x)$ by using a computer. We plot the zeros of the twisted (p,q)-Euler polynomials $E_{n,p,q,\omega}(x)$ for $x \in \mathbb{C}$(Figure 1). In Figure 1(top-left), we choose $n = 20, p = 1/2, q = 1/10$ and $\omega = e^{\frac{2\pi i}{2}}$. In Figure 1(top-right), we choose $n = 40, p = 1/2, q = 1/10$ and $\omega = e^{\frac{2\pi i}{2}}$. In Figure 1(bottom-left), we choose $n = 20, p = 1/2, q = 1/10$ and $\omega = e^{\frac{2\pi i}{4}}$. In Figure 1(bottom-right), we choose $n = 40, p = 1/2, q = 1/10$ and $\omega = e^{\frac{2\pi i}{4}}$.

3. Twisted (p,q)-Euler zeta function

By using twisted (p,q)-Euler numbers and polynomials, (p,q)-Euler zeta function and Hurwitz (p,q)-Euler zeta function is defined. These functions interpolate the twisted (p,q)-Euler numbers $E_{n,p,q,\omega}$, and polynomials $E_{n,p,q,\omega}(x)$, respectively. From (2.1), we note that

$$\begin{aligned} \left.\frac{d^k}{dt^k}F_{p,q,\omega}(t)\right|_{t=0} &= [2]_q \sum_{m=0}^{\infty}(-1)^n q^m \omega^m [m]_{p,q}^k \\ &= E_{k,p,q,\omega}, (k \in \mathbb{N}). \end{aligned}$$

By using the above equation, we are now ready to define twisted (p,q)-Euler zeta function.

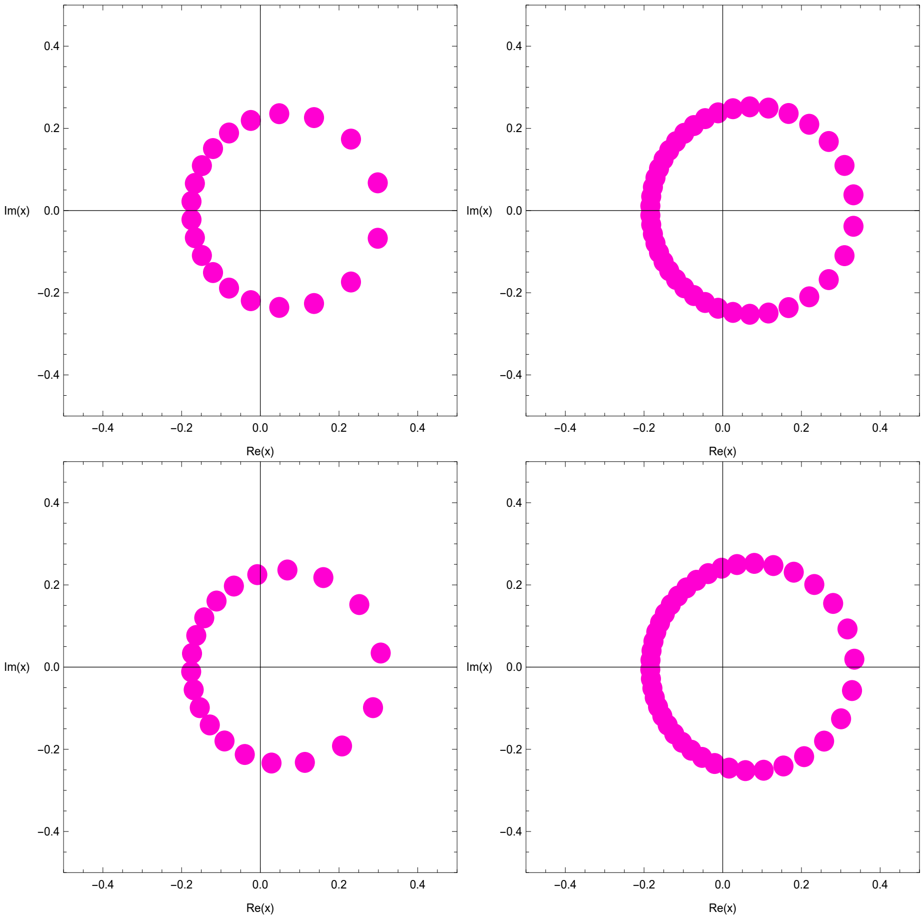

Figure 1: Zeros of $E_{n,p,q,\omega}(x)$

Definition 10. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$.

$$\zeta_{p,q,\omega}(s) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n \omega^n}{[n]_{p,q}^s}. \tag{3.1}$$

Note that $\zeta_{p,q,\omega}(s)$ is a meromorphic function on $\mathbb{C}$. Note that, if $p = 1, q \to 1$, then $\zeta_{p,q,\omega}(s) = \zeta_E(s)$ which is the Euler zeta functions(see [4]). Relation between $\zeta_{p,q,\omega}(s)$ and $E_{k,p,q,\omega}$ is given by the following theorem.

Theorem 11. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q,\omega}(-k) = E_{k,p,q,\omega}.$$

Observe that $\zeta_{p,q,\omega}(s)$ function interpolates $E_{k,p,q,\omega}$ numbers at non-negative integers. By using (2.2), we note that

$$\left.\frac{d^k}{dt^k} F_{p,q,\omega}(t,x)\right|_{t=0} = [2]_q \sum_{m=0}^{\infty} (-1)^m q^m \omega^m [m+x]_{p,q}^k \tag{3.2}$$

and

$$\left(\frac{d}{dt}\right)^k \left.\left(\sum_{n=0}^{\infty} E_{n,p,q}(x)\frac{t^n}{n!}\right)\right|_{t=0} = E_{k,p,q}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz (p,q)-Euler zeta function.

Definition 12. Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$ and $x \notin \mathbb{Z}_0^-$.

$$\zeta_{p,q,\omega}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n \omega^n}{[n+x]_{p,q}^s}. \tag{3.4}$$

Note that $\zeta_{p,q,\omega}(s,x)$ is a meromorphic function on $\mathbb{C}$. Obverse that, if $p = 1$ and $q \to 1$, then $\zeta_{p,q,\omega}(s,x) = \zeta_E(s,x)$ which is the Hurwitz Euler zeta functions(see [1, 3, 6]). Relation between $\zeta_{p,q,\omega}(s,x)$ and $E_{k,p,q,\omega}(x)$ is given by the following theorem.

Theorem 13. For $k \in \mathbb{N}$, we have

$$\zeta_{p,q,\omega}(-k,x) = E_{k,p,q,\omega}(x).$$

Observe that $\zeta_{p,q,\omega}(-k,x)$ function interpolates $E_{k,p,q,\omega}(x)$ numbers at non-negative integers.

Acknowledgment

Acknowledgement: This work was supported by 2020 Hannam University Research Fund.

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STABILITY OF SET-VALUED PEXIDER FUNCTIONAL EQUATIONS

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Abstract. In this paper, we investigate a set-valued solution of the following Pexider functional equation

$$F(ax+by)=\alpha G(x)+\beta H(y)$$

with three unknown functions F, G and H, where a, b, α, β are positive real scalars.

1. Introduction and preliminaries

Assume that Y is a topological vector space satisfying the T_0 separation axiom. For real numbers s, t and sets $A, B \subset Y$ we put $sA+tB := \{y \in Y; y = sa+tb, a \in A, b \in B\}$. Suppose that the space 2^Y of all subsets of Y is endowed with the Hausdorff topology (see [4]). A set-valued function $F : X \to 2^Y$ is said to be additive if it satisfies the Cauchy functional equation $F(x_1+x_2) = F(x_1)+F(x_2), x_1, x_2 \in X$. The family of all closed and convex subsets of Y will be denoted by $CC(Y)$, and the sets of all real, rational and positive integer numbers are denoted by $\mathbb{R}, \mathbb{Q}, \mathbb{N}$, respectively.

Lemma 1.1. [1] *Let λ and μ be real numbers. If A and B are nonempty subsets of a real vector space X, then*

$$\lambda(A+B)=\lambda A+\lambda B,$$
$$(\lambda+\mu)A \subseteq \lambda A+\mu B.$$

Moreover, if A is a convex set and $\lambda, \mu \geq 0$, then we have

$$(\lambda+\mu)A=\lambda A+\mu A.$$

Lemma 1.2. [3] *Let A, B be subsets of Y and assume that B is closed and convex. If there exists a bounded and nonempty set $C \subset Y$ such that $A+C \subset B+C$, then $A \subset B$.*

Lemma 1.3. *If $(A_n)_{n\in\mathbf{N}}$ and $(B_n)_{n\in\mathbf{N}}$ are decreasing sequences of compact subsets of Y, then $\bigcap_{n\in\mathbb{N}}(A_n+B_n) = \bigcap_{n\in\mathbb{N}} A_n + \bigcap_{n\in\mathbb{N}} B_n$.*

Lemma 1.4. *If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of compact subsets of Y, then $A_n \to \bigcap_{n\in\mathbb{N}} A_n$.*

Lemma 1.5. *If A is a bounded subset of Y and $(s_n)_{n\in\mathbb{N}}$ is a real sequence converging to an $s \in \mathbb{R}$, then $s_n A \to sA$.*

Lemma 1.6. *If $A_n \to A$ and $B_n \to B$, then $A_n+B_n \to A+B$.*

2010 *Mathematics Subject Classification.* Primary 54C60, 39B52, 47H04, 49J54.

Key words and phrases. Hyers-Ulam stability, additive set-valued functional equation, closed and convex subset, cone.

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Lemma 1.7. *If $A_n \to A$ and $A_n \to B$, then $clA = clB$.*

Lemma 1.3–1.7 are rather known and can be easily verified. The proofs of them can be found in [1, 2].

2. SET-VALUED SOLUTION OF THE PEXIDER FUNCTIONAL EQUATION

In the section, we give the solution of the Pexider functional equation.

Theorem 2.1. *Assume that $(X, +)$ is a vector space and Y is a T_0 topological vector space. If set-valued functions $F : X \to CC(Y), G : X \to CC(Y)$ and $H : X \to CC(Y)$ satisfy the functional equation*

$$F(ax + by) = \alpha G(x) + \beta H(y) \tag{2.1}$$

for all $x, y \in X$, where a, b, α and β are positive real numbers, then there exist an additive set-valued function $F_0 : X \to CC(Y)$ and sets $A, B \in CC(Y)$ such that

$$F(x) = \alpha F_0(x) + \alpha A + \beta B, \quad G(x) = F_0(ax) + A \quad \text{and} \quad H(x) = F_0(bx) + B$$

for all $x \in X$.

Proof. First, assume that $0 \in G(0)$ and $0 \in H(0)$. Then, for all $x, y \in X$, we have

$$\begin{aligned} F(x+y) &= F\left(a\frac{x}{a} + b\frac{y}{b}\right) = \alpha G\left(\frac{x}{a}\right) + \beta H\left(\frac{y}{b}\right) \\ &\subset \alpha G\left(\frac{x}{a}\right) + \beta H(0) + \alpha G(0) + \beta H\left(\frac{y}{b}\right) \\ &= F(x) + F(y). \end{aligned}$$

Letting $x = y$ in the above equation, we get $F(2x) \subset 2F(x)$, which implies that the sequence $(2^{-n}F(2^n x))_{n\in N}$ is decreasing. Put $F_0(x) := \bigcap_{n\in\mathbb{N}} 2^{-n}F(2^n x), x \in X$. It is clear that $F_0(x) \in CC(Y)$ for all $x \in X$. Similarly, we get

$$\begin{aligned} \alpha G(2x) + \beta H(0) &= F(2ax) = F\left(ax + b\left(\frac{ax}{b}\right)\right) \\ &= \alpha G(x) + \beta H\left(\frac{ax}{b}\right) \subset \alpha G(x) + \alpha G(0) + \beta H\left(\frac{ax}{b}\right) \\ &= \alpha G(x) + F(ax) = \alpha G(x) + \alpha G(x) + \beta H(0) = 2\alpha G(x) + \beta H(0). \end{aligned}$$

In view of Lemma 1.2, we obtain that $G(2x) \subset 2G(x)$, and consequently the sequence $(2^{-n}G(2^n x))_{n\in\mathbb{N}}$ is decreasing. Applying Lemma1.3 and this equality $F(a2^n x) = \alpha G(2^n x) + \beta H(0), n \in \mathbb{N}$, we obtain

$$F_0(ax) = \bigcap_{n\in\mathbb{N}} 2^{-n}F(a2^n x) = \alpha \bigcap_{n\in\mathbb{N}} 2^{-n}G(2^n x) + \beta \bigcap_{n\in\mathbb{N}} 2^{-n}H(0).$$

But $\bigcap_{n\in\mathbb{N}} 2^{-n}H(0) = \{0\}$, since the set $H(0)$ is bounded. Therefore $F_0(ax) = \alpha \bigcap_{n\in\mathbb{N}} 2^{-n}G(2^n x)$ for all $x \in X$. In an analogous way we show that the sequence $(2^{-n}H(2^n x))_{n\in\mathbb{N}}$ is decreasing and $F_0(bx) = \beta \bigcap_{n\in\mathbb{N}} 2^{-n}H(2^n x)$ for all $x \in X$. Hence,using once more Lemma 1.3, we get

$$\begin{aligned} F_0(x_1+x_2) &= \bigcap_{n\in\mathbb{N}} 2^{-n}F(2^n x_1 + 2^n x_2) = \bigcap_{n\in\mathbb{N}} 2^{-n}\left(\alpha G\left(\frac{2^n x_1}{a}\right) + \beta H\left(\frac{2^n x_2}{b}\right)\right) \\ &= \bigcap_{n\in\mathbb{N}} 2^{-n}\alpha G\left(\frac{2^n x_1}{a}\right) + \bigcap_{n\in\mathbb{N}} 2^{-n}\beta H\left(\frac{2^n x_2}{b}\right) \\ &= F_0(x_1) + F_0(x_2), x_1, x_2 \in X, \end{aligned}$$

which means that the set-valued function F_0 is additive.

Now observe that

$$F(nbx) + (n-1)\beta H(0) = F(bx) + (n-1)\beta H(x) \tag{2.2}$$

for all $x \in X$ and $n \in \mathbb{N}$. Indeed, for $n = 1$ the equality is trivial. Assume that it holds for a natural number k. Then, in virtue of (2.1), we obtain

$$\begin{aligned} &F((k+1)bx) + k\beta H(0) = \alpha G\left(\frac{kbx}{a}\right) + \beta H(x) + k\beta H(0) = F(kbx) + \beta H(x) + (k\beta - \beta)H(0) \\ &= F(bx) + (k-1)\beta H(x) + \beta H(x) = F(x) + k\beta H(x). \end{aligned}$$

which proves that (2.2) holds for $n = k+1$. Thus, by induction, it holds for all $n \in \mathbb{N}$. In particular, we have

$$F(2^n x) + (2^n - 1)H(0) = F(x) + (2^n - 1)H\left(\frac{x}{b}\right),$$

and so

$$2^{-n}F(2^n x) + (1 - 2^{-n})H(0) = 2^{-n}F(x) + (1 - 2^{-n})H\left(\frac{x}{b}\right)$$

for all $x \in X$. By Lemma 1.4, $2^{-n}F(2^n x) \to \bigcap_{n\in\mathbb{N}} 2^{-n}F(2^n x) = F_0(x)$.

On the other hand, by Lemma 1.5, $1-2^{-n}H(0) \to H(0), 2^{-n}F(x) \to \{0\}$ and $(1-2^{-n})H\left(\frac{x}{b}\right) \to H\left(\frac{x}{b}\right)$. Thus, using Lemmas 1.6 and 1.7, we get $cl[F_0(x) + H(0)] = clH\left(\frac{x}{b}\right)$, whence $H\left(\frac{x}{b}\right) = F_0(x) + H(0)$ for all $x \in X$. Similarly, we can obtain $G\left(\frac{x}{a}\right) = F_0(x) + G(0), x \in X$. Let $A := G(0)$ and $B := H(0)$. Then $G(x) = F_0(ax) + A$ and $H(x) = F_0(bx) + B$ for all $x \in X$. Moreover $F(x) = \alpha F_0(x) + \alpha A + \beta B, x \in X$. This finishes our proof in the case that $0 \in G(0)$ and $0 \in H(0)$.

In the opposite case, fix arbitrarily points $a \in G(0)$ and $b \in H(0)$, and consider the set-valued functions $F_1, G-1, H_1 : X \to CC(Y)$ defined by $F_1(x) := F(x) - \alpha a - \beta b, G_1(x) := G(x) - a$ and $H_1 := H(x) - b, x \in X$. These set-valued functions satisfy the equation (2.1) and moreover $0 \in G_1(0)$ and $0 \in H_1(0)$. Therefore, by what we have discussed previously, we can get the same result. This completes the proof. □

In [2], Nikodem proved that a set-valued function $F_0 : [0,\infty) \to CC(Y)$, where Y is a locally convex Hausdorff space, is additive if and only if there exists an additive function $f : [0,\infty) \to Y$ and a set $K \in CC(Y)$ such that $F_0(x) = f(x) + xK, x \in [0,\infty)$. Thus we can get the following.

Theorem 2.2. *Let Y be a locally convex Hausdorff space. The set-valued functions $F : [0,\infty) \to CC(Y), G : [0,\infty) \to CC(Y)$ and $H : [0,\infty) \to CC(Y)$ satisfy the functional equation (2.1) if and only if there exist an additive function $f : [0,\infty) \to Y$ and sets $K, A, B \in CC(Y)$ such that*

$$F(x) = \alpha f(x) + \alpha Kx + \alpha A + \beta B, G(x) = f(ax) + akx + A \quad \textit{and} \quad H(x) = f(bx) + bkx + B$$

for all $x \in [0, \infty)$.

Funding
This work was supported by National Natural Science Foundation of China (No.11761074), the projection of the Department of Science and Technology of JiLin Province (No. JJKH20170453KJ) and the Education Department of Jilin Province (No. 20170101052JC).

Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.3, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On symmetries and solutions of certain sixth order difference equations

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Abstract

We use the Lie group analysis method to investigate the invariance properties and the solutions of

$$x_{n+1}=\frac{x_{n-5}x_{n-3}}{x_{n-1}(a_n+b_nx_{n-5}x_{n-3})}.$$

We show that this equation has a two-dimensional Lie algebra and that its solutions can be presented in a unified manner. Besides presenting solutions of the recursive sequence above where a_n and b_n are sequences of real numbers, some specific cases are emphasized.

Key words: Difference equation; symmetry; reduction; group invariant solutions, periodicity
MSC 2010: 39A05, 39A23, 70G65

1 Introduction

Difference equations are important in mathematical modelling, especially where discrete time evolving variables are concerned. They also occur when studying discretization methods for differential equations. Countless results in the subject of difference equations have been recorded. For rational difference equations of order greater than 1, the study can be quite challenging at the same time rewarding. Rewarding in the sense that such a study lays ground for the theory of global properties of difference equations (not necessarily rational) of higher order.
In [4], the author developed an effective symmetry based algorithm to deal with the obtention of solutions of difference equations of any order. However, the calculation one deals with in this application to difference equations of order greater than one can become cumbersome but with great recompense often times. The method consists of finding a group of transformations that maps solutions onto themselves. Symmetry method is a valuable tool and it has been used to solve several difference equations [1–3, 7, 8].
In this paper, our objective is to obtain the symmetry operators of

$$x_{n+1}=\frac{x_{n-5}x_{n-3}}{x_{n-1}(a_n+b_nx_{n-5}x_{n-3})}\tag{1}$$

where a_n and b_n are real sequences and to find its solutions by way of symmetries. Without loss of generality, we equivalently study the forward difference equation

$$u_{n+6} = \frac{u_n u_{n+2}}{u_{n+4}(A_n + B_n u_n u_{n+2})}. \tag{2}$$

We refer the interested reader to [4, 9] for a deeper knowledge of Lie analysis.

2 Definitions and Notation

In this section, we briefly present some definitions and notation (largely from Hydon in [4]) indispensable for the understanding of Lie symmetry analysis of difference equations.

Definition 2.1 *Let G be a local group of transformations acting on a manifold M. A subset $\mathcal{S} \subset M$ is called G-invariant, and G is called symmetry group of $\mathcal{S}$, if whenever $x \in \mathcal{S}$, and $g \in G$ is such that $g \cdot x$ is defined, then $g \cdot x \in \mathcal{S}$.*

Definition 2.2 *Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\mathcal{V} : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\mathcal{V}) = 0 \qquad \text{for all} \qquad x \in M,$$

and every infinitesimal generator X of G.

Definition 2.3 *A parameterized set of point transformations,*

$$\Gamma_\varepsilon : x \mapsto \hat{x}(x; \varepsilon), \tag{3}$$

where $x = x_i$, $i = 1, \ldots, p$ are continuous variables, is a one-parameter local Lie group of transformations if the following conditions are satisfied:

1. *Γ_0 is the identity map if $\hat{x} = x$ when $\varepsilon = 0$*
2. *$\Gamma_a \Gamma_b = \Gamma_{a+b}$ for every a and b sufficiently close to 0*
3. *Each $\hat{x_i}$ can be represented as a Taylor series (in a neighborhood of $\varepsilon = 0$ that is determined by x), and therefore*

$$\hat{x_i}(x : \varepsilon) = x_i + \varepsilon \xi_i(x) + O(\varepsilon^2), i = 1, ..., p. \tag{4}$$

Assuming that the sixth-order difference equation has the form

$$u_{n+6} = \Psi(n, u_n, \ldots, u_{n+5}), \quad n \in D \tag{5}$$

for some smooth function Ω and a regular domain $D \subset \mathbb{Z}$. To deduce the symmetry group of (5), we search for a one parameter Lie group of point transformations

$$\Gamma_\varepsilon : (n, u_n) \mapsto (n, u_n + \varepsilon Q(n, u_n)), \tag{6}$$

in which ε is the parameter and Q a continuous function, referred to as characteristic. Let

$$X = Q(n, u_n)\frac{\partial}{\partial u_n} + Q(n+1, u_{n+1})\frac{\partial}{\partial u_{n+1}} + \cdots + Q(n+5, u_{n+5})\frac{\partial}{\partial u_{n+5}} \tag{7}$$

be the corresponding 'prolonged' infinitesimal generator and $S : n \mapsto n+1$ the shift operator. The linearized symmetry condition is given by

$$S^6 Q - X\Psi = 0. \tag{8a}$$

Upon knowledge of the characteristic Q, it is important to introduce the canonical coordinate

$$S_n = \int \frac{du_n}{Q(n, u_n)}, \tag{9}$$

a useful tool which allows one to obtain the invariant $\mathcal{V}$.

3 Main results

As earlier emphasized, our equation under study is

$$u_{n+6} = \Psi = \frac{u_n u_{n+2}}{u_{n+4}(A_n + B_n u_n u_{n+2})}. \tag{10}$$

Appliying the criterion of invariance (8) to (10), we get

$$\begin{aligned} & Q(n+6, u_{n+6}) + \frac{u_n u_{n+2}}{u_{n+4}^2 \left(A_n + B_n u_n u_{n+2}\right)} Q\left(n+4, u_{n+4}\right) \\ & - \frac{A_n u_n}{u_{n+4}\left(\left(A_n + B_n u_n u_{n+2}\right)^2\right.} Q\left(n+2, u_{n+2}\right) \\ & - \frac{A_n u_{n+2}}{u_{n+4}\left(A_n + B_n u_n u_{n+2}\right)^2} Q\left(n, u_n\right) = 0. \end{aligned} \tag{11}$$

In order to eliminate u_{n+3}, we invoke implicit differentiation with respect to u_n (regarding u_{n+4} as a function of u_n, u_{n+2} and u_{n+3}) via the operator

$$L = \frac{\partial}{\partial u_n} - \frac{\Psi_{u_n}}{\Psi_{u_{n+4}}}\frac{\partial}{\partial u_{n+4}}.$$

With some simplification, one gets

$$\begin{aligned} & \left(A_n + B_n u_n u_{n+2}\right) Q'\left(n+4, u_{n+4}\right) - \frac{\left(A_n + B_n u_n u_{n+2}\right)}{u_{n+4}} Q\left(n+4, u_{n+4}\right) \\ & + B_n u_n Q(n+2, u_{n+2}) - \left(A_n + B_n u_n u_{n+2}\right) Q'(n, u_n) \\ & + \left(2B_n u_{n+2} + \frac{A_n}{u_n}\right) Q\left(n, u_n\right) = 0. \end{aligned} \tag{12}$$

Note that the symbol $'$ stands for the derivative with respect to the continuous variable. After twice differentiating (12) with respect to u_n, keeping u_{n+2} and u_{n+4} fixed, we are led to the equation

$$\begin{aligned}&-B_n u_n u_{n+2} Q'''(n,u_n) - A_n Q'''(n,u_n) + \frac{A_n}{u_n} Q''(n,u_n) - \frac{2A_n}{{u_n}^2} Q'(n,u_n)\\ &+ \frac{2A_n}{{u_n}^3} Q(n,u_n) = 0 \qquad (13)\end{aligned}$$

Note that the characteristic in (13) is not a function of u_{n+2} and so we split (13) up with respect to u_{n+2} to get the system

$$1 : Q'''(n,u_n) - \frac{1}{u_n} Q''(n,u_n) + \frac{2}{{u_n}^2} Q'(n,u_n) - \frac{2}{{u_n}^3} Q(n,u_n) = 0 \quad (14a)$$

$$u_{n+2} : Q'''(n,u_n) = 0. \quad (14b)$$

We find that the solution to (14) is

$$Q(n,u_n) = \alpha_n {u_n}^2 + \beta_n u_n \quad (15)$$

for some arbitrary functions α_n and β_n that depend on n. Substituting (16) and its first, second and third shifts in (11), and then replacing the expression of u_{n+3} given in (10) in the resulting equation yields

$$\begin{aligned}&B_n {u_n}^2 {u_{n+2}}^2 {u_{n+4}}^2 \alpha_{n+4} + B_n {u_n}^2 {u_{n+2}}^2 u_{n+4} (\beta_{n+4} + \beta_{n+3})\\ &- A_n {u_n}^2 u_{n+2} u_{n+4} \alpha_n - A_n u_n {u_{n+2}}^2 u_{n+4} \alpha_{n+2} + A_n u_n u_{n+2} {u_{n+4}}^2 \alpha_{n+4}\\ &+ {u_n}^2 {u_{n+2}}^2 \alpha_{n+2} - A_n (\beta_n + \beta_{n+2} - \beta_{n+4} - \beta_{n+3}) = 0. \qquad (16)\end{aligned}$$

Equating all coefficients of all powers of shifts of u_n to zero and simplifying the resulting system, we get its reduced form

$$\alpha_n = 0, \quad (17)$$

$$\beta_n + \beta_{n+2} = 0. \quad (18)$$

The two independent solutions of the linear second-order difference equation above are given by

$$\beta_n = \beta^n \text{ and } \beta_n = \bar{\beta}^n, \quad (19)$$

where $\beta = \exp\{i\pi/2\}$ and $\bar{\beta} = -\exp\{i\pi/2\}$ is its complex conjugate. The characteristic functions are given by

$$Q_1(n,u_n) = \beta^n u_n \quad \text{and} \quad Q_2(n,u_n) = \bar{\beta}^n u_n, \quad (20)$$

and so the Lie algebra of (10) is generated by

$$X_1 = \beta^n u_n \frac{\partial}{\partial_{u_n}} + \beta^{n+2} u_{n+2} \frac{\partial}{\partial_{u_{n+2}}} + \beta^{n+4} u_{n+4} \frac{\partial}{\partial_{u_{n+4}}} \quad (21)$$

$$X_2 = \bar{\beta}^n u_n \frac{\partial}{\partial_{u_n}} + \bar{\beta}^{n+2} u_{n+2} \frac{\partial}{\partial_{u_{n+2}}} + \bar{\beta}^{n+4} u_{n+4} \frac{\partial}{\partial_{u_{n+4}}}. \quad (22)$$

Using the canonical coordinate

$$S_n = \int \frac{du_n}{Q_1(n, u_n)} = \int \frac{du_n}{\beta^n u_n} = \frac{1}{\beta^n} \ln |u_n| \tag{23}$$

and (17), we derive the invariant function $\tilde{\mathcal{V}}_n$ as follows:

$$\tilde{\mathcal{V}}_n = S_n \beta^n + S_{n+2} \beta^{n+2}. \tag{24}$$

Actually,

$$X_1(\tilde{\mathcal{V}}_n) = \beta^n + \beta^{n+2} = 0 \tag{25}$$

and

$$X_2(\tilde{\mathcal{V}}_n) = \bar{\beta}^n + \bar{\beta}^{n+2} = 0. \tag{26}$$

For the sake of convenience, we use

$$|\mathcal{V}_n| = \exp\{-\tilde{\mathcal{V}}_n\} \tag{27}$$

instead/ In other words, $\mathcal{V}_n = \pm 1/(u_n u_{n+2})$. Using (10) and (27), one can prove that

$$\mathcal{V}_{n+4} = A_n \mathcal{V}_n \pm B_n. \tag{28}$$

Utilizing the plus sign, the solution of (28) can be written as

$$\mathcal{V}_{4n+j} \quad = \mathcal{V}_j \left(\prod_{k_1=0}^{n-1} A_{4k_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{4l+j} \prod_{k_2=l+1}^{n-1} A_{4k_2+j} \right), \tag{29}$$

where $j = 0, 1, 2, 3$. From here, obtaining the solution of (10) is straightforward. We first employ (23) to get

$$|u_n| = \exp\left(\beta_n S_n\right). \tag{30}$$

Secondly, we employ (24) to obtain

$$|u_n| = \exp\left(\beta^n c_1 + \bar{\beta}^n c_2 - \frac{1}{2} \sum_{k_1=0}^{n-1} \beta^n \bar{\beta}^{k_1} \tilde{\mathcal{V}}_{k_1} - \frac{1}{2} \sum_{k_2=0}^{n-1} \bar{\beta}^n \beta^{k_2} \tilde{\mathcal{V}}_{k_2} \right). \tag{31}$$

Lastly, we use (27) to get

$$|u_n| = \exp\left(\beta^n c_1 + \bar{\beta}^n c_2 + \frac{1}{2} \sum_{k_1=0}^{n-1} \beta^n \bar{\beta}^{k_1} \ln |\mathcal{V}_{k_1}| + \frac{1}{2} \sum_{k_2=0}^{n-1} \bar{\beta}^n \beta^{k_2} \ln |\mathcal{V}_{k_2}| \right), \tag{32a}$$

in which V_k is given in (28) with $\gamma(n, k) = \beta^n \bar{\beta}^k$. Note that the constants c_1 and c_2 satisfy

$$c_1 + c_2 = \ln |u_0| \quad \text{and} \quad \beta(c_1 - c_2) = \ln |u_1|. \tag{32b}$$

Note. Equations in (32) give the solutions of (2) in a unified manner. On a further note, $\gamma(n,k)=\beta^n\bar{\beta}^k$ satisfies

$$\begin{aligned}&\gamma(0,1)=\bar{\beta},\gamma(0,2)=-1,\gamma(1,0)=\beta,\gamma(1,2)=-\beta,\gamma(1,3)=-1,\\&\gamma(n,n)=1,\gamma(n+2,k)=-\gamma(n,k),\\&\gamma(n,k+2)=-\gamma(n,k),\gamma(4n,k)=\gamma(0,k),\gamma(n,4k)=\gamma(n,0).\end{aligned}\tag{33}$$

From u_n given in (32a) and equation (33), observe that

$$|u_{4n+j}|=\exp\left(H_j+\sum_{k_1=0}^{4n+j-1}\mathrm{Re}(\gamma(0,k_1))\ln|\mathcal{V}_{k_1}|\right)\tag{34}$$

in which

$$H_j=\beta^jc_1+\bar{\beta}^jc_2.$$

For $j=0$, we have,

$$\begin{aligned}|u_{4n}|&=\exp(H_0+\ln|\mathcal{V}_0|-\ln|\mathcal{V}_2|+\ldots+\ln|\mathcal{V}_{4n-4}|-\ln|\mathcal{V}_{4n-2}|)\\&=\exp(H_0)\prod_{s=0}^{n-1}\left|\frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}}\right|.\end{aligned}\tag{35}$$

It can be shown that there is no need for the absolute values via the utilization of the fact that

$$\mathcal{V}_i=\frac{1}{u_iu_{i+2}}.\tag{36}$$

In order to deduce $\exp(H_0)$, we set $n=0$ in (35) and note that $|u_0|=\exp(H_0)$. Thus

$$u_{4n}=u_0\prod_{s=0}^{n-1}\frac{\mathcal{V}_{4s}}{\mathcal{V}_{4s+2}}.$$

Similarly, replacing n with $4n+j$ for $j=0,1,2,3$, we obtain

$$U_{4n+j}=u_j\prod_{s=0}^{n-1}\frac{\mathcal{V}_{4s+j}}{\mathcal{V}_{4s+j+2}}.\tag{37}$$

Nevertheless, from (28), using the plus sign we are led to

$$\mathcal{V}_{4n+j}\quad=\mathcal{V}_j\left(\prod_{k_1=0}^{n-1}A_{4k_1+j}\right)+\sum_{l=0}^{n-1}\left(B_{4l+j}\prod_{k_2=l+1}^{n-1}A_{4k_2+j}\right),\tag{38}$$

for $j = 0, 1, 2, 3$, where $\mathcal{V}_0 = \frac{1}{u_0 u_2}$. Thus, using (37) with $j = 0$, we get

$$\begin{aligned}
U_{4n} &= u_0 \prod_{s=0}^{n-1} \frac{V_{4s}}{V_{4s+2}} \\
&= u_0 \prod_{s=0}^{n-1} \frac{V_0 \left(\prod_{k_1=0}^{s-1} A_{4k_1} \right) + \sum_{l=0}^{s-1} \left(B_{4l} \prod_{k_2=l+1}^{s-1} A_{4k_2} \right)}{V_2 \left(\prod_{k_1=0}^{s-1} A_{4k_1+2} \right) + \sum_{l=0}^{s-1} \left(B_{4l+2} \prod_{k_2=l+1}^{s-1} A_{4k_2+2} \right)} \\
&= u_0 \prod_{s=0}^{n-1} \frac{u_4}{u_0} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1} \right) + u_0 u_2 \sum_{l=0}^{s-1} \left(B_{4l} \prod_{k_2=l+1}^{s-1} A_{4k_2} \right)}{\left(\prod_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum_{l=0}^{s-1} \left(B_{4l+2} \prod_{k_2=l+1}^{s-1} A_{4k_2+2} \right)} \\
&= u_0^{1-n} u_4^n \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1} \right) + u_0 u_2 \sum_{l=0}^{s-1} \left(B_{4l} \prod_{k_2=l+1}^{s-1} A_{4k_2} \right)}{\left(\prod_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum_{l=0}^{s-1} \left(B_{4l+2} \prod_{k_2=l+1}^{s-1} A_{4k_2+2} \right)}
\end{aligned}$$

For $j = 1$, we have

$$\begin{aligned}
U_{4n+1} &= u_1 \prod_{s=0}^{n-1} \frac{V_{4s+1}}{V_{4s+3}} \\
&= u_1^{1-n} u_5^n \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1+1} \right) + u_1 u_3 \sum_{l=0}^{s-1} \left(B_{4l+1} \prod_{k_2=l+1}^{s-1} A_{4k_2+1} \right)}{\left(\prod_{k_1=0}^{s-1} A_{4k_1+3} \right) + u_3 u_5 \sum_{l=0}^{s-1} \left(B_{4l+3} \prod_{k_2=l+1}^{s-1} A_{4k_2+3} \right)}
\end{aligned}$$

For $j = 2$, we have

$$\begin{aligned}
U_{4n+2} &= u_2 \prod_{s=0}^{n-1} \frac{V_{4s+2}}{V_{4s+4}} \\
&= u_0^n u_4^{-n} u_2 \prod_{s=0}^{n-1} \frac{\left(\prod_{k_1=0}^{s-1} A_{4k_1+2} \right) + u_2 u_4 \sum_{l=0}^{s-1} \left(B_{4l+2} \prod_{k_2=l+1}^{s-1} A_{4k_2+2} \right)}{\left(\prod_{k_1=0}^{s} A_{4k_1} \right) + u_0 u_2 \sum_{l=0}^{s} \left(B_{4l} \prod_{k_2=l+1}^{s} A_{4k_2} \right)}
\end{aligned}$$

For $j = 3$, we get

$$\begin{aligned} U_{4n+3} &= u_3 \prod_{s=0}^{n-1} \frac{V_{4s+3}}{V_{4s+5}} \\ &= u_1^n u_5^{-n} u_3 \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{s-1} A_{4k_1+3}\right) + u_3 u_5 \sum\limits_{l=0}^{s-1} \left(B_{4l+3} \prod\limits_{k_2=l+1}^{s-1} A_{4k_2+3}\right)}{\left(\prod\limits_{k_1=0}^{s} A_{4k_1+1}\right) + u_1 u_3 \sum\limits_{l=0}^{s} \left(B_{4l+1} \prod\limits_{k_2=l+1}^{s} A_{4k_2+1}\right)} \end{aligned}$$

Hence, our solution in terms of x_n $(n > 0)$ is given by

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^{n} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{s-1} a_{4k_1}\right) + x_{-5} x_{-3} \sum\limits_{l=0}^{s-1} \left(b_{4l} \prod\limits_{k_2=l+1}^{s-1} a_{4k_2}\right)}{\left(\prod\limits_{k_1=0}^{s-1} a_{4k_1+2}\right) + x_{-3} x_{-1} \sum\limits_{l=0}^{s-1} \left(b_{4l+2} \prod\limits_{k_2=l+1}^{s-1} a_{4k_2+2}\right)}, \tag{39}$$

$$x_{4n-4} = x_{-4}^{1-n} x_{0}^{n} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{s-1} a_{4k_1+1}\right) + x_{-4} x_{-2} \sum\limits_{l=0}^{s-1} \left(b_{4l+1} \prod\limits_{k_2=l+1}^{s-1} a_{4k_2+1}\right)}{\left(\prod\limits_{k_1=0}^{s-1} a_{4k_1+3}\right) + x_{-2} x_{0} \sum\limits_{l=0}^{s-1} \left(b_{4l+3} \prod\limits_{k_2=l+1}^{s-1} a_{4k_2+3}\right)}, \tag{40}$$

$$x_{4n-3} = x_{-5}^{n} x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{s-1} a_{4k_1+2}\right) + x_{-3} x_{-1} \sum\limits_{l=0}^{s-1} \left(b_{4l+2} \prod\limits_{k_2=l+1}^{s-1} a_{4k_2+2}\right)}{\left(\prod\limits_{k_1=0}^{s} a_{4k_1}\right) + x_{-5} x_{-3} \sum_{l=0}^{s} \left(b_{4l} \prod\limits_{k_2=l+1}^{s} a_{4k_2}\right)} \tag{41}$$

and

$$x_{4n-2} = x_{-4}^{n} x_{0}^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{\left(\prod\limits_{k_1=0}^{s-1} a_{4k_1+3}\right) + x_{-2} x_{0} \sum\limits_{l=0}^{s-1} \left(b_{4l+3} \prod\limits_{k_2=l+1}^{s-1} a_{4k_2+3}\right)}{\left(\prod\limits_{k_1=0}^{s} a_{4k_1+1}\right) + x_{-4} x_{-2} \sum\limits_{l=0}^{s} \left(b_{4l+1} \prod\limits_{k_2=l+1}^{s} a_{4k_2+1}\right)} \tag{42}$$

In the following sections, we specifically look at some special cases.

4 The case a_n , b_n are 1-periodic

Let $a_n = a$ and $b_n = b$, where a and b are non-zero constants.

4.1 Case: $a \neq 1$

We have

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^{n} \prod_{s=0}^{n-1} \frac{a^s + bx_{-5}x_{-3}\frac{1-a^s}{1-a}}{a^s + bx_{-3}x_{-1}\frac{1-a^s}{1-a}},$$

$$x_{4n-4} = x_{-4}^{1-n} x_{0}^{n} \prod_{s=0}^{n-1} \frac{a^s + bx_{-4}x_{-2}\frac{1-a^s}{1-a}}{a^s + bx_{-2}x_{0}\frac{1-a^s}{1-a}},$$

$$x_{4n-3} = x_{-5}^{n} x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{a^s + bx_{-3}x_{-1}\frac{1-a^s}{1-a}}{a^{s+1} + bx_{-5}x_{-3}\frac{1-a^{s+1}}{1-a}},$$

$$x_{4n-2} = x_{-4}^{n} x_{0}^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{a^s + bx_{-2}x_{0}\frac{1-a^s}{1-a}}{a^{s+1} + bx_{-4}x_{-2}\frac{1-a^{s+1}}{1-a}}$$

as long as any of the denominators does not vanish.

Case: $a = -1$

We have

$$x_{4n-5} = \begin{cases} x_{-5}^{1-n} x_{-1}^{n} \left(\frac{-1+bx_{-5}x_{-3}}{-1+bx_{-3}x_{-1}}\right)^{\lfloor \frac{n-1}{2} \rfloor}, & \text{if } n \text{ is odd;} \\ \\ x_{-5}^{1-n} x_{-1}^{n} \left(\frac{-1+bx_{-5}x_{-3}}{-1+bx_{-3}x_{-1}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases}$$

$$x_{4n-4} = \begin{cases} x_{-4}^{1-n} x_{0}^{n} \left(\frac{-1+bx_{-4}x_{-2}}{-1+bx_{-2}x_{0}}\right)^{\lfloor \frac{n-1}{2} \rfloor}, & \text{if } n \text{ is even;} \\ \\ x_{-4}^{1-n} x_{0}^{n} \left(\frac{-1+bx_{-4}x_{-2}}{-1+bx_{-2}x_{0}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \end{cases}$$

$$x_{4n-3} = \begin{cases} \frac{x_{-5}^{n} x_{-1}^{-n} x_{-3}}{-1+bx_{-5}x_{-3}} \left(\frac{-1+bx_{-3}x_{-1}}{-1+bx_{-5}x_{-3}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \\ \\ x_{-5}^{n} x_{-1}^{-n} x_{-3} \left(\frac{-1+bx_{-3}x_{-1}}{-1+bx_{-5}x_{-3}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases}$$

and

$$x_{4n-2} = \begin{cases} \frac{x_{-4}^{n} x_{0}^{-n} x_{-2}}{-1+bx_{-4}x_{-2}} \left(\frac{-1+bx_{-2}x_{0}}{-1+bx_{-4}x_{-2}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is odd;} \\ \\ x_{-4}^{n} x_{0}^{-n} x_{-2} \left(\frac{-1+bx_{-2}x_{0}}{-1+bx_{-4}x_{-2}}\right)^{\lfloor \frac{n-1}{2} \rfloor + 1}, & \text{if } n \text{ is even;} \end{cases}$$

where $bx_{-i}x_{2-i} \neq 1$ for $i = 2, 3, 4, 5$.

4.2 Case: $a = 1$

The solution is given by

$$x_{4n-5} = x_{-5}^{1-n}x_{-1}^{n}\prod_{s=0}^{n-1}\frac{1+bx_{-5}x_{-3}s}{1+bx_{-3}x_{-1}s}, x_{4n-4} = x_{-4}^{1-n}x_{0}^{n}\prod_{s=0}^{n-1}\frac{1+bx_{-4}x_{-2}s}{1+bx_{-2}x_{0}s},$$

$$x_{4n-3} = x_{-5}^{n}x_{-1}^{-n}x_{-3}\prod_{s=0}^{n-1}\frac{1+bx_{-3}x_{-1}s}{1+bx_{-5}x_{-3}(s+1)},$$

$$x_{4n-2} = x_{-4}^{n}x_{0}^{-n}x_{-2}\prod_{s=0}^{n-1}\frac{1+bx_{-2}x_{0}s}{1+bx_{-4}x_{-2}(s+1)}.$$

5 The case a_n, b_n are 2-periodic

In this case, we have $\{a_n\}_{n=0}^{\infty} = a_0, a_1, a_0, a_1, \ldots$, and similarly $\{b_n\}_{n=0}^{\infty} = b_0, b_1, b_0, b_1, \ldots$ where $a_0 \neq a_1$, and $b_0 \neq b_1$. Then the solution is given by

$$x_{4n-5} = x_{-5}^{1-n}x_{-1}^{n}\prod_{s=0}^{n-1}\frac{a_0^s + b_0x_{-5}x_{-3}\sum\limits_{l=0}^{s-1}a_0^l}{a_0^s + b_0x_{-3}x_{-1}\sum\limits_{l=0}^{s-1}a_0^l},$$

$$x_{4n-4} = x_{-4}^{1-n}x_{0}^{n}\prod_{s=0}^{n-1}\frac{a_1^s + b_1x_{-4}x_{-2}\sum\limits_{l=0}^{s-1}a_1^l}{a_1^s + b_1x_{-2}x_{0}\sum\limits_{l=0}^{s-1}a_1^l},$$

$$x_{4n-3} = x_{-5}^{n}x_{-1}^{-n}x_{-3}\prod_{s=0}^{n-1}\frac{a_0^s + b_0x_{-3}x_{-1}\sum\limits_{l=0}^{s-1}a_0^l}{a_0^{s+1} + b_0x_{-5}x_{-3}\sum\limits_{l=0}^{s}a_0^l}$$

and

$$x_{4n-2} = x_{-4}^{n}x_{0}^{-n}x_{-2}\prod_{s=0}^{n-1}\frac{a_1^s + b_1x_{-2}x_{0}\sum\limits_{l=0}^{s-1}a_1^l}{a_1^{s+1} + b_1x_{-4}x_{-2}\sum\limits_{l=0}^{s}a_1^l}$$

as long as any of the denominators does not vanish.

6 The case a_n, b_n are 4-periodic

We assume that $\{a_n\} = a_0, a_1, a_2, a_3, a_0, a_1, a_2, a_3, \ldots$ and $\{b_n\} = b_0, b_1, b_2, b_3, b_0, b_1, b_2, b_3, \cdots$. The solution is given by

$$x_{4n-5} = x_{-5}^{1-n} x_{-1}^{n} \prod_{s=0}^{n-1} \frac{a_0^s + b_0 x_{-5} x_{-3} \sum\limits_{l=0}^{s-1} a_0^l}{a_2^s + b_2 x_{-3} x_{-1} \sum\limits_{l=0}^{s-1} a_2^l}, \tag{43}$$

$$x_{4n-4} = x_{-4}^{1-n} x_{0}^{n} \prod_{s=0}^{n-1} \frac{a_1^s + b_1 x_{-4} x_{-2} \sum\limits_{l=0}^{s-1} a_1^l}{a_3^s + b_3 x_{-2} x_{0} \sum\limits_{l=0}^{s-1} a_3^l}, \tag{44}$$

$$x_{4n-3} = x_{-5}^{n} x_{-1}^{-n} x_{-3} \prod_{s=0}^{n-1} \frac{a_2^s + b_2 x_{-3} x_{-1} \sum\limits_{l=0}^{s-1} a_2^l}{a_0^{s+1} + b_0 x_{-5} x_{-3} \sum\limits_{l=0}^{s} a_0^l} \tag{45}$$

and

$$x_{4n-2} = x_{-4}^{n} x_{0}^{-n} x_{-2} \prod_{s=0}^{n-1} \frac{a_3^s + b_3 x_{-2} x_{0} \sum\limits_{l=0}^{s-1} a_3^l}{a_1^{s+1} + b_1 x_{-4} x_{-2} \sum\limits_{l=0}^{s} a_1^l} \tag{46}$$

as long as any of the denominators does not vanish.

7 Conclusion

In this paper, non-trivial symmetries for difference equations of the form (1) were found. Consequently, the results were used to find formulas for the solutions of the equations. Specific cases of the solutions were also discussed.

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Volume 29, Number 4 August 2021
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS, LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)
Editor in Chief: George Anastassiou
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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

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Exact Solitary Wave Solutions for Wick-type Stochastic (2+1)-dimensional Coupled KdV equations

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Abstract

Variable coefficients and Wick-type stochastic (2+1)-dimensional coupled KdV equations are investigated. By using the F-expansion method , Hermite transform and white noise theory, the white noise functional solutions for Wick-type stochastic (2+1)-dimensional coupled KdV equations are obtained. The exact travelling wave solutions are expressed in terms of Jacobi elliptic (JEF), trigonometric and hyperbolic functions.

Keywords: KdV equations; F-expansion method; Hermite transform; Wick product.
PACS No. : 05.40. ± a, 02.30.Jr.

1 Introduction

In this paper, we shall explore exact solutions for the following variable coefficients (2+1)-dimensional coupled KdV equations.

$$\begin{cases} u_t + \phi_1(t)uv_x + \phi_2(t)vu_x + \phi_3(t)u_{xxx} = 0, \\ \\ u_x + v_y = 0, \end{cases} \tag{1.1}$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ and $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Random wave is an important subject of stochastic partial differential equations (PDEs). Many authors have studied this subject. Wadati first introduced and studied the stochastic KdV equations and gave the diffusion of soliton of the KdV equation under Gaussian noise in [30, 32] and others [3, 4, 5, 25] also researched stochastic KdV-type equations. Xie first introduced Wick-type stochastic KdV equations on white noise space and showed the auto- Backlund transformation and the exact white noise functional solutions in [37]. Furthermore, Xie [38, 39, 40, 41], Ghany et al. [11, 12, 13, 15, 16, 17, 18, 19, 20] researched some Wick-type stochastic wave equations using white noise analysis.

In this paper we use F-expansion method for finding new periodic wave solutions of nonlinear evolution equations in mathematical physics, and we obtain some new periodic wave solutions for (2+1)-dimensional coupled KdV equations. This method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations arising in mathematical physics. The effort in finding exact solutions to nonlinear equations is important for the understanding of most nonlinear physical phenomena. For instance, the nonlinear wave phenomena observed in fluid dynamics, plasma, and optical fibers[24]. Many effective methods have been presented, such as tanh-function method [34, 42, 8], variational iteration method [6, 7], exp-function method [22, 23, 36, 43, 44] , homotopy perturbation method [10, 29, 35], homotopy analysis method [1], tanh-coth method [33, 34, 31], Jacobi elliptic function expansion method [27, 28, 9, 26] and F-expansion method [45, 46, 47, 48]. The main objective of this paper is using the F-expansion method to construct white noise functional solutions for wick-type stochastic (2+1)-dimensional coupled KdV equations via hermite transform, wick-type product and white noise analysis. If equation (1.1) is considered in a random environment, we can get stochastic (2+1)-dimensional coupled KdV equations. In order to give the exact solutions of stochastic (2+1)-dimensional coupled KdV equations, we only consider this problem in white noise environment. We shall study the following Wick-type stochastic (2+1)-dimensional coupled KdV equations.

$$\begin{cases} U_t + \Phi_1(t) \diamond U \diamond V_x + \Phi_2(t) \diamond V \diamond U_x + \Phi_3(t) \diamond U_{xxx} = 0, \\ U_x + V_y = 0, \end{cases} \tag{1.2}$$

where " $\diamond$ " is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$ which was defined in [21] and $\Phi_1(t), \Phi_2(t)$ and $\Phi_3(t)$ are $(\mathcal{S})_{-1}$ -valued functions.

2 Description of the F-expansion Method

In order to at the same time obtain more periodic wave solutions expressed by various Jacobi elliptic functions to nonlinear wave equations, we introduce an F-expansion method which can be thought of as a succinctly over-all generalization of Jacobi elliptic function expansion. We briefly show what is F-expansion method and how to use it to obtain various periodic wave solutions to nonlinear wave equations. Suppose a nonlinear wave equation for $u(t,x)$ is given by

$$\theta_1(u, u_t, u_x, u_y, u_{xx}, u_{xxx}, ...) = 0, \tag{2.1}$$

where $u = u(t,x)$ is an unknown function, θ_1 is a polynomial in u and its various partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following we give the main steps of a deformation F-expansion method.

Step 1. Look for traveling wave solution of Eq.(2.1) by taking

$$u(t,x,y) = u(\xi) \ \ , \xi(t,x,y) = kx + ly + \mu \int_0^t \omega(\tau)d\tau + c, \tag{2.2}$$

Hence, under the transformation (2.2). Eq.(2.1) can be transformed into the following ordinary differential equation (ODE) as following

$$\theta_2(u, \mu\omega u', ku', lu', k^2u'', k^3u''', ...) = 0, \tag{2.3}$$

Step 2. Suppose that $u(\xi)$ can be expressed by a finite power series of $F(\xi)$ of the form

$$u(t,x,y) = u(\xi) = \sum_{i=1}^{N} a_i F^i(\xi), \tag{2.4}$$

where $a_0, a_1, ..., a_N$ are constants to be determined later, while $F'(\xi)$ in(2.4) satisfy

$$[F'(\xi)]^2 = PF^4(\xi) + QF^2(\xi) + R, \tag{2.5}$$

and hence holds for $F(\xi)$

$$\begin{cases} F'F'' = 2PF^3F' + QFF', \\ F'' = 2PF^3 + QF, \\ F''' = 6PF^2F' + QF', \\ \cdots \end{cases} \tag{2.6}$$

where $P, Q,$ and R are constants.

Step 3. The positive integer N can be determined by considering the homogeneous balance between the highest derivative term and the nonlinear terms appearing in (2.3). Therefore, we can get the value of N in (2.4).

Step 4. Substituting (2.4) into (2.3) with the condition (2.5), we obtain polynomial in $F^i(\xi)[F'(\xi)]^j$, $(i = 0 \pm 1, \pm 2, ..., j = 0, 1)$. Setting each coefficient of this polynomial to be zero yields a set of algebraic equations for $a_0, a_1, ..., a_N, \mu$ and ω .

Step 5. Solving the algebraic equations with the aid of Maple we have $a_0, a_1, ..., a_N, \mu$ and ω can be expressed by (P, Q, R) . Substituting these results into F-expansion (2.4), then a general form of traveling wave solution of Eq. (2.1) can be obtained.

Step 6. Since the general solutions of (2.4) have been well known for us Choose properly (P, Q and R .) in ODE (2.5) such that the corresponding solution $F(\xi)$ of it is one of Jacobi elliptic functions. (See Appendices A, B and C .)[45, 46, 47]

3 New Exact Wave Solutions of Eq. (1.2)

Taking the Hermite transform, white noise theory, and F-expansion method to explore new exact wave solutions for Eq.(1.2). Applying Hermite transform to Eq.(1.2), we get the deterministic equation.

$$\begin{cases} \widetilde{U}_t(t,x,y,z) + \widetilde{\Phi_1}(t,z)\widetilde{U}(t,x,y,z)\widetilde{V}_x(t,x,y,z) + \widetilde{\Phi_2}(t,z)\widetilde{V}(t,x,y,z)\widetilde{U}_x(t,x,y,z) \\ +\widetilde{\Phi_3}(t,z)\widetilde{U}_{xxx}(t,x,y,z) = 0, \\ \\ \widetilde{U}_x(t,x,y,z) + \widetilde{V}_y(t,x,y,z) = 0, \end{cases} \tag{3.1}$$

where $z = (z_1, z_2, ...) \in (\mathbb{C}^{\mathbb{N}})$ is a vector parameter. To look for the travelling wave solution of Eq.(3.1), we make the transformations $\widetilde{\Phi_1}(t,z) := \phi_1(t,z)$, $\widetilde{\Phi_2}(t,z) := \phi_2(t,z)$, $\widetilde{\Phi_3}(t,z) := \phi_3(t,z)$, $\widetilde{U}(t,x,y,z) := u(t,x,y,z) = u(\xi(t,x,y,z))$ and $\widetilde{V}(t,x,y,z) := v(t,x,y,z) = v(\xi(t,x,y,z))$ with

$$\xi(t,x,y,z) = kx + ly + \mu\int_0^t \omega(\tau,z)d\tau + c,$$

where k,μ and c are arbitrary constants which satisfy $k\mu \neq 0$, $\omega(\tau,z)$ is a nonzero function of the indicated variables to be determined later. Hence, Eq.(3.1) can be transformed into the following (ODE).

$$\begin{cases} \mu\omega u' + k\phi_1 uv' + k\phi_2 vu' + k^3\phi_3 u''' = 0, \\ ku' + lv' = 0, \end{cases} \tag{3.2}$$

where the prime denote to the differential with respect to ξ . In view of F-expansion method, the solution of Eq. (3.1), can be expressed in the form.

$$\begin{cases} u(t,x,y,z) = u(\xi) = \Sigma_{i=1}^{N} a_i F^i(\xi), \\ v(t,x,y,z) = v(\xi) = \Sigma_{i=1}^{M} b_i F^i(\xi), \end{cases} \tag{3.3}$$

where a_i and b_i are constants to be determined later. considering homogeneous balance between the highest order nonlinear terms and the highest order partial derivative of u in (3.2), then we can obtain $N = M = 2$ so (3.3) can be rewritten as following

$$\begin{cases} u(t,x,y,z) = a_0 + a_1F(\xi) + a_2F^2(\xi), \\ v(t,x,y,z) = b_0 + b_1F(\xi) + b_2F^2(\xi), \end{cases} \tag{3.4}$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are constants to be determined later. Substituting (3.4) with the conditions (2.5),(2.6) into (3.2) and collecting all terms with the same power of

$F^i(\xi)[F'(\xi)]^j$, $(i = 0 \pm 1, \pm 2, ..., j = 0, 1)$. as following

$$\begin{cases} [\mu\omega a_1 + ka_0b_1\phi_1 + ka_1b_0\phi_2 + k^3a_1\phi_3Q]F' \\ +[2\mu\omega a_2 + 2ka_0b_2\phi_1 + ka_1b_1\phi_1 + 2ka_2b_0\phi_2 + ka_1b_1\phi_2 + 8k^3a_2\phi_3Q]FF' \\ +k[2a_1b_2\phi_1 + a_2b_1\phi_1 + 2a_2b_1\phi_2 + a_1b_1\phi_2 + 6k^2a_1\phi_3P]F^2F' \\ +2ka_2[b_2\phi_1 + b_2\phi_2 + 12k^2\phi_3P]F^2F' = 0, \\ (ka_1 + lb_1)F' + 2[ka_2 + lb_2]FF' = 0. \end{cases} \tag{3.5}$$

Setting each coefficients of $F^i(\xi)[F'(\xi)]^j$ to be zero, we get a system of algebraic equations which can be expressed by.

$$\begin{cases} \mu\omega a_1 + ka_0b_1\phi_1 + ka_1b_0\phi_2 + k^3a_1\phi_3Q = 0, \\ 2\mu\omega a_2 + 2ka_0b_2\phi_1 + ka_1b_1\phi_1 + 2ka_2b_0\phi_2 + ka_1b_1\phi_2 + 8k^3a_2\phi_3Q = 0, \\ k[2a_1b_2\phi_1 + a_2b_1\phi_1 + 2a_2b_1\phi_2 + a_1b_1\phi_2 + 6k^2a_1\phi_3P] = 0, \\ 2ka_2[b_2\phi_1 + b_2\phi_2 + 12k^2\phi_3P] = 0, \\ ka_1 + lb_1 = 0, \\ 2[ka_2 + lb_2] = 0. \end{cases} \tag{3.6}$$

with solving by *Maple* to get the following coefficients

$$\begin{cases} a_2 = b_2 = 0, \ a_0 \ , b_0 = \text{arbitrary constant}, \\ a_1 = \frac{6lk\phi_3(t,z)P}{\phi_2(t,z)}, \\ b_1 = -\frac{6k^2\phi_3(t,z)P}{\phi_2(t,z)}, \\ \omega = \frac{k^2a_0\phi_1(t,z) - lk[b_0\phi_2(t,z) + k^2\phi_3(t,z)Q]}{l\mu}. \end{cases} \tag{3.7}$$

Substituting by coefficient (3.7) into (3.4) yields general form solutions of Eq. (1.2).

$$u(t, x, y, z) = a_0 + \frac{6lk\phi_3(t, z)P}{\phi_2(t, z)} \ F(\xi), \tag{3.8}$$

$$v(t, x, y, z) = b_0 - \frac{6k^2\phi_3(t, z)P}{\phi_2(t, z)} \ F(\xi), \tag{3.9}$$

with

$$\xi(t, x, y, z) = kx + ly + \int_0^t \frac{k^2a_0\phi_1(\tau, z) - lk[b_0\phi_2(\tau, z) + k^2\phi_3(\tau, z)Q]}{l} \ d\tau.$$

From Appendix A, we give the special cases as following.

Case I:
If we take $P=\frac{1}{4}, Q=\frac{m^2-2}{2}$ and $R=\frac{m^2}{4}$, we have $F(\xi)\rightarrow ns(\xi)\pm ds(\xi)$,

$$u_1(t,x,y,z)=a_0+\frac{3lk\phi_3(t,z)}{2\phi_2(t,z)}\left[ns\left(\xi_1(t,x,y,z)\right)\pm ds\left(\xi_1(t,x,y,z)\right)\right], \tag{3.10}$$

$$v_1(t,x,y,z)=b_0-\frac{3k^2\phi_3(t,z)}{2\phi_2(t,z)}\left[ns\left(\xi_1(t,x,y,z)\right)\pm ds\left(\xi_1(t,x,y,z)\right)\right], \tag{3.11}$$

with

$$\xi_1(t,x,y,z)=kx+ly+\int_0^t\left\{\frac{2k^2a_0\phi_1(\tau,z)-lk[2b_0\phi_2(\tau,z)+k^2\phi_3(\tau,z)(m^2-2)]}{2l}\right\}d\tau.$$

In the limit case when $m\rightarrow o$, we have $ns(\xi)\pm ds(\xi)\rightarrow 2\csc(\xi)$, thus (3.10),(3.11) become.

$$u_2(t,x,y,z)=a_0+\frac{3lk\phi_3(t,z)}{\phi_2(t,z)}\csc\left(\xi_2(t,x,y,z)\right), \tag{3.12}$$

$$v_2(t,x,y,z)=b_0-\frac{3k^2\phi_3(t,z)}{\phi_2(t,z)}\csc\left(\xi_2(t,x,y,z)\right), \tag{3.13}$$

with

$$\xi_2(t,x,y,z)=kx+ly+\int_0^t\left\{\frac{k^2a_0\phi_1(\tau,z)-lk[b_0\phi_2(\tau,z)-k^2\phi_3(\tau,z)]}{l}\right\}d\tau.$$

In the limit case when $m\rightarrow 1$ we have $ns(\xi)\pm ds(\xi\rightarrow\coth(\xi)\pm(\xi)$, thus (3.10).(3.11) become.

$$u_3(t,x,y,z)=a_0+\frac{3lk\phi_3(t,z)}{2\phi_2(t,z)}\left[\coth\,\xi_3(t,x,y,z)\pm\,(\xi_3(t,x,y,z))\right], \tag{3.14}$$

$$v_3(t,x,y,z)=b_0-\frac{3k^2\phi_3(t,z)}{2\phi_2(t,z)}\left\{[\coth\,\xi_3(t,x,y,z)\pm\,(\xi_3(t,x,y,z))\right], \tag{3.15}$$

with

$$\xi_3(t,x,y,z)=kx+ly+\int_0^t\left\{\frac{2k^2a_0\phi_1(\tau,z)-lk[2b_0\phi_2(\tau,z)-k^2\phi_3(\tau,z)]}{2l}\right\}d\tau.$$

Case II:
If we take $P = 1, Q = -(1 + m^2)$ and $R = m^2$, then $F(\xi) \to ns(\xi)$,

$$u_4(t,x,y,z) = a_0 + \frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ ns\left(\xi_4(t,x,y,z)\right), \tag{3.16}$$

$$v_4(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ ns\left(\xi_4(t,x,y,z)\right), \tag{3.17}$$

with

$$\xi_4(t,x,y,z) = kx + ly + \int_0^t \left\{ \frac{2k^2a_0\phi_1(\tau,z) - lk[2b_0\phi_2(\tau,z) + k^2\phi_3(\tau,z)(m^2-2)]}{l} \right\} d\tau.$$

In the limit case when $m \to o$ we have $ns(\xi) \pm ds(\xi) \to \csc(\xi)$, thus (3.10),(3.11) become.

$$u_5(t,x,y,z) = a_0 + \frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ \csc\left(\xi_2(t,x,y,z)\right), \tag{3.18}$$

$$v_5(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ \csc\left(\xi_2(t,x,y,z)\right). \tag{3.19}$$

In the limit case when $m \to 1$ we have $ns(\xi) \to \coth(\xi)$, thus (3.10).(3.11) become.

$$u_6(t,x,y,z) = a_0 + \frac{6lk\phi_3(t,z)}{2\phi_2(t,z)}\ \coth\left(\xi_5(t,x,y,z)\right), \tag{3.20}$$

$$v_6(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{2\phi_2(t,z)}\ \coth\left(\xi_5(t,x,y,z)\right), \tag{3.21}$$

with

$$\xi_5(t,x,y,z) = kx + ly + \int_0^t \left\{ \frac{k^2a_0\phi_1(\tau,z) - lk[b_0\phi_2(\tau,z) - 2k^2\phi_3(\tau,z)]}{l} \right\} d\tau.$$

Case III:
If we take $P = 1, Q = (2 - m^2)$ and $R = 1 - m^2$, then $F(\xi) \to cs(\xi)$,

$$u_7(t,x,y,z) = a_0 + \frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ cs\left(\xi_6(t,x,y,z)\right), \tag{3.22}$$

$$v_7(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ cs\,(\xi_6(t,x,y,z)), \tag{3.23}$$

with

$$\xi_6(t,x,y,z) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \phi_1(\tau,z) - lk[2b_0\phi_2(\tau,z) + k^2\phi_3(\tau,z)(2-m^2)]}{l} \right\} d\tau.$$

In the limit case when $m \to o$ we have $cs(\xi) \to \cot(\xi)$, thus (3.10),(3.11) become.

$$u_8(t,x,y,z) = a_0 + \frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ \cot\,(\xi_7(t,x,y,z)), \tag{3.24}$$

$$v_8(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ \cot\,(\xi_7(t,x,y,z)), \tag{3.25}$$

$$\xi_7(t,x,y,z) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \phi_1(\tau,z) - lk[b_0\phi_2(\tau,z) + 2k^2\phi_3(\tau,z)]}{l} \right\} d\tau.$$

In the limit case when $m \to 1$ we have $cs(\xi) \to (\xi)$, thus (3.10).(3.11) become.

$$u_9(t,x,y,z) = a_0 + \frac{6lk\phi_3(t,z)}{\phi_2(t,z)}\ (\xi_8(t,x,y,z)), \tag{3.26}$$

$$v_9(t,x,y,z) = b_0 - \frac{6k^2\phi_3(t,z)}{\phi_2(t,z)}\ (\xi_8(t,x,y,z)), \tag{3.27}$$

with

$$\xi_8(t,x,y,z) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \phi_1(\tau,z) - lk[b_0\phi_2(\tau,z) + k^2\phi_3(\tau,z)]}{l} \right\} d\tau.$$

Obviously, there are another solutions for Eq.(1.2). These solutions come from setting different values for the coefficients P, Q and R . (see Appendix A, B and C.)[46, 47]. The above mentioned cases are just to clarify how far our technique is applicable.

4 White Noise Functional Solutions of Eq.(1.2)

In this section, we employ the results of the Section 3 by using Hermite transform to obtain exact white noise functional solutions for Wick-type stochastic (2+1)-dimensional coupled KdV equations (1.2). The properties of exponential and trigonometric functions yield that there exists a bounded open set $\mathbf{G} \subset \mathbb{R}_+ \times \mathbb{R}^2$, $\rho < \infty$, $\lambda > 0$ such that the solution $u(t,x,y,z)$ of Eq. (3.1) and all its partial derivatives which are involved in Eq. (3.1) are uniformly bounded for $(t,x,y,z) \in \mathbf{G} \times K_\rho(\lambda)$, continuous with respect to $(t,x,y) \in \mathbf{G}$ for all $z \in K_\rho(\lambda)$ and analytic with respect to $z \in K_\rho(\lambda)$, for all $(t,x,y) \in \mathbf{G}$. From Theorem 4.1.1 in [21], there exists $U(t,x,y,z) \in (\mathcal{S})_{-1}$ such that $u(t,x,y,z) = \widetilde{U}(t,x,y)(z)$ for all $(t,x,y,z) \in \mathbf{G} \times K_\rho(\lambda)$ and $U(t,x,y)$ solves Eq.(1.2) in $(\mathcal{S})_{-1}$. Hence, by applying the inverse Hermite transform to the results of Section 3, we get exact white noise functional solutions of Eq. (1.2) as follows.

- **White noise functional solutions of JEF type**:

$$U_1(t,x,y) = a_0 + \frac{3lk\Phi_3(t)}{2\Phi_2(t)} \diamond \Big[ns^{\diamond}\left(\Xi_1(t,x,y)\right) \pm ds^{\diamond}\left(\Xi_1(t,x,y)\right) \Big], \tag{4.1}$$

$$V_1(t,x,y) = b_0 - \frac{3k^2\Phi_3(t)}{2\Phi_2(t)} \diamond \Big[ns^{\diamond}\left(\Xi_1(t,x,y)\right) \pm ds^{\diamond}\left(\Xi_1(t,x,y)\right) \Big], \tag{4.2}$$

$$U_2(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond ns^{\diamond}\left(\Xi_2(t,x,y)\right), \tag{4.3}$$

$$V_2(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond ns^{\diamond}\left(\Xi_2(t,x,y)\right), \tag{4.4}$$

$$U_3(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond cs^{\diamond}\left(\Xi_3(t,x,y)\right), \tag{4.5}$$

$$V_3(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond cs^{\diamond}\left(\Xi_3(t,x,y)\right), \tag{4.6}$$

with

$$\Xi_1(t,x,y) = kx + ly + \int_0^t \left\{ \frac{2k^2a_0\Phi_1(\tau) - lk[2b_0\Phi_2(\tau) + k^2\phi_3(\tau)(m^2-2)]}{2l} \right\} d\tau,$$

$$\Xi_2(t,x,y) = kx + ly + \int_0^t \left\{ \frac{2k^2a_0\Phi_1(\tau) - lk[2b_0\Phi_2(\tau) + k^2\Phi_3(\tau)(m^2-2)]}{l} \right\} d\tau,$$

$$\Xi_3(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \Phi_1(\tau) - lk[2b_0\Phi_2(\tau) + k^2\Phi_3(\tau)(2-m^2)]}{l} \right\} d\tau.$$

- **White noise functional solutions of trigonometric type**:

$$U_4(t,x,y) = a_0 + \frac{3lk\Phi_3(t)}{\Phi_2(t)} \diamond \csc^{\diamond}(\Xi_4(t,x,y)), \tag{4.7}$$

$$V_4(t,x,y) = b_0 - \frac{3k^2\Phi_3(t)}{\Phi_2(t)} \diamond \csc^{\diamond}(\Xi_4(t,x,y)), \tag{4.8}$$

$$U_5(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond \csc^{\diamond}(\Xi_4(t,x,y)), \tag{4.9}$$

$$V_5(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond \csc^{\diamond}(\Xi_4(t,x,y)), \tag{4.10}$$

$$U_6(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond \cot^{\diamond}(\Xi_5(t,x,y)), \tag{4.11}$$

$$V_6(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond \cot^{\diamond}(\Xi_5(t,x,y)), \tag{4.12}$$

with

$$\Xi_4(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \Phi_1(\tau) - lk[b_0\Phi_2(\tau) - k^2\Phi_3(\tau)]}{l} \right\} d\tau,$$

$$\Xi_5(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \Phi_1(\tau) - lk[b_0\Phi_2(\tau) + 2k^2\Phi_3(\tau)]}{l} \right\} d\tau.$$

- **White noise functional solutions of hyperbolic type**:

$$U_7(t,x,y) = a_0 + \frac{3lk\Phi_3(t)}{2\Phi_2(t)} \diamond \Big[\coth^{\diamond}(\Xi_6(t,x,y)) \pm^{\diamond} (\Xi_6(t,x,y)) \Big], \tag{4.13}$$

$$V_7(t,x,y) = b_0 - \frac{3k^2\Phi_3(t)}{2\Phi_2(t)} \diamond \Big[\coth^{\diamond}(\Xi_6(t,x,y)) \pm^{\diamond} (\Xi_6(t,x,y)) \Big], \tag{4.14}$$

$$U_8(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{2\Phi_2(t)} \diamond \coth^{\diamond}(\Xi_7(t,x,y)), \tag{4.15}$$

$$V_8(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{2\Phi_2(t)} \diamond \coth^{\diamond}(\Xi_7(t,x,y)), \tag{4.16}$$

$$U_9(t,x,y) = a_0 + \frac{6lk\Phi_3(t)}{\Phi_2(t)} \diamond {}^{\diamond}(\Xi_8(t,x,y)), \tag{4.17}$$

$$V_9(t,x,y) = b_0 - \frac{6k^2\Phi_3(t)}{\Phi_2(t)} \diamond {}^{\diamond}(\Xi_8(t,x,y)), \tag{4.18}$$

with

$$\Xi_6(t,x,y) = kx + ly + \int_0^t \left\{ \frac{2k^2 a_0 \Phi_1(\tau) - lk[2b_0\Phi_2(\tau) - k^2\Phi_3(\tau)]}{2l} \right\} d\tau,$$

$$\Xi_7(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \Phi_1(\tau) - lk[b_0\Phi_2(\tau) - 2k^2\Phi_3(\tau)]}{l} \right\} d\tau,$$

$$\Xi_8(t,x,y) = kx + ly + \int_0^t \left\{ \frac{k^2 a_0 \Phi_1(\tau) - lk[b_0\Phi_2(\tau) + k^2\Phi_3(\tau)]}{l} \right\} d\tau.$$

We observe that, for different forms of Φ_1, Φ_2 and Φ_3, we can get different exact white noise functional solutions of Eq. (1.2) from Eqs. (4.1)-(4.18).

5 Example

It is well known that Wick version of function is usually difficult to evaluate. So, in this section, we give non-Wick version of solutions of Eq. (1.2). Let $W_t = \dot{B}_t$ be the Gaussian white noise, where B_t is the Brownian motion. We have the Hermite transform $\widetilde{W_t}(z) = \sum_{i=1}^{\infty} z_i \int_0^t \eta_i(s)ds$ [21]. Since $\exp^{\diamond}(B_t) = \exp(B_t - \frac{t^2}{2})$, we have $\cot^{\diamond}(B_t) = \cot(B_t - \frac{t^2}{2})$, $\csc^{\diamond}(B_t) = \csc(B_t - \frac{t^2}{2})$, $\coth^{\diamond}(B_t) = \coth(B_t - \frac{t^2}{2})$ and ${}^{\diamond}(B_t) = (B_t - \frac{t^2}{2})$. Suppose that. $\Phi_1(t) = \psi_1\Phi_3(t), \Phi_2(t) = \psi_2\Phi_3(t)$ and $\Phi_3(t) = \Gamma(t) + \psi_3 W_t$ where ψ_1, ψ_2 and ψ_3 are arbitrary constants and $\Gamma(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$. Therefore, for $\Phi_1(t)\Phi_2(t)\Phi_3(t) \neq 0$. thus exact white noise functional solutions of Eq. (1.2) are as follows.

$$U_{10}(t,x,y)=a_0+\frac{3lk}{\psi_2}\ \csc\left(\Omega_1(t,x,y)\right), \tag{5.1}$$

$$V_{10}(t,x,y)=b_0-\frac{3k^2}{\psi_2}\ \csc\left(\Omega_1(t,x,y)\right), \tag{5.2}$$

$$U_{11}(t,x,y)=a_0+\frac{6lk}{\psi_2}\ \csc\Omega_1(t,x,y), \tag{5.3}$$

$$V_{11}(t,x,y)=b_0-\frac{6k^2}{\psi_2}\ \csc\left(\Omega_1(t,x,y)\right), \tag{5.4}$$

$$U_{12}(t,x,y)=a_0+\frac{6lk}{\psi_2}\ \cot\left(\Omega_2(t,x,y)\right), \tag{5.5}$$

$$V_{12}(t,x,y)=b_0-\frac{6k^2}{\psi_2}\ \cot\left(\Omega_2(t,x,y)\right), \tag{5.6}$$

with

$$\Omega_1(t,x,y)=kx+ly+\big(\frac{k^2a_0\psi_1-lk[b_0\psi_2-k^2]}{l}\big)\left\{\int_0^t\Gamma(\tau)d\tau+\psi_3[B_t-\frac{t^2}{2}]\right\},$$

$$\Omega_2(t,x,y)=kx+ly+\big(\frac{k^2a_0\psi_1-lk[b_0\psi_2+2k^2]}{l}\big)\left\{\int_0^t\Gamma(\tau)d\tau+\psi_3[B_t-\frac{t^2}{2}]\right\},$$

and

$$U_{13}(t,x,y)=a_0+\frac{3lk}{2\psi_2}\ \left[\coth\left(\Omega_3(t,x,y)\right)\pm\ \left(\Omega_3(t,x,y)\right)\right], \tag{5.7}$$

$$V_{13}(t,x,y)=b_0-\frac{3k^2}{2\psi_2}\ \left[\coth\left(\Omega_2(t,x,y)\right)\pm\ \left(\Omega_3(t,x,y)\right)\right], \tag{5.8}$$

$$U_{14}(t,x,y)=a_0+\frac{6lk}{2\psi_2}\ \coth\left(\Omega_4(t,x,y)\right), \tag{5.9}$$

$$V_{14}(t,x,y) = b_0 - \frac{6k^2}{2\psi_2}\ \coth(\Omega_4(t,x,y)), \tag{5.10}$$

$$U_{15}(t,x,y) = a_0 + \frac{6lk}{\psi_2}\ (\Omega_5(t,x,y)), \tag{5.11}$$

$$V_{15}(t,x,y) = b_0 - \frac{6k^2}{\psi_2}\ (\Omega_5(t,x,y)), \tag{5.12}$$

with

$$\Omega_3(t,x,y) = kx + ly + (\frac{2k^2a_0\psi_1 - lk[2b_0\psi_2 - k^2]}{2l})\left\{\int_0^t \Gamma(\tau)d\tau + \psi_3[B_t - \frac{t^2}{2}]\right\},$$

$$\Omega_4(t,x,y) = kx + ly + (\frac{k^2a_0\psi_1 - lk[b_0\psi_2 - 2k^2]}{l})\left\{\int_0^t \Gamma(\tau)d\tau + \psi_3[B_t - \frac{t^2}{2}]\right\},$$

$$\Omega_5(t,x,y) = kx + ly + (\frac{k^2a_0\psi_1 - lk[b_0\psi_2 + k^2]}{l})\left\{\int_0^t \Gamma(\tau)d\tau + \psi_3[B_t - \frac{t^2}{2}]\right\}.$$

6 Conclusion

We have discussed the solutions of (SPDEs) driven by Gaussian white noise. There is a unitary mapping between the Gaussian white noise space and the Poisson white noise space. This connection was given by Benth and Gjerde [2]. By the aid of this connection, we can derive some stochastic exact soliton solutionsfor our problem. In this paper, using Hermite transformation, white noise theory and F-expansion method, we study the white noise functional solutions of the Wick-type stochastic (2+1)-dimensional coupled KdV equations. This paper shows that the F-expansion method is sufficient to solve many stochastic nonlinear equations in mathematical physics. The method which we have proposed in this paper is standard, direct and computerized method, which allows us to do complicated and tedious algebraic calculation. It is shown that the algorithm can be also applied to other nonlinear (PDEs) in mathematical physics such as modified Hirota-Satsuma coupled KdV, KdV-Burgers, modified KdV Burgers, Sawada-Kotera, Zhiber-Shabat equations and Benjamin-Bona-Mahony equations. Since the equation (1.2) has other solutions if select other values of P, Q and R (see Appendices A, B, C), and there are many other of exact solutions for wick-type stochastic (2+1)-dimensional coupled KdV equations.

Appendix A. The ODE and Jacobi Elliptic Functions

Relation between values of (P , Q , R) and corresponding $F(\xi)$ in ODE.

$$(F')^2(\xi) = PF^4(\xi) + QF^2(\xi) + R,$$

P	Q	R	$F(\xi)$
m^2	$-1-m^2$	1	$\mathrm{sn}\xi,\ \mathrm{cd}\xi = \frac{cn\xi}{dn\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{cn}\xi$
-1	$2-m^2$	m^2-1	$\mathrm{dn}\xi$
1	$-1-m^2$	m^2	$\mathrm{ns}\xi = \frac{1}{\mathrm{sn}\xi},\ \mathrm{dc}\xi = \frac{\mathrm{dn}\xi}{\mathrm{cn}\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{nc}\xi = \frac{1}{\mathrm{cn}\xi}$
m^2-1	$2-m^2$	-1	$\mathrm{nd}\xi = \frac{1}{\mathrm{dn}\xi}$
$1-m^2$	$2-m^2$	1	$\mathrm{sc}\xi = \frac{\mathrm{sn}\xi}{\mathrm{cn}\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{sd}\xi = \frac{\mathrm{sn}\xi}{\mathrm{dn}\xi}$
1	$2-m^2$	$1-m^2$	$\mathrm{cs}\xi = \frac{\mathrm{cn}\xi}{\mathrm{sn}\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{ds}\xi = \frac{\mathrm{dn}\xi}{\mathrm{sn}\xi}$
$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi},\ \frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\pm\mathrm{dn}\xi}$
$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{sn}\xi \pm i\mathrm{cn}\xi,\ \frac{\mathrm{dn}\xi}{i\sqrt{1-m^2}\mathrm{sn}\xi\pm\mathrm{cn}\xi},\ \frac{\mathrm{m\,sn}\xi}{1\pm\mathrm{dn}\xi}$
$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\mathrm{ns}\xi \pm\ \mathrm{cs}\xi,\ \frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi\pm\mathrm{dn}\xi},\ \frac{\mathrm{sn}\xi}{1\pm\mathrm{cn}\xi},$
$\frac{m^2-1}{4}$	$\frac{m^2+1}{2}$	$\frac{m^2-1}{4}$	$\frac{\mathrm{dn}\xi}{1\pm m\mathrm{sn}\xi}$
$\frac{1-m^2}{4}$	$\frac{m^2+1}{2}$	$\frac{1-m^2}{4}$	$\mathrm{nc}\xi \pm i\mathrm{sc}\xi\ \frac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}$
$\frac{-1}{4}$	$\frac{m^2+1}{2}$	$\frac{-(1-m^2)^2}{4}$	$m\mathrm{cn}\xi \pm \mathrm{dn}\xi$
$\frac{1}{4}$	$\frac{m^2+1}{2}$	$\frac{(1-m^2)^2}{4}$	$\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}$
$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$	$\mathrm{ns}\xi \pm \mathrm{ds}\xi$

Appendix B.

the jacobi elliptic functions degenerate into trigonometric functions when $m \to 0$.

$$sn\xi \to \sin\xi, cn\xi \to \cos\xi, dn\xi \to 1, sc\xi \to \tan\xi, sd\xi \to \sin\xi, cd\xi \to \cos\xi,$$
$$ns\xi \to \csc\xi, nc\xi \to \sec\xi, nd\xi \to 1, cs\xi \to \cot\xi, ds\xi \to \csc\xi, dc\xi \to \sec\xi.$$

Appendix C.
the jacobi elliptic functions degenerate into hyperbolic functions when $m \to 1$.

$$\begin{aligned} &sn\xi \to \tan\xi, cn\xi \to \xi, dn\xi \to \xi, sc\xi \to \sinh\xi, sd\xi \to \sinh\xi, cd\xi \to 1,\\ &ns\xi \to \coth\xi, nc\xi \to \cosh\xi, nd\xi \to \cosh, cs\xi \to \xi, ds\xi \to \xi, dc\xi \to 1. \end{aligned}$$

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Exact Solutions for Stochastic Fractional Zhiber-Shabat Equations

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Abstract

This paper is devoted to give exact solutions of the variable coefficient fractional Zhiber -Shabat equation with space-time-fractional derivatives. Moreover, by using the Hermite transform and the homogeneous balance principle, the white noise functional solutions for the Wick-type stochastic fractional Zhiber-Shabat equation are explicitly shown. Detailed computations and implemented examples are explicitly provided.

Keywords: Fractional Zhiber-Shabat equations; White noise; Stochastic; Hermite transform.

MSC: 60H30; 60H15; 35R60

1 Introduction

The main task of this paper is to explore exact solutions for the following fractional Zhiber-Shabat equation with variable coefficients:

$$\partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}u + p(t)e^{u} + q(t)e^{-u} + r(t)e^{-2u} = 0 \tag{1.1}$$

where $\partial_{x^{\alpha_1}}, \partial_{t^{\alpha_2}} (0 < \alpha_1, \alpha_2 < 0)$ are the modified Riemann-Liouville fractional derivatives defined by Jumarie [6] and $q(t), p(t)$ and $r(t)$ are bounded measurable or integrable functions on $\mathbb{R}_+$. Random waves is an important subject of random fractional partial differential equations. Recently, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear integer-order differential equation. In the past decades, an important progress has been made in the research of the exact solutions of nonlinear partial differential equations (PDEs). To seek various exact solutions of multifarious physical models described by nonlinear

PDEs, various methods have been proposed. There are many authers studied this subject. Wadati first introduced and studied the stochastic KdV equation and gave the diffusion of soliton of the KdV equation under Gaussian noise in ([10]-[12]). Xie firstly researched Wick-type stochastic KdV equation on white noise space and showed the auto-Bachlund transformation and the exact white noise functional solutions in [14], furthermore, Chen and Xie ([1]-[3]) and Xie ([15]-[17]) researched some Wick-type stochastic wave equations using white noise analysis method. Recently, Uğurlu and Kaya[9] gave the tanh function method, Wazzan [13] showed the modified tanh-coth method, these methods have been applied to derive nonlinear transformations and exact solutions of nonlinear PDEs in mathematical physics. If Eqn.(1.1) is considered in random environment, we can get random fractional Zhiber-Shabat equation with space-fractional derivatives. In order to give the exact solutions of random fractional Zhiber-Shabat equation with space-fractional derivatives, we only consider this problem in white noise environment. Wick-type stochastic generalized fractional Zhiber-Shabat equations with space-fractional derivatives is the perturbation of Eqn.(1.1) by random force $W(t) \diamond R^{\diamond}(U, U_{xt})$, which represented by:

$$\partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}U + P(t)\diamond e^{\diamond U} + Q(t)\diamond e^{\diamond(-U)} + R(t)e^{\diamond(-2U)} = W(t)\diamond R^{\diamond}(U, U_{x^{\alpha_1}t^{\alpha_2}}) \tag{1.2}$$

where $W(t)$ is Gaussian white noise, i.e., $W(t) = B^{\cdot}(t)$ and B(t) is a Brownian motion, $R(u, u_{x^{\alpha_1}t^{\alpha_2}}) = -\beta_1\partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}u - \beta_2 e^{u} - \beta_3 e^{-u} - \beta_4 e^{-2u}$ is a functional of $u, \partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}u := \frac{\partial^{\alpha_1+\alpha_2}u}{\partial x^{\alpha_1}\partial x^{\alpha_2}} = u_{x^{\alpha_1}x^{\alpha_2}}$ for some constants $\beta_1, ..., \beta_4$ and $R^{\diamond}$ is the Wick version of the functional R. " $\diamond$ " is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$ and $P(t), Q(t)$ and $R(t)$ are white noise functionals. Eqn.(1.2) can be seen as the perturbation of the coefficients $p(t), q(t)$ and $r(t)$ of Eqn.(1.1) by white noise functionals.

This paper is devoted to give white noise functional solution for Wick-type stochastic generalized fractional Zhiber-Shabat equations with space-fractional derivatives. Moreover, the Hermite transform and the homogenous balance principle are employed to find the exact solution for stochastic fractional Zhiber-Shabat equation with variable coefficient. Finally, implemented examples are explicitly shown.

2 Preliminaries

There are different definitions for fractional derivatives, for more details (see [5, 6]). In our paper we use the modified Riemann-Liouville derivative defined by Jumarie [6]:

$$D_x^{\alpha}f(x) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\int_0^x (x-y)^{-\alpha-1}[f(y)-f(0)]dy, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\int_0^x (x-y)^{-\alpha}[f(y)-f(0)]dy, & 0 < \alpha < 1, \\ \left[f^{(\alpha-n)}(x)\right]^{(n)}, & n \le \alpha < n+1, \quad n \in \mathbb{N} \end{cases} \tag{2.1}$$

which has merits over the original one, for example, the α-order derivative of a constant is zero. Some properties of the modified Riemann-Liouville derivative were summarized in [5] , three useful formulas of them are

$$\begin{cases} D_x^\alpha x^\beta = \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \quad \beta > 0, \\ D_x^\alpha (u(x)v(x)) = u(x) D_x^\alpha v(x) + v(x) D_x^\alpha u(x), \\ D_x^\alpha [f(u(x)] = \dfrac{df}{du} D_x^\alpha u(x) = \left(\dfrac{du}{dx}\right)^\alpha D_u^\alpha f(u). \end{cases} \tag{2.2}$$

Now, we outline the main idea of the modified fractional sub-equation method. Many authors considered nonlinear FPDE, say, in two variables

$$F(u, u_x, u_t, D_x^\alpha u, D_t^\alpha u, ...) = 0, \qquad 0 < \alpha \leq 1 \tag{2.3}$$

where F is a nonlinear function with respect to the indicated variables. To determine the solution $u = u(x,t)$ explicitly, we first introduce the following transformation

$$u = u(\xi), \qquad \xi = \xi(x,t) \tag{2.4}$$

which converts Eq.(2.3) into a fractional ordinary differential equation

$$G(u, u', u'', D_\xi^\alpha u, D_\xi^{2\alpha} u, ...) = 0. \tag{2.5}$$

Next we introduce a new variable $Y = Y(\xi)$ which is a solution of the fractional Riccati equation

$$D_\xi^\alpha Y = h_0 + h_1 Y + h_2 Y^2, \qquad 0 < \alpha \leq 1, \tag{2.6}$$

where h_0, h_1 and h_2 are arbitrary constants. Eq.(2.6) is the fractional Riccati differential equation, where α is a parameter describing the order of the fractional derivative. In the case of $\alpha = 1$ Eq.(2.6) is reduced to the classical Riccati differential equation. The importance of this equation usually arises in the optimal control problems. The feed back gain of the linear quadratic optimal control depends on a solution of a Riccati differential equation which has to be found for the whole time horizon of the control process [18, 19]. Then we propose the following series expansion as a solution of Eq.(2.3)

$$u(x,t) = u(\xi) = \sum_{k=0}^{n} a_k(x,t) Y^k(\xi) + \sum_{k=1}^{n} b_k(x,t) Y^{-k}(\xi), \tag{2.7}$$

where $a_k (k = 0, 1, ..., n), b_k (k = 1, ..., n)$ are functions to be determined later and n is a positive integer which can be determined via the balancing of the highest derivative term with the nonlinear term in equation Eq.(2.5). Inserting Eq.(2.7) into Eq.(2.5) and using Eq.(2.6) will give an algebraic equation in powers of Y . Since all coefficients of Y^k must vanish, this will give a system of algebraic equations with respect to a_k and b_k . With the aid of Mathematica, we can determine a_k and b_k . According to the recent paper by Zhang et al. [19], we can deduce the following set of solutions of Eq.(2.6).

$$\begin{cases} Y_1(\xi) = E_\alpha(\xi) - 1, & h_0 = h_1 = 1,\ h_2 = 0, \\ Y_2(\xi) = \coth_\alpha(\xi) \pm \operatorname{csch}_\alpha(\xi),\ Y_3(\xi) = \tanh_\alpha(\xi) \pm i \operatorname{sech}_\alpha(\xi), & h_0 = -h_2 = \frac{1}{2},\ h_1 = 0, \\ Y_4(\xi) = \frac{1}{2} \tan_\alpha(2\xi),\ Y_5(\xi) = \frac{1}{2} \cot_\alpha(2\xi), & h_0 = \frac{1}{4} h_2 = 1,\ h_1 = 0, \end{cases} \tag{2.8}$$

with the generalized hyperbolic and trigonometric functions

$$\tanh_\alpha(x) = \frac{\sinh_\alpha(x)}{\cosh_\alpha(x)},\ \coth_\alpha(x) = \frac{\cosh_\alpha(x)}{\sinh_\alpha(x)},\ \mathrm{csch}_\alpha(x) = \frac{1}{\sinh_\alpha(x)},\ \mathrm{sech}_\alpha(x) = \frac{1}{\cosh_\alpha(x)},$$

$$\sinh_\alpha(x) = \frac{E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)}{2},\ \cosh_\alpha(x) = \frac{E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)}{2},\ \tan_\alpha(x) = \frac{\sin_\alpha(x)}{\cos_\alpha(x)},$$

$$\cot_\alpha(x) = \frac{\cos_\alpha(x)}{\sin_\alpha(x)},\ \sin_\alpha(x) = \frac{E_\alpha(ix^\alpha) - E_\alpha(-ix^\alpha)}{2i},\ \cos_\alpha(x) = \frac{E_\alpha(ix^\alpha) + E_\alpha(-ix^\alpha)}{2},$$

defined by the Mittag-Leffler function $E_\alpha(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(1+j\alpha)}$. For more details about the generalized exponential, hyperbolic and trigonometric functions see [8].

3 Exact Solutions of Eqn. (1.2).

Many authors considered nonlinear equations of the form

$$P(u, u_t, u_x, u_{xt}, u_{xx}, u_{xxx}, ...) = 0 \tag{3.1}$$

where P is a nonlinear function with respect to the indicated variables. Introducing the one wave variable $\zeta = x - ct$ carry out the two independent partial differential equation (3.1) into an ODE

$$N(u, u', u'', u''', ...) = 0 \tag{3.2}$$

Equation (3.2) is then integrated as long as all terms contain derivatives. The tanh technique is based on the priori assumption that the travelling wave solutions can be expressed in terms of the tanh function [7]. We therefor introduce a new independent variable

$$Y = tanh(\mu\zeta)$$

that leads to the change of derivatives:

$$\frac{d}{d\zeta} = \mu(1 - Y^2)\frac{d}{dY},$$

$$\frac{d^2}{d\zeta^2} = \mu^2(1 - Y^2)(-2Y\frac{d}{dY} + (1 - Y^2\frac{d^2}{dY^2}))$$

The solution can be proposed by the tanh method as a finite power series in Y in the form:

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{3.3}$$

limiting them to solitary and shock wave profiles. However, the extended tanh method admits the use of the finite expansion

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} a_k Y^{-k}, \tag{3.4}$$

where M is a positive integer, in most cases, that will be determined. Expansion (3.4) reduces to the standard tanh method [7] for $a_k = 0, 1 \leqslant k \leqslant M$. Substituting (3.3) or (3.4) into the ODE (3.2) results in an algebraic equation in powers of Y. In this section, we will give exact solutions of Eqn(3.2). Taking the Hermite transform of Eqn.(3.2), we get

$$\partial_{x^{\alpha_1}}\partial_{t^{\alpha_2}}\widetilde{U}(x,t,z)+\lambda_2(t,z)e^{\widetilde{U}(x,t,z)}+\lambda_2(t,z)e^{-\widetilde{U}(x,t,z)}+\lambda_3(t,z)e^{-2\widetilde{U}(x,t,z)}=0 \tag{3.5}$$

where $z=(z_1,z_2,...)\in C^{\mathbb{N}}$ is a parameter. Using the transformation

$$\zeta=\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)}+\frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)},$$

that will carry out Eqn.(3.5) into

$$\lambda_1\widetilde{U}_{\zeta\zeta}+\lambda_2(t,z)e^{\widetilde{U}(\zeta,z)}+\lambda_3(t,z)e^{-\widetilde{U}(\zeta,z)}+\lambda_4(t,z)e^{-2\widetilde{U}(\zeta,z)}=0. \tag{3.6}$$

where, $\lambda_1=\mu\nu$, $\lambda_2=:\lambda_2(t,z)=\frac{1}{1+\beta_1}\{\widetilde{P}(t,z)+\beta_2\}$, $\lambda_3=:\lambda_3(t,z)=\frac{1}{1+\beta_1}\{\widetilde{Q}(t,z)+\beta_3\}$ and $\lambda_4=:\lambda_4(t,z)=\frac{1}{1+\beta_1}\{\widetilde{R}(t,z)+\beta_4\}$. Denote $u(\zeta,z)=\widetilde{U}(\zeta,z)$ and assume that the solutions of (3.6) is the form

$$u(\zeta,z)=\frac{\partial^2F(\phi(\zeta,z))}{\partial\zeta^2}+V(\zeta,z)$$

Let $v(\zeta,z)=e^{u(\zeta,z)}$, then Eqn.(3.6) becomes

$$\lambda_1\{vv''-v'^2\}+\lambda_2v^3+\lambda_3v+\lambda_4=0; \tag{3.7}$$

Considering the homogeneous balance between vv'' and v^3 in (3.7), gives M=2, hence we set the tanh-coth assumption by

$$v(x,t,z)=S(Y)=a_0(t,z)+a_1(t,z)Y(\zeta)+a_2(t,z)Y^2(\zeta)+b_1(t,z)Y^{-1}(\zeta)+b_2(t,z)Y^{-2}(\zeta) \tag{3.8}$$

where $Y(\zeta)$ satisfies the Riccati equation

$$Y'=c_1+c_2Y+c_3Y^2, \tag{3.9}$$

and c_1,c_2,c_3 are constant to be prescribed later. By virtue of (3.8) and (3.9) with observation of the linear independence of $Y^n(n=-6,-5,...,6)$, Eqn.(3.7) implies the following system of linear equations

$$\begin{cases}
\lambda_4+\lambda_3a_0+\lambda_2[a_0(a_0^2+2a_1b_1+2a_2b_2)+a_1(2a_0b_1+2a_1b_2)+a_2(b_1^2+2a_0b_2)+\\
b_1(2a_0a_1+2a_2b_1)+b_2(a_1^2+2a_0a_2)]+\lambda_1[D_0a_0+D_8a_1+D_7a_2+D_1b_1+D_2b_2-(a_1c_1-b_1c_3)^2+\\
(a_1c_2+2a_2c_1)(b_1c_2+2b_2c_3)+(a_1c_3+2a_2c_2)(b_1c_1+2b_2c_2)+4a_2c_3b_2c_1]=0;\\
\lambda_3a_1+\lambda_2[a_0(2a_0a_1+2a_2b_1)+a_1(a_0^2+2a_1b_1+2a_2b_2)+a_2(2a_0b_1+2a_1b_2)+\\
b_1(a_1^2+2a_0a_2)+2a_1a_2b_2]+\lambda_1[D_0a_1+D_1a_0+D_8a_2+D_2b_1+D_3b_2\\
-(a_1c_1-b_1c_3)(a_1c_2+2a_2c_1)+(a_1c_3+2a_2c_2)(b_1c_2+2b_2c_3)]=0;\\
\lambda_3a_2+\lambda_2[a_0(a_1^2+2a_0a_2)+a_1(2a_0a_1+2a_2b_1)+a_2(a_0^2+2a_1b_1+2a_2b_2)+2a_1a_2b_1+a_2^2b_2]+\\
\lambda_1[D_2a_0++D_1a_1+D_3b_1+D_4b_2-(a_1c_1-b_1c_3)(a_1c_3+2a_2c_2)\\
-(a_1c_2+2a_2c_1)^2+2a_2c_3(b_1c_2+2b_2c_3)]=0;\\
\lambda_2[2a_0a_1a_2+a_1(a_2^2+2a_0a_2)+a_2(2a_0a_1+2a_2b_1)+b_1a_2^2]+\\
\lambda_1[D_3a_0+D_2a_1+D_1a_2+D_4b_1-2a_2c_3(a_1c_1-b_1c_3)-(a_1c_2+2a_2c_1)(a_1c_3+2a_2c_2)]=0;\\
\lambda_2[a_0a_2^2+2a_1^2a_2+a_2(a_1^2+2a_0a_2)]+\lambda_1[D_4a_0+D_3a_1+D_2a_2-2a_2c_3(a_1c_2+2a_2c_1)\\
-(a_1c_3+2a_2c_2)^2]=0;\\
\lambda_2[a_1a_2^2+2a_1a_2^2]+\lambda_1[D_4a_1+D_3a_2-2a_2c_3(a_1c_3+2a_2c_2)]=0;\\
\lambda_2[a_2^3]+\lambda_1[D_4a_2-4a_2^2c_3^2]=0;\\
\lambda_3b_2+\lambda_2[a_0(b_1^2+2a_0b_2)+b_1(2a_0b_1+2a_1b_2)+b_2(a_0^2+2a_1b_1+2a_2b_2)+\\
2a_1b_1b_2+a_2b_2^2]+\lambda_1[D_0b_2+D_7a_0+D_6a_1+D_5a_2+D_8b_1-(b_1c_2+2b_2c_3)^2+\\
2b_2c_1(a_1c_2+2a_2c_1)+(a_1c_1-b_1c_3)(b_1c_1+2b_2c_2)]=0;\\
\lambda_3b_1+\lambda_2[a_0(2a_0b_1+2a_1b_2)+b_1(a_0^2+2a_1b_1+2a_2b_2)+b_2(2a_0a_1+2a_2b_1)+\\
a_1(b_1^2+2a_0b_2)+2a_2b_2b_1]+\lambda_1[D_0b_1+D_8a_0+D_7a_1+D_6a_2+\\
D_1b_2+(a_1c_1-b_1c_3)(b_1c_2+2b_2c_3)+(a_1c_2+2a_2c_1)(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[2a_0b_1b_2+b_1(b_1^2+2a_0b_2)+b_2(2a_0b_1+2a_1b_2)+a_1b_2^2]+\lambda_1[D_6a_0+D_5a_1+D_7b_1+D_8b_2\\
+2b_2c_1(a_1c_1-b_1c_3)-(b_1c_2+2b_2c_3)(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[a_0b_2^2+2a_0b_1^2b_2+b_2(b_1^2+2a_0b_2)]+\lambda_1[D_5a_0+D_6b_1+D_7b_2-2b_2c_1(b_1c_2+2b_2c_3)\\
-(b_1c_1+2b_2c_2)^2]=0;\\
\lambda_2[b_1b_2^2+2a_0b_1b_2^2]+\lambda_1[D_5b_1+D_6b_2-2b_2c_1(b_1c_1+2b_2c_2)]=0;\\
\lambda_2[b_2^3]+\lambda_1[D_5b_2-4b_2^2c_1^2]=0.
\end{cases}$$

where, $D_0=c_1(a_1c_2+2a_2c_1)+c_3(b_1c_2+2b_2c_3)$, $D_1=c_2(a_1c_2+2a_2c_1)+2c_1(a_1c_3+2a_2c_2)$, $D_2=c_3(a_1c_2+2a_2c_1)+2c_2(a_1c_3+2a_2c_2)+6a_2c_3c_1$, $D_3=2c_3(a_1c_3+2a_2c_2)+6a_2c_3c_2$, $D_4=6a_2c_3^2$, $D_5=6a_2b_2c_1^2$, $D_6=2c_1(b_1c_1+2b_2c_2)+6b_2c_1c_2$, $D_7=c_1(b_1c_2+2b_2c_3)+2c_2(b_1c_1+2b_2c_2)+6b_2c_3c_1$, and $D_8=c_2(b_1c_2+2b_2c_3)+2c_3(b_1c_1+2b_2c_2)$. In the remaining part of this section we will discuss and solve our problem for some special cases for the Riccati equation as follows:

A. $c_1=c_2=1, c_3=0$.

This choice for the constants implies that

$$Y_1(\zeta)=exp(\zeta)-1 \tag{3.10}$$

By the aid of Maple 12, the above system of equations can be solve for the following cases:

Case 1: $\lambda_4 = a_1 = a_2 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_0 = \pm i\sqrt{\frac{\lambda_3}{\lambda_2}}$; $b_1 = \frac{3}{\lambda_2 \pm i\sqrt{\lambda_2\lambda_3}}$; $b_2 = -\frac{2\lambda_1}{\lambda_2}$. By virtue of Eqn.(3.8), then Eqn.(3.5) have the solution

$$u_1 = ln\{\pm i\sqrt{\frac{\lambda_3}{\lambda_2}} + \frac{3}{\lambda_2 \pm i\sqrt{\lambda_2\lambda_3}} \times \frac{1}{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1} - \frac{2\lambda_1}{\lambda_2(exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1)^2}\} \tag{3.11}$$

Case 2: For $\lambda_4 = a_1 = a_2 = b_1 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_0 = \pm i\sqrt{\frac{\lambda_3}{\lambda_2}}$; $b_2 = -\frac{2\lambda_1}{\lambda_2}$. Eqn.(3.5) have the solution

$$u_2 = ln\{\pm i\sqrt{\frac{\lambda_3}{\lambda_2}} - \frac{2\lambda_1}{\lambda_2(exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1)^2}\} \tag{3.12}$$

B. $c_1 = -c_3 = 0.5, c_2 = 0$.

This choice for the constants implies that

$$Y_2(\zeta) = coth(\zeta) \pm csch(\zeta) \tag{3.13}$$

or

$$Y_3(\zeta) = tanh(\zeta) \pm isech(\zeta) \tag{3.14}$$

By the aid of Maple 12, the above system of equations can be solve for the following cases:

Case 3: $\lambda_4 = a_0 = a_1 = a_2 = b_1 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $b_2 = -\frac{\lambda_1}{2\lambda_2}$. By virtue of Eqn.(3.8), then Eqn.(3.5) have the solution

$$u_3 = ln\{-\frac{\lambda_1}{2\lambda_2(coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2}\} \tag{3.15}$$

or

$$u_4 = ln\{-\frac{\lambda_1}{2\lambda_2(tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2}\} \tag{3.16}$$

Case 4: For $\lambda_4 = a_0 = a_1 = b_1 = b_2 = 0, \lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0$; $a_2 = -\frac{\lambda_1}{2\lambda_2}$. Eqn.(3.5) have the solution

$$u_5 = ln\{-\frac{\lambda_1}{2\lambda_2}(coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2\} \tag{3.17}$$

or

$$u_6 = ln\{-\frac{\lambda_1}{2\lambda_2}(tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}))^2\} \tag{3.18}$$

4 White noise functional solutions of (1.2)

In this section, we will use Theorem 2.1 of Xie [17] for $d = 1$ to obtain white noise functional solutions of Eqs.(1.2). The properties of hyperbolic functions yield that there exists a bounded open set $\mathbf{S} \subset \mathbb{R}_+ \times \mathbb{R}, m > 0$ and $n > 0$ such that $u(x,t,z), u_{xt}(x,t,z)$ are uniformally bounded for all $(t,x,z) \in \mathbf{S} \times \mathbb{K}_m(n)$, continuous with respect to $(t,x) \in \mathbf{S}$ for all $z \in \mathbb{K}_m(n)$ and analytic with respect to $z \in \mathbb{K}_m(n)$ for all $(t,x) \in \mathbf{S}$. Using Theorem 2.1 of Xie [17], there exists a stochastic process $U(t,x)$ such that the Hermite transformation of $U(t,x)$ is $u(t,x,z)$ for all $\mathbf{S} \times \mathbb{K}_m(n)$, and $U(t,x)$ is the solution of (1.2). This implies that $U(t,x)$ is the inverse Hermite transformation of $u(t,x,z)$. Hence, for $\Lambda_1\Lambda_2\Lambda_3 \neq 0$ the white noise functional solutions of Eqn.(1.2) as follows:

$$U_1(x,t) = ln^{\diamond}\{\pm i\sqrt{\frac{\Lambda_3(t)}{\Lambda_2(t)}} + \frac{3\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-1}}{\Lambda_2(t) \pm i\sqrt{\Lambda_2(t)\Lambda_3(t)}} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \tag{4.1}$$

$$U_2(x,t) = ln^{\diamond}\{\pm i\sqrt{\frac{\Lambda_3(t)}{\Lambda_2(t)}} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \tag{4.2}$$

$$U_3(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \tag{4.3}$$

$$U_4(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \tag{4.4}$$

$$U_5(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \tag{4.5}$$

$$U_6(x,t) = ln^{\diamond}\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm isech^{\diamond}(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu t^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \tag{4.6}$$

We observe that for different form of $\Lambda_2(t)$ and $\Lambda_3(t)$, we can get different solutions of (1.2) from (3.1)-(3.6).

5 Example and Concluding Remarks

Let $\underset{\sim}{B_t^{\cdot}}$ be the Gaussian white noise, where B_t is Brown motion. We have the Hermite transform $\widetilde{B}^{\cdot}(t,z) = \sum_{k=1}^{\infty} z_k \int_0^t \eta_k(s)ds$. Science $\exp^{\diamond}(B_t) = \exp(B_t - t^2/2)$, we have $tanh^{\diamond}(B_t) = tanh(B_t - t^2/2)$, $coth^{\diamond}(B_t) = cot(B_t - t^2/2)$, $sech^{\diamond}(B_t) = sech(B_t - t^2/2)$ and $csch^{\diamond}(B_t) = csch(B_t - t^2/2)$. Suppose $\Lambda_3(t) = \alpha\Lambda_2(t)$ and $\Lambda_2(t) = \lambda_2(t) + \beta B_t^{\cdot}$, where α, β are arbitrary constants and $\lambda_2(t)$ is integrable or bounded measurable function on $\mathbb{R}_+$. The white noise functional solutions of (1.2) are as follows: If $\Lambda_1(t)\Lambda_2(t)\Lambda_3(t) \neq 0$

$$U_7(x,t) = ln\{\pm i\sqrt{\alpha} + \frac{3\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-1}}{\Lambda_2(t)(1 + \pm i\sqrt{\alpha})} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \quad (5.1)$$

$$U_8(x,t) = ln\{\pm i\sqrt{\alpha} - \frac{2\mu\nu}{\Lambda_2(t)}\{exp(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) - 1\}^{-2}\} \quad (5.2)$$

$$U_9(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \quad (5.3)$$

$$U_{10}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm i\, sech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{-2}\} \quad (5.4)$$

$$U_{11}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{coth(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm csch(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \quad (5.5)$$

$$U_{12}(x,t) = ln\{-\frac{\mu\nu}{2\Lambda_2(t)}\{tanh(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)}) \pm i\, sech(\frac{\mu x^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{\nu(t-\beta B_t+0.5\beta t^2)^{\alpha_2}}{\Gamma(1+\alpha_2)})\}^{2}\} \quad (5.6)$$

Finally, we remark that for $\alpha_1 = \alpha_2 = 0$, $p(t) =1$ and $q(t) = r(t) = 0$, Eqn.(1.1) reduces to the Liouville equation. For $\alpha_1 = \alpha_2 = 0$, $r(t) =0$ and $q(t) = p(t) = 1$, Eqn.(1.1) reduces to the Sinh-Gordon equation. For $\alpha_1 = \alpha_2 = 0$, $p(t) = r(t) =1$ and $q(t) = 0$, Eqn.(1.1) reduces to the the well known Dodd-Bullough-Mikhailov equation. Moreover, for $\alpha_1 = \alpha_2 = 0$, $p(t) =0$, $q(t) = -1$ and $r(t) = 1$, gives Tzitzeica-Dodd-Bullough equation. Hence, our results in this work can be considered as a continuation of our results in our previous papers [4,5], this work gives directly exact solutions for wick-type stochastic form to each one of the above equations. Also, we remark that, since the Riccati equation has other solution if select other values of c_1, c_2 and c_3 , there are many other exact solutions of variable coefficient and wick-type stochastic Zhiber-Shabat equations

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Invariance, solutions, periodicity and asymptotic behavior of a class of fourth order difference equations

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Abstract

We construct Lie symmetry generators of some fourth order difference equations. We use these generators to derive similarity variables that make it possible to obtain exact solutions. In some cases, we study periodicity and asymptotic behavior of the solutions.

2010 Mathematics Subject Classification: 39A11, 39A05.
Key words: Difference equation; symmetry; reduction; group invariant solutions

1 Introduction

Several years back, Sophus Lie studied the invariance property of equations under a group of transformations. The approach used was later known as Lie symmetry method. This method has been used to solve differential equations, and recently it has been applied to difference equations. Although Maeda studied difference equations via Lie symmetry analysis in twentieth century [9, 10], it is Hydon who really rekindled the interest for solving difference equations via symmetry. For Hydon's work, refer to [8].

Most often, difference equations arise as a result of discretizing differential equations, especially in phenomena that depend on time. There are many ways in which a differential equation can be discretized (see [4]). Difference equations have numerous applications. For example, biological systems, population dynamics, economics, physics (see [1, 2]). Although difference equations appear simple, finding their solutions can be incredibly difficult. The symmetry approach to finding solutions of difference equations is recent and the reader can refer to [8] and some recent articles [5–7, 11, 12] for further knowledge on this method.

In this paper, we consider the system of difference equations

$$x_{n+4} = \frac{x_n x_{n+1}}{x_{n+3}(a_n + b_n x_n x_{n+1})} \tag{1}$$

where $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are non-zero sequences of real numbers. For equation (1), we derive all Lie point symmetries and give formulas for solutions in closed form. We also discuss periodicity and asymptotic behavior of solutions in some cases.

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1.1 Preliminaries

In this section, we give a background on symmetry methods for difference equations. Our definitions and notation come from [3, 8, 13].
Consider the difference equations

$$x_{n+4} = \Omega(x_n, x_{n+1}, x_{n+3}), \tag{2}$$

where n denotes the independent variable; x_n the dependent variable. In this case u_{n+i} denotes the 'i-th shift' of u_n.
Consider the group of transformations

$$(n, x_n) \mapsto (n, \tilde{x}_n = x_n + \varepsilon Q_1(n, x_n) + O(\varepsilon^2)), \tag{3}$$

where Q is the characteristic of the group of point transformations. Let

$$X = Q(n, x_n)\frac{\partial}{\partial x_n} \tag{4}$$

be the corresponding infinitesimal generator. The group of transformations (3) is a symmetry group if and only if

$$Q(n+4, \Omega) - \mathcal{X}(\Omega) = 0, \tag{5}$$

whenever (2) holds. Here,

$$\mathcal{X} = Q(n, x_n)\frac{\partial}{\partial x_n} + Q(n, x_{n+1})\frac{\partial}{\partial x_{n+1}} + Q(n+3, x_{n+3})\frac{\partial}{\partial x_{n+3}}$$

denotes the prolongation of X to all shifts of x_n appearing in the right hand sides of equations in (2). Equation (5), known as the linearized symmetry condition, can be solved for Q by applying the appropriate differential operators. The characteristic, together with the canonical coordinate

$$s = \int \frac{dx_n}{Q(n, x_n)}, \tag{6}$$

are necessary in the reduction of order of (2). The following definition can be used to check if a given function is invariant under a given group of transformations.

Definition 1 *[13] Let G be a connected group of transformations acting on a manifold M. A smooth real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X(\zeta) = 0 \qquad \text{for all} \qquad x \in M.$$

2 Main results

2.1 Symmetry and difference invariant

To obtain the the criterion which gives the Lie point symmetries of (1), we force (5) on

$$x_{n+4} = \frac{x_n x_{n+1}}{x_{n+3}(a_n + b_n x_n x_{n+1})}. \tag{7}$$

This leads to

$$\begin{aligned} Q(n+4, x_{n+4}) &+ \frac{x_n x_{n+1}(a_n + b_n x_n x_{n+1}) Q(n+3, x_{n+3})}{{x_{n+3}}^2 (a_n + b_n x_n x_{n+1})^2} \\ &- \frac{a_n[x_n Q(n+1, x_{n+1}) + x_{n+1} Q(n, x_n)]}{x_{n+3}(a_n + b_n x_n x_{n+1})^2} = 0. \end{aligned} \tag{8}$$

The methodology of solving these functional equations is given as follows:

- Firstly, apply the differential operator $\frac{\partial}{\partial x_n} + \frac{x_{n+1}}{x_n}\frac{\partial}{\partial x_{n+1}}$ on equation (8). This leads (after simplification) to

  $$x_{n+1}Q'(n+1, x_{n+1}) - x_{n+1}Q'(n, x_n) - Q(n+1, x_{n+1}) + \frac{a_n}{x_n} Q(n, x_n) = 0.$$

- Secondly, differentiate with respect to x_n, separate by powers of x_{n+1} and solve the resulting system of over determining equations for Q. This gives

  $$Q(n, x_n) = \alpha(n) x_n + \beta(n)$$

  for some functions α and β of n.

- Lastly, substitute the latter in (8) to eliminate any dependency among the arbitrary functions that appear in Q. This leads to the constraints

  $$\alpha(n) + \alpha(n+1) = 0 \text{ and } \quad \beta(n) = 0. \tag{9}$$

We have omitted the details in the computation. The constraints in (9) are readily solved ($\alpha(n) = (-1)^n$) and we have

$$Q = (-1)^n x_n. \tag{10}$$

Consequently, Equation (1) admits a one dimensional Lie algebra:

$$X = (-1)^n x_n \frac{\partial}{\partial x_n}. \tag{11}$$

The canonical coordinate is given by

$$s_n = \int \frac{dx_n}{(-1)^n x_n} = (-1)^n \ln |x_n| \tag{12}$$

and the difference invariant which is inspired by the form of the final constraints (9) is given by

$$\mathbf{u}_n = (-1)^n s_n + (-1)^{n+1} s_{n+1}. \tag{13}$$

It is not difficult to verify, using Definition 1 together with (11), that (13) is indeed invariant under the group of transformations of (1). For simplicity, we prefer using the compatible variable

$$|u_n| = \exp(-\mathbf{u}_n) \tag{14}$$

which is also invariant. This gives a convenient choice of the change variables which does not require lucky guesses. With this variable u_n, it follows that

$$u_{n+3} = a_n u_n + b_n \tag{15}$$

whose solution is given by

$$u_{3n+j} = u_j \left(\prod_{k_1=0}^{n-1} a_{3k_1+j} \right) + \sum_{l=0}^{n-1} \left(b_{3l+j} \prod_{k_2=l+1}^{n-1} a_{3k_2+j} \right), \quad j = 0, 1, 2. \tag{16}$$

To obtain the solutions of (1), we go up the hierarchy created by the changes of variables. By evaluating (13) as a telescoping series, we have

$$(-1)^n s_n = (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{k_1} \mathbf{u}_{k_1} + (-1)^n s_0 \quad (= \ln|x_n| \quad \text{from} \quad (12)), \tag{17}$$

i.e.

$$x_n = \exp\left\{ (-1)^{n-1} \sum_{k_1=0}^{n-1} (-1)^{k_1} \mathbf{u}_{k_1} + (-1)^n s_0 \right\}, \tag{18}$$

$$= \exp\{ \sum_{k_1=0}^{n-1} (-1)^{n+k_1} \ln u_{k_1} + \ln x_0 \}, \tag{19}$$

where all the u_{k_1}'s are obtained using (16).
Note. Equation (19) gives the closed form solution of (1) in a unified manner. Looking at the form of u_l in (16), we rephrase (19) as follows:

$$x_{6n+j} = \exp\left\{ \sum_{k_1=0}^{6n+j-1} (-1)^{6n+j+k_1} \ln u_{k_1} + \ln x_0 \right\}, \tag{20}$$

$$= x_j \prod_{i=0}^{n-1} \left(\prod_{r=0}^{2} \frac{u_{6i+j+2r}}{u_{6i+j+2r+1}} \right), \tag{21}$$

$j = 0, 1, \ldots, 5$. More clearly,

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{u_{3(2i)}}{u_{3(2i)+1}} \frac{u_{3(2i)+2}}{u_{3(2i+1)}} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}}, \tag{22a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{u_{3(2i)+1}}{u_{3(2i)+2}} \frac{u_{3(2i+1)}}{u_{3(2i+1)+1}} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}}, \tag{22b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{u_{3(2i)+2}}{u_{3(2i+1)}} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}} \frac{u_{3(2i+2)}}{u_{3(2i+2)+1}}, \tag{22c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)}}{u_{3(2i+1)+1}} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}} \frac{u_{3(2i+2)+1}}{u_{3(2i+2)+2}}, \tag{22d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)+1}}{u_{3(2i+1)+2}} \frac{u_{3(2i+2)}}{u_{3(2i+2)+1}} \frac{u_{3(2i+2)+2}}{u_{3(2i+3)}}, \tag{22e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{u_{3(2i+1)+2}}{u_{3(2i+2)}} \frac{u_{3(2i+2)+1}}{u_{3(2i+2)+2}} \frac{u_{3(2i+3)}}{u_{3(2i+3)+1}}. \tag{22f}$$

We then substitute the expressions given in (16) in (22) to get

$$\begin{aligned} x_{6n} =& x_0 \prod_{i=0}^{n-1} \frac{u_0 \prod\limits_{l_1=0}^{2i-1} a_{3l_1} + \sum\limits_{j=0}^{2i-1} b_{3j} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2}}{u_1 \prod\limits_{l=0}^{2i-1} a_{3l+1} + \sum\limits_{j=0}^{2i-1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+1}} \frac{u_2 \prod\limits_{l=0}^{2i-1} a_{3l+2} + \sum\limits_{j=0}^{2i-1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i-1} a_{kl_2+2}}{u_0 \prod\limits_{l=0}^{2i} a_{3l} + \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{kl_2}} \\ & \frac{u_1 \prod\limits_{l=0}^{2i} a_{3l+1} + \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{kl_2+1}}{u_2 \prod\limits_{l=0}^{2i} a_{3l+2} + \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{kl_2+2}}, \end{aligned} \tag{23a}$$

$$\begin{aligned} x_{6n+1} =& x_1 \prod_{i=0}^{n-1} \frac{u_1 \prod\limits_{l_1=0}^{2i-1} a_{3l_1+1} + \sum\limits_{j=0}^{2i-1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+1}}{u_2 \prod\limits_{l=0}^{2i-1} a_{3l+2} + \sum\limits_{j=0}^{2i-1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+2}} \frac{u_0 \prod\limits_{l=0}^{2i} a_{3l} + \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{3l_2}}{u_1 \prod\limits_{l=0}^{2i} a_{3l+1} + \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}} \\ & \frac{u_2 \prod\limits_{l=0}^{2i} a_{3l+2} + \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod\limits_{l=0}^{2i+1} a_{3l} + \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}}, \end{aligned} \tag{23b}$$

$$\begin{aligned} x_{6n+2} =& x_2 \prod_{i=0}^{n-1} \frac{u_2 \prod_{l_1=0}^{2i-1} a_{3l_1+2} + \sum_{j=0}^{2i-1} b_{3j+2} \prod_{l_2=j+1}^{2i-1} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i} a_{3l} + \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{3l_2}} \frac{u_1 \prod_{l=0}^{2i} a_{3l+1} + \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i} a_{3l+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}} \\ & \frac{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}{u_1 \prod_{l=0}^{2i+1} a_{3l+1} + \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}, \end{aligned} \tag{23c}$$

$$\begin{aligned} x_{6n+3} =& x_3 \prod_{i=0}^{n-1} \frac{u_0 \prod_{l_1=0}^{2i} a_{3l_1} + \sum_{j=0}^{2i} b_{3j} \prod_{l_2=j+1}^{2i} a_{3l_2}}{u_1 \prod_{l=0}^{2i} a_{3l+1} + \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}} \frac{u_2 \prod_{l=0}^{2i} a_{3l+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}} \\ & \frac{u_1 \prod_{l=0}^{2i+1} a_{3l+1} + \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i+1} a_{3l+2} + \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}}, \end{aligned} \tag{23d}$$

$$\begin{aligned} x_{6n+4} =& x_4 \prod_{i=0}^{n-1} \frac{u_1 \prod_{l_1=0}^{2i} a_{3l_1+1} + \sum_{j=0}^{2i} b_{3j+1} \prod_{l_2=j+1}^{2i} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i} a_{3l+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}} \frac{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}}{u_1 \prod_{l=0}^{2i+1} a_{3l+1} + \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{kl_2+1}} \\ & \frac{u_2 \prod_{l=0}^{2i+1} a_{3l+2} + \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i+2} a_{3l} + \sum_{j=0}^{2i+2} b_{3j} \prod_{l_2=j+1}^{2i+2} a_{3l_2}}, \end{aligned} \tag{23e}$$

$$\begin{aligned} x_{6n+5} =& x_5 \prod_{i=0}^{n-1} \frac{u_2 \prod_{l_1=0}^{2i} a_{3l_1+2} + \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{u_0 \prod_{l=0}^{2i+1} a_{3l} + \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}} \frac{u_1 \prod_{l=0}^{2i+1} a_{3l+1} + \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}{u_2 \prod_{l=0}^{2i+1} a_{3l+2} + \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}} \\ & \frac{u_0 \prod_{l=0}^{2i+2} a_{3l} + \sum_{j=0}^{2i+2} b_{3j} \prod_{l_2=j+1}^{2i+2} a_{3l_2}}{u_1 \prod_{l=0}^{2i+2} a_{3l+1} + \sum_{j=0}^{2i+2} b_{3j+1} \prod_{l_2=j+1}^{2i+2} a_{3l_2+1}}. \end{aligned} \tag{23f}$$

We rewrite (23) in terms of initial conditions only as follows:

$$\begin{aligned}x_{6n} =& x_0 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i-1} a_{3l_1} + x_0x_1 \sum\limits_{j=0}^{2i-1} b_{3j} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2}}{\prod\limits_{l=0}^{2i-1} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i-1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+1}} \frac{\prod\limits_{l=0}^{2i-1} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i-1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i-1} a_{kl_2+2}}{\prod\limits_{l=0}^{2i} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{kl_2}} \\ & \frac{\prod\limits_{l=0}^{2i} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{kl_2+1}}{\prod\limits_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{kl_2+2}},\end{aligned} \tag{24a}$$

$$\begin{aligned}x_{6n+1} =& x_1 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i-1} a_{3l_1+1} + x_1x_2 \sum\limits_{j=0}^{2i-1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+1}}{\prod\limits_{l=0}^{2i-1} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i-1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+2}} \frac{\prod\limits_{l=0}^{2i} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{3l_2}}{\prod\limits_{l=0}^{2i} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}} \\ & \frac{\prod\limits_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}}{\prod\limits_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}},\end{aligned} \tag{24b}$$

$$\begin{aligned}x_{6n+2} =& x_2 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i-1} a_{3l_1+2} + x_2x_3 \sum\limits_{j=0}^{2i-1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i-1} a_{3l_2+2}}{\prod\limits_{l=0}^{2i} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{3l_2}} \frac{\prod\limits_{l=0}^{2i} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}}{\prod\limits_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}} \\ & \frac{\prod\limits_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}}{\prod\limits_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i+1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+1}},\end{aligned} \tag{24c}$$

$$\begin{aligned}x_{6n+3} =& x_3 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i} a_{3l_1} + x_0x_1 \sum\limits_{j=0}^{2i} b_{3j} \prod\limits_{l_2=j+1}^{2i} a_{3l_2}}{\prod\limits_{l=0}^{2i} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}} \frac{\prod\limits_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}}{\prod\limits_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}} \\ & \frac{\prod\limits_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i+1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+1}}{\prod\limits_{l=0}^{2i+1} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i+1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+2}},\end{aligned} \tag{24d}$$

$$\begin{aligned}x_{6n+4} =& x_4 \prod_{i=0}^{n-1} \frac{\prod\limits_{l_1=0}^{2i} a_{3l_1+1} + x_1x_2 \sum\limits_{j=0}^{2i} b_{3j+1} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+1}}{\prod\limits_{l=0}^{2i} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i} b_{3j+2} \prod\limits_{l_2=j+1}^{2i} a_{3l_2+2}} \frac{\prod\limits_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i+1} b_{3j} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2}}{\prod\limits_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum\limits_{j=0}^{2i+1} b_{3j+1} \prod\limits_{l_2=j+1}^{2i+1} a_{kl_2+1}} \\ & \frac{\prod\limits_{l=0}^{2i+1} a_{3l+2} + x_2x_3 \sum\limits_{j=0}^{2i+1} b_{3j+2} \prod\limits_{l_2=j+1}^{2i+1} a_{3l_2+2}}{\prod\limits_{l=0}^{2i+2} a_{3l} + x_0x_1 \sum\limits_{j=0}^{2i+2} b_{3j} \prod\limits_{l_2=j+1}^{2i+2} a_{3l_2}},\end{aligned} \tag{24e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{\prod_{l_1=0}^{2i} a_{3l_1+2} + x_2x_3 \sum_{j=0}^{2i} b_{3j+2} \prod_{l_2=j+1}^{2i} a_{3l_2+2}}{\prod_{l=0}^{2i+1} a_{3l} + x_0x_1 \sum_{j=0}^{2i+1} b_{3j} \prod_{l_2=j+1}^{2i+1} a_{3l_2}} \frac{\prod_{l=0}^{2i+1} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i+1} b_{3j+1} \prod_{l_2=j+1}^{2i+1} a_{3l_2+1}}{\prod_{l=0}^{2i+1} a_{3l+2} + x_2x_3 \sum_{j=0}^{2i+1} b_{3j+2} \prod_{l_2=j+1}^{2i+1} a_{3l_2+2}} \frac{\prod_{l=0}^{2i+2} a_{3l} + x_0x_1 \sum_{j=0}^{2i+2} b_{3j} \prod_{l_2=j+1}^{2i+2} a_{3l_2}}{\prod_{l=0}^{2i+2} a_{3l+1} + x_1x_2 \sum_{j=0}^{2i+2} b_{3j+1} \prod_{l_2=j+1}^{2i+2} a_{3l_2+1}}, \tag{24f}$$

where $x_4 = x_0x_1/(x_3(a_0 + b_0x_0x_1))$ and $x_5 = x_2x_3(a_0 + b_0x_0x_1)/(x_0(a_1 + b_1x_1x_2))$. In the following subsections, we study some special cases.

2.2 The case where (a_n) and (b_n) are 3 periodic sequences

Let $a_n = \{a_0, a_1, a_2, a_0, a_1, a_2, \dots\}$ and $b_n = \{b_0, b_1, b_2, b_0, b_1, b_2, \dots\}$. Equations in (23) reduce to

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{a_0^{2i} + b_0x_0x_1 \sum_{j=0}^{2i-1} a_0^j}{a_1^{2i} + b_1x_1x_2 \sum_{j=0}^{2i-1} a_1^j} \frac{a_2^{2i} + b_2x_2x_3 \sum_{j=0}^{2i-1} a_2^j}{a_0^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a_0^j} \frac{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j},$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{a_1^{2i} + b_1x_1x_2 \sum_{j=0}^{2i-1} a_1^j}{a_2^{2i} + b_2x_2x_3 \sum_{j=0}^{2i-1} a_2^j} \frac{a_0^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a_0^j}{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j} \frac{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j},$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{a_2^{2i} + b_2x_2x_3 \sum_{j=0}^{2i-1} a_2^j}{a_0^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a_0^j} \frac{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j} \frac{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j}{a_1^{2i+2} + b_1x_1x_2 \sum_{j=0}^{2i+1} a_1^j},$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{a_0^{2i+1} + b_0x_0x_1 \sum_{j=0}^{2i} a_0^j}{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j} \frac{u_2a_2^{2i+1} + b_2 \sum_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j} \frac{a_1^{2i+2} + b_1x_1x_2 \sum_{j=0}^{2i+1} a_1^j}{a_2^{2i+2} + b_2x_2x_3 \sum_{j=0}^{2i+1} a_2^j},$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{a_1^{2i+1} + b_1x_1x_2 \sum_{j=0}^{2i} a_1^j}{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j} \frac{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j}{a_1^{2i+2} + b_1x_1x_2 \sum_{j=0}^{2i+1} a_1^j} \frac{a_2^{2i+2} + b_2x_2x_3 \sum_{j=0}^{2i+1} a_2^j}{a_0^{2i+3} + b_0x_0x_1 \sum_{j=0}^{2i+2} a_0^j},$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{a_2^{2i+1} + b_2x_2x_3 \sum_{j=0}^{2i} a_2^j}{a_0^{2i+2} + b_0x_0x_1 \sum_{j=0}^{2i+1} a_0^j} \frac{a_1^{2i+2} + b_1x_1x_2 \sum_{j=0}^{2i+1} a_1^j}{a_2^{2i+2} + b_2x_2x_3 \sum_{j=0}^{2i+1} a_2^j} \frac{a_0^{2i+3} + b_0x_0x_1 \sum_{j=0}^{2i+2} a_0^j}{a_1^{2i+3} + b_1x_1x_2 \sum_{j=0}^{2i+2} a_1^j}.$$

2.3 The case where (a_n) and (b_n) are real constants

Let $a_n = a$ and $b_n = b$. Equations in (23) give rise to

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_0x_1\sum_{j=0}^{2i-1} a^j}{a^{2i} + bx_1x_2\sum_{j=0}^{2i-1} a^j} \frac{a^{2i} + bx_2x_3\sum_{j=0}^{2i-1} a^j}{a^{2i+1} + bx_0x_1\sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + bx_1x_2\sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3\sum_{j=0}^{2i} a^j}, \tag{25a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_1x_2\sum_{j=0}^{2i-1} a^j}{a^{2i} + b_2x_2x_3\sum_{j=0}^{2i-1} a^j} \frac{a^{2i+1} + bx_0x_1\sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_1x_2\sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + bx_2x_3\sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1\sum_{j=0}^{2i+1} a^j}, \tag{25b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{a^{2i} + bx_2x_3\sum_{j=0}^{2i-1} a^j}{a^{2i+1} + bx_0x_1\sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + b_1x_1x_2\sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3\sum_{j=0}^{2i} a^j} \frac{a^{2i+2} + bx_0x_1\sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_1x_2\sum_{j=0}^{2i+1} a^j}, \tag{25c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{a^{2i+1} + b_0x_0x_1\sum_{j=0}^{2i} a^j}{a^{2i+1} + b_1x_1x_2\sum_{j=0}^{2i} a^j} \frac{a^{2i+1} + b_2x_2x_3\sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1\sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_1x_2\sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_2x_3\sum_{j=0}^{2i+1} a^j}, \tag{25d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{a^{2i+1} + bx_1x_2\sum_{j=0}^{2i} a^j}{a^{2i+1} + bx_2x_3\sum_{j=0}^{2i} a^j} \frac{a^{2i+2} + bx_0x_1\sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_1x_2\sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_2x_3\sum_{j=0}^{2i+1} a^j}{a^{2i+3} + bx_0x_1\sum_{j=0}^{2i+2} a^j}, \tag{25e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{a^{2i+1} + bx_2x_3\sum_{j=0}^{2i} a^j}{a^{2i+2} + bx_0x_1\sum_{j=0}^{2i+1} a^j} \frac{a^{2i+2} + bx_1x_2\sum_{j=0}^{2i+1} a^j}{a^{2i+2} + bx_2x_3\sum_{j=0}^{2i+1} a^j} \frac{a^{2i+3} + bx_0x_1\sum_{j=0}^{2i+2} a^j}{a^{2i+3} + bx_1x_2\sum_{j=0}^{2i+2} a^j}. \tag{25f}$$

2.3.1 The case where $a = 1$

Equations in (25) simplify to

$$x_{6n} = x_0 \prod_{i=0}^{n-1} \frac{1 + 2ibx_0x_1}{1 + 2ibx_1x_2} \frac{1 + 2ibx_2x_3}{1 + (2i+1)bx_0x_1} \frac{1 + (2i+1)bx_1x_2}{1 + (2i+1)bx_2x_3}, \tag{26a}$$

$$x_{6n+1} = x_1 \prod_{i=0}^{n-1} \frac{1 + 2ibx_1x_2}{1 + 2ibx_2x_3} \frac{1 + (2i+1)bx_0x_1}{1 + (2i+1)bx_1x_2} \frac{1 + (2i+1)bx_2x_3}{1 + (2i+2)bx_0x_1}, \tag{26b}$$

$$x_{6n+2} = x_2 \prod_{i=0}^{n-1} \frac{1 + 2ibx_2x_3}{1 + (2i+1)bx_0x_1} \frac{1 + (2i+1)bx_1x_2}{1 + (2i+1)bx_2x_3} \frac{1 + (2i+2)bx_0x_1}{1 + (2i+2)bx_1x_2}, \tag{26c}$$

$$x_{6n+3} = x_3 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_0x_1}{1+(2i+1)bx_1x_2} \frac{1+(2i+1)bx_2x_3}{1+(2i+2)bx_0x_1} \frac{1+(2i+2)bx_1x_2}{1+(2i+2)bx_2x_3}, \tag{26d}$$

$$x_{6n+4} = x_4 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_1x_2}{1+(2i+1)bx_2x_3} \frac{1+(2i+2)bx_0x_1}{1+(2i+2)bx_1x_2} \frac{1+(2i+2)bx_2x_3}{1+(2i+3)bx_0x_1}, \tag{26e}$$

$$x_{6n+5} = x_5 \prod_{i=0}^{n-1} \frac{1+(2i+1)bx_2x_3}{1+(2i+2)bx_0x_1} \frac{1+(2i+2)bx_1x_2}{1+(2i+2)bx_2x_3} \frac{1+(2i+3)bx_0x_1}{1+(2i+3)bx_1x_2}. \tag{26f}$$

2.3.2 The case where $a = -1$

Let $a_n = -1$ and $b_n = b$. Equations in (23) result in

$$x_{6n} = \frac{x_0(x_1x_2b-1)^n}{(x_0x_1b-1)^n(x_2x_3b-1)^n}, x_{6n+1} = \frac{x_1(x_0x_1b-1)^n(x_2x_3b-1)^n}{(x_1x_2b-1)^n},$$

$$x_{6n+2} = \frac{x_2(x_1x_2b-1)^n}{(x_0x_1b-1)^n(x_2x_3b-1)^n}, x_{6n+3} = \frac{x_3(x_0x_1b-1)^n(x_2x_3b-1)^n}{(x_1x_2b-1)^n},$$

$$x_{6n+4} = \frac{x_0x_1(x_1x_2b-1)^n}{x_3(x_0x_1b-1)^{n+1}(x_2x_3b-1)^n}, x_{6n+5} = \frac{x_2x_3(x_0x_1b-1)^{n+1}(x_2x_3b-1)^n}{x_0(x_1x_2b-1)^{n+1}}.$$

2.4 Existence of six periodic solutions

From (26), if $a = 1$ and $b = 0$, then the solution of (1) is periodic with period six as long as $u_0 \neq x_2$ or $x_1 \neq x_3$. It should also be noted that the solutions are periodic with period two when $x_0 = x_2$ and $x_1 = x_3$.
The graphs below are cases where the solutions are six periodic.

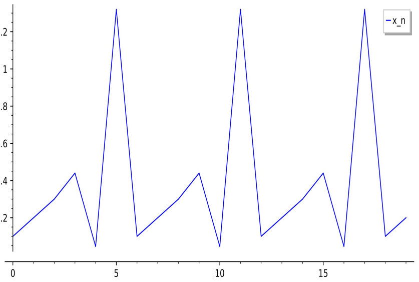

Figure 1: $a = 1, b = 0, x_0 = 0.1, x_1 = 0.2, x_2 = 0.3, x_3 = 0.44$.

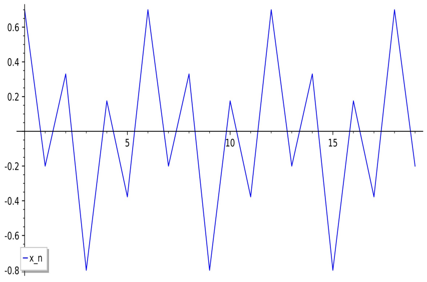

Figure 2: $a = 1, b = 0, x_0 = 0.7, x_1 = -0.2, x_2 = 0.33, x_3 = -0.8$.

2.5 Existence of 12-periodic solutions

Using (27), we have that if $a = -1$ and $b = 0$, then the solution of (1) is periodic with period twelve.
The graphs below are cases where the solutions are twelve periodic.

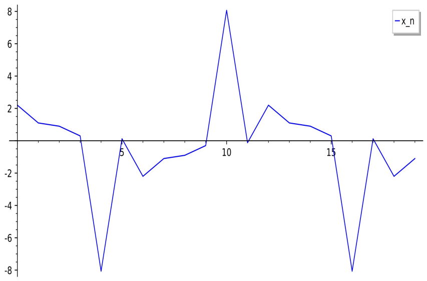

Figure 3: $a = -1, b = 0, x_0 = 2.2, x_1 = 1.1, x_2 = 0.9, x_3 = 0.3$.

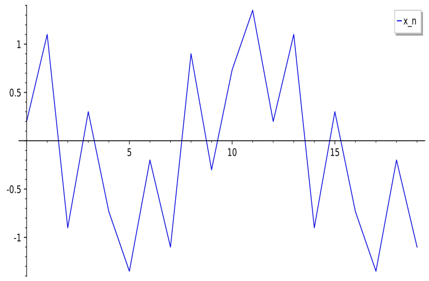

Figure 4: $a = -1, b = 0, x_0 = 0.2, x_1 = 1.1, x_2 = -0.9, x_3 = 0.3$.

3 Asymptotic behavior of the solutions for constant coefficients

Theorem 1 *Let $\{x_n\}_{n\in\mathbb{N}}$ be the solution to the sequence in* (1) *where $a_n = 1$ for all $n \geq 0$ and $b_n = b \neq 0$. Then*

$$\lim_{n\to\infty} x_n = 0.$$

Proof 1 *Using* (26)*, we have that*

$$\begin{aligned}x_{6n} =& x_0 \prod_{i=0}^{n-1} \frac{1+2ibx_0x_1}{1+2ibx_1x_2} \frac{1+2ibx_2x_3}{1+(2i+1)bx_0x_1} \frac{1+(2i+1)bx_1x_2}{1+(2i+1)bx_2x_3}\\ =& x_0 \prod_{i=0}^{n-1} \frac{1+2ibx_0x_1}{1+(2i+1)bx_0x_1} \frac{1+2ibx_2x_3}{1+(2i+1)bx_2x_3} \frac{1+(2i+1)bx_1x_2}{1+(2i)bx_1x_2}\\ =& x_0 \prod_{i=0}^{n-1} \left(1+\frac{bx_0x_1}{1+2ibx_0x_1}\right)^{-1} \left(1+\frac{bx_2x_3}{1+2ibx_2x_3}\right)^{-1} \left(1+\frac{bx_1x_2}{1+2ibx_1x_2}\right)\end{aligned}$$

We know that $1 + 2ix_kx_{k+1} \to \infty$ as $i \to \infty$. Hence, there is a sufficiently large integer t such that for $i \geq t$, we have

$$1 + 2ix_kx_{k+1} \sim 2ix_kx_{k+1}.$$

Thus

$$\begin{aligned} x_{6n} =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\left(1+\frac{1}{2i}\right)^{-1}\left(1+\frac{1}{2i}\right)^{-1}\left(1+\frac{1}{2i}\right) \\ =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\exp\left[\ln\left(1+\frac{1}{2i}\right)^{-1}+\ln\left(1+\frac{1}{2i}\right)^{-1}+\ln\left(1+\frac{1}{2i}\right)\right], \end{aligned}$$

where

$$\Gamma(t)=\prod_{i=0}^{t}\left(1+\frac{bx_0x_1}{1+2ibx_0x_1}\right)^{-1}\left(1+\frac{bx_2x_3}{1+2ibx_2x_3}\right)^{-1}\left(1+\frac{bx_1x_2}{1+2ibx_1x_2}\right).$$

Utilizing the expansion $\ln(1+x)=x+O(x^2), (1+x)^{-1}=1-x+O(x^2)$, *for* $x\to 0$, *we obtain*

$$\begin{aligned} x_{6n} =& x_0\Gamma(t)\prod_{i=t+1}^{n-1}\exp\left[-\frac{1}{2i}+O\left(\frac{1}{i^2}\right)\right] \\ =& x_0\Gamma(t)\exp\left[-\sum_{i=t+1}^{n-1}\left(\frac{1}{2i}\right)\right]\prod_{i=t+1}^{n-1}\exp\left[O\left(\frac{1}{i^2}\right)\right]. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} x_{6n}=0 \quad as \quad n\to\infty.$$

Similarly,

$$\lim_{n\to\infty} x_{6n+j}=0 \quad as \quad n\to\infty,$$

for $j=1,2,3,4,5$.

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GENERALIZED ZWEIER $\mathcal{I}$-CONVERGENT SEQUENCE SPACES OF FUZZY NUMBERS

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Abstract. In the present paper we introduce Zweier ideal convergent sequences spaces of fuzzy numbers by using lacunary sequence, infinite matrix and generalized difference matrix operator A_i^p. We study some topological and algebraic properties of these sequence spaces. Some inclusion relations related to these spaces are also establish.

1. Introduction and Preliminaries

Initially the idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al.[10]. Gurdal [7] studied the ideal convergence sequences in 2-normed spaces. Later on, it was further studied by Savas [21], Savas and Hazarika [8], Tripathy and Dutta [25], Tripathy and Hazarika [26], Raj et al.[17]. Let X be a non-empty set, then a family of sets $\mathcal{I} \subset 2^X$ is called an ideal iff for each $X_1, X_2 \in \mathcal{I}$, we have $X_1 \cup X_2 \in \mathcal{I}$ and for each $X_1 \in \mathcal{I}$ and each $X_2 \subset X_1$, we have $X_2 \in \mathcal{I}$. A non-empty family of sets $U \subset 2^X$ is a filter on X iff $\phi \notin U$, for each $X_1, X_2 \in U$, we have $X_1 \cap X_2 \in U$ and each $X_1 \in U$ and each $X_1 \subset X_2$, we have $X_2 \in U$. An ideal $\mathcal{I}$ is said to be non-trivial ideal if $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$. Clearly, $\mathcal{I} \subset 2^X$ is a non-trivial ideal iff $U = U(\mathcal{I}) = \{X - X_1 : X_1 \in \mathcal{I}\}$ is a filter on X. A non-trivial ideal $\mathcal{I} \subset 2^X$ is said to be admissible iff $\{x : x \in X\} \subset \mathcal{I}$. A non-trivial ideal is called maximal if there cannot exists any non-trivial ideal $\mathcal{J} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset.

A sequence $x = (x_k)$ of points in $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to a real number x_0 if

$$\{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\} \in \mathcal{I},$$

for every $\epsilon > 0$ (see [10]). We denote it by $\mathcal{I} - \lim x_k = x_0$.

Kızmaz [9] introduced the notion of difference sequence spaces and studied $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. Further this notion generalized by Et and Çolak [5] by introducing the spaces $l_\infty(\Delta^i)$, $c(\Delta^i)$ and $c_0(\Delta^i)$. The new type of generalization of the difference sequence spaces was introduced by Tripathy and Esi [27] who studied the spaces $l_\infty(\Delta_v^i)$, $c(\Delta_v^i)$ and $c_0(\Delta_v^i)$. Let i, v be non-negative integers, then for $Z = l_\infty,\ c,\ c_0$ we have sequence spaces

$$Z(\Delta_v^i) = \{x = (x_k) \in w : (\Delta_v^i x_k) \in Z\},$$

where $\Delta_v^i x = (\Delta_v^i x_k) = (\Delta_v^{i-1} x_k - \Delta_v^{i-1} x_{k+1})$ and $\Delta_v^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_v^i x_k = \sum_{n=0}^{i} (-1)^n \begin{pmatrix} i \\ n \end{pmatrix} x_{k+vn}.$$

Başar and Atlay [2] introduced and studied the generalized difference matrix $A(m,n) =$

2010 *Mathematics Subject Classification.* 40A05, 40A30.

Key words and phrases. Musielak-Orlicz function, ideal convergence, generalized difference matrix operator, fuzzy real number.

$(a_{rs}(m,n))$ which is a generalization of $\Delta^1_{(1)}$-difference operator as follows:

$$a_{rs}(m,n) = \begin{cases} m, & (s = r); \\ n, & (s = r-1); \\ 0, & 0 \le s \le r-1 \text{ or } s > r. \end{cases}$$

for all $r, s \in \mathbb{N}$ and $m, n \in \mathbb{R} - \{0\}$.
Başarir and Kayikçi [3] introduced the generalized difference matrix A^p of order p and the binomial representation of this operator is

$$A^p(x_k) = \sum_{v=0}^{p} \binom{p}{v} m^{p-v} n^v x_{k-v},$$

where $m, n \in \mathbb{R} - \{0\}$ and $r \in \mathbb{N}$.
Recently, Başarir et al.[4] studied the following generalized difference sequence spaces

$$Z(A^p_i) = \{x = (x_k) \in w : (A^p_i x_k) \in Z\},$$

for $Z = l_\infty,\ \bar{c},\ \bar{c_0}$, where $\bar{c}, \bar{c_0}$ are the sets of statistically convergent and statistically null convergent respectively and the binomial representation of operator A^p_i is as follows:

$$A^p_i(x_k) = \sum_{v=0}^{p} \binom{p}{v} m^{p-v} n^v x_{k-iv}.$$

Şengönül [22] defined the sequence $y = (y_k)$ which is frequently used as the $Z-$transformation of the sequence $x = (x_k)$ that is,

$$y_k = \beta x_k + (1-\beta) x_{k-1},$$

where $x_{-1} = 0,\ k \ne 0,\ 1 < k < \infty$ and Z denotes the matrix $Z = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} \beta, & (i = k); \\ 1-\beta, & (i-1 = k)(i, k \in \mathbb{N}); \\ 0, & \text{otherwise}. \end{cases}$$

Şengönül [22] introduced the Zweier sequence spaces $\mathscr{Z}$ and $\mathscr{Z}_0$ as follows:

$$\mathscr{Z} = \{x = (x_k) \in w : Z(x) \in c\}$$

and

$$\mathscr{Z}_0 = \{x = (x_k) \in w : Z(x) \in c_0\}.$$

An Orlicz function $M : [0,\infty) \to [0,\infty)$ is convex, continuous and non-decreasing function which also satisfy $M(0) = 0,\ M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Lindenstrauss and Tzafriri [11] used the idea of Orlicz function to define the following sequence space:

$$\ell_M = \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

which is called as an Orlicz sequence space. An Orlicz function is said to satisfy Δ_2-condition if for a constant R, $M(Qx) \le RQM(x)$ for all values of $x \ge 0$ and for $Q > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called as Musielak-Orlicz function.To know more about sequence spaces see ([1], [15], [16], [24], [18], [19] and [28]) and references therein.
An increasing non-negative integer sequence $\theta = (k_r)$ with $k_0 = 0$ and $k_r - k_{r-1} \to \infty$ as $r \to \infty$ is known as lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$. We write $h_r = k_r - k_{r-1}$ and q_r denotes the ratio $\frac{k_r}{k_{r-1}}$. The space of

lacunary strongly convergent sequence was defined by Freedman et al. [6] as follows:

$$N_\theta = \Big\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0, \text{ for some } L\Big\}.$$

The space N_θ is a $BK-$ space with the norm

$$\|x\| = \sup\left(\frac{1}{h_r} \sum_{k\in I_r} |x_k|\right).$$

Let $\lambda = (\lambda_{nk})$ be an infinite matrix of real or complex numbers λ_{nk}, where $n, k \in$ N. Then a matrix transformation of $x = (x_k)$ is denoted as λx and $\lambda x = \lambda_n(x))$ if $\lambda_n(x) = \sum_{k=1}^{\infty} \lambda_{nk} x_k$ converges for each $n \in$ N.

The concept of fuzzy numbers and arithmetic operations with these numbers were first introduced and investigated by Zadeh [29] in 1965. Subsequently many authors have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events and fuzzy mathematical programming. The theory of sequences of fuzzy numbers was first studied by Matloka [12]. He studied some of their properties and showed that every convergent sequences of fuzzy numbers is bounded. Later on Nanda [13] introduced sequences of fuzzy numbers and studied that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Further, the theory of sequences of fuzzy numbers have been discussed by Savas and Mursaleen [20], Tripathy and Nanda [23], Hazarika and Savas [8] and many more.

Let B denote the set of all closed bounded intervals $U = [u_1, u_2]$ on the real line $\mathbb{R}$. For $U, V \in B$, we define $U \leq V$ iff $u_1 \leq v_1$ and $u_2 \leq v_2$ and we define

$$d(U, V) = \max\{|u_1 - v_1|, |u_2, v_2|\}.$$

It is well known that d defines a metric on B and (B, d) is a complete metric space (see [14]).

A fuzzy number is a function $U : \mathbb{R} \to [0, 1]$, which satisfy the following conditions:

(i) U is normal i.e there exits an x_0 such that $U(x_0) = 1$,

(ii) U is convex i.e for $x, y \in \mathbb{R}$ and $0 \leq \tau \leq 1$,

$$U(\tau x + (1 - \tau)y) \geq \min\{U(x), U(y)\},$$

(iii) U is upper semi-continuous,

(iv) the closure of the set supp(U) is compact, where supp$(U) = \{x \in \mathbb{R} : U(x) > 0\}$ and it is denoted by $[U]^0$.

The set of all fuzzy numbers are denoted by $\mathbb{R}_{\mathbb{F}}$. Let $[U]^0 = \overline{x \in \mathbb{R} : u(x) > 0}$ and the r-level set is $[U]^r = \{x \in \mathbb{R} : u(x) \geq r\}, (0 \leq r \leq 1)$. The set $[U]^r$ is a closed and bounded interval of $\mathbb{R}$. For any $U, V \in \mathbb{R}_{\mathbb{F}}$ and $\lambda \in \mathbb{R}$, it is positive to define uniquely the sum $U \oplus V$ and the product $U \odot V$ as follows:

$$[U \oplus V]^r = [U]^r + [V]^r \text{ and } [\lambda \odot U]^r = \lambda[U]^r.$$

Now, denote the interval $[U]^r$ by $[u_1^{(r)}, u_2^{(r)}]$, where $u_1^{(r)} \leq u_2^{(r)}$ and $u_1^{(r)}, u_2^{(r)} \in \mathbb{R}$, for $r \in [0, 1]$.

Now, define $\hat{d} : \mathbb{R}_{\mathbb{F}} \times \mathbb{R}_{\mathbb{F}} \to \mathbb{R}$ by

$$\hat{d}(U, V) = \sup_{r\in[0,1]} d([U]^r, [V]^r).$$

Definition 1.1. A sequence $x = (x_k)$ of fuzzy numbers is said to be convergent to a fuzzy number x_0 if for every $\epsilon > 0$ there exist a positive integer n_0 such that

$$\hat{d}(x_k, x_0) < \epsilon, \text{ for } k > n_0.$$

Definition 1.2. A sequence $x = (x_k)$ of fuzzy numbers is said to be $\mathcal{I}$- convergent to a fuzzy number x_0 if for every $\epsilon > 0$ such that

$$\{k \in \mathbb{N} : \hat{d}(x_k, x_0) \geq \epsilon\} \in \mathcal{I}.$$

Throughout the article, we denote Zweier fuzzy number sequence $Z(x)$ by x' for $x \in \omega^F$.

Let $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ be a bounded sequence of positive real numbers, $\lambda = (\lambda_{nk})$ be an infinite matrix, θ be a lacunary sequence and ω^F is the set of all sequences of fuzzy real numbers. In the present paper we define lacunary Zweier $\mathcal{I}-$convergent, lacunary Zweier $\mathcal{I}-$null and lacunary Zweier $\mathcal{I}-$bounded sequence spaces of fuzzy numbers as follows:

$$\mathcal{Z}^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Bigg\{ x = (x_k) \in \omega^F : \Bigg\{ n \in \mathbb{N} : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Bigg[M_k \Bigg(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho} \Bigg) \Bigg]^{q_k} \geq \epsilon \Bigg\} \in \mathcal{I} \text{ for some } \rho > 0 \text{ and } x_0 \in \mathbb{R}_{\mathbb{F}} \Bigg\},$$

$$\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Bigg\{ x = (x_k) \in \omega^F : \Bigg\{ n \in \mathbb{N} : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Bigg[M_k \Bigg(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \Bigg) \Bigg]^{q_k} \geq \epsilon \Bigg\} \in \mathcal{I} \text{ for some } \rho > 0 \Bigg\}$$

and

$$\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q] = \Bigg\{ x = (x_k) \in \omega^F : \exists K > 0 \text{ s.t. } \Bigg\{ n \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \Bigg[M_k \Bigg(\frac{\hat{d}(A_i^p x_k', \bar{0})}{\rho} \Bigg) \Bigg]^{q_k} \geq K \Bigg\} \in \mathcal{I} \text{ for some } \rho > 0 \Bigg\},$$

where,

$$\bar{0}(t) = \begin{cases} 1, & if \;\; t = 0; \\ 0, & \text{otherwise.} \end{cases}$$

If $0 < q_k \leq \sup q_k = D, C = \max(1, 2^{D-1})$. Then

$$|c_k + d_k|^{q_k} \leq C(|c_k|^{q_k} + |d_k|^{q_k}), \tag{1.1}$$

for all $c_k, d_k \in \mathbb{R}$ and for all $k \in \mathbb{N}$.

The main purpose of this paper is to study some classes of lacunary Zweier sequences of fuzzy numbers defined by means of generalized difference matrix operator, Musielak-Orlicz function and infinite matrix. We shall make an effort to study some interesting algebraic and topological properties of concerning sequence spaces. Also, we examine some interrelations between these sequence spaces.

2. Main Results

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ be a bounded sequence of positive real numbers and θ be a lacunary sequence. Then the sequence spaces $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$, $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ and $\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ are closed under addition and scalar multiplication.*

Proof. Consider $x = (x_k), y = (y_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ and α, β are scalars. Then there exist positive numbers $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \in \mathcal{I}$$

and

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y_k', y_0)}{\rho_2}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \in \mathcal{I}.$$

Since A_i^p is linear and by using the continuity of Musielak-Orlicz function $\mathcal{M}$, we have the following inequality:

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p(\alpha(x_k') + \beta(y_k')))}{|\alpha|\rho_1 + |\beta|\rho_2}\right)\right]^{q_k} \\
&\quad\leq D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[\frac{|\alpha|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1}\right)\right]^{q_k} \\
&\quad+ D \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[\frac{|\beta|}{|\alpha|\rho_1 + |\beta|\rho_2} M_k\left(\frac{\hat{d}(A_i^p y_k', y_0)}{\rho_2}\right)\right]^{q_k} \\
&\quad\leq DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1}\right)\right]^{q_k} \\
&\quad+ DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y_k', y_0)}{\rho_2}\right)\right]^{q_k},
\end{aligned}$$

where $K = \max\left\{1, \frac{|\alpha|\rho_1}{|\alpha|\rho_1+|\beta|\rho_2}, \frac{|\beta|\rho_2}{|\alpha|\rho_1+|\beta|\rho_2}\right\}$.

Thus, we have

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p(\alpha(x_k') + \beta(y_k')))}{|\alpha|\rho_1 + |\beta|\rho_2}\right)\right]^{q_k} \geq \epsilon\right\} \\
&\quad\subseteq \left\{n \in \mathbb{N} : DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho_1}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\} \\
&\quad\cup \left\{n \in \mathbb{N} : DK \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \left[M_k\left(\frac{\hat{d}(A_i^p y_k', y_0)}{\rho_2}\right)\right]^{q_k} \geq \frac{\epsilon}{2}\right\}.
\end{aligned}$$

Since the sets on right hand side of above relation belong to $\mathcal{I}$. Thus, the sequence space $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P, \theta, \lambda, \mathcal{M}, q]$ is closed under addition and scalar multiplication . Similarly, we can prove others. □

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $q = (q_k)$ and $v = (v_k)$ be two bounded sequences of positive real numbers with $0 < q_k \leq v_k$ for each k and $(\frac{v_k}{q_k})$ be bounded. Then*

(i) $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$,
(ii) $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$,
(iii)$\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},v],\subseteq \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$.

Proof. The proof of the theorem is straightforward, so we omit it. □

Theorem 2.3. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function and $q=(q_k)$ be a bounded sequence of positive numbers. Then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q],\subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}$ $[A_i^P,\theta,\lambda,\mathcal{M},q]$.*

Proof. We know that the first inclusion is obvious. Next, we show that $\mathcal{Z}^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^P,\theta,\lambda,\mathcal{M},q]$. Let $(x_k)\in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]$. Then we have

$$
\begin{aligned}
\tfrac{1}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k} & \\
\leq\ & \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',x_0)}{\rho}\Big)\Big]^{q_k}\\
+\ & \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(x_0,\bar{0})}{\rho}\Big)\Big]^{q_k}\\
\leq\ & \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',x_0)}{\rho}\Big)\Big]^{q_k}\\
+\ & C\max\Big\{1,\sup\Big(\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(x_0,\bar{0})}{\rho}\Big)\Big]\Big)^D\Big\},
\end{aligned}
$$

where $\sup q_k = D$ and $C=\max(1,2^{D-1})$. Therefore, $(x_k)\in \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]$. This completes the proof of the theorem. □

Theorem 2.4. *Let $\mathcal{M}=(M_k)$ and $\mathcal{M}'=(M_k')$ be two Musielak-Orlicz functions. Then the folowing inclusions holds:*
(i) $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$,
(ii) $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$,
(iii) $\mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]\subset \mathcal{Z}_\infty^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}+\mathcal{M}',q]$.

Proof. Suppose $(x_k)\in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M},q]\bigcap \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p,\theta,\lambda,\mathcal{M}',q]$. Then, we have

$$
\begin{aligned}
\lambda_{nk}\Big[(M_k+M_k')\Big(\tfrac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k} & \\
\leq\ & C\Big[\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k}+C\Big[\lambda_{nk}\Big[M_k'\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k},
\end{aligned}
$$

which consequently implies that

$$
\begin{aligned}
\tfrac{1}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[(M_k+M_k')\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k} & \\
\leq\ & \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k}\\
+\ & \frac{C}{h_r}\sum_{k\in I_r}\lambda_{nk}\Big[M_k'\Big(\frac{\hat{d}(A_i^p x_k',\bar{0})}{\rho}\Big)\Big]^{q_k}.
\end{aligned}
$$

This implies $(x_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M} + \mathcal{M}', q]$. We can prove the other cases in the same way. □

Theorem 2.5. *Let* $\mathcal{M} = (M_k)$ *and* $\mathcal{M}' = (M'_k)$ *be two Musielak-Orlicz functions. Then the folowing inclusion holds:*

$$\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}', q] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}.\mathcal{M}', q].$$

Proof. For given $\epsilon > 0$ and choose ϵ_0 such that $\sup_n \left(\sum_{k \in I_r} \lambda_{nk} \right) \max\{\epsilon_0^h, \epsilon_0^D\} < \epsilon$. Choose $0 < \varphi < 1$ such that $M_k(t) < \epsilon_0$, for all $k \in \mathbb{N}$. Let $x = (x_k) \in \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}', q]$. Then for some $\rho > 0$, we have

$$B_1 = \left\{ n \in \mathbb{N} : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M'_k \left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho} \right) \right]^{q_k} \geq \varphi^D \right\} \in \mathcal{I}.$$

If $n \notin B_1$, then we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M'_k \left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho} \right) \right]^{q_k} < \varphi^D.$$

This implies

$$\left[M'_k \left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho} \right) \right]^{q_k} < \varphi^D \quad \text{for all } k \in \mathbb{N}.$$

Hence,

$$M'_k \left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho} \right)^{q_k} < \varphi \quad \text{for all } k \in \mathbb{N}.$$

Therefore,

$$M_k \left(M'_k \left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho} \right) \right)^{q_k} < \epsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Thus, we get

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k \left(M'_k \left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho} \right) \right) \right]^{q_k} < \sup_n \left(\sum_{k \in I_r} \lambda_{nk} \right) \max\{\epsilon_0^h, \epsilon_0^D\} < \epsilon.$$

Now, we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k \left(M'_k \left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho} \right) \right) \right]^{q_k} < \epsilon.$$

This implies

$$\left\{ n \in \mathbb{N} : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k \left(M'_k \left(\frac{\hat{d}(A_i^p x'_k, \bar{0})}{\rho} \right) \right) \right]^{q_k} \geq \epsilon \right\} \subset B_1 \in \mathcal{I}.$$

This completes the proof. □

Theorem 2.6. *If* $\lim q_k > 0$ *and* $x = (x_k) \to x_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$, *then* x_0 *is unique.*

Proof. Let $\lim q_k = u_0$. Consider that $(x_k) \to x_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$ and $(x_k) \to y_0(\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q])$. So, there exist $\rho_1, \rho_2 > 0$, such that

$$X_1 = \left\{ n \in \mathbb{N} : \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \lambda_{nk} \left[M_k \left(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho_1} \right) \right]^{q_k} \geq \frac{\epsilon}{2} \right\} \in \mathcal{I} \tag{2.1}$$

and

$$X_2 = \Big\{ n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, y_0)}{\rho_2} \Big) \Big]^{q_k} \geq \frac{\epsilon}{2} \Big\} \in \mathcal{I}. \tag{2.2}$$

Define $\rho = \max\{2\rho_1, 2\rho_2\}$. Then we have

$$\sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(x_0, y_0)}{\rho} \Big) \Big]^{q_k}$$

$$\leq \quad D \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho} \Big) \Big]^{q_k} + D \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, y_0)}{\rho} \Big) \Big]^{q_k}.$$

Then from (2.1) and (2.2), we have

$$\Big\{ n \in \mathbb{N} : \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(x_0, y_0)}{\rho} \Big) \Big]^{q_k} \geq \epsilon \Big\}$$

$$\begin{aligned} &\subseteq \quad \Big\{ n \in \mathbb{N} : D \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho_1} \Big) \Big]^{q_k} \geq \frac{\epsilon}{2} \Big\} \\ &\cup \quad \Big\{ n \in \mathbb{N} : D \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, y_0)}{\rho_2} \Big) \Big]^{q_k} \geq \frac{\epsilon}{2} \Big\} \\ &\subseteq \quad X_1 \cup X_2 \in \mathcal{I}. \end{aligned}$$

Also,

$$\Big[M_k\Big(\frac{\hat{d}(x_0, y_0)}{\rho} \Big) \Big]^{q_k} \to \Big[M_k\Big(\frac{\hat{d}(x_0, y_0)}{\rho} \Big) \Big]^{u_0} \quad \text{as } \ k \to \infty.$$

Then, we have

$$\lim_{k\to\infty} \Big[M_k\Big(\frac{\hat{d}(x_0, y_0)}{\rho} \Big) \Big]^{q_k} = \Big[M_k\Big(\frac{\hat{d}(x_0, y_0)}{\rho} \Big) \Big]^{u_0} = 0.$$

Thus, $x_0 = y_0$. ☐

Theorem 2.7. *Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $q = (q_k)$ be a bounded sequence of positive real numbers,*
(a) If $0 < \inf q_k \leq q_k \leq 1$ for all k, then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$ and $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q] \subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$.
(b) If $1 \leq q_k \leq \sup q_k = D < \infty$ for all k, then $\mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}] \subseteq \mathcal{Z}_0^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$ and $\mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}] \subseteq \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$.

Proof. (a) Suppose $(x_k) \in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}, q]$. Since $0 < \inf q_k \leq q_k \leq 1$, then we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho} \Big) \Big]$$

$$\leq \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk} \Big[M_k\Big(\frac{\hat{d}(A_i^p x'_k, x_0)}{\rho} \Big) \Big]^{q_k}.$$

Thus,

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho}\right)\right] \geq \epsilon\right\}$$

$$\subseteq \quad \left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho}\right)\right]^{q_k} \geq \epsilon\right\} \in \mathcal{I}.$$

The other part can be proved in the same way.

(ii) Suppose $(x_k) \in \mathcal{Z}^{\mathcal{I}(F)}[A_i^p, \theta, \lambda, \mathcal{M}]$. Since $1 \leq q_k \leq \sup q_k = D < \infty$. Then for each $0 < \epsilon < 1$, there exists a positive integer m_0 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho}\right)\right] \leq \epsilon < 1,$$

for all $n \geq m_0$. This implies

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho}\right)\right]^{q_k}$$

$$\leq \quad \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho}\right)\right].$$

Thus,

$$\left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho}\right)\right]^{q_k} \geq \epsilon\right\}$$

$$\subseteq \quad \left\{n \in \mathbb{N} : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} \lambda_{nk}\left[M_k\left(\frac{\hat{d}(A_i^p x_k', x_0)}{\rho}\right)\right] \geq \epsilon\right\} \in \mathcal{I}.$$

The other part can be proved in the same way. ☐

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Some convergence results using K^* iteration process in $CAT(0)$ spaces

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Abstract. In this paper, some strong and Δ-convergence results for Suzuki generalized nonexpansive mappings in the setting of complete $CAT(0)$ spaces are proved. We are using newly introduced K^* iteration process for approximation of fixed point. We also give an example to show the efficiency of the K^* iteration process. Our results are extension, improvement and generalization of many well known results in the literature of fixed point theory in $CAT(0)$ spaces.

Mathematics Subject Classification (2010). Primary 47H09, 47H10.

Keywords. Suzuki generalized nonexpansive mapping; $CAT(0)$ space; K^* iterative process; Δ-convergence; strong convergence.

1. Introduction

It is well-known that several mathematics problems are naturally formulated as fixed point problem $Tx = x$, where T is some suitable mapping, may be nonlinear. For example, for given functions $\zeta : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ and $\xi : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$, the solution of following nonlinear integral equation

$$x(c) = \zeta(c) + \int_a^b \xi(c, r, x(r))dr,$$

where $x \in C[a, b]$ (the set of all continuous real-valued functions defined on $[a, b] \subseteq \mathbb{R}$), is equivalently to fixed point problems for the following mapping $T : C[a, b] \to C[a, b]$ defined by

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This work was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2017R1D1A1B04032937). We would like to thank Prof. Balwant Singh Thakur for technical assistance.

$$(Tx)(c) = \zeta(c) + \int_a^b \xi(c, r, x(r))dr$$

for all $x \in C[a, b]$.

The well-known Banach contraction theorem uses the Picard iteration process for approximation of fixed point. Many iterative processes have been developed to approximate fixed points of contraction type of mapping in $CAT(0)$ type spaces of ground spaces. Some of the other well-known iterative processes are those of Mann [17], Ishikawa [10], Noor [8], Abbas [1], Agarwal [2], Phuengrattana and Suantai [19], Karahan and Ozdemir [11], Chugh, Kumar and Kumar [6], Sahu and Petrusel [20], Khan [14], Gursoy and Karakaya [9], Thakur, Thakur and Postolache [22] and so on. See also [13, 23, 25] for more information on $CAT(0)$ spaces and applications. Recently, Ullah and Arshad [24] introduced a new three steps iteration process as the K^* iteration process and proved that it is strong and converges fast as compared to all above mentioned iteration processes. They use uniformly convex Banach space as a ground space.

Motivated by above, in this paper, first we develop an example of Suzuki generalized nonexpansive mappings is given which is not nonexpansive. We compare the speed of convergence of the K^* iteration process with the leading two steps S-iteration process and leading three steps Picard-S-iteration process for Suzuki generalized nonexpansive mappings, and graphic representation is also given.

Finally, we prove some strong and Δ-convergence theorems for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ spaces.

2. Preliminaries

Let (X, d) be a metric space. A geodesic from x to y in X is a mapping c from closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t^{'})) = |t - t^{'}|$ for all $t, t^{'} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image of c is called a geodesic (or metric) segment joining x and y. The space (X, d) is said to be a geodesic space if every two points of X is joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each x, $y \in X$, which we denote by $[x, y]$, called the segment joining x to y.

A geodesic triangle $\Delta(x_{1,}x_{2,}x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the triangle $\Delta(x_{1,}x_{2,}x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_{1,}x_{2,}x_3) := \Delta(\bar{x}_{1,}\bar{x}_{2,}\bar{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said be a $CAT(0)$ space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

$CAT(0)$: Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the $CAT(0)$ inequality if for $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x,y) \leq d_{E^2}(\bar{x}, \bar{y}).$$

If $x, y_{1,} y_2$ are points in $CAT(0)$ space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the $CAT(0)$ inequality implies

$$d(x,y_0)^2 \leq \frac{1}{2}d(x,y_1)^2 + \frac{1}{2}d(x,y_2)^2 - \frac{1}{4}d(y_1,y_2)^2. \tag{CN}$$

This is the (CN) inequality of Burhat and Tits [5].

We recall the following result from Dhompongsa and Panyanak [8].

Lemma 2.1. ([8]) *For $x, y \in X$ and $\alpha \in [0,1]$, there exists a unique point $z \in [x,\, y]$ such that*

$$d(x,z) = \alpha d(x,y) \text{ and } d(y,z) = (1-\alpha)d(x,y). \tag{2.1}$$

The notation $((1-\alpha)x \oplus \alpha y)$ is used for the unique point z satisfying (2.1).

$CAT(0)$ space may be regarded as a metric version of Hilbert space. For example, in $CAT(0)$ space we have the following extended version of parallelogram law:

$$d(z, \alpha x \oplus (1-\alpha)y)^2 = \alpha d(x,z)^2 + (1-\alpha)d(z,y)^2 - \alpha(1-\alpha)d(x,y)^2 \tag{2.2}$$

for any $\alpha \in [0,1]$, $x, y \in X$.

If $\alpha = \frac{1}{2}$, then the inequality (2.2) becomes the (CN) inequality.

In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality (cf. [5]). Complete $CAT(0)$ spaces are often called Hadmard spaces. For more on these spaces, please refer to [3, 4].

Lemma 2.2. ([14, Lemma 2.4]) *For $x, y, z \in X$ and $\alpha \in [0,1]$, we have*

$$d(z, \alpha x \oplus (1-\alpha)y) \leq \alpha d(z,x) + (1-\alpha)d(z,y).$$

Let C be a nonempty closed convex subset of a $CAT(0)$ space X let $\{x_n\}$ be a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius of $\{x_n\}$ relative to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}$$

and the asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}.$$

It is well known that, in a complete $CAT(0)$ space, $A(C, \{x_n\})$ consists of exactly one point.

We now recall the definition of Δ-convergence in $CAT(0)$ space.

Definition 2.3. A sequence $\{x_n\}$ in a $CAT(0)$ space X is said to be Δ-convergent to $x \in X$ if x is the unique asymptotic center of $\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$.

In this case, we write $\Delta\text{-}\lim_n x_n = x$ and call x the Δ-lim of $\{x_n\}$.

Recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r\{u_x\}$ for every subsequence $\{u_x\}$ of $\{x_n\}$.

Since in a $CAT(0)$ space every regular sequence Δ-converges, we see that every bounded sequence in X has a Δ-convergent subsequence.

A $CAT(0)$ space X is said to satisfy the Opial′s property [17] if for each sequence $\{x_n\}$ in X, Δ-converges to $x \in X$, we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y)$$

for all $y \in X$ such that $y \neq x$.

Definition 2.4. A point p is called a fixed point of a mapping T if $T(p) = p$ and $F(T)$ represents the set of all fixed points of the mapping T.

Definition 2.5. Let C be a nonempty subset of a $CAT(0)$ space X.

(i) A mapping $T : C \to C$ is called a contraction if there exists $\alpha \in (0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y)$$

for all $x, y \in C$.

(ii) A mapping $T : C \to C$ is called nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in C$.

(iii) A mapping is a quasi-nonexpansive if for all $x \in C$ and $p \in F(T)$, we have

$$d(Tx, p) \leq d(x, p).$$

In 2008, Suzuki [21] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called condition (C). A mapping $T : C \to C$ is said to satisfy condition (C) if for all $x, y \in C$, we have

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq d(x, y).$$

Suzuki [21] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness. The mapping satisfying condition (C) is called a Suzuki generalized nonexpansive mapping.

Suzuki [21] obtained fixed point theorems and convergence theorems for Suzuki generalized nonexpansive mapping. In 2011, Phuengrattana [18] proved convergence theorems for Suzuki generalized nonexpansive mappings using the Ishikawa iteration in uniformly convex Banach spaces and $CAT(0)$ spaces. Recently, fixed point theorems for Suzuki generalized nonexpansive mapping have been studied by a number of authors, see, e.g., [22] and references therein.

The following are some basic properties of Suzuki generalized nonexpansive mappings whose proofs in the setup of $CAT(0)$ spaces follow the same lines as those of [12, Propostions 11, 14, 19] and therefore we omit them.

Proposition 2.6. *Let C be a nonempty subset of a $CAT(0)$ space X and $T : C \to C$ be any mapping.*

(i) [21, Proposition 1] *If T is nonexpansive, then T is a Suzuki generalized nonexpansive mapping.*

(ii) [21, Proposition 2] *If T is a Suzuki generalized nonexpansive mapping and has a fixed point, then T is a quasi-nonexpansive mapping.*

(iii) [21, Lemma 7] *If T is a Suzuki generalized nonexpansive mapping, then*

$$d(x, Ty) \leq 3d(Tx, x) + d(x, y)$$

for all $x, y \in C$.

Lemma 2.7. [21, Theorem 5] *Let C be a weakly compact convex subset of a $CAT(0)$ space X. Let T be a mapping on C. Assume that T is a Suzuki generalized nonexpansive mapping. Then T has a fixed point.*

Lemma 2.8. [16, Lemma 2.9] *Suppose that X is a complete $CAT(0)$ space and $x \in X$. If $\{t_n\}$ is a sequence in $[b, c]$ for some $b, c \in (0, 1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that for some $r \geq 0$, we have*

$$\begin{aligned} \lim_{n\to\infty} \sup d(x_n, x) &\leq r, \\ \lim_{n\to\infty} \sup d(y_n, x) &\leq r, \\ \lim_{n\to\infty} \sup d(t_n x_n + (1 - t_n)y_n, x) &= r, \end{aligned}$$

then

$$\lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 2.9. [7, Proposition 2.1] *If C is a closed comvex subset of a complete $CAT(0)$ space X and if $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ is in C.*

Lemma 2.10. [15] *Every bounded sequence in a complete $CAT(0)$ space always has a Δ-convergent subsequence.*

Lemma 2.11. [15, Proposition 3.7] *Let C is a closed comvex subset of a complete $CAT(0)$ space X and $T : C \to X$ be a Suzuki generalized nonexpansive mapping. Then the conditions $\{x_n\}$ Δ-converges to x and $d(Tx_n, x_n) \to 0$ imply $x \in C$ and $Tx = x$.*

The following is an example of Suzuki generalized nonexpansive mapping which is not nonexpansive.

Example 1. Define a mapping $T : [0, 1] \to [0, 1]$ by

$$Tx = \begin{cases} 1 - x \text{ if } x \in \left[0, \frac{1}{6}\right) \\ \frac{x+5}{6} \text{ if } x \in \left[\frac{1}{6}, 1\right]. \end{cases}$$

We need to prove that T is a Suzuki generalized nonexpansive but not nonexpansive.

If $x=\frac{15}{96}$ and $y=\frac{1}{6}$, then we have

$$\begin{aligned} d(Tx,Ty) &= |Tx-Ty| \\ &= \left|1-\frac{15}{96}-\frac{31}{36}\right| \\ &= \frac{5}{288} \\ &> \frac{1}{96} \\ &= d(x,y). \end{aligned}$$

Hence T is not a nonexpansive mapping.

To verify that T is a Suzuki generalized nonexpansive mapping, consider the following cases:

Case I: Let $x\in\left[0,\frac{1}{6}\right)$. Then $\frac{1}{2}d(\,x,Tx)=\frac{1-2x}{2}\in\left(\frac{1}{3},\frac{1}{2}\right]$. For $\frac{1}{2}d(x,Tx)\leq d(x,y)$, we have $\frac{1-2x}{2}\leq y-x$, i.e., $\frac{1}{2}\leq y$ and hence $y\in\left[\frac{1}{2},1\right]$. We have

$$d(Tx,Ty)=\left|\frac{y+5}{6}-(1-x)\right|=\left|\frac{y+6x-1}{6}\right|<\frac{1}{6}$$

and

$$d(x,y)=|x-y|>\left|\frac{1}{6}-\frac{1}{2}\right|=\frac{2}{6}.$$

Hence $\frac{1}{2}d(x,Tx)\leq d(x,y)\Longrightarrow d(Tx,Ty)\leq d(x,y)$.

Case II: Let $x\in\left[\frac{1}{6},1\right]$. Then $\frac{1}{2}d(\,x,Tx)=\frac{1}{2}\left|\frac{x+5}{6}-x\right|=\frac{5-5x}{12}\in\left[0,\frac{25}{72}\right]$. For $\frac{1}{2}d(x,Tx)\leq d(x,y)$, we have $\frac{5-5x}{12}\leq|y-x|$, which gives two possibilities:

(a) Let $x<y$. Then $\frac{5-5x}{12}\leq y-x\Longrightarrow y\geq\frac{5+7x}{12}\Longrightarrow y\in\left[\frac{37}{72},1\right]\subset\left[\frac{1}{6},1\right]$. So

$$d(Tx,Ty)=\left|\frac{x+5}{6}-\frac{y+5}{6}\right|=\frac{1}{6}d(x,y)\leq d(x,y).$$

Hence $\frac{1}{2}d(x,Tx)\leq d(x,y)\Longrightarrow d(Tx,Ty)\leq d(x,y)$.

(b) Let $x>y$. Then $\frac{5-5x}{12}\leq x-y\Longrightarrow y\leq x-\frac{5-5x}{12}=\frac{17x-5}{12}\Longrightarrow y\in\left[-\frac{13}{72},1\right]$. Since $y\in[0,1]$, $y\leq\frac{17x-5}{12}\Longrightarrow x\in\left[\frac{5}{12},1\right]$. So the case is $x\in\left[\frac{5}{12},1\right]$ and $y\in[0,1]$.

Now the case that $x\in\left[\frac{5}{12},1\right]$ and $y\in\left[\frac{1}{6},1\right]$ is the same case as that of (a). So let $x\in\left[\frac{5}{12},1\right]$ and $y\in\left[0,\frac{1}{6}\right)$. Then

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{x+5}{6}-(1-y)\right| \\ &= \left|\frac{x+6y-1}{6}\right|. \end{aligned}$$

For convenience, first we consider $x\in\left[\frac{5}{12},\frac{1}{2}\right]$ and $y\in\left[0,\frac{1}{6}\right)$. Then $d(Tx,Ty)\leq\frac{1}{12}$ and $d(x,y)\geq\frac{3}{12}$. Hence $d(Tx,Ty)\leq d(x,y)$.

TABLE 1. Some values produced by S, Picard-S and K^* IP

	K^*	Picard-S	S
x_0	0.9	0.9	0.9
x_1	0.99809713998382	0.99722222222222	0.98333333333333
x_2	0.99997729192914	0.99993300629392	0.99758822658104
x_3	0.99999985210113	0.99999849779947	0.99967552468466
x_4	0.99999999971662	0.99999996779523	0.99995826261755
x_5	1	0.99999999933035	0.99999479283092
x_6	1	0.99999999998638	0.99999936458953
x_7	1	0.99999999999973	0.99999992375668
x_8	1	0.99999999999999	0.99999999097156
x_9	1	1	0.99999999894221
x_{10}	1	1	0.99999999987715

TABLE 2. Some values produced by S, Picard-S and K^* IP

	K^*	Picard-S	S
x_0	0.5	0.5	0.5
x_1	0.99048569991909	0.99722222222222	0.98333333333333
x_2	0.99988645964572	0.99993300629392	0.99758822658104
x_3	0.99999926050565	0.99999926050566	0.99967552468466
x_4	0.99999999858311	0.99999996779523	0.99995826261755
x_5	1	0.99999999933035	0.99999479283092
x_6	1	0.99999999998638	0.99999936458953
x_7	1	0.99999999999973	0.99999992375668
x_8	1	0.99999999999999	0.99999999097156
x_9	1	1	0.99999999894221
x_{10}	1	1	0.99999999987715

Next consider $x \in \left[\frac{1}{2}, 1\right]$ and $y \in \left[0, \frac{1}{6}\right)$. Then $d(Tx, Ty) \leq \frac{1}{6}$ and $d(x, y) \geq \frac{2}{6}$. Hence $d(Tx, Ty) \leq d(x, y)$. So

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Longrightarrow d(Tx, Ty) \leq d(x, y).$$

Hence T is a Suzuki generalized nonexpansive mapping.

In order to show the efficiency of K^* iteration process, we use Example 1 with $x_0 = 0.9, x_0 = 0.5$ and get the above Tables 1 and 2. Graphic representation is given in Figure 1.

Let $n \geq 0$ and $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0, 1]$. Ullah and Arshad [24] introduced a new iteration process known as the K^* iteration process

$$\begin{cases} x_0 \in C \\ z_n = (1 - \beta_n)x_n + \beta_n T x_n \\ y_n = T((1 - \alpha_n)z_n + \alpha_n T z_n) \\ x_{n+1} = T y_n. \end{cases}$$

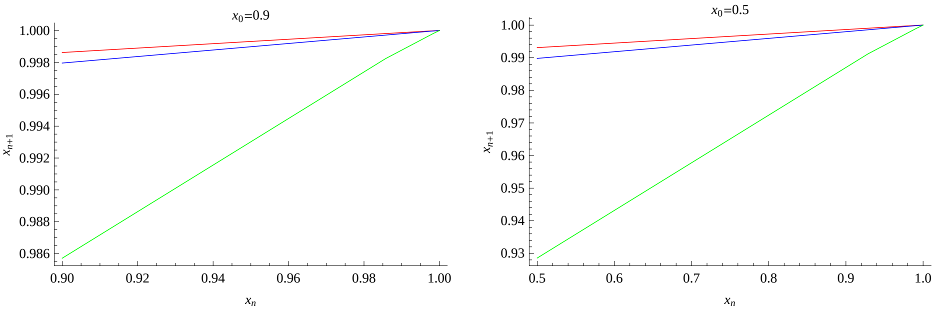

FIGURE 1. Convergence of iterative sequences generated by K^* (red line), Picard-S (blue line) and S (green line) iteration process to the fixed point 1 of the mapping T defined in Example 1.

They also proved that the K^* iteration process is faster than the Picard-S iteration and S-iteration processes with the help of a numerical example.

3. Convergence results for Suzuki generalized nonexpansive mappings

In this section, we prove some strong and Δ-convergence theorems of a sequence generated by a K^* iteration process for Suzuki generalized nonexpansive mappings in the setting of $CAT(0)$ space. The K^* iteration process in the language of $CAT(0)$ space is given by

$$\begin{array}{c} x_0 \in C \\ z_n = (1-\beta_n)x_n \oplus \beta_n T x_n \\ y_n = T((1-\alpha_n)z_n \oplus \alpha_n T z_n) \\ x_{n+1} = T y_n \end{array} . \tag{3.1}$$

Lemma 3.1. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. For arbitrarily chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1). Then $\lim_{n\to\infty} d(x_n, p)$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$ and $z \in C$. Since T is a Suzuki generalized nonexpansive mapping,

$$\frac{1}{2}d(p, Tp) = 0 \leq d(p, z) \text{ implies that } d(Tp, Tz) \leq d(p, z).$$

By Proposition 2.6 (ii), we have

$$\begin{aligned} d(z_n,p) &= d(((1-\beta_n)x_n \oplus \beta_n Tx_n),p) \\ &\leq (1-\beta_n)d(x_n,p)+\beta_n d(Tx_n,p) \\ &\leq (1-\beta_n)d(x_n,p)+\beta_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.2}$$

Using (3.2), we get

$$\begin{aligned} d(y_n,p) &= d((T(1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq d(((1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(z_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(x_n,p) \\ &= d(x_n,p). \end{aligned} \tag{3.3}$$

Similarly by using (3.3), we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\leq d(y_n,p) \\ &\leq d(x_n,p). \end{aligned}$$

This implies that $\{d(x_n,p)\}$ is bounded and nonincreasing for all $p \in F(T)$. Hence $\lim\limits_{n\to\infty} d(x_n,p)$ exists, as required. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a,b]$ for some a,b with $0 < a \leq b < 1$. Then $F(T) \neq \emptyset$ if and only if $\{x_n\}$ is bounded and $\lim\limits_{n\to\infty} d(Tx_n,x_n) = 0$.*

Proof. Suppose $F(T) \neq \emptyset$ and let $p \in F(T)$. Then, by Theorem 3.2, $\lim\limits_{n\to\infty} d(x_n,p)$ exists and $\{x_n\}$ is bounded. Put

$$\lim_{n\to\infty} d(x_n,p) = r. \tag{3.4}$$

From (3.2) and (3.4), we have

$$\limsup_{n\to\infty} d(z_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = r. \tag{3.5}$$

By Proposition 2.6 (ii) we have

$$\limsup_{n\to\infty} d(y_n,p) \leq \limsup_{n\to\infty} d(x_n,p) = r. \tag{3.6}$$

On the other hand, by using (3.2), we have

$$\begin{aligned} d(x_{n+1},p) &= d(Ty_n,p) \\ &\leq d(y_n,p) \\ &= d((T(1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq d(((1-\alpha_n)z_n \oplus \alpha_n Tz_n),p) \\ &\leq (1-\alpha_n)d(z_n,p)+\alpha_n d(Tz_n,p) \\ &\leq (1-\alpha_n)d(x_n,p)+\alpha_n d(z_n,p) \\ &= d(x_n,p)-\alpha_n d(x_n,p)+\alpha_n d(z_n,p). \end{aligned}$$

This implies that

$$\frac{d(x_{n+1},p)-d(x_n,p)}{\alpha_n} \leq d(z_n,p)-d(x_n,p).$$

So

$$d(x_{n+1},p)-d(x_n,p) \leq \frac{d(x_{n+1},p)-d(x_n,p)}{\alpha_n} \leq d(z_n,p)-d(x_n,p),$$

which implies that

$$d(x_{n+1},p) \leq d(z_n,p).$$

Therefore,

$$r \leq \liminf_{n\to\infty} d(z_n,p). \tag{3.7}$$

By (3.5) and (3.7), we get

$$\begin{aligned} r &= \lim_{n\to\infty} d(z_n,p) \\ &= \lim_{n\to\infty} d(((1-\beta_n)x_n+\beta_n Tx_n),p) \\ &= \lim_{n\to\infty} d(\beta_n(Tx_n,p)+(1-\beta_n)(x_n,p)). \end{aligned} \tag{3.8}$$

From (3.4), (3.6), (3.8) and Lemma 2.8, we have that $\lim_{n\to\infty} d(Tx_n,x_n)=0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n,x_n)=0$. Let $p\in A(C,\{x_n\})$. By Proposition 2.6 (iii), we have

$$\begin{aligned} r(Tp,\{x_n\}) &= \limsup_{n\to\infty} d(x_n,Tp) \\ &\leq \limsup_{n\to\infty} (3d(Tx_n,x_n)+d(x_n,p)) \\ &\leq \limsup_{n\to\infty} d(x_n,p) \\ &= r(p,\{x_n\}). \end{aligned}$$

This implies that $Tp\in A(C,\{x_n\})$. Since X is uniformly convex, $A(C,\{x_n\})$ is a singleton and hence we have $Tp=p$. So $F(T)\neq\emptyset$. □

Now we are in the position to prove Δ-convergence theorem.

Theorem 3.3. *Let C be a nonempty closed convex subset of a complete $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{t_n\}$ and $\{s_n\}$ be sequences in $[0, 1]$ such that $\{t_n\} \in [a, b]$ and $\{s_n\} \in [0, b]$ or $\{t_n\} \in [a, 1]$ and $\{s_n\} \in [a, b$ for some a, b with $0 < a \leq b < 1$. For an arbitrary element $x_1 \in C$, $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Since $F(T) \neq \emptyset$, by Theorem 3.3, we have that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. We now let $w_w\{x_n\} := \bigcup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $w_w\{x_n\} \subset F(T)$. Let $u \in w_w\{x_n\}$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.9 and 2.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n$ $\{v_n\} = v \in C$. Since $\lim_{n\to\infty} d(v_n, Tv_n) = 0$, $v \in F(T)$ by Lemma 2.11. We claim that $u = v$. Suppose not. Since T is a Suzuki generalized nonexpansive mapping and $v \in F(T)$, $\lim_n$ $d(x_n, v)$ exists by Theorem 3.2. Then by uniqueness of asymptotic centers,

$$\begin{aligned}
\lim_{n\to\infty} sup d(v_n, v) &< \lim_{n\to\infty} sup d(v_n, u)\\
&\leq \lim_{n\to\infty} sup d(u_n, u)\\
&< \lim_{n\to\infty} sup d(u_n, v)\\
&= \lim_{n\to\infty} sup d(x_n, v))\\
&= \lim_{n\to\infty} sup d(v_n, v),
\end{aligned}$$

which is a contradiction and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it is sufices to show that $w_w\{x_n\}$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 2.9 and 2.10, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n$ $\{v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that $c \in F(T)$. We can complete the proof by showing that $x = v$. Suppose not. Since $\{d(x_n, v)\}$ is convergent, by the uniqueness of asymptotic centers,

$$\begin{aligned}
\lim_{n\to\infty} sup d(v_n, v) &< \lim_{n\to\infty} sup d(v_n, x)\\
&\leq \lim_{n\to\infty} sup d(x_n, x)\\
&< \lim_{n\to\infty} sup d(x_n, v)\\
&= \lim_{n\to\infty} sup d(v_n, v),
\end{aligned}$$

which is a contradiction and hence the conclusion follows. □

Next we prove the strong convergence theorem.

Theorem 3.4. *Let C be a nonempty compact convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by (3.1) for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.7, we have that $F(T) \neq \emptyset$ and so by Theorem 3.2 we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Since C is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to p for some $p \in C$. By Proposition 2.6 (iii), we have

$$d(x_{n_k}, Tp) \leq 3d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p), \text{ for all } n \geq 1.$$

Letting $k \to \infty$, we get $Tp = p$, *i.e.*, $p \in F(T)$. By Theorem 3.2, $\lim_{n\to\infty} d(x_n, p)$ exists for every $p \in F(T)$ and so $\{x_n\}$ converges strongly to p. □

Senter and Dotson [22] introduced the notion of a mappings satisfying condition (I) as follows.

A mapping $T : C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf_{p\in F(T)} d(x, p)$.

Now we prove the strong convergence theorem using condition (I).

Theorem 3.5. *Let C be a nonempty closed convex subset of a $CAT(0)$ space X and $T : C \to C$ be a Suzuki generalized nonexpansive mapping. For arbitrary chosen $x_0 \in C$, let the sequence $\{x_n\}$ be generated by* (3.1) *for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers in $[a, b]$ for some a, b with $0 < a \leq b < 1$ such that $F(T) \neq \emptyset$. If T satisfies condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.1, we see that $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and so $\lim_{n\to\infty} d(x_n, F(T))$ exists. Assume that $\lim_{n\to\infty} d(x_n, p) = r$ for some $r \geq 0$. If $r = 0$, then the result follows. Suppose $r > 0$. Then from the hypothesis and condition (I),

$$f(d(x_n, F(T))) \leq d(Tx_n, x_n). \tag{3.9}$$

Since $F(T) \neq \emptyset$, by Theorem 3.3, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. So (3.9) implies that

$$\lim_{n\to\infty} f(d(x_n, F(T))) = 0. \tag{3.10}$$

Since f is a nondecreasing function, from (3.10), we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Thus we have a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{y_k\}$, $y_k \in F(T)$, such that

$$d(x_{n_k}, y_k) < \frac{1}{2^k} \text{ for all } k \in \mathbb{N}.$$

So using (3.4), we get

$$d(x_{n_{k+1}}, y_k) \leq d(x_{n_k}, y_k) < \frac{1}{2^k}.$$

Hence

$$\begin{aligned} d(y_{k+1}, y_k) &\leq d(y_{k+1}, x_{k+1}) + d(x_{k+1}, y_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}} \to 0, \text{ as } k \to \infty. \end{aligned}$$

This shows that $\{y_k\}$ is a Cauchy sequence in $F(T)$ and so it converges to a point p. Since $F(T)$ is closed, $p \in F(T)$ and then $\{x_{n_k}\}$ converges strongly to p. Since $\lim_{n\to\infty} d(x_n, p)$ exists, we have that $x_n \to p \in F(T)$. Hence the proof is complete. ☐

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Nonlinear Discrete Inequalities Method for the Ulam Stability of First Order Nonlinear Difference Equations

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Abstract

In this paper, first we derive some nonlinear discrete inequalities, and then as an application, we study the Ulam stability of the first order nonlinear difference equation

$$\Delta y(n) = f(n, y(n)),\ n \geq n_0,$$

where f is a given function. The obtained result on Ulam stability is new to the literature in the sense that our approach does not require the explicit form of solutions of the investigated equations.

2010 Mathematics Subject Classification: 39A30,39B82

Keywords and Phrases: Ulam stability, discrete inequality, nonlinear difference equation.

1. Introduction

In the passed years, the Ulam stability of functional equations received a great attention.In general, we say that an equation is stable in the sense of Ulam if for every approximate solution of that equation there exists an exact solution of the equation near it. For more details on Ulam stability, one can refer to [13].

The problem of the Ulam stability of difference equations is related to the notion of the perturbation of discrete dynamical systems. In [2–5, 7–9, 11, 12, 14, 17], the authors studied Ulam stability of linear difference equations and in [16], the authors obtained some results on Ulam stability for some second order linear difference equations. In all these papers, the authors studied the Ulam stability of first and second order linear difference equations and it seems that no results dealing with Ulam stability for the nonlinear difference equations are available in the literature.

Therefore the purpose of this paper is to study that Ulam stability of the following first order nonlinear difference equation

$$\Delta y(n) = f(n, y(n)),\ n \geq \mathbb{N}, \tag{1.1}$$

where $f \in C(\mathbb{N}, \mathbb{R})$ and $\mathbb{N}$ denotes the set of all non-negative integers, without using the explicit form of the solutions.

Next, we present the definition of the Ulam stability for difference equations.

Definition 1.1. *The equation* (1.1) *is called stable in Ulam sense if there exists a constant* $L \geq 0$ *such that for every* $\epsilon > 0$ *and every* $\{y(n)\}$ *in* $\mathbb{R}$ *satisfying*

$$|\Delta y(n) - f(n, y(n))| \leq \epsilon,\ n \geq 0 \tag{1.2}$$

there exists a sequence $\{x(n)\}$ *in* $\mathbb{R}$ *with the properties*

$$\Delta x(n) = f(n, x(n)),\ n \geq 0 \tag{1.3}$$

and

$$|y(n) - x(n)| \leq L\epsilon,\ n \geq 0. \tag{1.4}$$

A sequence $\{y(n)\}$ which satisfies (1.2) for some $\epsilon > 0$ is called an approximate solution of the nonlinear difference equation (1.1), and we reformulate the above definition as: the equation (1.1) is called Ulam stable if for every approximate solution of it there exists an exact solutions close to it. If in Definition 1.1, the number ϵ is replaced by a sequence of positive numbers $\{\epsilon(n)\}$ and $L\epsilon$ from (1.4)by a sequence of positive numbers $\{\eta(n)\}$ the equation (1.1) is called genelized stable in the Ulam sense.

In this paper first we derive some nonlinear discrete inequalities, and as an application we investigate the Ulam stability of equations (1.1).

2. Nonlinear Discrete Inequalities

In this section, we present some nonlinear discrete inequalities which provide us a powerful tool for investigating the Ulam stability of a nonlinear first order difference equations.

We begin with the following results which can be found in: [[6], Theorem 41, pp.39].

Lemma 2.1. *If* $a > 0$ *and* $0 < \alpha \leq 1$, *then*

$$a^{\alpha} \leq \alpha a + (1 - \alpha)$$

and the equality holds if $\alpha = 1$.

Theorem 2.2. *Let* $\{u(n)\}$, $\{f(n)\}$, $\{g(n)\}$ *and* $\{h(n)\}$ *be nonnegative real sequences defined for all* $n \in \mathbb{N}$, *and*

$$u(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s) u^{\alpha}(s), \tag{2.1}$$

where $0 < \alpha \leq 1$. *Then*

$$u(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s)(\alpha f(s) + (1 - \alpha)) \exp\left(\sum_{t=s+1}^{n-1} \alpha f(t) g(t) \right). \tag{2.2}$$

Proof. Defining a sequence $R(n)$ by

$$R(n) = \sum_{s=0}^{n-1} h(s) u^{\alpha}(s),$$

then $R(0) = 0$ and $u(n) \leq f(n) + g(n)R(n)$. Now using Lemma 2.1, one can obtain

$$\begin{aligned} \Delta R(n) &= h(n)u^{\alpha}(n) \leq h(n)(f(n) + g(n)R(n))^{\alpha} \\ &\leq (\alpha h(n) f(n) + (a - \alpha) h(n)) + \alpha h(n) g(n) R(n) \end{aligned}$$

or

$$R(n+1)-(1+\alpha h(n)g(n))R(n)\leq h(n)(\alpha f(n)+(1-\alpha)). \tag{2.3}$$

Multiplying (2.3) by $\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}$, we have

$$\Delta\left(R(n)\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}\right)\leq h(n)(\alpha f(n)+(1-\alpha))\prod_{s=0}^{n}(1+\alpha h(s)g(s))^{-1}.$$

Summing up the last inequality from 0 to $n-1$, we obtain

$$\begin{aligned} R(n) &\leq \sum_{s=0}^{n-1} h(s)(\alpha f(s)+(1-\alpha))\prod_{t=s+1}^{n-1}(1+\alpha h(t)g(t)) \\ &\leq \sum_{s=0}^{n-1} h(s)(\alpha f(s)+(1-\alpha))\left(\exp\sum_{t=s+1}^{n-1}\alpha h(t)g(t)\right). \end{aligned} \tag{2.4}$$

Using (2.4) in $u(n)\leq f(n)+g(n)R(n)$, we have the desired inequality (2.2). This completes the proof. ■

Corollary 2.3. *Let $u(n)$ and $p(n)$ be non-negative real sequences defined for all $n\in\mathbb{N}$ such that*

$$u(n)\leq c+\sum_{s=0}^{n-1}p(s)u^{\alpha}(s) \tag{2.5}$$

where $c\geq 0$ and $0<\alpha\leq 1$. Then

$$u(n)\leq\left(\frac{c\alpha+(1-\alpha)}{\alpha}\right)\exp\left(\sum_{s=0}^{n-1}\alpha p(s)\right). \tag{2.6}$$

Proof. Let $f(n)=c\geq 0$, $g(n)=1$ and $h(n)=p(n)$ in (2.2), we have

$$\begin{aligned} u(n) &\leq c+\sum_{s=0}^{n-1}p(s)(\alpha c+1-\alpha)\prod_{t=s+1}^{n-1}(1+\alpha p(t)) \\ &= c+\frac{(\alpha c+(1-\alpha))}{\alpha}\sum_{s=0}^{n-1}\alpha p(s)\prod_{t=s+1}^{n-1}(1+\alpha p(t)) \\ &= c+\frac{(\alpha c+(1-\alpha))}{\alpha}\left(\prod_{s=0}^{n-1}(1+\alpha p(s))-1\right) \\ &\leq \left(\frac{\alpha c+(1-\alpha)}{\alpha}\right)\exp\left(\sum_{s=0}^{n-1}\alpha p(s)\right). \end{aligned}$$

The proof is now complete. ■

Theorem 2.4. *Let* $u(n), p(n)$ *and* $h(n)$ *be non-negative real sequences for all* $n \in \mathbb{N}$ *and*

$$u(n) \leq c + \sum_{s=0}^{n-1} p(s)u(s) + \sum_{s=0}^{n-1} h(s)u^{\alpha}(s), \tag{2.7}$$

where $c \geq 0$ *and* $0 < \alpha \leq 1$. *Then*

$$u(n) \leq \left(c + (1-\alpha) \sum_{s=0}^{n-1} h(s) \right) \exp\left(\sum_{s=0}^{n-1} (p(s) + \alpha h(s)) \right). \tag{2.8}$$

Proof. Let $R(n)$ be the right hand side of (2.7). Then $R(0) = c$ and $u(n) \leq R(n)$ and

$$\begin{aligned} \Delta R(n) &= p(n)u(n) + h(n)u^{\alpha}(n) \\ &\leq p(n)R(n) + h(n)R^{\alpha}(n) \\ &\leq p(n)R(n) + h(n)(\alpha R(n) + (1-\alpha)) \\ &= (p(n) + \alpha h(n))R(n) + (1-\alpha)h(n) \end{aligned} \tag{2.9}$$

where we have used Lemma 2.1. Now from (2.9), we have

$$R(n+1) - (1 + p(n) + \alpha h(n))R(n) \leq (1-\alpha)h(n).$$

Arguing as in the proof of Theorem 2.2, one can easily obtain the desired result and hence the details are omitted. ■

Remark 2.1. *(a) If* $\alpha = 1$ *in Theorem 2.2, then it reduced to the well-known Pachpatte inequality [10], in 2002. For* $0 < \alpha < 1$ *the estimate* (2.2) *of Theorem 2.2 is new to the literature.*

(b) If $\alpha = 1$ *and* $g(n) \equiv 1$, *then Theorem 2.2 reduced to a well-known result due to Sugiyama [15], in 1969.*

Remark 2.2. *If* $\alpha = 1$ *in Corollary 2.3, then it reduced to the discrete analogue of the well-known Gronwall-Bellman inequality [1].*

Remark 2.3. *The result obtained in Theorem 2.4 is different from that one by Willet and Wong [18] for the case* $0 < \alpha < 1$.

3. ULAM STABILITY

As an application of the discrete inequalities established in Section 2, we investigate the Ulam stability of equation (1.1).

Theorem 3.1. *Let $p(n)$ be a positive real sequence for all $n \in \mathbb{N}$ such that*

$$|f(n,u) - f(n,v)| \leq p(n)|u-v|^{\alpha} \tag{3.1}$$

where $0 < \alpha \leq 1$, and

$$\sum_{n=0}^{\infty} p(n) < \infty. \tag{3.2}$$

If for a positive real sequence $\phi(n)$ such that $\sum_{n=0}^{\infty} \phi(n) < \infty$, and

$$|\Delta y(n) - f(n, y(n))| \leq \phi(n) \tag{3.3}$$

then there exists a real sequence $x(n)$ and a constant $k > 0$ satisfying

$$\Delta x(n) = f(n, x(n)) \tag{3.4}$$

such that $|y(n) - x(n)| \leq k$; that is, equation (1.1) has the Ulam stability.

Proof. From the inequality (3.3), we have

$$y(n) \leq y(0) + \sum_{s=0}^{n-1} f(s, y(s)) + \sum_{s=0}^{n-1} \phi(s) \tag{3.5}$$

and from the equation (3.4), we obtain

$$x(n) = x(0) + \sum_{s=0}^{n-1} f(s, x(s)). \tag{3.6}$$

Combining (3.5) and (3.6) yields

$$|y(n) - x(n)| \leq |y(0) - x(0)| + \sum_{s=0}^{n-1} |f(s, y(s)) - f(s, x(s))| + \sum_{s=0}^{n-1} \phi(s).$$

Using the condition (3.1) in the above inequality, we have

$$|y(n) - x(n)| \leq M_1 + \sum_{s=0}^{n-1} p(s)|y(s) - x(s)|^{\alpha} + M_2 \tag{3.7}$$

where $M_1 = |y(0) - x(0)|$ and $\sum_{n=0}^{\infty} \phi(n) \le M_2$ by hypothesis. Now applying Corollary 2.3 in (3.7), we obtain

$$|y(n) - x(n)| \le \frac{((M_1 + M_2)\alpha + (1 - \alpha))}{\alpha} \exp\left(\sum_{s=0}^{n-1} \alpha p(s)\right). \tag{3.8}$$

It follows from (3.2) that there is a constant $M_3 > 0$ such that $\sum_{n=0}^{\infty} p(n) \le M_3$, and using this in (3.8), one obtains

$$|y(n) - x(n)| \le k$$

where $k = \frac{((M_1+M_2)\alpha+(1-\alpha))}{\alpha} \exp(\alpha M_3)$. This completes the proof. ■

4. CONCLUSION

In this paper, first we have obtained some new nonlinear discrete inequalities and then as an application we investigate the Ulam stability of a nonlinear first order difference equation. In this approach, we do not need to require the explicit form of the solution of the studied equation, where as in [3,4,7-9,11,12,14] the authors used the explicit form of the solutions to prove their established results.

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Algebras and Smarandache Types

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Abstract. In this paper we introduce the notion of a BQ-algebra and show that is is equivalent to an abelian group. For deep investigations of several algebraic structures, we introduce the notions of a Smarandache V-algebra-type U-algebra and a Smarandache V-algebra-trans-type U-algebra, and apply the notions to several algebras.

1. Introduction

W. B. Vasantha Kandasamy ([8]) studied the concept of Smarandache groupoids, ideals of groupoids, Smarandache Bol groupoids and strong Bol groupoids, and obtained many interesting results about them. Smarandace semigroups are very important for the study of congruences, and it was studied by R. Padilla ([18]). It will be very interesting to study the Smarandache structure in general algebraic structures. Kim et al. ([11]) defined the concept of a Smarndache d-algebra and investigated some related properties of it. Seo et al. ([19]) introduced the concept of a Smarndache fuzzy BCI-algebra and investigated some related properties of it. Neggers et al. ([17]) defined the notion of a B-algebra and investigated some related properties of it. Some properties of B-algebra are studied in ([3, 12, 13]).

In this paper, we introduce the notion of a BQ-algebras and show that it is equivalent to an abelian group. Moreover, we introduce the notions of a Smarandache V-algebra-type U-algebra and a Smarandache V-algebra-trans-type U-algebra, and apply the notions to several algebras.

2. Preliminaries

A *B-algebra* ([17]) is a non-empty set X with a selected point 0 and a binary operation "$*$" satisfying the following axioms: (i) $x*x=0$, (ii) $x*0=x$, (iii) $(x*y)*z=x*(z*(0*y))$ for any $x,y,z\in X$. A B-algebra $(X,*,0)$ is said to be *0-commutative* ([2]) if $x*(0*y)=y*(0*x)$ for any $x,y\in X$. Let $(X,*,0)$ be a B-algebra and let $g\in X$. We define $g^{[0]}:=0, g^{[1]}:=g^{[0]}*(0*g)=0*(0*g)=g$ and $g^{[n]}:=g^{[n-1]}*(0*g)$ where $n\geq 1$.

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[0]**2010 Mathematics Subject Classification**: 06F35; 03G25.

[0]**Keywords**: Smarandache algebra, point algebra, p-derived algebra.

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Theorem 2.1. *Let* $(X, *, 0)$ *be a* B*-algebra and let* $g \in X$*. Then*

$$g^{[m]} * g^{[n]} = \begin{cases} g^{[m-n]} & \text{if } m \geq n, \\ 0 * g^{[n-m]} & \text{otherwise.} \end{cases}$$

Theorem 2.2. ([10]) *Every 0-commutative* B*-algebra is a* BCI*-algebra.*

Theorem 2.3. ([10]) *The following are equivalent:*

(i) X *is an abelian group,*
(ii) X *is a p-semisimple* BCI*-algebra,*
(iii) X *is a 0-commutative* B*-algebra.*

Let $(X, *, 0)$ be a B-algebra. Given $x, y \in X$, we define $x *^{\langle 1 \rangle} y := x * y, x *^{\langle 2 \rangle} y := (x * y) * y, x *^{\langle n \rangle} y := (x *^{\langle n-1 \rangle} y) * y$ where $n \geq 3$. For general references for BCK/BCI-algebras, we refer to [5, 6, 14].

3. Several algebras

Let $(X, *)$ be a groupoid (or a binary system, an algebra), i.e., X is a set and "$*$" is a binary operation on X. If we take an element p in X which plays an important role in $(X, *)$, then we say that p is a *selected point* and we write it by $(X, *, p)$. Such an algebra $(X, *, p)$ is said to be a *pointed algebra*.

Example 3.1. *Let* $(X, *)$ *be a group with identity* e*. The identity element* e *plays an important role in* $(X, *)$ *and hence we may write it by* $(X, *, e)$ *and* e *becomes a selected point in* $(X, *)$*.*

We regard all algebras below as pointed algebras without loss of generality. For simplicity's sake, we shall write $p = 0$, not intending 0 to have the usual meaning. Thus, in Example 3.1, $(X, *, e)$ becomes $(X, *, 0)$ unless it is important to distinguish the algebra $(X, *, 0)$ which contains the subalgebra (not necessary a subgroup) $(Y, *)$ with its selected point p to produce $(Y, *, p)$.

Example 3.2. Consider $X := \{a, b, c, d\}$ with the following table:

$*$	a	b	c	d
a	a	a	a	a
b	a	a	a	b
c	a	b	c	c
d	a	b	c	d

Then $(X, *, d)$ is an pointed algebra and the selected point d is the right identity. Consider $Y := \{a, c\}$ and $Z := \{a, d\}$ with the following tables:

$*$	a	c
a	a	a
c	a	c

$*$	a	d
a	a	a
d	a	d

Then $(Y, *, c)$ is a pointed algebra with a selected point c is the right identity, and $(Z, *, d)$ is also a pointed algebra with a special point d is the left identity.

Definition 3.3. Let $(X, *, p)$ be a pointed algebra. Define a binary operation "$\bullet$" on X by

$$x \bullet y := x * (p * y)$$

for any $x, y \in X$. Then the algebra $(X, \bullet, p)$ is called a *p-derived algebra from* $(X, *, p)$.

Example 3.4. (i) Let $(X, *, e)$ be a group with identity e. If $(X, \bullet, e)$ is an e-derived algebra of $(X, *, e)$, then $(X, \bullet) = (X, *)$, since e is the identity, we have $x \bullet y = x * (e * y) = x * y$ for all $x, y \in X$.
(ii) Let $(X, *, p)$ be a left-zero-semigroup with a selected point p. If $(X, \bullet, p)$ is a p-derived algebra of $(X, *, p)$, then $(X, *) = (X, \bullet)$.

Let X be a d-algebra and $x \in X$. Define $x * X := \{x * a | a \in X\}$. X is said to be *edge* ([16]) if for any $x \in X$, $x * X = \{x, 0\}$.

Lemma 3.5. ([16]) *Let X be an edge d-algebra. Then*

(i) $x * 0 = x$ *for all* $x \in X$.
(ii) $(x * (x * y)) * y = 0$ *for all* $x, y \in X$.

Example 3.6. (i) Let $(X, *, 0)$ be an edge d-algebra. If $(X, \bullet, 0)$ is an e-derived algebra of $(X, *, 0)$, then $(X, \bullet)$ is a left-zero-semigroup.
(ii) Let $(X, *, 0)$ be a BCK-algebra. If $(X, \bullet, 0)$ is an e-derived algebra of $(X, *, 0)$, then $(X, \bullet)$ is a left-zero-semigroup. In fact, $x \bullet y = x * (0 * y) = x * 0 = x$ for all $x, y \in X$.

In terms of list of axioms to be used to describe the various algebra types we note the following section of axioms:

(1) $x * x = 0$ for all $x \in X$.
(2) $x * 0 = x$ for all $x \in X$.
(3) $0 * x = x$ for all $x \in X$.
(4) $x * y = y * x$ for all $x, y \in X$.
(5) $x * y = y * x = 0 \Leftrightarrow x = y$ for all $x, y \in X$.
(6) $x * y = y * x = 0 \Rightarrow x = y$ for all $x, y \in X$.
(7) $x * y = y * x \Rightarrow x = y$ for all $x, y \in X$.
(8) $0 * x = 0$ for all $x \in X$.

(9) $(x*y)*z=(x*z)*y$ for all $x,y,z\in X$.
(10) $(x*y)*z=x*(z*y)$ for all $x,y,z\in X$.
(11) $(x*y)*z=x*(z*(0*y))$ for all $x,y,z\in X$.
(12) $(x*y)*z=(x*z)*(y*z)$ for all $x,y,z\in X$.
(13) $(x*y)*(0*y)=x$ for all $x,y\in X$.
(14) $x*(y*z)=(x*y)*z$ for all $x,y,z\in X$.
(15) $(x*(x*y))*y=0$ for all $x,y\in X$.
(16) $((x*y)*(x*z))*(x*y)=0$ for all $x,y,z\in X$.
(17) for any $x\in X$, there exists $y\in X$ with $x*y=0$.
(18) for any $x\in X$, there exists $y\in X$ with $y*x=0$.

An algebra $(X,*,0)$ is called a *group* if it satisfies (2),(3), (14), (17), and (18). An algebra $(X,*)$ is called a *semigroup* if it satisfies (14). An algebra $(X,*,0)$ is called a *semigroup with identity* if it satisfies (2),(3), and (14). An algebra $(X,*,0)$ is called a *B-algebra* ([17]) if it satisfies (1),(2), and (11). An algebra $(X,*,0)$ is called a *BG-algebra* ([9]) if it satisfies (1),(2), and (13). An algebra $(X,*,0)$ is called a *BH-algebra* ([7]) if it satisfies (1), (2), and (6). An algebra $(X,*,0)$ is called a *Q-algebra* ([15]) if it satisfies (1), (2), and (9). An algebra $(X,*,0)$ is called a *d-algebra* ([16]) if it satisfies (1), (5), and (8). An algebra $(X,*,0)$ is called a *BCK-algebra* ([14]) if it satisfies (1), (5), (8), (15), and (16). An algebra $(X,*,0)$ is called a *$gBCK$-algebra* ([4]) if it satisfies (1), (2), (9) and (12). An algebra $(X,*,0)$ is called an *abelian group* if it satisfies (2), (3), (4), (14), (17), and (18). An algebra $(X,*,0)$ is called a *commutative semigroup* if it satisfies (4) and (14).

4. BQ-ALGEBRAS

In this section, we introduce the notion of a BQ-algebra and we show that it is equivalent to an abelian group. An algebra $(X,*,0)$ is said to be a *BQ-algebra* if it satisfies the conditions (1), (2), (9) and (11).

Theorem 4.1. *Let $(X,*,0)$ be a BQ-algebra. If we define $x\bullet y:=x*(0*y)$ for any $x,y\in X$, then $(X,\bullet,0)$ is an abelian group.*

Proof. Since $(X,*,0)$ is a BQ-algebra, it is both a B-algebra and a Q-algebra. It was proved that if $(X,*,0)$ is a B-algebra, then $(X,\bullet,0)$ is a group ([1]). By (9) we obtain $(x*(0*y))*(0*z)=(x*(0*z))*(0*y)$ for any $x,y,z\in X$. It follows that $(x\bullet y)\bullet z=(x\bullet z)\bullet y$ for any $x,y,z\in X$. If we take $x:=0$, then $(0\bullet y)\bullet z=(0\bullet z)\bullet y$. Since $(X,*,0)$ is a B-algebra, we have $0\bullet y=0*(0*y)=y$ for any $y\in X$. Hence we obtain $y\bullet z=z\bullet y$ for any $y,z\in X$. This proves that $(X,\bullet,0)$ is an abelian group. ☐

Theorem 4.2. *Let $(X,\bullet,0)$ be an abelian group. If we define $x*y:=x\bullet y^{-1}$ for any $x,y\in X$, then $(X,*,0)$ is a BQ-algebra.*

Proof. (1) For any $x \in X$, we have $x*x = x\bullet x^{-1} = 0$. (2) For any $x \in X$, $x*0 = x\bullet 0^{-1} = x\bullet 0 = x$. (9) Given $x, y, z \in X$, since $(X, \bullet, 0)$ is a group, we obtain

$$\begin{aligned}(x * y) * z &= (x \bullet y^{-1}) \bullet z^{-1}\\ &= (x \bullet z^{-1}) \bullet y^{-1}\\ &= (x * z) * y.\end{aligned}$$

(11) Given $x, y, z \in X$, since $(X, \bullet, 0)$ is a group, we have

$$\begin{aligned}x * (z * (0 * y)) &= x \bullet [z \bullet [(y^{-1})^{-1} \bullet 0^{-1}]]^{-1}\\ &= x \bullet [z \bullet (y \bullet 0)]^{-1}\\ &= x \bullet (z \bullet y)^{-1}\\ &= x \bullet (y^{-1} \bullet z^{-1}).\end{aligned}$$

Similarly, we prove that $(x * y) * z = (x \bullet y^{-1}) \bullet z^{-1}$. Since $(X, \bullet, 0)$ is a group, we obtain $(x * y) * z = x * (z * (0 * y))$. Hence $(X, *, 0)$ is a BQ-algebra. ☐

By Theorems 4.1 and 4.2, we conclude that *the class of all BQ-algebras is equivalent to the class of all abelian groups.*

The interesting fact to note is that we are able to take advantage of the relationship $x \bullet y = x * (0 * y)$ to understand better what the meaning of the class of BQ-algebra is. Other such questions around in this setting as well as others. E.q., what class of B-algebras corresponds to the class of solvable groups ? Can it be considered to be of the form: B"V"-algebras corresponds to solvable groups where "V"-algebras is some nicely identifiable class, as the same as the class for BQ-algebras ?

5. SMARANDACHE TYPES

Let $(X, *)$ be an U-algebra. Then $(X, *)$ is said to be a *Smarandache V-algebra-type U-algebra* if there exists $Y \subseteq X$ such that $(Y, *)$ is a non-trivial subalgebra of $(X, *)$ and $|Y| \geq 2$, and $(Y, *)$ is a V-algebra. For example, a B-algebra $(X, *, 0)$ is said to be a *Smarandache Q-algebra-type B-algebra* it it contains a non-trivial sub-B-algebra $(Y, *, 0)$ of $(X, *, 0)$ and $|Y| \geq 2$, and $(Y, *, 0)$ is a Q-algebra. Similarly, a Q-algebra $(X, *, 0)$ is called a *Smarandache group-type Q-algebra* if it contains a non-trivial sub-Q-algebra $(Y, *, 0)$ of $(X, *, 0)$, and $(Y, *, 0)$ is a group where $|Y| \geq 2$.

Theorem 5.1. *There is no Smarandache d-algebra-type commutative groupoid.*

Proof. Assume that there is a Smarandache d-algebra-type commutative groupoid $(X, *, 0)$. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial subgroupoid of a commutative groupoid $(X, *, 0)$, $|Y| \geq 2$ and $(Y, *, 0)$ is a d-algebra. It follows that $0 * y = 0$ for all $y \in Y$. Since $(X, *, 0)$

is a commutative groupoid and $Y \subseteq X$, we obtain $0 * y = y * 0 = 0$ for all $y \in Y$. Since $(X, *, 0)$ is a d-algebra and $Y \subseteq X$, we obtain $y = 0$, i.e., $|Y| = 1$, a contradiction. ☐

Theorem 5.2. *There is no Smarandache semigroup-type d-algebra.*

Proof. Assume that there is a Smarandache semigroup-type d-algebra $(X, *, 0)$. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial subalgebra of a d-algebra $(X, *, 0)$, $|Y| \geq 2$ and $(Y, *, 0)$ is a semigroup. It follows that $0 * (y * 0) = 0$ for any $y \in Y$, since $Y \subset X$ and $(X, *, 0)$ is a d-algebra. Hence

$$\begin{aligned} y * 0 &= y * (0 * (y * 0)) \\ &= (y * 0) * (y * 0) \\ &= 0. \end{aligned}$$

Since $Y \subseteq X$ and $(X, *, 0)$ is a d-algebra, we obtain $0 * y = 0$ for all $y \in Y$. By (6), we have $y = 0$, i.e., $|Y| = 1$, a contradiction. ☐

Theorem 5.3. *A Smarandache group-type B-algebra is equal to a Smarandache Boolean-group-type B-algebra.*

Proof. Since every Boolean group is a group, it is enough to show that every Smarandache group-type B-algebra is a Smarandache Boolean-group-type B-algebra. Assume $(X, *, 0)$ is a Smarandache group-type B-algebra. Then there exists $Y \subseteq X$ such that $|Y| \geq 2$, $(Y, *, 0)$ is a non-trivial subalgebra of a B-algebra and $(Y, *, 0)$ is a group. For any $y \in Y$, since $Y \subseteq X$ and $(X, *, 0)$ is a B-algebra, we obtain $y * y = 0$. Since $(Y, *)$ is a group, the order of y is 2 in the group $(Y, *)$ for any $y \neq 0$ in Y and hence $(Y, *, 0)$ is a Boolean group, proving the theorem. ☐

Corollary 5.4. *A Smarandache group-type Q-algebra is equal to a Smarandache Boolean-group-type Q-algebra.*

Proof. Every Q-algebra has also the condition (1), and the proof is similar to the proof of Theorem 5.3. ☐

Theorem 5.5. *Every Smarandache B-algebra-type group is a Smarandache Boolean-group-type group.*

Proof. Let $(X, *, 0)$ be a Smarandache B-algebra-type group. Then there exists $Y \subseteq X$ such that $|Y| \geq 2$, $(Y, *, 0)$ is a non-trivial subgroup of a group $(X, *, 0)$ and $(Y, *, 0)$ is a B-algebra. It follows that $y * y = 0$ for all $y \in Y$. Since $Y \subseteq X$ and $(X, *, 0)$ is a group, we obtain $y = y^{-1}$ in the group. Hence $x * y^{-1} = x * y \in Y$, which shows that $(Y, *)$ is a subgroup of $(X, *)$ and the order of y is 2. Thus $(Y, *)$ is a Boolean group. This proves that $(X, *, 0)$ is a Smarandache Boolean-group-type group. ☐

Theorem 5.6. *Let $(X,*,0)$ be a Smarandache L-algebra-type M-algebra. If every L-algebra is an N-algebra, then $(X,*,0)$ is a Smarandache N-algebra-type M-algebra.*

Proof. It is easy and omit the proof. ☐

Theorem 5.7. *Let $(X,*,0)$ be a Smarandache 0-commutative-B-algebra-type M-algebra. Then $(X,*,0)$ is a Smarandache BCI-algebra-type M-algebra, where M-algebra is any algebra.*

Proof. By applying Theorems 2.2 and 5.6, we prove the theorem. ☐

Theorem 5.8. *Let $(X,*,0)$ be an M-algebra. Then the following are equivalent:*

(i) *X is a Smarandache abelian-group-type M-algebra*
(ii) *X is a Smarandache p-semisimple BCI-algebra-type M-algebra,*
(iii) *X is a Smarandache 0-commutative B-algebra-type M-algebra.*

Proof. It follows immediately from Theorems 2.3 and 5.6. ☐

Proposition 5.9. *If $(X,*,0)$ is a Smarandache Q-algebra-type group, then it is a Smarandache Boolean-group-type group.*

Proof. Let $(X,*,0)$ be a Smarandache Q-algebra-type group. Then there exists $Y \subseteq X$ such that $|Y| \geq 2$, $(Y,*,0)$ is a non-trivial subgroup of a group $(X,*,0)$ and $(Y,*,0)$ is a Q-algebra. Since Y is a Q-algebra, we have $y*y=0$ for any $y \in Y$. This means the order of y is 2 in the group $(Y,*)$, i.e., $y=y^{-1}$, which shows that $(Y,*,0)$ is a Boolean-group. Hence $(X,*,0)$ is a Smarandache Boolean-group-type group. ☐

Theorem 5.10. *Any non-trivial d-algebra cannot be a Smarandache group-type d-algebra.*

Proof. Assume there exists a Smarandache group-type d-algebra $(X,*,0)$. Then there exists $Y \subseteq X$ such that $(Y,*,0)$ is a non-trivial sub-d-algebra of $(X,*,0)$ and $(Y,*,0)$ is a group where $|Y| \geq 2$. Since $(Y,*,0)$ is a group and $(X,*,0)$ is a d-algebra, we have $y=0*y=0$ for all $y \in Y$. It follows that $|Y|=1$, a contradiction. ☐

Theorem 5.11. *Any non-trivial group cannot be a Smarandache d-algebra-type group.*

Proof. Assume that there exists a Smarandache d-algebra-type group $(X,*,0)$. Then there exists $Y \subseteq X$ such that $(Y,*,0)$ is a non-trivial subgroup of a group $(X,*,0)$, and $(Y,*,0)$ is a d-algebra and $|Y| \geq 2$. Then $0*x=0$ for all $x \in Y$. Since $(Y,*,0)$ is a group, we obtain $x=0$ for all $x \in Y$, proving that $|Y|=1$, a contradiction. ☐

Theorem 5.12. *Any non-trivial $gBCK$-algebra cannot be a Smarandache group-type $gBCK$-algebra.*

Proof. Let $(X,*,0)$ be a Smarandache group-type $gBCK$-algebra. Then there exists $Y \subseteq X$ such that $(Y,*,0)$ is a non-trivial sub-$gBCK$-algebra of $(X,*,0)$, and $(Y,*,0)$ is a group and $|Y| \geq 2$. Since $Y \subseteq X$ and $(X,*,0)$ is a $gBCK$-algebra, we obtain $y*y=0$ for all $y \in Y$. It follows from

$(Y, *, 0)$ is a group that the order of y is 2, i.e., $(Y, *, 0)$ is a Boolean group. Now, since $(Y, *, 0)$ is a $gBCK$-algebra, we have $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$. It follows that $(x * x) * x = (x * x) * (x * x)$ for all $x \in X$. Since $(X, *, 0)$ is a group, we obtain $x = 0$ for all $x \in X$, proving that $|X| = 1$, a contradiction. ☐

Corollary 5.13. *Any non-trivial group cannot be a Smarandache $gBCK$-algebra-type group.*

Proof. The proof is similar to Theorem 5.12, and we omit it. ☐

Definition 5.14. Let $(X, *, p)$ be an L-algebra and let $(Y, *, p)$ be both a sub-L-algebra of $(X, *, p)$ and an M-algebra. $(X, *, p)$ is said to be a *Smarandache N-algebra-trans-type L-algebra* if $(Y, *, p)$ is isomorphic with an N-algebra $(Y, \odot, q)$.

$$\begin{tikzcd}
(X, *, p) \arrow[d, dash] & \\
(Y, *, p) \arrow[r, "\cong"] & (Y, \odot, q)
\end{tikzcd}$$

where $L-, M-, N-$ algebras are arbitrary algebras.

Theorem 5.15. *If $(X, *, 0)$ is a Smarandache B-algebra-type Q-algebra, then it is a Smarandache abelian-group-trans-type Q-algebra.*

Proof. Let $(X, *, 0)$ be a Smarandache B-algebra-type Q-algebra. Then there exists $Y \subseteq X$ such that $(Y, *, 0)$ is a non-trivial sub-Q-algebra of a Q-algebra $(X, *, 0)$, $|Y| \geq 2$ and $(Y, *, 0)$ is a B-algebra. Define $x \bullet y := x * (0 * y)$ for any $x, y \in Y$. Then $(Y, \bullet, 0)$ is an abelian group. In fact, since Y is both a Q-algebra and B-algebra, $(Y, *, 0)$ is a BQ-algebra. By Theorem 4.1, $(Y, \bullet, 0)$ is an abelian group. By Theorems 4.1 and 4.2, $(Y, *, 0) \cong (Y, \bullet, 0)$. This shows that $(X, *, 0)$ is a Smarandache abelian-group-trans-type Q-algebra. ☐

Corollary 5.16. *If $(X, *, 0)$ is a Smaradache Q-algebra-type B-algebra, then it is a Smarandache abelian-group-trans-type B-algebra.*

Proof. It is similar to Theorem 5.15. ☐

Proposition 5.17. *Every B-algebra is a Smarandache BQ-algebra-trans-type B-algebra.*

Proof. Let $(X, *, 0)$ be a B-algebra. Define $x \bullet y := x * (0 * y)$ for all $x, y \in X$. Then $(X, \bullet, 0)$ is a group. Let $x \in X$ such that $x \neq 0$. Let $\langle x \rangle$ be a cyclic group generated by x. Then $\langle x \rangle$ is a non-trivial abelian subgroup of $(X, \bullet, 0)$. If we let $Y_x := \{x *^{\langle n \rangle} (0 * x) \,|\, n \in \mathbf{Z}\}$, then $Y_x \cong \langle x \rangle$. By Theorems 4.1 and 4.2, Y_x is a non-trivial BQ-algebra. This shows that X is a Smarandache BQ-algebra-trans-type B-algebra. ☐

6. Conclusion

We introduced the notion of a BQ-algebra and proved that it is equivalent to an abelian group. For detailed investigations among several algebraic structures, we introduced the notions of a Smarandache V-type U-algebra and a Smarandache V-trans-type U-algebra, and applied this notions to several algebras. For further investigations, we will apply the notions of a hyper structure theory and several fuzzy related algebras to the notions of a Smarandache V-type U-algebra and a Smarandache V-trans-type U-algebra.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Nonlinear differential equations associated with degenerate (h, q)-tangent numbers

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Abstract : In this paper, we study nonlinear differential equations arising from the generating functions of degenerate (h, q)-tangent numbers. We give explicit identities for the degenerate (h, q)-tangent numbers.

Key words : Nonlinear differential equations, (h, q)-tangent numbers and polynomials, degenerate tangent numbers, degenerate (h, q)-tangent numbers, higher-order degenerate tangent numbers.

AMS Mathematics Subject Classification : 05A19, 11B83, 34A30, 65L99.

1. Introduction

Recently, many mathematicians have studied in the area of the degenerate Euler numbers and polynomials, degenerate Bernoulli numbers and polynomials, degenerate Genocchi numbers and polynomials, and degenerate tangent numbers and polynomials(see [1, 2, 3, 4, 5, 6, 7]). In [1], L. Carlitz introduced the degenerate Bernoulli polynomials. Recently, Feng Qi *et al.*[2] studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. The degenerate (h, q)-tangent numbers $\mathcal{T}_{n,q}^{(h)}(\lambda)$ are defined by the generating function:

$$\sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(h)}(\lambda)\frac{t^n}{n!} = \frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}. \tag{1.1}$$

The degenerate (h, q)-tangent numbers of higher order, $\mathcal{T}_{n,\lambda,q}^{(k,h)}$ are defined by means of the following generating function

$$\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^k = \sum_{n=0}^{\infty} \mathcal{T}_{n,q}^{(k,h)}(\lambda)\frac{t^n}{n!}. \tag{1.2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations(see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. We also have

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \text{ and } \sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.3}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.4}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.5}$$

Many mathematicians have studied in the area of the linear and nonlinear differential equations arising from the generating functions of special numbers and polynomials in order to give explicit identities for special polynomials. In this paper, we study nonlinear differential equations arising from the generating functions of degenerate (h,q)-tangent numbers . We give explicit identities for the degenerate (h,q)-tangent numbers .

2. Nonlinear differential equations associated with degenerate (h,q)-tangent numbers

In this section, we study nonlinear differential equations arising from the generating functions of degenerate twisted (h,q)-tangent numbers. Let

$$F=F(t,\lambda,q,h)=\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}=\sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(h)}(\lambda)\frac{t^n}{n!}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned}F^{(1)}&=\frac{\partial}{\partial t}F(t,\lambda,q,h)=\frac{\partial}{\partial t}\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)\\&=\frac{1}{1+\lambda t}\left(\frac{-4}{q^h(1+\lambda t)^{2/\lambda}+1}\right)+\frac{1}{1+\lambda t}\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^2\\&=\frac{-2F+F^2}{1+\lambda t}.\end{aligned} \tag{2.2}$$

By (2.2), we have

$$F^2=2F+(1+\lambda t)F^{(1)}. \tag{2.3}$$

Taking the derivative with respect to t in (2.3), we obtain

$$\begin{aligned}2FF^{(1)}&=2F^{(1)}+\lambda F^{(1)}+(1+\lambda t)F^{(2)}\\&=(\lambda+2)F^{(1)}+(1+\lambda t)F^{(2)}.\end{aligned} \tag{2.4}$$

From (2.2), (2.3), and (2.4), we have

$$2F^3=4F+(1+\lambda)(1+\lambda t)F^{(1)}+(1+\lambda t)^2F^{(2)}.$$

Continuing this process, we can guess that

$$N!F^{N+1}=\sum_{i=0}^{N}a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)},\quad (N=0,1,2,\ldots), \tag{2.5}$$

where $F^{(i)}=\left(\frac{\partial}{\partial t}\right)^i F(t,\lambda,q,h)$. Differentiating (2.5) with respect to t, we have

$$(N+1)!F^NF^{(1)}=\sum_{i=0}^{N}i\lambda a_i(N,\lambda,q,h)(1+\lambda t)^{i-1}F^{(i)}+\sum_{i=0}^{N}a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i+1)} \tag{2.6}$$

and

$$(N+1)!F^NF^{(1)}=(N+1)!F^N\left(\frac{-2F+F^2}{1+\lambda t}\right)=(N+1)!\left(\frac{F^{N+2}-2F^{N+1}}{1+\lambda t}\right). \tag{2.7}$$

By (2.5), (2.6), and (2.7), we have

$$\begin{aligned}(N+1)!F^{N+2} &= 2(N+1)!F^{N+1}\\ &\quad+\sum_{i=0}^{N}\lambda i a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}+\sum_{i=0}^{N}a_i(N,\lambda)(1+\lambda t)^{i+1}F^{(i+1)}\\ &=2(N+1)\sum_{i=0}^{N}a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}\\ &\quad+\sum_{i=0}^{N}\lambda i a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}+\sum_{i=0}^{N}a_i(N,\lambda,q,h)(1+\lambda t)^{i+1}F^{(i+1)}\\ &=\sum_{i=0}^{N}\left(2(N+1)+\lambda i\right)a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}+\sum_{i=1}^{N+1}a_{i-1}(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}.\end{aligned} \tag{2.8}$$

Now replacing N by $N+1$ in (2.5), we find

$$(N+1)!F^{N+2}=\sum_{i=0}^{N+1}a_i(N+1,\lambda,q,h)(1+\lambda t)^iF^{(i)}. \tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{aligned}\sum_{i=0}^{N+1}a_i(N+1,\lambda,q,h)(1+\lambda t)^iF^{(i)}&=\sum_{i=0}^{N}\left(2(N+1)+\lambda i\right)a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}\\ &\quad+\sum_{i=1}^{N+1}a_{i-1}(N,\lambda,q,h)(1+\lambda t)^iF^{(i)}.\end{aligned} \tag{2.10}$$

Comparing the coefficients on both sides of (2.10), we obtain

$$\begin{aligned}2(N+1)a_0(N,\lambda,q,h)&=a_0(N+1,\lambda,q,h),\\ a_{N+1}(N+1,\lambda,q,h)&=a_N(N,\lambda,q,h),\end{aligned} \tag{2.11}$$

and

$$a_i(N+1,\lambda,q,h)=\left(2(N+1)+\lambda i\right)a_i(N,\lambda,q,h)+a_{i-1}(N,\lambda,q,h),(1\le i\le N). \tag{2.12}$$

In addition, by (2.5), we have

$$F=a_0(0,\lambda,q,h)F, \tag{2.13}$$

which gives

$$a_0(0,\lambda,q,h)=1. \tag{2.14}$$

It is not difficult to show that

$$F^2=a_0(1,\lambda,q,h)F+a_1(1,\lambda,q,h)(1+\lambda t)F^{(1)}=2F+(1+\lambda t)F^{(1)}. \tag{2.15}$$

Thus, by (2.15), we also find

$$a_0(1,\lambda,q,h)=2,\quad a_1(1,\lambda,q,h)=1. \tag{2.16}$$

From (2.11), we note that

$$\begin{aligned}a_0(N+1,\lambda,q,h)&=2(N+1)a_0(N,\lambda,q,h)=4(N+1)Na_0(N-1,\lambda,q,h)\\ &=\cdots=2^{N+1}(N+1)!,\end{aligned} \tag{2.17}$$

and

$$a_{N+1}(N+1,\lambda,q,h)=a_N(N,\lambda,q,h)=\cdots=1. \tag{2.18}$$

For $i=1,2,3$ in (2.11), then we find that

$$\begin{aligned}
a_1(N+1,\lambda,q,h)&=\sum_{k=0}^{N}2^k\left(N+1+\frac{\lambda}{2}\right)_k a_0(N-k,\lambda,q,h),\\
a_2(N+1,\lambda,q,h)&=\sum_{k=0}^{N-1}2^k\left(N+1+\frac{\lambda}{2}\times 2\right)_k a_1(N-k,\lambda,q,h),\\
a_3(N+1,\lambda,q,h)&=\sum_{k=0}^{N-2}2^k\left(N+1+\frac{\lambda}{2}\times 3\right)_k a_2(N-k,\lambda,q,h).
\end{aligned}$$

Continuing this process, we can deduce that, for $1\le i\le N$,

$$a_i(N+1,\lambda,q,h)=\sum_{k=0}^{N-i+1}2^k\left(N+1+\frac{\lambda}{2}\times i\right)_k a_{i-1}(N-k,\lambda,q,h). \tag{2.19}$$

Note that, here the matrix $a_i(j,\lambda,q,h)_{0\le i,j\le N+1}$ is given by

$$\begin{pmatrix}
1 & 2 & 2!2^2 & 3!2^3 & \cdots & (N+1)!2^{N+1}\\
0 & 1 & \cdot & \cdot & \cdots & \cdot\\
0 & 0 & 1 & \cdot & \cdots & \cdot\\
0 & 0 & 0 & 1 & \cdots & \cdot\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1,\lambda,q,h)$. By (2.17), (2.18), and (2.19), we have

$$\begin{aligned}
a_1(N+1,\lambda,q,h)&=\sum_{k_1=0}^{N}2^{k_1}\left(N+1+\frac{\lambda}{2}\right)_{k_1} a_0(N-k_1,\lambda,q,h)\\
&=\sum_{k_1=0}^{N}2^{N}(N-k_1)!\left(N+1+\frac{\lambda}{2}\right)_{k_1},
\end{aligned}$$

$$\begin{aligned}
&a_2(N+1,\lambda,q,h)=\sum_{k_2=0}^{N-1}2^{k_2}\left(N+1+\frac{\lambda}{2}\times 2\right)_{k_2} a_1(N-k_2,\lambda,q,h)\\
&=\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}2^{N-1}(N-k_2-k_1-1)!\left(N+1+\frac{\lambda}{2}\times 2\right)_{k_2}\left(N-k_2+\frac{\lambda}{2}\right)_{k_1},
\end{aligned}$$

and

$$\begin{aligned}
&a_3(N+1,\lambda,q,h)=\sum_{k_3=0}^{N-2}2^{k_3}\left(N+1+\frac{\lambda}{2}\times 3\right)_{k_3} a_2(N-k_3,\lambda,q,h)\\
&=\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2}2^{N-2}(N-k_3-k_2-k_1-2)!\left(N+1+\frac{\lambda}{2}\times 3\right)_{k_3}\\
&\quad\times\cdots\times\left(N-k_3+\frac{\lambda}{2}\times 2\right)_{k_2}\left(N-k_3-k_2-1+\frac{\lambda}{2}\right)_{k_1}
\end{aligned}$$

Continuing this process, we have

$$\begin{aligned} a_i(N+1,\lambda,q,h) &= \sum_{k_i=0}^{N-i+1}\sum_{k_{i-1}=0}^{N-k_i-i+1}\cdots\sum_{k_1=0}^{N-k_{i-1}-\cdots-k_2-i+1} 2^{N-i+1} \\ &\times (N-k_i-k_{i-1}-\cdots-k_2-k_1-i+1)! \\ &\times \left(N+1+\frac{\lambda}{2}\times i\right)_{k_i}\left(N-k_i+\frac{\lambda}{2}\times(i-1)\right)_{k_{i-1}} \\ &\times \left(N-k_i-k_{i-1}-1+\frac{\lambda}{2}\times(i-2)\right)_{k_{i-2}} \\ &\times \left(N-k_i-k_{i-1}-k_{i-2}-2+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots \\ &\times \left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+2+\frac{\lambda}{2}\right)_{k_1}. \end{aligned} \tag{2.20}$$

Therefore, by (2.20), we obtain the following theorem.

Theorem 1. For $N=0,1,2,\ldots$, the nonlinear functional equation

$$N!F^{N+1}=\sum_{i=0}^{N} a_i(N,\lambda,q,h)(1+\lambda t)^i F^{(i)}$$

has a solution

$$F=F(t,\lambda,q,h)=\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1},$$

where

$$\begin{aligned} &a_0(N,\lambda,q,h)=2^N N!, \\ &a_N(N,\lambda,q,h)=1, \\ &a_i(N,\lambda,q,h)=\sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}(2q^h-x)^{N-i} \\ &\times (N-k_i-k_{i-1}-\cdots-k_2-k_1-i)! \\ &\times \left(N+\frac{\lambda}{2}\times i\right)_{k_i}\left(N-k_i-1+\frac{\lambda}{2}\times(i-1)\right)_{k_{i-1}} \\ &\times \left(N-k_i-k_{i-1}-2+\frac{\lambda}{2}\times(i-2)\right)_{k_{i-2}} \\ &\times \left(N-k_i-k_{i-1}--k_{i-2}-3+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots \\ &\times \left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+1+\frac{\lambda}{2}\right)_{k_1}. \end{aligned}$$

From (1.1) and (1.2), we note that

$$N!F^{N+1}=N!\left(\frac{2}{q^h(1+\lambda t)^{2/\lambda}+1}\right)^{N+1}=N!\sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(N+1,h)}(\lambda)\frac{t^n}{n!}. \tag{2.21}$$

From (2.5), we note that

$$F^{(i)}=\left(\frac{\partial}{\partial t}\right)^i F(t,\lambda,q,h)=\sum_{l=0}^{\infty}\mathcal{T}_{i+l,q}^{(h)}(\lambda)\frac{t^l}{l!}. \tag{2.22}$$

From Theorem 1, (1.5), (2.21), and (2.22), we can derive the following equation:

$$\begin{aligned}
N!F^N\sum_{n=0}^{\infty}\mathcal{T}_{n,q}^{(N+1,h)}(\lambda)\frac{t^n}{n!} &= \sum_{i=0}^{N}a_i(N,\lambda,q,h)(1+\lambda t)^iF^{(i)} \\
&= \sum_{i=0}^{N}a_i(N,\lambda,q,h)\sum_{k=0}^{\infty}(i)_k\lambda^k\frac{t^k}{k!}\sum_{l=0}^{\infty}\mathcal{T}_{i+l,q}^{(h)}(\lambda)\frac{t^l}{l!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}a_i(N,\lambda,q,h)(i)_k\lambda^k\mathcal{T}_{n-k+i,q}^{(h)}(\lambda)\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.23}$$

By comparing the coefficients on both sides of (2.23), we obtain the following theorem.

Theorem 2. For $k, N = 0, 1, 2, \ldots$, we have

$$N!\mathcal{T}_{n,q}^{(N+1,h)}(\lambda) = \sum_{i=0}^{N}\sum_{k=0}^{n}\binom{n}{k}a_i(N,\lambda,q,h)(i)_k\lambda^k\mathcal{T}_{n-k+i,q}^{(h)}(\lambda),$$

where

$$\begin{aligned}
&a_0(N,\lambda) = N!2^N, \quad a_N(N,\lambda) = 1, \\
&a_i(N,\lambda) = \sum_{k_i=0}^{N-i}\sum_{k_{i-1}=0}^{N-k_i-i}\cdots\sum_{k_1=0}^{N-k_i-\cdots-k_2-i}2^{N-i} \\
&\quad\times(N-k_i-k_{i-1}-\cdots-k_2-k_1-i)! \\
&\quad\times\left(N-k_i-k_{i-1}--k_{i-2}-3+\frac{\lambda}{2}\times(i-3)\right)_{k_{i-3}}\cdots \\
&\quad\times\left(N-k_i-k_{i-1}-k_{i-2}-\cdots-k_2-i+1+\frac{\lambda}{2}\right)_{k_1}.
\end{aligned}$$

Acknowledgement: This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On the symmetries of the second kind (h, q)-Bernoulli polynomials

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Abstract : In this paper, by applying the symmetry of the fermionic p-adic q-integral on $\mathbb{Z}_p$, we give recurrence identities the second kind (h, q)-Bernoulli polynomials and the sums of powers of consecutive (h, q)-odd integers.

Key words : Bernoulli numbers and polynomials, the second kind Bernoulli numbers and polynomials, the second kind q-Bernoulli numbers and polynomials, the second kind (h, q)-Bernoulli numbers and polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Bernoulli numbers, Bernoulli polynomials, q-Bernoulli numbers, q-Bernoulli polynomials, the second kind Bernoulli number and the second kind Bernoulli polynomials were studied by many authors(see [1-8]). Bernoulli numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In [5], by using the second kind Bernoulli numbers B_j and polynomials $B_j(x)$, we investigated the q-analogue of sums of powers of consecutive odd integers(see [6]). Let k be a positive integer. Then we obtain

$$O_k(n-1) = \sum_{i=0}^{n-1}(2i+1)^{k-1} = \frac{B_k(2n) - B_k}{2k}.$$

In [4], we introduced the second kind (h, q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$. By using computer, we observed an interesting phenomenon of 'scattering' of the zeros of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$ in complex plane. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind (h, q)-Bernoulli polynomials $B_{n,q}^{(h)}(x)$. In this paper, we give recurrence identities the second kind (h, q)-Bernoulli polynomials and the sums of powers of consecutive (h, q)-odd integers.

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the p-adic q-integral was defined by [2, 5]

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]} \sum_{x=0}^{p^N-1} g(x) q^x.$$

The bosonic integral was considered from a physical point of view to the bosonic limit $q \to 1$, as follows:

$$I_1(g) = \lim_{q\to 1} I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} g(x) \text{ (see [2]).} \tag{1.1}$$

By (1.1), we easily see that

$$I_1(g_1) = I_1(g) + g'(0), \tag{1.2}$$

where $g_1(x) = g(x+1)$ and $g'(0) = \frac{dg(x)}{dx}\Big|_{x=0}$.

First, we introduce the second kind Bernoulli numbers B_n and polynomials $B_n(x)$. The second kind Bernoulli numbers B_n and polynomials $B_n(x)$ are defined by means of the following generating functions (see [3]):

$$\frac{2te^t}{e^{2t}-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

and

$$\left(\frac{2te^t}{e^{2t}-1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

respectively.

The second kind (h,q)-Bernoulli polynomials, $B_{n,q}^{(h)}(x)$ are defined by means of the generating function:

$$\left(\frac{(h\log q + 2t)e^t}{q^h e^{2t}-1}\right) e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{1.3}$$

The second kind (h,q)-Bernoulli numbers $E_{n,q}^{(h)}$ are defined by means of the generating function:

$$\frac{(h\log q + 2t)e^t}{q^h e^{2t}-1} = \sum_{n=0}^{\infty} B_{n,q}^{(h)} \frac{t^n}{n!}. \tag{1.4}$$

In (1.2), if we take $g(x) = q^{hx}e^{(2x+1)t}$, then we have

$$\int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x) = \frac{(h\log q + 2t)e^t}{q^h e^{2t}-1}. \tag{1.5}$$

for $|t| \le p^{-\frac{1}{p-1}}, h \in \mathbb{Z}$. In (1.2), if we take $g(x) = e^{2nxt}$, then we also have

$$\int_{\mathbb{Z}_p} e^{2nxt} d\mu_1(x) = \frac{2nt}{e^{2nt}-1}. \tag{1.6}$$

for $|t| \le p^{-\frac{1}{p-1}}$. It will be more convenient to write (1.2) as the equivalent bosonic integral form

$$\int_{\mathbb{Z}_p} g(x+1) d\mu_1(x) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) + g'(0), \quad \text{(see [2]).} \tag{1.7}$$

For $n \in \mathbb{N}$, we also derive the following bosonic integral form by (1.7),

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_1(x) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x) + \sum_{k=0}^{n-1} g'(k), \text{ where } g'(k) = \frac{dg(x)}{dx}\Big|_{x=k}. \tag{1.8}$$

In [4], we introduced the second kind (h,q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$ and investigate their properties. The following elementary properties of the second kind (h,q)-Bernoulli numbers $B_{n,q}^{(h)}$ and polynomials $B_{n,q}^{(h)}(x)$ are readily derived form (1.1), (1.2), (1.3) and (1.4). We, therefore, choose to omit details involved.

Theorem 1. For $h \in \mathbb{Z}, q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, we have

$$\begin{aligned} B_{n,q}^{(h)} &= \int_{\mathbb{Z}_p} q^{hx}(2x+1)^n d\mu_1(x), \\ B_{n,q}^{(h)}(x) &= \int_{\mathbb{Z}_p} q^{hy}(x+2y+1)^n d\mu_1(y). \end{aligned}$$

Theorem 2. For any positive integer n, we have

$$B_{n,q}^{(h)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_{k,q}^{(h)} x^{n-k}.$$

Theorem 3. For any positive integer m, we obtain

$$B_{n,q}^{(h)}(x) = m^{n-1} \sum_{i=0}^{m-1} q^{hi} B_{n,q^m}^{(h)}\left(\frac{2i+x+1-m}{m}\right) \text{ for } n \geq 0.$$

2. On the symmetries of the second kind (h,q)-Bernoulli polynomials

In this section, we assume that $q \in \mathbb{C}_p$ and $h \in \mathbb{Z}$. We investigate interesting properties of symmetry p-adic invariant integral on $\mathbb{Z}_p$ for the second kind (h,q)-Bernoulli polynomials. W also obtain recurrence identities the second kind (h,q)-Bernoulli polynomials.

By (1.7), we obtain

$$\begin{aligned} &\frac{1}{h\log q + 2t}\left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x)\right) \\ &\qquad = \frac{n\int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x)}{\int_{\mathbb{Z}_p} q^{hnx} e^{2ntx} d\mu_1(x)} \end{aligned} \tag{2.1}$$

By (1.8), we obtain

$$\begin{aligned} &\frac{1}{h\log q + 2t}\left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x)\right) \\ &\qquad = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} q^{hi}(2i+1)^k\right) \frac{t^k}{k!}. \end{aligned} \tag{2.2}$$

For each integer $k \geq 0$, let

$$O_{k,q}^{(h)}(n) = 1^k + q^h 3^k + q^{2h} 5^k + q^{3h} 7^k + \cdots + q^{nh}(2n+1)^k.$$

The above sum $O_{k,q}^{(h)}(n)$ is called the sums of powers of consecutive (h,q)-odd integers. From the above and (2.2), we obtain

$$\begin{aligned} &\frac{1}{h\log q + 2t}\left(\int_{\mathbb{Z}_p} q^{hx} q^{hn} e^{(2x+2n+1)t} d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx} e^{(2x+1)t} d\mu_1(x)\right) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} O_{k,q}^{(h)}(n-1) \frac{t^k}{k!}. \end{aligned} \tag{2.3}$$

Thus, we have

$$\sum_{k=0}^{\infty}\left(q^{hn}\int_{\mathbb{Z}_p} q^{hx}(2x+2n+1)^k d\mu_1(x) - \int_{\mathbb{Z}_p} q^{hx}(2x+1)^k d\mu_1(x)\right)\frac{t^k}{k!} = \sum_{k=0}^{\infty}(h\log q + 2t) O_{k,q}^{(h)}(n-1)\frac{t^k}{k!}$$

By comparing coefficients $\frac{t^k}{k!}$ in the above equation, we have

$$\begin{aligned}&(h\log q+2t)O_{k,q}^{(h)}(n-1)\\&=\int_{\mathbb{Z}_p}q^{hx}(2x+2n+1)^k d\mu_1(x)-\int_{\mathbb{Z}_p}q^{hx}(2x+1)^k d\mu_1(x)^{\cdot}\end{aligned}$$

By using the above equation we arrive at the following theorem:

Theorem 4. Let k be a positive integer. Then we obtain

$$q^{hn}B_{n,q}^{(h)}(2n)-B_{n,q}^{(h)}=h\log qO_{k,q}^{(h)}(n-1)+2kO_{k-1,q}^{(h)}(n-1). \tag{2.4}$$

Remark 5. For the alternating sums of powers of consecutive integers, we have

$$\begin{aligned}\lim_{q\to 1}\left(h\log qO_{k,q}^{(h)}(n-1)+2kO_{k-1,q}^{(h)}(n-1)\right)&=\sum_{i=0}^{n-1}(2i+1)^{k-1}\\&=\frac{B_k(2n)-B_k}{2k},\ \text{for } k\in\mathbb{N}.\end{aligned}$$

By using (2.1) and (2.3), we arrive at the following theorem:

Theorem 6. Let n be positive integer. Then we have

$$\frac{n\int_{\mathbb{Z}_p}q^{hx}e^{(2x+1)t}d\mu_1(x)}{\int_{\mathbb{Z}_p}q^{hnx}e^{2ntx}d\mu_1(x)}=\sum_{m=0}^{\infty}\left(O_{m,q}^{(h)}(n-1)\right)\frac{t^m}{m!}. \tag{2.5}$$

Let w_1 and w_2 be positive integers. By using (1.5) and (1.6), we have

$$\begin{aligned}&\frac{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}q^{h(w_1x_1+w_2x_2)}e^{(w_1(2x_1+1)+w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_1)d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2xt}d\mu_1(x)}\\&=\frac{(h\log q+2t)e^{w_1t}e^{w_2t}e^{w_1w_2xt}(q^{hw_1w_2}e^{2w_1w_2t}-1)}{(q^{hw_1}e^{2w_1t}-1)(q^{hw_2}e^{2w_2t}-1)}\end{aligned} \tag{2.6}$$

By using (2.4) and (2.6), after calculations, we obtain

$$\begin{aligned}S&=\left(\frac{1}{w_1}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(w_1(2x_1+1)+w_1w_2x)t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}\sum_{m=0}^{\infty}B_{m,q^{w_1}}^{(h)}(w_2x)w_1^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_2}}^{(h)}(w_1-1)w_2^m\frac{t^m}{m!}\right).\end{aligned} \tag{2.7}$$

By using Cauchy product in the above, we have

$$S=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B_{j,q^{w_1}}^{(h)}(w_2x)w_1^{j-1}O_{m-j,q^{w_2}}^{(h)}(w_1-1)w_2^{m-j}\right)\frac{t^m}{m!} \tag{2.8}$$

By using the symmetry in (2.7), we have

$$\begin{aligned}S&=\left(\frac{1}{w_2}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(w_2(2x_2+1)+w_1w_2x)t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}\sum_{m=0}^{\infty}B_{m,q^{w_2}}^{(h)}(w_1x)w_2^m\frac{t^m}{m!}\right)\left(\sum_{m=0}^{\infty}O_{m,q^{w_1}}^{(h)}(w_2-1)w_1^m\frac{t^m}{m!}\right).\end{aligned}$$

Thus we have

$$S=\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\binom{m}{j}B^{(h)}_{j,q^{w_2}}(w_1x)w_2^{j-1}O^{(h)}_{m-j,q^{w_1}}(w_2-1)w_1^{m-j}\right)\frac{t^m}{m!} \tag{2.9}$$

By comparing coefficients $\dfrac{t^m}{m!}$ in the both sides of (2.8) and (2.9), we arrive at the following theorem:

Theorem 7. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\binom{m}{j}B^{(h)}_{j,q^{w_1}}(w_2x)w_1^{j-1}O^{(h)}_{m-j,q^{w_2}}(w_1-1)w_2^{m-j}\\&=\sum_{j=0}^{m}\binom{m}{j}B^{(h)}_{j,q^{w_2}}(w_1x)w_2^{j-1}O^{(h)}_{m-j,q^{w_1}}(w_2-1)w_1^{m-j},\end{aligned}$$

where $B^{(h)}_{k,q}(x)$ and $O^{(h)}_{m,q}(k)$ denote the second kind (h,q)-Bernoulli polynomials and the sums of powers of consecutive (h,q)-odd integers, respectively.

By using Theorem 2, we have the following corollary:

Corollary 8. Let w_1 and w_2 be positive integers. Then we have

$$\begin{aligned}&\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{m-k}w_2^{j-1}x^{j-k}B^{(h)}_{k,q^{w_2}}O^{(h)}_{m-j,q^{w_1}}(w_2-1)\\&=\sum_{j=0}^{m}\sum_{k=0}^{j}\binom{m}{j}\binom{j}{k}w_1^{j-1}w_2^{m-k}x^{j-k}B^{(h)}_{k,q^{w_1}}O^{(h)}_{m-j,q^{w_2}}(w_1-1),\end{aligned}$$

By using (2.6), we have

$$\begin{aligned}S&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\frac{w_1\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)(w_2t)}d\mu_1(x_2)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_1}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)w_1t}d\mu_1(x_1)\right)\left(\sum_{j=0}^{w_1-1}q^{w_2hj}e^{(2j+1)(w_2t)}\right)\\&=\sum_{j=0}^{w_1-1}q^{w_2hj}\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{\left(2x_1+1+w_2x+(2j+1)\frac{w_2}{w_1}\right)(w_1t)}d\mu_1(x_1)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_1-1}q^{w_2hj}B^{(h)}_{n,q^{w_1}}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^{n-1}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.10}$$

By using the symmetry property in (2.10), we also have

$$\begin{aligned}S&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\frac{w_2\int_{\mathbb{Z}_p}q^{hw_1x_1}e^{(2x_1+1)(w_1t)}d\mu_1(x_1)}{\int_{\mathbb{Z}_p}q^{hw_1w_2x}e^{2w_1w_2tx}d\mu_1(x)}\right)\\&=\left(\frac{1}{w_2}e^{w_1w_2xt}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{(2x_2+1)w_2t}d\mu_1(x_2)\right)\left(\sum_{j=0}^{w_2-1}q^{w_1hj}e^{(2j+1)(w_1t)}\right)\\&=\sum_{j=0}^{w_2-1}q^{w_1hj}\int_{\mathbb{Z}_p}q^{hw_2x_2}e^{\left(2x_2+1+w_1x+(2j+1)\frac{w_1}{w_2}\right)(w_2t)}d\mu_1(x_2)\\&=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{w_2-1}q^{w_1hj}B^{(h)}_{n,q^{w_2}}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^{n-1}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.11}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.10) and (2.11), we have the following theorem.

Theorem 9. Let w_1 and w_2 be positive integers. Then we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1} q^{w_2hj} B_{n,q^{w_1}}^{(h)}\left(w_2x+(2j+1)\frac{w_2}{w_1}\right)w_1^{n-1}\\ &=\sum_{j=0}^{w_2-1} q^{w_1hj} B_{n,q^{w_2}}^{(h)}\left(w_1x+(2j+1)\frac{w_1}{w_2}\right)w_2^{n-1}.\end{aligned} \tag{2.12}$$

Observe that if $h=1$, then (2.12) reduces to Theorem 5 in [9](see [5, 9]). Substituting $w_1=1$ into (2.12), we arrive at the following corollary.

Corollary 10. Let w_2 be positive integer. Then we obtain

$$B_{n,q}^{(h)}(x)=w_2^{n-1}\sum_{j=0}^{w_2-1} q^{hj} B_{n,q^{w_2}}^{(h)}\left(\frac{x-w_2+2j+1}{w_2}\right). \tag{2.13}$$

The Corollary 10 is shown to yield the known distribution relation of the second kind (h,q)-Bernoulli polynomials(see Theorem 3). Note that if $q\to 1$, then (2.13) reduces to distribution relation of the second kind Bernoulli polynomials(see [8]).

Corollary 11. Let w_2 be positive integer. Then we have

$$B_n(x)=w_2^{n-1}\sum_{j=0}^{w_2-1} B_n\left(\frac{x-w_2+2j+1}{w_2}\right).$$

Acknowledgement: This work was supported by 2020 Hannam University Research Fund.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

SOME NEW FUZZY BEST PROXIMITY POINT THEOREMS IN NON-ARCHIMEDEAN FUZZY METRIC SPACES

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ABSTRACT. In this paper, we define fuzzy weak P-property. Then we prove a fuzzy best proximity point theorems for γ-contractions with condition fuzzy weak P-property. Later, we give definition of fuzzy isometric distance between two functions in non-Archimedean fuzzy metric spaces. Also, we introduce γ-proximal contraction type-1 and type-2 contraction respectively via functions preserving fuzzy isometric distance and providing fuzzy isometry. Then, we obtain some fuzzy best proximity results for γ-proximal contractions types in non-Archimedean fuzzy metric spaces. Finally, we present some examples to illustrate the validity of the definitions and results obtained in the paper.

1. INTRODUCTION AND PRELIMINARIES

The Banach contraction principle found by Banach has an important resonance in mathematics as well as in other fields [1]. Later, the subject of fixed point theory attracted the attention of many aouthors and caused this subject to be discussed in different areas of mathematics and different topological spaces. Then, authors intensively introduced many works regarding the fixed point theory. On the other hand, the concept of fuzzy metric space was introduced in different ways by some authors (see [2, 7]). Importantly, Gregori and Sapena [5] introduced the notion of fuzzy contractive mapping and gave some fixed point theorems for complete fuzzy metric spaces in the sense of George and Veeramani, and also for Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense. At the same time, there are presented by many authors by expanding the Banach's result in the literature (see [9–11, 14, 16, 20, 21]).

In this work, we prove some fuzzy best proximity point results for mappings providing γ-proximal contractions. Then, we give some examples are supplied in order to support the useability of our results. Also, we show that our main results are more general than known results in the existing literature.

2010 *Mathematics Subject Classification.* 47H10,54H25.

Key words and phrases. γ-proximal contraction, fuzzy best proximity point, non-Archimedean fuzzy metric space.

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Definition 1. *[12] A binary operation* $*:[0,1]\times[0,1]\to[0,1]$ *is called a continuous triangular norm (in short, continuous* $t-$*norm) if it satisfies the following conditions:*

(TN-1) $*$ *is commutative and associative,*
(TN-2) $*$ *is continuous,*
(TN-3) $*(a,1)=a$ *for every* $a\in[0,1]$,
(TN-4) $*(a,b)\le *(c,d)$ *whenever* $a\le c$, $b\le d$ *and* $a,b,c,d\in[0,1]$.

An arbitrary $t-$norm $*$ can be extended (by associativity) in a unique way to an nary operator taking for $(x_1,x_2,...,x_n)\in[0,1]^n$, $n\in N$, the value $*(x_1,x_2,...,x_n)$ is defined, in [4], by $*_{I=1}^{0}x_i=1$, $*_{I=1}^{n}x_i=*(*_{I=1}^{n-1}x_i,x_n)=*(x_1,x_2,...,x_n)$.

Definition 2. *[3] A fuzzy metric space is an ordered triple* $(X,M,*)$ *such that* X *is a nonempty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2\times(0,\infty)$, *satisfying the following conditions, for all* $x,y,z\in X,\ s,t>0:$

(FM-1) $M(x,y,t)>0$,
(FM-2) $M(x,y,t)=1$ *iff* $x=y$,
(FM-3) $M(x,y,t)=M(y,x,t)$,
(FM-4) $M(x,z,t+s)\ge M(x,y,t)*M(y,z,s)$,
(FM-5) $M(x,y,\cdot):(0,\infty)\to[0,1]$ *is continuous.*

If, in the above definition, the triangular inequality (FM-4) is replaced by
(NA) $M(x,z,max\{t,s\})\ge M(x,y,t)*M(y,z,s)$ for all $x,y,z\in X,\ s,t>0$, or equivalently,
$M(x,z,t)\ge M(x,y,t)*M(y,z,t)$
then the triple $(X,M,*)$ is called a non-Archimedean fuzzy metric space [6].

Definition 3. *Let* $(X,M,*)$ *be a fuzzy metric space (or non-Archimedean fuzzy metric space). Then*

(i) *A sequence* $\{x_n\}$ *in* X *is said to converge to* x *in* X, *denoted by* $x_n\to x$, *if and only if* $\lim\limits_{n\to\infty}M(x_n,x,t)=1$ *for all* $t>0$, *i.e. for each* $r\in(0,1)$ *and* $t>0$, *there exists* $n_0\in N$ *such that* $M(x_n,x,t)>1-r$ *for all* $n\ge n_0$ *[7, 13].*

(ii) *A sequence* $\{x_n\}$ *is a M-Cauchy sequence if and only if for all* $\varepsilon\in(0,1)$ *and* $t>0$, *there exists* $n_0\in N$ *such that* $M(x_n,x_m,t)\ge 1-\varepsilon$ *for all* $m>n\ge n_0$ *[3, 13]. A sequence* $\{x_n\}$ *is a G-Cauchy sequence if and only if* $\lim\limits_{n\to\infty}M(x_n,x_{n+p},t)=1$ *for any* $p>0$ *and* $t>0$ *[4, 5, 15].*

(iii) *The fuzzy metric space* $(X,M,*)$ *is called M-complete (G-complete) if every M-Cauchy (G-Cauchy)sequence is convergent.*

Definition 4. *[18,19] Let A, B be a non-empty subset of a non-Archimedean fuzzy metric space $(X, M, *)$. The mapping $g : A \to A$ is said to be a fuzzy isometric if*

$$M(gx_1, gx_2, t) = M(x_1, x_2, t)$$

for all $x_1, x_2 \in A$.

Definition 5. *[17] For $t > 0$, a non-empty subset A of a fuzzy metric space $(X, M, *)$ is said to be t-approximatively compact if for each x in X and each sequence y_n in A with $M(y_n, x, t) \longrightarrow M(A, x, t)$, there exists a subsequence y_{n_k} of y_n converging to an element y_0 in A.*

Definition 6. *[22] Let $\gamma : [0, 1) \to \mathbb{R}$ be a strictly increasing, continuous mapping and for each sequence $\{a_n\}_{n \in N}$ of positive numbers $\lim_{n\to\infty} a_n = 1$ if and only if $\lim_{n\to\infty} \gamma(a_n) = +\infty$. Let Γ is the family of all γ functions.*
A mapping $T : X \to X$ is said to be a γ-contraction if there exists a $\delta \in (0, 1)$ such that

$$M(Tx, Ty, t) < 1 \Rightarrow \gamma(M(Tx, Ty, t)) \geq \gamma(M(x, y, t)) + \delta \tag{1.1}$$

for all $x, y \in X$ and $\gamma \in \Gamma$.

2. Main Results

In this section, we present some definitions and deduce some best proximity point results in non-Archimedean fuzzy metric spaces.

Let $A_0(t)$ and $B_0(t)$ two nonempty subsets of a fuzzy metric space $(X, M, *)$. We will use the following notations:

$$M(A, B, t) = \sup\{M(x, y, t) : x \in A, y \in B\};$$
$$A_0(t) = \{x \in A : M(x, y, t) = M(A, B, t) \text{ for some } y \in B\};$$
$$B_0(t) = \{y \in B : M(x, y, t) = M(A, B, t) \text{ for some } x \in A\}.$$

Now, let us state our main results.

Definition 7. *Let (A, B) be a pair of nonempty subsets of a non-Archimedean fuzzy metric space X with $A_0 \neq 0$. Then the pair (A, B) is said to have the fuzzy weak P-property if and ony if*

$$\begin{cases} M(x_1, y_1, t) = M(A, B, t) \\ M(x_2, y_2, t) = M(A, B, t) \end{cases} \Longrightarrow M(x_1, x_2, t) \geq M(y_1, y_2, t)$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

Example 8. *Let* $X = R \times R$ *and* $M : X \times X \times (0, \infty) \to (0, 1]$ *be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$*, where* $d : X \times X \to [0, \infty)$ *is the standart metric* $d(x, y) = |x - y|$ *for all* $x \in X$*. Let* $A = \{(0, 0)\}$*,* $B = \{(1, 0), (-1, 0)\}$*. Then here,* $d(A, B) = 1$ *and* $M(A, B, t) = \frac{t}{t+1}$*. Let us consider*

$$\begin{aligned} M(u_1, x_1, t) &= M(A, B, t) \\ M(u_2, x_2, t) &= M(A, B, t). \end{aligned}$$

Herefrom, we have
$(u_1, x_1) = ((0, 0), (1, 0))$ *and* $(u_2, x_2) = ((0, 0), (-1, 0))$

$$M(u_1, u_2, t) = M((0, 0), (0, 0), t) = 1 > \frac{t}{t + 2} = M(x_1, x_2, t).$$

Then it is easy to see that (A, B) *is said to have the fuzzy weak P-property.*

Definition 9. *Let* A, B *be a nonempty subset of a non-Archimedean fuzzy metric space* $(X, M, *)$*. Given* $T : A \to B$ *and a fuzzy isometry* $g : A \to A$*, the mapping* T *is said to preserve fuzzy isometric distance with respect to* g *if*

$$M(Tgx_1, Tgx_2, t) = M(Tx_1, Tx_2, t)$$

for all $x_1, x_2 \in A$*.*

Example 10. *Let* $X = R\times [0, 1]$ *and* $M : X \times X \times (0, \infty) \to (0, 1]$ *be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$*, where* $d : X \times X \to [0, \infty)$ *is the standart metric* $d(x, y) = |x - y|$ *for all* $x \in X$*. Let* $A = \{(x, 0) :$ *for all* $x \in R\}$*. Define* $g : A \to A$ *by* $g(x, 0) = (-x, 0)$*. Then* $M(x, y, t) = \frac{t}{t+d(x,y)} = M(gx, gy, t)$*, where* $x = (x_1, 0)$ *and* $y = (y_1, 0) \in A$*. Therefore,* g *is a fuzzy isometry.*

Theorem 11. *Let* A *and* B *be two nonempty, closed subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Let* $T : A \to B$ *be* γ*-contraction such that* $T(A_0(t)) \subseteq B_0(t)$*. Suppose that the pair* (A, B) *has the fuzzy P-property. Then, there exists a unique* x^* *in* A *such that* $M(x^*, Tx^*, t) = M(A, B, t)$*.*

Proof. Let we choose an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$, we can find $x_1 \in A_0(t)$ such that $M(x_1, Tx_0, t) = M(A, B, t)$. Further, since $T(A_0(t)) \subseteq B_0(t)$, it follows that there is an element x_2 in $A_0(t)$ such that $M(x_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(x_{n+1}, Tx_n, t) = M(A, B, t), \quad \text{for all } n \in N. \tag{2.1}$$

(A, B) satisfies the fuzzy weak P-property, therefore from (2.1) we obtain

$$M(x_n, x_{n+1}, t) \geq M(Tx_{n-1}, Tx_n, t), \quad \text{for all } n \in N. \tag{2.2}$$

Now we will prove that the sequence $\{x_n\}$ is convergent in $A_0(t)$. If there exists $n_0 \in N$ such that $M(Tx_{n_0-1}, Tx_{n_0}, t) = 1$, then by (2.2) we get $M(x_{n_0}, x_{n_0+1}, t) = 1$ which implies $x_{n_0} = x_{n_0+1}$. Therefore, we get

$$Tx_{n_0} = Tx_{n_0+1} \Longrightarrow M(Tx_{n_0}, Tx_{n_0+1}, t) = 1. \tag{2.3}$$

From (2.2) and (2.3), we have that

$$M(x_{n_0+2}, x_{n_0+1}, t) \geq M(Tx_{n_0+1}, Tx_{n_0}, t) = 1 \Longrightarrow x_{n_0+2} = x_{n_0+1}.$$

Therefore, $x_n = x_{n_0}$, for all $n \geq n_0$ and $\{x_n\}$ is convergent in $A_0(t)$. Also, we obtain

$$M(x_{n_0}, Tx_{n_0}, t) = M(x_{n_0+1}, Tx_{n_0}, t) = M(A, B, t).$$

This shows that x_{n_0} is a fuzzy best proximity point of T and the proof is completed. Due to this reason, we suppose that $M(Tx_{n-1}, Tx_n, t) \neq 1$, for all $n \in N$. In view of (1.1) and by (2.2), we get

$$\begin{aligned}
\gamma(M(x_n, x_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\
&\geq \gamma(M(x_{n-2}, x_{n-1}, t)) + 2\delta \\
&\quad \ldots \\
&\geq \gamma(M(x_0, x_1, t)) + n\delta.
\end{aligned} \tag{2.4}$$

Letting $n \to \infty$, from (2.4) we get

$$\lim_{n\to\infty} \gamma(M(x_n, Tx_{n+1}, t)) = +\infty.$$

Then, we have

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1. \tag{2.5}$$

Now, we want to show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there are $\varepsilon \in (0, 1)$ and $t_0 > 0$ such that for all

$k \in N$ there exist $n(k), m(k) \in N$ with $n(k) > m(k) > k$ and

$$M(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \varepsilon \ . \tag{2.6}$$

Assume that $m(k)$ is the least integer exceeding $n(k)$ satisfying the inequality (2.6). Then, we have

$$M(x_{m(k)-1}, x_{n(k)}, t_0) > 1 - \varepsilon$$

and so, for all $k \in N$, we get

$$\begin{aligned} 1 - \varepsilon &\geq M(x_{n(k)}, x_{m(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) * M(x_{m(k)-1}, x_{n(k)}, t_0) \\ &\geq M(x_{m(k)-1}, x_{m(k)}, t_0) * (1 - \varepsilon\ ). \end{aligned} \tag{2.7}$$

By taking $k \to \infty$ in (2.7) and using (2.5), we obtain

$$\lim_{k \to \infty} M(x_{n(k)}, x_{m(k)}, t_0) = 1 - \varepsilon. \tag{2.8}$$

From (FM-4), we get

$$\begin{aligned} M(x_{m(k)+1}, x_{n(k)+1}, t_0) &\geq M(x_{m(k)+1}, x_{m(k)}, t_0) \\ &\quad * M(x_{m(k)}, x_{n(k)}, t_0) \\ &\quad * M(x_{n(k)1}, x_{n(k)+1}, t_0). \end{aligned} \tag{2.9}$$

Taking the limit as $k \to \infty$ in (2.9), we obtain

$$\lim_{k \to \infty} M(x_{n(k)+1}, x_{m(k)+1}, t_0) = 1 - \varepsilon\ . \tag{2.10}$$

By applying the inequality (1.1) with $x = x_{m(k)}$ and $y = x_{n(k)}$

$$\gamma(M(x_{n(k)+1}, x_{m(k)+1}, t)) \geq \gamma(M(x_{n(k)}, x_{m(k)}, t)) + \delta. \tag{2.11}$$

Taking the limit as $k \to \infty$ in (2.11), applying (1.1), from (2.8), (2.10) and continuitiy of γ, we obtain

$$\gamma(1 - \varepsilon) \geq \gamma(1 - \varepsilon) + \delta$$

which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X. Since $A_0(t)$ is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $x^* \in A_0(t)$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

Since T is continuous, we obtain $Tx_n \to Tx^*$. Also, from continuity of the fuzzy metric function M, we have $M(x_{n+1}, Tx_n, t) = M(x^*, Tx^*, t)$. From (2.1), $M(x^*, Tx^*, t) = M(A, B, t)$. So, we prove that x^* is a fuzzy best proximity point of T. The uniqueness of the best proximity point of T. From the condition that T is γ-contraction, we get

$$x_1, x_2 \in A \text{ such that } x_1 \neq x_2 \text{ and } M(x_1, Tx_1, t) = M(x_2, Tx_2, t) = M(A, B, t).$$

Then by the fuzzy weak P-property of (A, B), we have $M(x_1, x_2, t) \geq M(Tx_1, Tx_2, t)$. Also

$$x_1 \neq x_2 \Longrightarrow M(x_1, x_2, t) \neq 1.$$

Hence,

$$\gamma(M(x_1, x_2, t)) \geq \gamma(M(Tx_1, Tx_2, t)) \geq \gamma(M(x_1, x_2, t)) + \delta > \gamma(M(x_1, x_2, t))$$

which is a contradiction. Therefore the fuzzy best proximity point is unique. □

Corollary 12. *Let $(X, M, *)$ be a non-Archimedean fuzzy metric space and $A_0(t)$ a nonempty closed subsets of X. Let $T : A \to A$ be a γ-contraction. Then, there exists a unique x^* in A.*

Example 13. *Let $X = [0, 1] \times R$ and $M : X \times X \times (0, \infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by as in Example 10. Let $A = \{(0, x) : \text{for all } x \in R\}$, $B = \{(1, y) : \text{for all } y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{t+1}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{1-x}$ for all $x \in X$. Now, define $T : A \to B$ by $T(0, x) = (1, \frac{x}{6})$. Then, we get $T(A_0(t)) = B_0(t)$. Let us consider*

$$\begin{aligned} M(u_1, Tx_1, t) &= M(A, B, t) \\ M(u_2, Tx_2, t) &= M(A, B, t). \end{aligned}$$

Herefrom, we have $(u_1, x_1) = ((0, -\frac{z_1}{6}), (0, -z_1))$ *or* $(u_2, x_2) = ((0, -\frac{z_2}{6}), (0, -z_2))$. *Then from (1.1), we obtain,*

$$\begin{aligned} \gamma(M(u_1,u_2,t)) &= \gamma(M((0,\tfrac{-z_1}{6}),(0,\tfrac{-z_2}{6}),t)) = \gamma(\frac{t}{t+\frac{|z_1-z_2|}{6}}) \\ &= \frac{1}{1-\frac{t}{t+\frac{|z_1-z_2|}{6}}} > \frac{1}{1-\frac{t}{t+|z_1-z_2|}} = \gamma(\frac{t}{t+|z_1-z_2|}) \\ &= \gamma(M(x_1,x_2,t)). \end{aligned}$$

That is,

$$\gamma(M(u_1,u_2,t)) > \gamma(M(x_1,x_2,t)).$$

Therefore, there exixts a $\delta \in (0,1)$ *such that*

$$\gamma(M(u_1,u_2,t)) \geq \gamma(M(x_1,x_2,t)) + \delta$$

Then it is easy to see that T *is a* γ*-contraction and* $(0,0)$ *is a unique fuzzy best proximity point of* T.

Definition 14. *(* γ*-proximal contraction of Type-1) Let* A *and* B *be two nonempty subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that a mapping* $T : A \to B$ *is said to be a* γ*-proximal contraction if there exists a* $\delta \in (0,1)$ *for all* $u_1, u_2, x_1, x_2 \in X$ *such that*

$$\left\{ \begin{array}{c} M(u_1,Tx_1,t) = M(A,B,t) \\ M(u_2,Tx_2,t) = M(A,B,t) \\ M(u_1,u_2,t), M(x_1,x_2,t) < 1 \end{array} \right. \Longrightarrow \gamma(M(u_1,u_2,t)) \geq \gamma(M(x_1,x_2,t)) + \delta. \quad (2.12)$$

Definition 15. *(*γ*-proximal contraction of Type-2) Let* A *and* B *be two nonempty subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that a mapping* $T : A \to B$ *is said to be a* γ*-proximal contraction if there exists a* $\delta \in (0,1)$ *for all* $u_1, u_2, x_1, x_2 \in X$ *such that*

$$\left\{ \begin{array}{c} M(u_1,Tx_1,t) = M(A,B,t) \\ M(u_2,Tx_2,t) = M(A,B,t) \\ M(Tu_1,Tu_2,t), M(Tx_1,Tx_2,t) < 1 \end{array} \right. \Longrightarrow \gamma(M(Tu_1,Tu_2,t)) \geq \gamma(M(Tx_1,Tx_2,t)) + \delta. \quad (2.13)$$

Theorem 16. *Let* A *and* B *be two nonempty, closed subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Suppose that* $T : A \to B$ *and* $g : A \to A$ *satisfy the following conditions:*

(i) $T(A_0(t)) \subseteq B_0(t)$,

(ii) $T : A \to B$ *is a continuous* γ*-proximal contraction of type-1,*

(iii) g *is a fuzzy isometry,*

(iv) $A_0(t) \subseteq g(A_0(t))$.

Then, there exists a unique element x *in* A *such that* $M(gx, Tx, t) = M(A, B, t)$.

Proof. Let we choose an element x_0 in $A_0(t)$. Since $T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$. Further, since $Tx_1 \in T(A_0(t)) \subseteq B_0(t)$ and and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t), \qquad \text{for all } n \in \mathbb{N}. \tag{2.14}$$

Now we will prove that the sequence $\{x_n\}$ is convergent in $A_0(t)$. If there exists $n_0 \in N$ such that $M(gx_{n_0}, Tx_{n_0+1}, t) = 1$, then it is clear that sequence $\{x_n\}$ is convergent. Hence, let $M(gx_{n_0}, gx_{n_0+1}, t) \neq 1$, for all $n \in N$. From T is a γ-proximal contraction of type-1 and (2.14), we have

$$\begin{aligned} \gamma(M(gx_n, gx_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\ \Rightarrow \gamma(M(x_n, x_{n+1}, t)) &\geq \gamma(M(x_{n-1}, x_n, t)) + \delta \\ &\cdots \\ &\geq \gamma(M(x_0, x_1, t)) + n\delta. \end{aligned} \tag{2.15}$$

Letting $n \to \infty$, from (2.15) we get

$$\lim_{n\to\infty} \gamma(M(x_n, Tx_{n+1}, t)) = +\infty.$$

Then, if we similarly continue as the process in the proof of Theorem 11, we have $\{x_n\}$ is a Cauchy sequence.

Since is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $x \in A_0(t)$ such that $\lim_{n\to\infty} x_n = x$.

Since T,g and M are continuous, passing to the limit $n \to \infty$, we have

$$M(gx, Tx, t) = M(A, B, t).$$

Let x^* be in $A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$. Now, we will show that $x = x^*$. Suppose to the contrary, let $x \neq x^*$. Therefore, $M(x, x^*, t) \neq 1$. Since T is a γ-proximal contraction of type-1 and g is an isometry, we have

$$\gamma(M(x, x^*, t)) = \gamma(M(gx, gx^*, t)) \geq \gamma(M(x, x^*, t)) + \delta > \gamma(M(x, x^*, t))$$

which is a contradiction. Hence, $x = x^*$. Therefore, the proof of Theorem 16 is completed. □

If we take g is the identity mapping, we obtain the following result.

Corollary 17. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X, M, *)$ such that $A_0(t)$ is nonempty. Assume that A is approximatively compact with respect to B. Also, suppose that $T : A \to B$ satisfy the following conditions:*

(i) $T(A_0(t)) \subseteq B_0(t)$,

(ii) $T : A \to B$ *is a continuous γ-proximal contraction of type-1,*

Then, T has a unique fuzzy best proximity point in A.

Example 18. *Let $X = R\times[-2, 2]$ and $M : X \times X \times (0,\infty) \to (0, 1]$ be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, where $d : X \times X \to [0,\infty)$ is the standart metric $d(x, y) = |x - y|$ for all $x \in X$. Let $A = \{(x, -2) : \text{for all } x \in R\}$, $B = \{(y, 2) : \text{for all } y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 4$ and $M(A, B, t) = \frac{t}{t+4}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{1-x^2}$ for all $x \in X$. Now, define $T : A \to B$ and $g : A \to A$ by

$$T(x, -2) = (\frac{x}{2}, 2) \text{ and } g(x, -2) = (-x, -2)$$

Clearly, g is fuzzy isometry. Then, we have, we get $T(A_0(t)) = B_0(t)$ and $A_0(t) = g(A_0(t))$. Let us consider

$$\begin{aligned} M(gu_1, Tx, t) &= M(A, B, t) \\ M(gu_2, Tx, t) &= M(A, B, t). \end{aligned}$$

Herefrom, we have $(u_1, x_1) = ((-\frac{z_1}{2}, -2), (z_1, -2))$ or $(u_2, x_2) = ((-\frac{z_2}{2}, -2), (z_2, -2))$. We claim that T is a γ-proximal contraction type-1. Now, putting $u_1 = (-\frac{z_1}{2}, -2), x_1 = (z_1, -2)$, $u_2 = (-\frac{z_2}{2}, -2)$ and $x_2 = (z_2, -2)$ in (2.12), we have

$$\begin{aligned} \gamma(M(gu_1, gu_2, t)) &= \gamma(M((\tfrac{z_1}{2}, -2), (\tfrac{z_2}{2}, -2), t) = \gamma(\frac{t}{t + \frac{|z_1-z_2|}{2}}) \\ &= \frac{1}{1 - (\frac{t}{t+\frac{|z_1-z_2|}{2}})^2} > \frac{1}{1 - (\frac{t}{t+|z_1-z_2|})^2} = \gamma(\frac{t}{t + |z_1 - z_2|}) \\ &= \gamma(M(x_1, x_2, t)). \end{aligned}$$

That is, we have

$$\gamma(M(u_1, u_2, t)) > \gamma(M(x_1, x_2, t)).$$

Therefore, there exixts a $\delta \in (0,1)$ *such that*

$$\gamma(M(u_1, u_2, t) \geq \gamma(M(x_1, x_2, t)) + \delta.$$

Then it is easy to see that T *is a* γ*-proximal contraction type-1. It now follows from Theorem 16 that* $(0,-2)$ *is a unique fuzzy best proximity point of* T.

Theorem 19. *Let* A *and* B *be two nonempty, closed subsets of a non-Archimedean fuzzy metric space* $(X, M, *)$ *such that* $A_0(t)$ *is nonempty. Assume that* A *is approximatively compact with respect to* B*. Also, suppose that* $T : A \to B$ *and* $g : A \to A$*satisfy the following conditions:*

(*i*) $T(A_0(t)) \subseteq B_0(t)$,
(*ii*) $T : A \to B$ *is a continuous* γ*-proximal contraction of type-2,*
(*iii*) g *is a fuzzy isometry,*
(*iv*) $A_0(t) \subseteq g(A_0(t))$,
(*v*) T *preserves fuzzy isometric distance with respect to* g.

Then, there exists an element x *in* A *such that* $M(gx, Tx, t) = M(A, B, t)$*. Moreover, if* x^* *is another element of* A *such that* $M(gx^*, Tx^*, t) = M(A, B, t)$.

Proof. Let we choose an element Tx_0 in $T(A_0(t))$. Since $Tx_0 \in T(A_0(t)) \subseteq B_0(t)$ and $A_0(t) \subseteq g(A_0(t))$, we can find $x_1 \in A_0(t)$ such that $M(gx_1, Tx_0, t) = M(A, B, t)$. Further, since $T(A_0(t)) \subseteq B_0(t)$ and and $A_0(t) \subseteq g(A_0(t))$, it follows that there is an element x_2 in $A_0(t)$ such that $M(gx_2, Tx_1, t) = M(A, B, t)$. Recursively, we obtain a sequence $\{x_n\}$ in $A_0(t)$ satisfying

$$M(gx_{n+1}, Tx_n, t) = M(A, B, t), \qquad \text{for all } n \in N. \tag{2.16}$$

Now we will prove that the sequence $\{Tx_n\}$ is convergent in B. If there exists $n_0 \in N$ such that $M(Tgx_{n_0}, Tgx_{n_0+1}, t) = 1$, then it is clear that sequence $\{Tx_n\}$ is convergent. Hence, let $M(Tgx_{n_0}, Tgx_{n_0+1}, t) \neq 1$, for all $n \in N$. From T is a γ-proximal contraction of type-2, T preserves fuzzy isometric distance with respect to g and (2.16), we have

$$\begin{aligned} \gamma(M(Tgx_n, Tgx_{n+1}, t)) &\geq \gamma(M(Tx_{n-1}, Tx_n, t)) + \delta \\ \Rightarrow \gamma(M(Tx_n, Tx_{n+1}, t)) &\geq \gamma(M(Tx_{n-1}, Tx_n, t)) + \delta \\ &\cdots \\ &\geq \gamma(M(Tx_0, Tx_1, t)) + n\delta. \end{aligned} \tag{2.17}$$

Letting $n \to \infty$, from (2.17) we get

$$\lim_{n\to\infty} \gamma(M(Tx_n, Tx_{n+1}, t)) = +\infty.$$

Then, if we similarly continue as the process in the proof of Theorem 11, we have $\{Tx_n\}$ is a Cauchy sequence in B.

Since B is a closed subset of the complete non-Archimedean fuzzy metric space $(X, M, *)$, there exists $y \in B$ such that $\lim_{n\to\infty} Tx_n = y$. From the triangular inequality, we obtain

$$\begin{aligned} M(y, A, t) \geq M(y, gx_n, t) \geq M(y, Tx_{n-1}, t) * M(Tx_{n-1}, gx_n, t) \\ = M(y, Tx_{n-1}, t) * M(A, B, t) \\ \geq M(y, Tx_{n-1}, t) * M(y, A, t). \end{aligned} \tag{2.18}$$

Passing to the limit as $n \to \infty$ in (2.18), we have

$$\lim_{n\to\infty} M(y, gx_n, t) = M(y, A, t).$$

Since $A_0(t)$ is approximatively compact with respect to B, there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that converges to some z in $A_0(t)$. Therefore, we have

$$M(z, y, t) = \lim_{k\to\infty} M(gx_{n_k}, Tgx_{n_k-1}, t) = M(y, A, t).$$

Hence, it implies that $z \in A_0(t)$. Since $A_0(t) \subseteq g(A_0(t))$, there exists $x \in A_0(t)$ such that $z = gx$. Taking to the limit as $\lim_{k\to\infty} gx_{n_k} = gx$ and g is a fuzzy isometry, we obtain

$$\lim_{k\to\infty} x_{n_k} = x.$$

Since T is continuous and $\{Tx_n\}$ is convergent to y, we have

$$\lim_{k\to\infty} Tx_{n_k} = Tx = y.$$

Hence, it follows that

$$M(gx, Tx, t) = \lim_{k\to\infty} M(gx_{n_k}, Tgx_{n_k}, t) = M(A, B, t).$$

Let x^* be in $A_0(t)$ such that $M(gx^*, Tx^*, t) = M(A, B, t)$. Now, we will show that $Tx = Tx^*$. Suppose to the contrary, let $Tx \neq Tx^*$. Therefore, $M(x, Tx^*, t) \neq 1$. Since T is a γ-proximal contraction of type-2 and T preserves fuzzy isometric distance with respect to g, we have

$$\gamma(M(Tx, Tx^*, t)) = \gamma(M(Tgx, Tgx^*, t)) \geq \gamma(M(x, x^*, t)) + \delta > \gamma(M(x, x^*, t))$$

which is a contradiction. Hence, $Tx = Tx^*$. Therefore, the proof of Theorem 19 is completed. □

If we take g is the identity mapping, we obtain the following result.

Corollary 20. *Let A and B be two nonempty, closed subsets of a non-Archimedean fuzzy metric space $(X, M, *)$ such that $A_0(t)$ is nonempty. Assume that A is approximatively compact with respect to B. Also, suppose that $T : A \to B$ satisfy the following conditions:*

(i) $T(A_0(t)) \subseteq B_0(t)$,
(ii) $T : A \to B$ *is a continuous* γ*-proximal contraction of type-2,*

Then, T has a unique fuzzy best proximity point in A. Moreover, if x^* is another fuzzy best proximity point T, then $Tx = Tx^*$.

Example 21. *Let $X = [0,1] \times R$ and $M : X \times X \times (0,\infty) \to (0,1]$ be the non-Archimedean fuzzy metric given by*

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$, where $d : X \times X \to [0,\infty)$ is the standart metric $d(x, y) = |x - y|$ for all $x \in X$. Let $A = \{(0, x) : \text{for all } x \in R\}$, $B = \{(1, y) : \text{for all } y \in R\}$. Then here $A_0(t) = A$, $B_0(t) = B$, $d(A, B) = 1$ and $M(A, B, t) = \frac{t}{t+1}$. Let $\gamma : [0, 1) \to \mathbb{R}$ such that $\gamma = \frac{1}{\sqrt{1-x}}$ for all $x \in X$. Now, define $T : A \to B$ and $g : A \to A$ by

$$T(0, x) = (1, \frac{x}{3}) \text{ and } g(0, x) = (0, -x)$$

Clearly, g is a fuzzy isometry. Then, we have, we get $T(A_0(t)) = B_0(t)$ and $A_0(t) = g(A_0(t))$. Let us consider

$$\begin{aligned} M(gu_1, Tx_1, t) &= M(A, B, t) \\ M(gu_2, Tx_2, t) &= M(A, B, t). \end{aligned}$$

. Clearly, T is preserve isometric distance with respect to g. That is $M(Tgx_1, Tgx_2, t) = M(Tx_1, Tx_2, t)$. We claim that T is a $\gamma-$proximal contraction type-2. Now, putting $u_1 = (0, -\frac{z_1}{3})$, $x_1 = (0, z_1)$, $u_2 = (0, -\frac{z_2}{3},)$ and $x_2 = (0, z_2)$ in (2.13), we have

$$\begin{aligned} \gamma(M(Tgu_1, Tgu_2, t) &= \gamma(M(1, \tfrac{z_1}{9}), (1, \tfrac{z_2}{9}), t) = \gamma(\frac{t}{t + \frac{|z_1 - z_2|}{9}}) \\ &= \frac{1}{\sqrt{1 - \frac{t}{t + \frac{|z_1 - z_2|}{9}}}} > \frac{1}{\sqrt{1 - \frac{t}{t + \frac{|z_1 - z_2|}{3}}}} = \gamma(\frac{t}{t + \frac{|z_1 - z_2|}{3}}) \\ &= \gamma(M(Tx_1, Tx_2, t)). \end{aligned}$$

Since, T preserves isometric distance with respect to g, we have

$$\gamma(M(Tu_1, Tu_2, t)) > \gamma(M(Tx_1, Tx_2, t)).$$

Therefore, there exixts a $\delta \in (0,1)$ *such that*

$$\gamma(M(Tu_1, Tu_2, t)) \geq \gamma(M(Tx_1, Tx_2, t)) + \delta.$$

Then it is easy to see that T *is a* γ*-proximal contraction type-2. It now follows from Theorem 19 that* $(0,0)$ *is a unique fuzzy best proximity point of* T*.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Exact solutions of conformable fractional Harry Dym equation

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Abstract: The aim of this paper is to find exact solutions for the conformable fractional Harry Dym Equation. In this work we deal with three different forms of conformable fractional Harry Dym Equation and for each form a suitable wave variable substitution is found. Each substitution transform its corresponding problem to an ordinary differential equation, What is more, the resulted ordinary differential equations in the three cases are the same. General solutions are obtained by applying the direct integration method on the resulted ordinary differential equation. These obtained solutions are found for some particular choices for the constants values. The behavior of every solution is discussed and illustrated in graphs. The tedious integrals and difficult computations associated with calculations in this paper are performed and simplified by using Mathematica 9.0.

Keywords: Conformable fractional derivative, Harry Dym Equation, Conformable Harry Dym Equation, Exact solutions.

1. Introduction

Recently, differential equations with fractional derivatives attracted the interest of many researchers; since such equations describe effectively many phenomena in applied sciences such as physics, biology, technology, and engineering [3, 7, 14].

Harry Dym equation (HD) was so named related to the name of its discoverer Harry Dym in his unpublished paper 1973-1974, although it appeared to first time in Kruskal and Moser [9]. HD equation represents a system which gathers non-linearity and dispersion, also it is a completely integrable nonlinear evolution equation which obeys an infinite number of conservation laws, but it does not have the Painleve property. More properties for HD equation discussed in details can be found in the reference [4]. Moreover HD equation can be connected to the Korteweg-ge Vries equation which has many applications in hydrodynamics [4, 15].

Many efforts have been done to find exact and approximate solutions for both HD equation and fractional HD equation like algebraic geometric solution of the HD equation[13], solitions solutions of the (2+1) dimensional HD equation via Darboux transformation [2], explicit solutions for HD equation [1], exact solution of the HD equation [12], an efficient approach for fractional HD equation by using sumudu transform [10], symmetries and exact solutions of the time fractional HD equation with Rieman-Liouville derivative [5], and a fractional model of HD equation and its approximate solution [11].

Fractional derivatives have many definitions [14] but the most used of these definitions are Riemann-Liouville derivative and Caputo derivative. They were defined as follows:

(i) Riemann - Liouville Definition. For $\alpha \in$ [n-1, n), the α derivative of f is:

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(\mathrm{n}-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx$$

(ii) Caputo Definition. For $\alpha \in$ [n-1, n), the α derivative of f is:

$$D_a^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx$$

Recently, a new definition called conformable fractional derivative was introduced by authors in [6], Since then the interest of it keeps growing and many equations were solved using such definition [8]. In this paper we intend to find exact solutions for fractional HD equation in the sense of this definition rather than Rieman-Liouville definition or Caputo definition. The rest of the paper is organized as follows: Basics of conformable fractional derivative are stated in section 2, in section 3 solutions for conformable fractional HD equation are found, in section 4 some examples are discussed.

2. Basic results on conformable fractional derivatives.

Now, Let us summarize the basic properties of the conformable fractional derivative definition.

Definition [6]: Given a function$f : [0, \infty) \to \mathbb{R}$. And $t > 0, \alpha \in (0, 1]$, then the conformable fractional derivative of order α is defined as

$$T_{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon},$$

T_{α} is called the conformable fractional derivative of f of order α .

Let $f^{\alpha}(t)$ stands for $T_{\alpha}(f)(t) = \frac{d^{\alpha} f}{dt^{\alpha}}$.

If f is α-differentiable in some$(0, b), b > 0$, and $\lim_{t \to 0^+} f^{\alpha}(t)$ exists, then by definition:

$$f^{\alpha}(0) = \lim_{t \to 0^+} f^{\alpha}(t)$$

Theorem 1 [6]: Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then

1. $T_{\alpha}(af + bg) = a\, T_{\alpha}(f) + b\, T_{\alpha}(g),\ for\ all\ a, b \in \mathbb{R}.$
2. $T_{\alpha}(t^p) = pt^{p-\alpha}\ for\ all\ p \in \mathbb{R}.$
3. $T_{\alpha}(\lambda) = 0\ for\ all\ constants\ functions\ f(t) = \lambda.$
4. $T_{\alpha}(fg) = f\, T_{\alpha}(g) + g\, T_{\alpha}(f).$
5. $T_{\alpha}\left(\frac{f}{g}\right) = \frac{g\, T_{\alpha}(f) - f\, T_{\alpha}(g)}{g^2}$.
6. If, in addition, f is differentiable, then $T_{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}$.

Theorem 2 [8]: let f be an α-differentiable function in conformable sense and differentiable and suppose that g is also differentiable and defined in the range of f. Then

$$T_\alpha\,(fog)\,(t) = t^{1-\alpha}\, g^{'}(t) f^{'}\big(g(t)\big).$$

More properties, definitions and theorems as Roll’s Theorem and Mean Value Theorem for conformable fractional derivative are expressed in the work [6],

3. Fractional Harry Dym Equation.

The classical HD equation is:

$$u_t \ = u^3 u_{xxx} \qquad\qquad (*)$$

Where $u(x,t)$ is a function of two real variables $x\ and\ t$.

Let us write:

$$u_t^\alpha = T_t^\alpha u = \frac{\partial^\alpha u}{\partial t^\alpha}\ , \qquad u_x^\alpha = T_x^\alpha u = \frac{\partial^\alpha u}{\partial x^\alpha}\ , \qquad u_x^{(3\alpha)} = T_x^{(3\alpha)} u = T_x^\alpha\, T_x^\alpha\, T_x^\alpha u\,.$$

Now we will solve three fractional forms of$(*)$:

(i) $u_t^\alpha = u^3 u_{xxx}$. (1)

(ii) $u_t \ = u^3 u_x^{(3\alpha)}$. (2)

(iii) $u_t^\alpha = u^3 u_x^{(3\alpha)}$. (3)

Where $\alpha \in (0,1]$.

Using suitable wave variable substitution in each form will transform the equation to an ordinary differential equation as follows:

1. For form (i) let the wave variable substitution $\eta = x + \frac{c}{\alpha} t^\alpha$ and $u(x,t) = v(\,\eta)$. So one can write $u = v \circ \eta$, now apply Theorem 2 to find u_t^α . You will get that $u_t^\alpha = t^{1-\alpha}\,\eta^{'}(t) v^{'}\big(\eta(t)\big) = cv^{'}, also\ u^3 = v^3 and\ u_{xxx} = v^{'''}$. Hence equation (1) is transformed to:

$$cv^{'} = v^3 v^{'''} \qquad\qquad (4)$$

2. For form (ii) let the wave variable substitution $\eta = \frac{1}{\alpha} x^\alpha + ct$ and $u(x,t) = v(\,\eta) = v \circ \eta.$ so

$u_t^\alpha = cv^{'}, u^3 = v^3 and\ u_x^{(3\alpha)} = v^{'''}$. Then equation (2) is transformed to:

$$cv^{'} = v^3 v^{'''} \qquad\qquad (4)$$

3. For form (iii) let the wave variable substitution $\eta = \frac{1}{\alpha}x^{\alpha} + \frac{c}{\alpha}t^{\alpha}$ and $u(x,t) = v(\eta)$. so $u_t^{\alpha} = cv', u^3 = v^3 and\ u_x^{(3\alpha)} = v'''$. Then equation (3) is transformed to:

$$cv' = v^3 v''' \qquad (4)$$

Now to solve the resulted ordinary differential equation (4), rewrite it as:

$$v''' + \left(\frac{c}{2v^2}\right)' = 0 \qquad (5)$$

Integrate (5) with respect to η, gets

$$v'' + \frac{c}{2v^2} = \frac{c_1}{2} \qquad (6)$$

Multiply (6) by v' then integrate with respect to η yields

$$(v')^2 = \frac{c}{v} + c_1 v + c_2 \qquad (7)$$

Using the separation of variables changes (7) to

$$d\eta = \pm\sqrt{\frac{v}{c_1 v^2 + c_2 v + c}}\ dv \qquad (8)$$

Integrate both sides of (8) using Mathematica 9.0 you will obtain

$$\eta = \pm\int\sqrt{\frac{v}{c_1 v^2 + c_2 v + c}}\,dv + c_3 \qquad (9)$$

$$\eta = \pm i\ \frac{ABCD}{c_1 G}[Elliptic\ E(i\ \sinh^{-1}(G), K) - Elliptic\ F(i\ \sinh^{-1}(G), K)] + c_3 \qquad (10)$$

Where: A $= \sqrt{\frac{v}{c_1 v^2 + c_2 v + c}}$, $B = -c_2 + \sqrt{-4cc_1 + {c_2}^2}$, $C = \sqrt{1 + \frac{2c_1 v}{c_2 - \sqrt{-4cc_1 + {c_2}^2}}}$

$$D=\sqrt{1+\frac{2c_1v}{c_2+\sqrt{-4cc_1+{c_2}^2}}}\quad,\ G=\sqrt{\frac{2c_1v}{c_2+\sqrt{-4cc_1+{c_2}^2}}}\quad \text{and}\ K=\frac{c_2+\sqrt{-4cc_1+{c_2}^2}}{c_2-\sqrt{-4cc_1+{c_2}^2}}\quad .$$

$Elliptic\ F$ and $Elliptic\ E$ are elliptic integrals of the first and second kind respectively.

For some particular choices to the constants c, c_1 and c_2 in equation (9) one can get simpler solutions as follows:

- Let $c_1=c_2=0$, then $\eta=\pm\frac{2}{3}\,v\sqrt{\frac{v}{c}}\ +c_3$, hence

$$v=(c_3\pm\tfrac{3}{2}\sqrt{c}\ \eta)^{\frac{2}{3}}\qquad (11)$$

- Let $c_1=0, c_2\neq 0$, then $\eta=\pm\left(\frac{\sqrt{cv+c_2v^2}}{c_2}\ -\frac{c}{{c_2}^{\frac{3}{2}}}\log(\,2c_2\sqrt{v}+2\sqrt{v{c_2}^2+c_2c}\ )\right)+c_3$

Other suggested constants are:

1. Let $c_2=2\sqrt{cc_1}$.
2. Let $c_2=-2\sqrt{cc_1}$.

You can easily using Mathematica 9.0 to perform the integration of equation (9) to get formula of η after you determine the suggested constants, however the difficulty that faces is how to get v with respect to η explicitly , except the formula in (11), this what was discussed in [12], Hence it seems that formula (11) is the only explicit solution for equations (1), (2) and (3). So results can be summarized as follows:

· The solution of equation (1) is $u(x,t)=\ (c_3\pm\frac{3}{2}\sqrt{c}\ \left(x+\frac{c}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$.

· The solution of equation (2) is $u(x,t)=\ (c_3\pm\frac{3}{2}\sqrt{c}\,(\frac{1}{\alpha}x^{\alpha}+ct)\,)^{\frac{2}{3}}$.

· The solution of equation (3) is $u(x,t)=\ (c_3\pm\frac{3}{2}\sqrt{c}\,(\frac{1}{\alpha}x^{\alpha}+\frac{c}{\alpha}t^{\alpha})\,)^{\frac{2}{3}}$.

Remarks:

1. The same ordinary differential equation is obtained from the three different forms of conformable fractional Harry Dym- Equation after using special wave variable for each form.

2. A function could be α-differentiable at a point but not differentiable, illustrating example was discussed in [6].

4. Examples.

Example 1: Let $\alpha = 0.7$, for the graph of equation (1) solution $u(x,t) = (c_3 + \frac{3}{2}\sqrt{c}\left(x + \frac{c}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4$ and $c = 1$ see Figure 1.

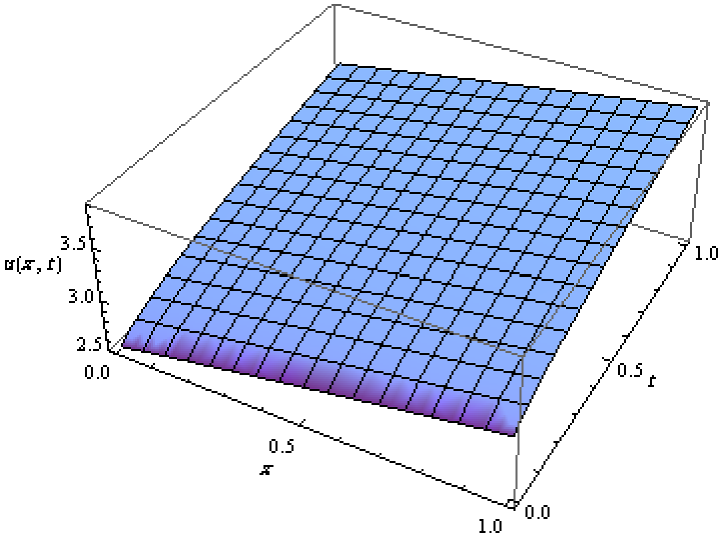

Fig. 1 The graph of $u(x,t) = (4 + \frac{3}{2}\left(x + \frac{1}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ at $\alpha = 0.7$ for example 1

Example 2: The graph of equation (1) solution $u(x,t) = (c_3 + \frac{3}{2}\sqrt{c}\left(x + \frac{c}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ versus x at $t = 1$, $c_3 = 4$ and $c = 1$ for different values of α is in Figure 2.

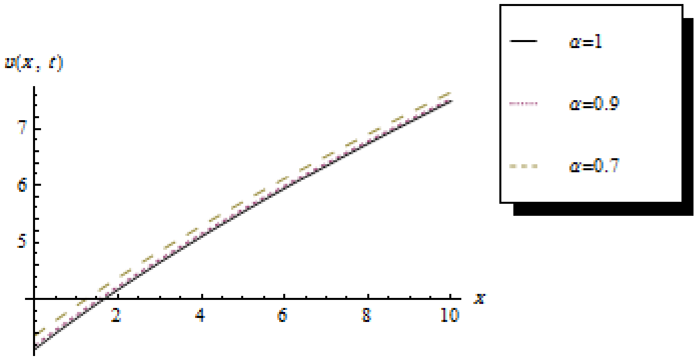

Fig. 2 The graph of $u(x,t) = (4 + \frac{3}{2}\left(x + \frac{1}{\alpha}\right))^{\frac{2}{3}}$ versus x at $t = 1$ at $\alpha = 1, 0.9$ and 0.7 for example 2

Example 3: Let $\alpha = 0.9$, for the graph of equation (2) solution $u(x,t) = (\mathrm{c}_3 + \frac{3}{2}\sqrt{c}\,(\frac{1}{\alpha}x^{\alpha} + ct)\,)^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4$ and $c = 1$ see Figure 3.

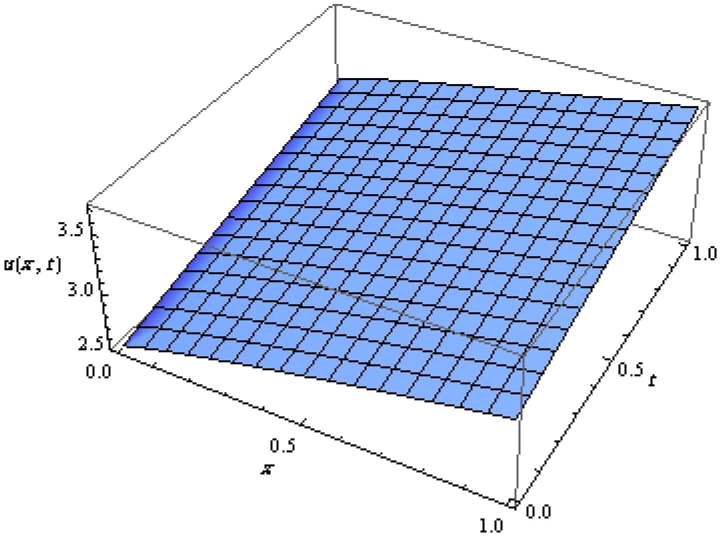

Fig. 3 The graph of $u(x,t) = (4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + t\right))^{\frac{2}{3}}$ at $\alpha = 0.9$ for example 3

Example 4: The graph of equation (2) solution $u(x,t) = \left(4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + t\right)\right)^{\frac{2}{3}}$ versus x at $t = 0, c_3 = 4$ and $c = 1$ for different values of α is in Figure 4.

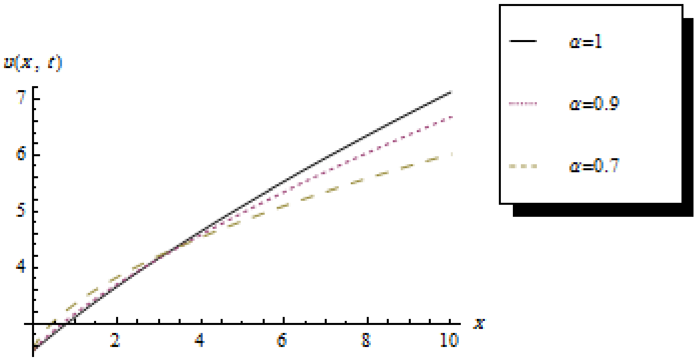

Fig. 4 The graph of $u(x,t) = (4 + \frac{3}{2}\left(\frac{1}{\alpha}\ x^{\alpha} + t\right))^{\frac{2}{3}}$ versus x at t = 0 $at\ \alpha = 1, 0.9\ and\ 0.7$ for example 4

Example 5: Let $\alpha = 0.9$, for the graph of equation (3) solution $u(x,t) = (\mathrm{c}_3 + \frac{3}{2}\sqrt{c}\ (\frac{1}{\alpha}x^{\alpha} + \frac{c}{\alpha}t^{\alpha})\ )^{\frac{2}{3}}$ with respect to x and t, with $c_3 = 4\ and\ c = 1$ see Figure 5.

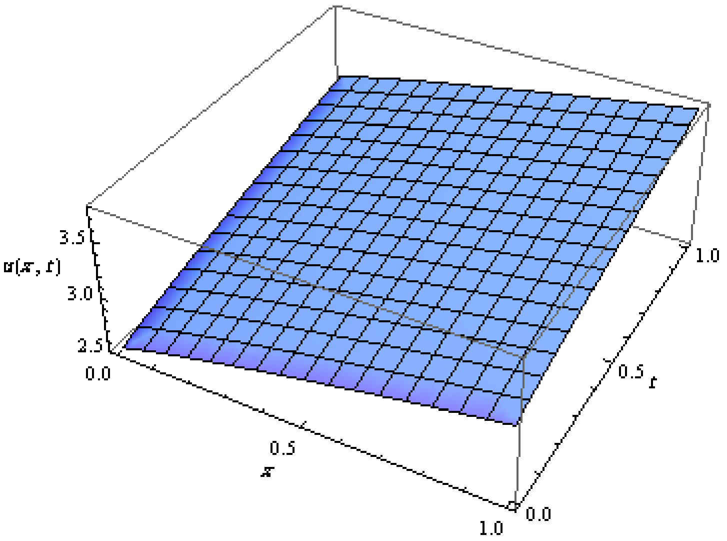

Fig. 5 The graph of $u(x,t) = \ (4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + \frac{1}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ at $\alpha = 0.9$ for example 5

Example 6: The graph of equation (3) solution $u(x,t) = (c_3 + \frac{3}{2}\sqrt{c}\,(\frac{1}{\alpha}x^{\alpha} + \frac{c}{\alpha}t^{\alpha})\,)^{\frac{2}{3}}$ versus x at $t = 1$, $c_3 = 4\ and\ c = 1$ for different values of α is in Figure 6.

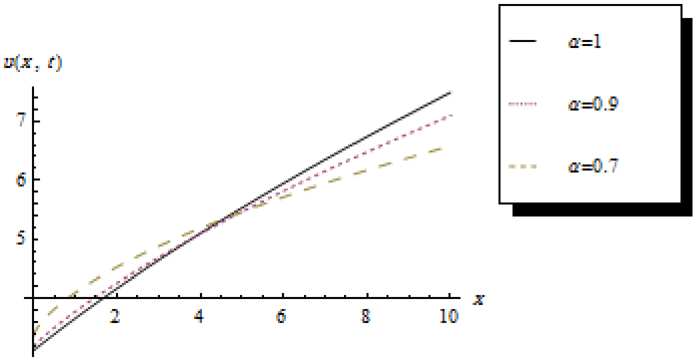

Fig. 6 The graph of $u(x,t) = (4 + \frac{3}{2}\left(\frac{1}{\alpha}x^{\alpha} + \frac{1}{\alpha}t^{\alpha}\right))^{\frac{2}{3}}$ versus x at t $= 1\ at\ \alpha = 1, 0.9\ and\ 0.7$ for example 6

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Some Properties of the q-Exponential Functions

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Abstract. This paper aims to investigate some striking properties of the q-exponential functions more profoundly. To achieve this, at first, the Gauss q-binomial formula is generalized and based on the formula, important properties of the q-exponential functions are established.

Keywords. q-Exponential function, q-Binomial formula.
Mathematics Subject Classification. 11B65, 05A30.

1 Introduction

The q-analogue of any real number t is defined as $[t]_q = \frac{1-q^t}{1-q}$ and the q-factorial, denoted by $[n]_q!$, is defined [1, 2] as

$$[n]_q! = \begin{cases} 1 & \text{if } n = 0, \\ [n]_q \times [n-1]_q \times \cdots \times [1]_q & \text{if } n = 1, 2, \dots. \end{cases} \tag{1}$$

The q-analogue of $(a+x)^n$, denoted by $(a+x)_q^n$, is defined [3] as

$$(a+x)_q^n = \begin{cases} 1 & n = 0, \\ \prod_{m=0}^{n-1}(a+q^m x) & n = 1, 2, \dots. \end{cases} \tag{2}$$

It is also defined for any complex number α as

$$(a+x)_q^\alpha = \frac{(a+x)_q^\infty}{(a+q^\alpha x)_q^\infty}, \tag{3}$$

where $(a+x)_q^\infty := \lim_{n\to\infty} \prod_{m=0}^{n}(a+q^m x)$, and the principal value of q^α is considered, $0 < q < 1$. Yet, the q-Maclaurin series expansion of $(a+x)_q^n$ is

$$(a+x)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q a^{n-k} x^k q^{\binom{k}{2}} \tag{4}$$

where $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ are called q-binomial coefficients. Expression (4) is called Gauss q-binomial formula (see [3], p. 15). In the q-binomial coefficients, if $|q| < 1$ and n tends to infinity (see [3], p. 30) we obtain $\lim_{n\to\infty} \binom{n}{k}_q = \frac{1}{(1-q)_q^k}$. More details about the identities involving q-binomial coefficients can be found in reference [4].

One can also recall definitions of the q-functions [2, 5, 6] as follows:

$$e_q^x = \frac{1}{(1-(1-q)x)_q^\infty} = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n, \qquad |x| < 1, \tag{5}$$

$$E_q^x = (1+(1-q)x)_q^\infty = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} x^n q^{\binom{n}{2}}, \qquad x \in \mathbb{C}. \tag{6}$$

It can be seen that $e_q^x E_q^{-x} = 1$ and $e_{q^{-1}}^x = E_q^x$. The product of the two functions are investigated in a more detailed way in [6, 7, 8]. The contribution of the corresponding references can be summarized in the following theorem:

Theorem 1. *For all $x, y \in \mathbb{C}$ the following equation holds*

$$e_q^x \, E_q^y = \sum_{n=0}^{\infty} \frac{1}{[n]_q!} (x+y)_q^n = e_q^{(x+y)_q} \tag{7}$$

where $(x+y)_q^n$ is defined in (4).

In the light of aforementioned preliminaries, this paper aims at studying about the q-exponential functions more closely. At first, the Gauss q-binomial formula is generalized and based on the formula, some properties of the q-exponential functions are established.

2 q-Exponential Functions

First, let us generalize the q-binomial formula given in (4). The generalization of the q-binomial can then be carried out as follows.

Theorem 2. *For any $x, y, z \in \mathbb{C}$ and positive integer n, the following identity holds:*

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z+y)_q^{n-k}. \tag{8}$$

Proof. The induction is used to prove the theorem. Equation (8) is valid for $n = 1$. Assuming that (8) holds for any n and we show that it holds for $n+1$. Then

$$\begin{aligned}(x+y)_q^{n+1} &= (x+y)_q^n \big(q^k(z+q^{n-k}y) + (x-q^k z)\big)\\ &= \sum_{k=0}^{n} \binom{n}{k}_q q^k (x-z)_q^k (z+y)_q^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^{k+1} (z+y)_q^{n-k}\\ &= (z+y)_q^{n+1} + (x-z)_q^{n+1} + \sum_{k=1}^{n} \binom{n}{k}_q q^k (x-z)_q^k (z+y)_q^{n+1-k}\\ &\qquad + \sum_{k=1}^{n} \binom{n}{k-1}_q (x-z)_q^k (z+y)_q^{n+1-k}\\ &= \sum_{k=0}^{n+1} \binom{n+1}{k}_q (x-z)_q^k (z+y)_q^{n+1-k}.\end{aligned}$$

Thus, the proof is complete. □

It is realized that the identity in Theorem 2 can be re-written as

$$(x+y)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^{n-k} (z+y)_q^{k}. \tag{9}$$

Its proof can be readily derived form the proof of Theorem 2.

Theorem 2 and its re-expression (9) allow one to conclude the striking identities given as follows:

- For $y = 0$ and $z = 1$, the q-Taylor expansion of x^n about $x = 1$, (see [3], p. 23) becomes

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q (x-1)_q^k.$$

- For $x = 1$, $y = -ab$ and $z = a$, the following identity (see [2], p. 25) is obtained

$$(1-ab)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q a^{n-k} (1-a)_q^k (1-b)_q^{n-k}.$$

- For $y = -x$, the identity

$$\sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z-x)_q^{n-k} = 0.$$

  is found.

- For the case of $z = 0$ in (9), the q-binomial formula in (4) is reached.
- For $x = 1$, $y = -ab$ and $z = b$ in (9); the identity (see [2], p. 25)

$$(1-ab)_q^n = \sum_{k=0}^{n} \binom{n}{k}_q b^k (1-a)_q^k (1-b)_q^{n-k}$$

  is stated.

Theorem 3. *For $x, y, z \in \mathbb{C}$, the following equations hold*

$$\frac{(x+y)_q^\infty}{(z+y)_q^\infty} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(x-z)_q^k}{(1-q)^k} \frac{1}{z^k} = e_q^{\frac{(x-z)_q}{(1-q)z}}, \tag{10}$$

and

$$\frac{(x+y)_q^\infty}{(x-z)_q^\infty} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(z+y)_q^k}{(1-q)^k} \frac{1}{x^k} = e_q^{\frac{(z+y)_q}{(1-q)x}}. \tag{11}$$

Proof. As $n \to \infty$ in equation (8), it is arrived at

$$\begin{aligned}(x+y)_q^\infty &= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z+y)_q^{n-k} \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k \frac{(z+y)_q^n}{(z+yq^{n-k})_q^k} \\ &= \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{1}{(1-q)^k} (x-z)_q^k \frac{(z+y)_q^\infty}{z^k}.\end{aligned}$$

Dividing both sides of the last equation by $(z+y)_q^\infty$ gives

$$\frac{(x+y)_q^\infty}{(z+y)_q^\infty} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(x-z)_q^k}{(1-q)^k} \frac{1}{z^k}.$$

By using Theorem 1, the right hand side of the previous equation can be re-written as $e_q^{\frac{(x-z)_q}{(1-q)z}}$ which completes the proof of equation (10). In a similar manner, the latter can be proven. □

Example 1. *If we take $x = 1$ and $y = -az$ in equation (11), we will get (see [2], p. 8)*

$$\frac{(1-az)_q^\infty}{(1-z)_q^\infty} = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \frac{(z-az)_q^k}{(1-q)^k} = \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} z^k = {}_1\phi_0\left(a; -; q, z\right).$$

The function on the right hand side of the above equation is called *basic hypergeometric series* and more details about it can be found in [2].
Now we concentrate about the q-exponential functions. At first, product of the q-exponential functions is given in the next theorem and then some properties of the q-exponential functions are derived.

Remark 1. *For $|x| < 1$ and $|q| < 1$, the following identity holds*

$$\frac{(1-y)_q^\infty}{(1-x)_q^\infty} = \sum_{k=0}^{\infty} \frac{(x-y)_q^k}{(1-q)_q^k}. \tag{12}$$

Theorem 4. *For $x, y, z \in \mathbb{C}$, the following identity holds*

$$e_q^{(x+y)_q} = e_q^{(x-z)_q}\ e_q^{(z+y)_q}. \tag{13}$$

Proof. The identity (7) is taken to expand the q-exponential functions on the right hand side of (13), and thus

$$\begin{aligned}e_q^{(x-z)_q}\ e_q^{(z+y)_q} &= \Big(\sum_{n=0}^{\infty} \frac{1}{[n]_q!}(x-z)_q^n\Big)\Big(\sum_{n=0}^{\infty} \frac{1}{[n]_q!}(z+y)_q^n\Big) \\ &= \sum_{n=0}^{\infty} \frac{1}{[n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q (x-z)_q^k (z+y)_q^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{[n]_q!}(x+y)_q^n = e_q^{(x+y)_q}.\end{aligned}$$

□

Corollary 1. *For $x, z, \in \mathbb{C}$, the following identity holds*

$$e_q^{-(x+z)_q} = \frac{1}{e_q^{(z+x)_q}}.$$

Proof. By taking $y := -x$ and $z := -z$ in Theorem 4, the requirement can be easily carried out. ☐

Theorem 5. *For $x \in \mathbb{C}$ and $m, n \in \mathbb{Z}$, the following identity*

$$e_q^{(m-n)_q x} = \begin{cases} \prod_{j=n}^{m-1} e_q^{((j+1)-j)_q x} & if\ m > n \\ \prod_{j=m}^{n-1} e_q^{(j-(j+1))_q x} & if\ m < n \end{cases}$$

holds.

Proof. First, consider the case of $m > n$. The theorem is proven by induction. For the basis step, $m = n + 1$, the theorem is valid. Take the case $m = k$, $k > n$. Then it needs to be proven that it holds for the case $m = k + 1$. By using identity (13) and the induction, it can be reached

$$e_q^{((k+1)-n)_q x} = e_q^{((k+1)-k)_q x} e_q^{(k-n)_q x} = e_q^{((k+1)-k)_q x} \prod_{j=n}^{k-1} e_q^{((j+1)-j)_q x} = \prod_{j=n}^{k} e_q^{((j+1)-j)_q x}$$

which completes the proof of the first part.
For the case of $m < n$, Corollary 1 is used. Then the result of the first part is applied to get

$$e_q^{(m-n)_q x} = \frac{1}{e_q^{(n-m)_q (x)}} = \frac{1}{\prod_{j=m}^{n-1} e_q^{((j+1)-j)_q x}} = \prod_{j=m}^{n-1} e_q^{(j-(j+1))_q x}$$

which completes the proof. ☐

Corollary 2. *For $x \in \mathbb{C}$, and positive integers m and n, the following identities hold:*

$$e_q^{mx} = \prod_{j=0}^{m-1} e_q^{((j+1)-j)_q x}, \tag{14}$$

$$E_q^{-nx} = \prod_{j=0}^{n-1} e_q^{(j-(j+1))_q x} \tag{15}$$

Proof. Consideration of (7) with $n = 0$ and m any positive integer in Theorem 5 leads to the complete proof of the first identity. Replacing m and n values between each other in the first identity gives the proof of the second one. ☐

Now then, the n-*th* q-derivative of the q-exponential functions is found in the next theorem.

Theorem 6. *For $\alpha, \beta, x \in \mathbb{C}$ and positive integer n,*

$$D_q^n e_q^{(\alpha+\beta)_q x} = (\alpha+\beta)_q^n \ e_q^{(\alpha+q^n\beta)_q x}. \tag{16}$$

Proof. We use the induction to prove the theorem. For the case of $n = 1$, we need to get the q-derivative of $e_q^{(\alpha+\beta)_q x}$. So we use equation (7) and then take the q-derivative to obtain

$$D_q e_q^{(\alpha+\beta)_q x} = D_q \Big(\sum_{k=0}^{\infty} \frac{1}{[k]_q!} (\alpha+\beta)_q^k x^k \Big) = (\alpha+\beta) \sum_{k=0}^{\infty} \frac{1}{[k]_q!} (\alpha+q\beta)_q^k x^k = (\alpha+\beta) \ e_q^{(\alpha+q\beta)_q x}.$$

Assuming that (16) holds for a given k and to prove that it holds for $k + 1$, we need to obtain the q-derivative of $D_q^k e_q^{(\alpha+\beta)_q x}$. Hence

$$D_q^{k+1} e_q^{(\alpha+\beta)_q x} = D_q \Big(D_q^k e_q^{(\alpha+\beta)_q x} \Big) = (\alpha+\beta)_q^k \ D_q \Big(e_q^{(\alpha+q^k\beta)_q x} \Big) = (\alpha+\beta)_q^{k+1} \ e_q^{(\alpha+q^{k+1}\beta)_q x}.$$

Thus the proof is complete. ☐

Theorem 7. *For $|x| < 1$, $|q| < 1$ and any arbitrary α, the following identity holds*

$$e_q^{(1-q^\alpha)_q x} = \frac{1}{(1-(1-q)x)_q^\alpha} \tag{17}$$

Proof. To prove the theorem, we use equations (3), (5), (6) and (7). Then we have

$$e_q^{(1-q^\alpha)_q x} = e_q^x \; E_q^{-q^\alpha x} = \frac{1}{(1-(1-q)x)_q^\infty} \; (1-(1-q)q^\alpha x)_q^\infty = \frac{1}{(1-(1-q)x)_q^\alpha}$$

which completes the proof. ☐

Remark 2. *Equation (17) can be rewritten as* $e_q^{(q^\alpha-1)_q x} = (1-(1-q)x)_q^\alpha$.

In the next example, we show that the q-binomial theorem (see: [1] P. 247 or [9] P. 488) can be proven shortly by using Theorem 1.

Example 2. *For $|x| < 1$ and $|q| < 1$,*

$$\sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} x^k = \sum_{k=0}^{\infty} \frac{(1-a)_q^k}{[k]_q!} \Big(\frac{x}{1-q}\Big)^k = e_q^{\frac{(1-a)_q x}{(1-q)}} = e_q^{(\frac{x}{1-q})} E_q^{(\frac{-ax}{1-q})} = \frac{(1-ax)_q^\infty}{(1-x)_q^\infty}.$$

Note that to reach this result; (7) in the second and third equations, and (5) and (6) in the last equation have been considered.

3 Conclusions and Recommendation

Some striking properties of the q-exponential functions have been analyzed in detail. In doing so, the Gauss q-binomial identity has generalized and based on it, remarkable properties of the q-exponential have been established. For further studies, similar discussion can be carried out for q-trigonometric functions.

4 Acknowledgment

I am thankful to Dr. M. Sari of Yildiz Technical University for patiently helping, advising me and also spending his valuable time to revise the paper.

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BCI-implicative ideals of BCI-algebras using neutrosophic quadruple structure

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Abstract. Neutrosophic quadruple structure is used to study BCI-implicative ideal in BCI-algebra. The conceot of neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra is introduced, and their related properties are investigated. Relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal are consulted. Conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal are provided. A characterization of a neutrosophic quadruple BCI-implicative ideal is displayed, and the extension property of neutrosophic quadruple BCI-implicative ideal is established.

1. Introduction

In [14], Smarandche has introduced the neutrosophic quadruple numbers for the first time. Using the notion of Smarandache's neutrosophic quadruple numbers, Akinleye et al. [2] presented the notion of neutrosophic quadruple algebraic structures. In particular, they studied neutrosophic quadruple rings. Agboola et al. [1] studied neutrosophic quadruple algebraic hyperstructures, in particular, they developed neutrosophic quadruple semihypergroups, neutrosophic quadruple canonical hypergroups and neutrosophic quadruple hyperrings. Using BCK/BCI-algebras, Jun et al. [7] have established neutrosophic quadruple BCK/BCI-algebra, and have studied neutrosophic quadruple (positive implicative) ideal in neutrosophic quadruple BCK-algebra and neutrosophic quadruple closed ideal in neutrosophic quadruple BCI-algebra. Muhiuddin et al. [13] have studied neutrosophic quadruple q-ideal and (regular) neutrosophic quadruple ideal in neutrosophic quadruple BCI-algebra. Muhiuddin et al. [12] also have studied implicative neutrosophic quadruple ideal in neutrosophic quadruple BCK-algebra.

In this article, we study BCI-implicative ideal in BCI-algebra using neutrosophic quadruple structure. We define neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra, and investigate their related properties. We consult relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal. We provide conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal. We discuss a characterization of an neutrosophic quadruple BCI-implicative ideal, and establish the extension property of neutrosophic quadruple BCI-implicative ideal.

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[0] **2010 Mathematics Subject Classification**: 06F35, 03G25, 08A72.

[0] **Keywords**: neutrosophic quadruple BCK/BCI-algebra; neutrosophic quadruple BCI-implicative ideal; neutrosophic quadruple BCI-positive implicative ideal; neutrosophic quadruple BCI-commutative ideal.

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2. **Preliminaries**

A BCK/BCI-algebra, which is an important class of logical algebras, is introduced by K. Iséki (see [4, 5]) and it is being studied by many researchers.

A *BCI-algebra* is a set X with a binary operation "$\cdot$" and a special element "0" that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x \cdot y) \cdot (x \cdot z)) \cdot (z \cdot y) = 0)$,

(II) $(\forall x, y \in X)\ ((x \cdot (x \cdot y)) \cdot y = 0)$,

(III) $(\forall x \in X)\ (x \cdot x = 0)$,

(IV) $(\forall x, y \in X)\ (x \cdot y = 0,\ y \cdot x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 \cdot x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x \cdot 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x \cdot z \leq y \cdot z,\ z \cdot y \leq z \cdot x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x \cdot y) \cdot z = (x \cdot z) \cdot y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x \cdot z) \cdot (y \cdot z) \leq x \cdot y) \tag{2.4}$$

where $x \leq y$ if and only if $x \cdot y = 0$.

Any BCI-algebra X satisfies the following conditions (see [3]):

$$(\forall x, y \in X)(x \cdot (x \cdot (x \cdot y)) = x \cdot y), \tag{2.5}$$

$$(\forall x, y \in X)(0 \cdot (x \cdot y) = (0 \cdot x) \cdot (0 \cdot y)), \tag{2.6}$$

$$(\forall x, y \in X)(0 \cdot (0 \cdot (x \cdot y)) = (0 \cdot y) \cdot (0 \cdot x)). \tag{2.7}$$

An element a in a BCI-algebra X is said to be *minimal* (see [3]) if the following assertion is valid.

$$(\forall x \in X)(x \leq a\ \Rightarrow\ x = a). \tag{2.8}$$

Note that the zero element 0 in a BCI-algebra X is minimal (see [3]).

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x \cdot y \in S$ for all $x, y \in S$. A subset G of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in G, \tag{2.9}$$

$$(\forall x \in X)\,(\forall y \in G)\,(x \cdot y \in G\ \Rightarrow\ x \in G)\,. \tag{2.10}$$

A subset G of a BCI-algebra X is called

- a *closed ideal* of X (see [3]) if it is an ideal of X which satisfies:

$$(\forall x \in X)(x \in G\ \Rightarrow\ 0 \cdot x \in G), \tag{2.11}$$

- a *BCI-positive implicative ideal* of X (see [8, 9]) if it satisfies (2.9) and

$$(\forall x, y, z \in X)\,(((x \cdot z) \cdot z) \cdot (y \cdot z) \in G,\ y \in G\ \Rightarrow\ x \cdot z \in G)\,, \tag{2.12}$$

- a *BCI-commutative ideal* of X (see [10]) if it satisfies (2.9) and
$$\begin{aligned}&(x\cdot y)\cdot z\in G,\ z\in G\\ &\Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in G\end{aligned}\tag{2.13}$$
for all $x,y,z\in X$,
- a *BCI-implicative ideal* of X (see [8]) if it satisfies (2.9) and
$$\begin{aligned}&(((x\cdot y)\cdot y)\cdot(0\cdot y))\cdot z\in G,\ z\in G\\ &\Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in G\end{aligned}\tag{2.14}$$
for all $x,y,z\in X$.

Note that every BCI-implicative ideal is an ideal, but the converse is not true (see [8]).

Lemma 2.1 ([8]). *A subset K of X is a BCI-implicative ideal of a BCI-algebra X if and only if it is an ideal of X that satisfies the following condition.*

$$((x\cdot y)\cdot y)\cdot(0\cdot y)\in K\ \Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in K\tag{2.15}$$

for all $x,y\in X$.

Lemma 2.2 ([10]). *An ideal K of X is a BCI-commutative ideal of X if and only if it satisfies:*

$$x\cdot y\in K\ \Rightarrow\ x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in K\tag{2.16}$$

for all $x,y,z\in X$.

Lemma 2.3 ([9]). *An ideal K of X is a BCI-positive implicative ideal of X if and only if it satisfies:*

$$((x\cdot y)\cdot y)\cdot(0\cdot y)\in K\ \Rightarrow\ x\cdot y\in K\tag{2.17}$$

for all $x,y,z\in X$.

We refer the reader to the books [3, 11] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Let X be a set. A *neutrosophic quadruple X-number* is an ordered quadruple (a,xT,yI,zF) where $a,x,y,z\in X$ and T, I, F have their usual neutrosophic logic meanings (see [7]).

The set of all neutrosophic quadruple X-numbers is denoted by $N_q(X)$, that is,

$$N_q(X):=\{(a,xT,yI,zF)\mid a,x,y,z\in X\},$$

and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $N_q(X)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let X be a BCK/BCI-algebra. We define a binary operation $\boxdot$ on $N_q(X)$ by

$$(a,xT,yI,zF)\boxdot(b,uT,vI,wF)=(a\cdot b,(x\cdot u)T,(y\cdot v)I,(z\cdot w)F)$$

for all (a,xT,yI,zF), $(b,uT,vI,wF)\in N_q(X)$. Given $a_1,a_2,a_3,a_4\in X$, the neutrosophic quadruple BCK/BCI-number (a_1,a_2T,a_3I,a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a}=(a_1,a_2T,a_3I,a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

Then $(N_q(X); \boxdot, \tilde{0})$ is a BCK/BCI-algebra (see [7]), which is called *neutrosophic quadruple BCK/BCI-algebra*, and it is simply denoted by $N_q(X)$.

We define an order relation "$\ll$" and the equality "$=$" on $N_q(X)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \le y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in N_q(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $N_q(X)$.

Let X be a BCK/BCI-algebra. Given nonempty subsets K and J of X, consider the set

$$N_q(K, J) := \{(a, xT, yI, zF) \in N_q(X) \mid a, x \in K \ \& \ y, z \in J\},$$

which is called the neutrosophic quadruple set based on K and J.

The set $N_q(K, K)$ is denoted by $N_q(K)$, and it is called the neutrosophic quadruple set based on K.

3. Neutrosophic quadruple BCI-implicative ideals

In what follows, let X denote a BCI-algebra unless otherwise specified.

Definition 3.1. Let K and J be nonempty subsets of X. Then the neutrosophic quadruple set based on K and J is called a *neutrosophic quadruple BCI-implicative ideal* (briefly, *NQ-BCI-implicative ideal*) over (X, K, J) if it is a BCI-implicative ideal of $N_q(X)$. If $K = J$, then we say that it is an *NQ-BCI-implicative ideal* over (X, K).

Example 3.2. Consider a BCI-algebra $X = \{0, 1, 2, 3, 4, 5\}$ with the binary operation $\cdot$, which is given in Table 1.

TABLE 1. Cayley table for the binary operation "$\cdot$"

$\cdot$	0	1	2	3	4	5
0	0	0	0	3	3	3
1	1	0	1	3	3	3
2	2	2	0	3	3	3
3	3	3	3	0	0	0
4	4	3	4	1	0	0
5	5	3	5	1	1	0

Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 6^4 elements. If we take $K = \{0, 1, 2\}$, then the neutrosophic quadruple set based on K has 81-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\zeta}_i \mid i = 1, 2, \cdots, 80\},$$

and it is an NQ-BCI-implicative ideal over (X, K) where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\zeta}_1 = (0, 0T, 0I, 1F)$, $\tilde{\zeta}_2 = (0, 0T, 0I, 2F)$,
$\tilde{\zeta}_3 = (0, 0T, 1I, 0F)$, $\tilde{\zeta}_4 = (0, 0T, 1I, 1F)$, $\tilde{\zeta}_5 = (0, 0T, 1I, 2F)$,
$\tilde{\zeta}_6 = (0, 0T, 2I, 0F)$, $\tilde{\zeta}_7 = (0, 0T, 2I, 1F)$, $\tilde{\zeta}_8 = (0, 0T, 2I, 2F)$,
$\tilde{\zeta}_9 = (0, 1T, 0I, 0F)$, $\tilde{\zeta}_{10} = (0, 1T, 0I, 1F)$, $\tilde{\zeta}_{11} = (0, 1T, 0I, 2F)$,

$\tilde{\zeta}_{12} = (0, 1T, 1I, 0F)$, $\tilde{\zeta}_{13} = (0, 1T, 1I, 1F)$, $\tilde{\zeta}_{14} = (0, 1T, 1I, 2F)$,
$\tilde{\zeta}_{15} = (0, 1T, 2I, 0F)$, $\tilde{\zeta}_{16} = (0, 1T, 2I, 1F)$, $\tilde{\zeta}_{17} = (0, 1T, 2I, 2F)$,
$\tilde{\zeta}_{18} = (0, 2T, 0I, 0F)$, $\tilde{\zeta}_{19} = (0, 2T, 0I, 1F)$, $\tilde{\zeta}_{20} = (0, 2T, 0I, 2F)$,
$\tilde{\zeta}_{21} = (0, 2T, 1I, 0F)$, $\tilde{\zeta}_{22} = (0, 2T, 1I, 1F)$, $\tilde{\zeta}_{23} = (0, 2T, 1I, 2F)$,
$\tilde{\zeta}_{24} = (0, 2T, 2I, 0F)$, $\tilde{\zeta}_{25} = (0, 2T, 2I, 1F)$, $\tilde{\zeta}_{26} = (0, 2T, 2I, 2F)$,
$\tilde{\zeta}_{27} = (1, 0T, 0I, 0F)$, $\tilde{\zeta}_{28} = (1, 0T, 0I, 1F)$, $\tilde{\zeta}_{29} = (1, 0T, 0I, 2F)$,
$\tilde{\zeta}_{30} = (1, 0T, 1I, 0F)$, $\tilde{\zeta}_{31} = (1, 0T, 1I, 1F)$, $\tilde{\zeta}_{32} = (1, 0T, 1I, 2F)$,
$\tilde{\zeta}_{33} = (1, 0T, 2I, 0F)$, $\tilde{\zeta}_{34} = (1, 0T, 2I, 1F)$, $\tilde{\zeta}_{35} = (1, 0T, 2I, 2F)$,
$\tilde{\zeta}_{36} = (1, 1T, 0I, 0F)$, $\tilde{\zeta}_{37} = (1, 1T, 0I, 1F)$, $\tilde{\zeta}_{38} = (1, 1T, 0I, 2F)$,
$\tilde{\zeta}_{39} = (1, 1T, 1I, 0F)$, $\tilde{\zeta}_{40} = (1, 1T, 1I, 1F)$, $\tilde{\zeta}_{41} = (1, 1T, 1I, 2F)$,
$\tilde{\zeta}_{42} = (1, 1T, 2I, 0F)$, $\tilde{\zeta}_{43} = (1, 1T, 2I, 1F)$, $\tilde{\zeta}_{44} = (1, 1T, 2I, 2F)$,
$\tilde{\zeta}_{45} = (1, 2T, 0I, 0F)$, $\tilde{\zeta}_{46} = (1, 2T, 0I, 1F)$, $\tilde{\zeta}_{47} = (1, 2T, 0I, 2F)$,
$\tilde{\zeta}_{48} = (1, 2T, 1I, 0F)$, $\tilde{\zeta}_{49} = (1, 2T, 1I, 1F)$, $\tilde{\zeta}_{50} = (1, 2T, 1I, 2F)$,
$\tilde{\zeta}_{51} = (1, 2T, 2I, 0F)$, $\tilde{\zeta}_{52} = (1, 2T, 2I, 1F)$, $\tilde{\zeta}_{53} = (1, 2T, 2I, 2F)$,
$\tilde{\zeta}_{54} = (2, 0T, 0I, 0F)$, $\tilde{\zeta}_{55} = (2, 0T, 0I, 1F)$, $\tilde{\zeta}_{56} = (2, 0T, 0I, 2F)$,
$\tilde{\zeta}_{57} = (2, 0T, 1I, 0F)$, $\tilde{\zeta}_{58} = (2, 0T, 1I, 1F)$, $\tilde{\zeta}_{59} = (2, 0T, 1I, 2F)$,
$\tilde{\zeta}_{60} = (2, 0T, 2I, 0F)$, $\tilde{\zeta}_{61} = (2, 0T, 2I, 1F)$, $\tilde{\zeta}_{62} = (2, 0T, 2I, 2F)$,
$\tilde{\zeta}_{63} = (2, 1T, 0I, 0F)$, $\tilde{\zeta}_{64} = (2, 1T, 0I, 1F)$, $\tilde{\zeta}_{65} = (2, 1T, 0I, 2F)$,
$\tilde{\zeta}_{66} = (2, 1T, 1I, 0F)$, $\tilde{\zeta}_{67} = (2, 1T, 1I, 1F)$, $\tilde{\zeta}_{68} = (2, 1T, 1I, 2F)$,
$\tilde{\zeta}_{69} = (2, 1T, 2I, 0F)$, $\tilde{\zeta}_{70} = (2, 1T, 2I, 1F)$, $\tilde{\zeta}_{71} = (2, 1T, 2I, 2F)$,
$\tilde{\zeta}_{72} = (2, 2T, 0I, 0F)$, $\tilde{\zeta}_{73} = (2, 2T, 0I, 1F)$, $\tilde{\zeta}_{74} = (2, 2T, 0I, 2F)$,
$\tilde{\zeta}_{75} = (2, 2T, 1I, 0F)$, $\tilde{\zeta}_{76} = (2, 2T, 1I, 1F)$, $\tilde{\zeta}_{77} = (2, 2T, 1I, 2F)$,
$\tilde{\zeta}_{78} = (2, 2T, 2I, 0F)$, $\tilde{\zeta}_{79} = (2, 2T, 2I, 1F)$, $\tilde{\zeta}_{80} = (2, 2T, 2I, 2F)$.

Theorem 3.3. *Every NQ-BCI-implicative ideal is a neutrosophic quadruple ideal.*

Proof. It is straightforward since every BCI-implicative ideal is an ideal in BCI-algebras. □

The converse of Theorem 3.3 is not true in general as seen in the following example.

Example 3.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation $\cdot$, which is given in Table 2.

TABLE 2. Cayley table for the binary operation "·"

$\cdot$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	0	0	4
2	2	2	0	0	4
3	3	3	2	0	4
4	4	4	4	4	0

Then X is a BCI-algebra (see [8]), and the neutrosophic quadruple BCI-algebra $N_q(X)$ has 625 elements. If we take $K = \{0, 1\}$, then the neutrosophic quadruple set based on K has 16-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\zeta}_i \mid i = 1, 2, \cdots, 15\},$$

and it is a neutrosophic quadruple ideal over (X, K) where
$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\zeta}_1 = (0, 0T, 0I, 1F)$,
$\tilde{\zeta}_2 = (0, 0T, 1I, 0F)$, $\tilde{\zeta}_3 = (0, 0T, 1I, 1F)$,
$\tilde{\zeta}_4 = (0, 1T, 0I, 0F)$, $\tilde{\zeta}_5 = (0, 1T, 0I, 1F)$,
$\tilde{\zeta}_6 = (0, 1T, 1I, 0F)$, $\tilde{\zeta}_7 = (0, 1T, 1I, 1F)$,
$\tilde{\zeta}_8 = (1, 0T, 0I, 0F)$, $\tilde{\zeta}_9 = (1, 0T, 0I, 1F)$,
$\tilde{\zeta}_{10} = (1, 0T, 1I, 0F)$, $\tilde{\zeta}_{11} = (1, 0T, 1I, 1F)$,
$\tilde{\zeta}_{12} = (1, 1T, 0I, 0F)$, $\tilde{\zeta}_{13} = (1, 1T, 0I, 1F)$,
$\tilde{\zeta}_{14} = (1, 1T, 1I, 0F)$, $\tilde{\zeta}_{15} = (1, 1T, 1I, 1F)$.
If we take $\tilde{x} = (2, 2T, 2I, 2F)$ and $\tilde{y} = (3, 3T, 3I, 3F)$ in $N_q(X)$, then

$$\begin{aligned}
&(((\tilde{y} \boxdot \tilde{x}) \boxdot \tilde{x}) \boxdot (\tilde{0} \boxdot \tilde{x})) \boxdot \tilde{0} \\
&= ((((3, 3T, 3I, 3F) \boxdot (2, 2T, 2I, 2F)) \boxdot (2, 2T, 2I, 2F)) \boxdot \\
&\quad ((0, 0T, 0I, 0F) \boxdot (2, 2T, 2I, 2F))) \boxdot (0, 0T, 0I, 0F) \\
&= (0, 0T, 0I, 0F) \in N_q(K).
\end{aligned}$$

But

$$\begin{aligned}
&\tilde{y} \boxdot ((\tilde{x} \boxdot (\tilde{x} \boxdot \tilde{y})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{y} \boxdot \tilde{x})))) \\
&= (3, 3T, 3I, 3F) \boxdot (((2, 2T, 2I, 2F) \boxdot ((2, 2T, 2I, 2F) \boxdot (3, 3T, 3I, 3F))) \boxdot \\
&\quad ((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot ((3, 3T, 3I, 3F) \boxdot (2, 2T, 2I, 2F))))) \\
&= (2, 2T, 2I, 2F) \notin N_q(K).
\end{aligned}$$

Hence $N_q(K)$ is not a BCI-implicative ideal of $N_q(X)$, and so it is not an NQ-BCI-implicative ideal over (X, K).

Lemma 3.5 ([7]). *If K and J are ideals of X, then the neutrosophic quadruple set based on K and J is a neutrosophic quadruple ideal over (X, K, J).*

Theorem 3.6. *The neutrosophic quadruple set based on BCI-implicative ideals K and J of X is an NQ-BCI-implicative ideal over (X, K, J).*

Proof. Let K and J be BCI-implicative ideals of X. Since $0 \in K \cap J$, we get $\tilde{0} \in N_q(K, J)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $N_q(X)$ such that

$$(((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y})) \boxdot \tilde{z} \in N_q(K, J)$$

and $\tilde{z} \in N_q(K,J)$. Then $\tilde{z} = (z_1,\, z_2T,\, z_3I,\, z_4F) \in N_q(K,J)$ and

$$\begin{aligned}
&(((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y})) \boxdot \tilde{z} \\
&= ((((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot (y_1, y_2T, y_3I, y_4F))) \boxdot (z_1, z_2T, z_3I, z_4F) \\
&= (((((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1)) \cdot z_1), ((((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2)) \cdot z_2)T, \\
&((((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3)) \cdot z_3)I, ((((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4)) \cdot z_4)F) \\
&\in N_q(K,J).
\end{aligned}$$

Hence $z_i \in K$ and $(((x_i \cdot y_i) \cdot y_i) \cdot (0 \cdot y_i)) \cdot z_i \in K$ for $i = 1, 2$; and $z_j \in J$ and $(((x_j \cdot y_j) \cdot y_j) \cdot (0 \cdot y_j)) \cdot z_j \in K$ for $j = 3, 4$. Since K and J are BCI-implicative ideals of X, it follows that $x_i \cdot ((y_i \cdot (y_i \cdot x_i)) \cdot (0 \cdot (0 \cdot (x_i \cdot y_i)))) \in K$ and $x_j \cdot ((y_j \cdot (y_j \cdot x_j)) \cdot (0 \cdot (0 \cdot (x_j \cdot y_j)))) \in J$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\begin{aligned}
&\tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \cdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y})))) \\
&= (x_1, x_2T, x_3I, x_4F) \cdot (((y_1, y_2T, y_3I, y_4F) \cdot ((y_1, y_2T, y_3I, y_4F) \cdot \\
&(x_1, x_2T, x_3I, x_4F))) \cdot ((0, 0T, 0I, 0F) \cdot ((0, 0T, 0I, 0F) \cdot \\
&((x_1, x_2T, x_3I, x_4F) \cdot (y_1, y_2T, y_3I, y_4F))))) \\
&= (x_1 \cdot ((y_1 \cdot (y_1 \cdot x_1)) \cdot (0 \cdot (0 \cdot (x_1 \cdot y_1)))), \\
&(x_2 \cdot ((y_2 \cdot (y_2 \cdot x_2)) \cdot (0 \cdot (0 \cdot (x_2 \cdot y_2)))))T, \\
&(x_3 \cdot ((y_3 \cdot (y_3 \cdot x_3)) \cdot (0 \cdot (0 \cdot (x_3 \cdot y_3)))))I, \\
&(x_4 \cdot ((y_4 \cdot (y_4 \cdot x_4)) \cdot (0 \cdot (0 \cdot (x_4 \cdot y_4)))))F) \\
&\in N_q(K,J).
\end{aligned}$$

Hence $N_q(K,J)$ is a BCI-implicative ideal of $N_q(X)$, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J). ☐

Corollary 3.7. *The neutrosophic quadruple set based on a BCI-implicative ideal K of X is an NQ-BCI-implicative ideal over (X, K).*

Proposition 3.8. *Every neutrosophic quadruple set based on BCI-implicative ideals K and J of X satisfies the following condition.*

$$\begin{aligned}
&((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K,J) \\
&\;\Rightarrow \tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y})))) \in N_q(K,J).
\end{aligned} \tag{3.1}$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in N_q(X)$.

Proof. Let $((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K, J)$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in N_q(X)$. Then

$$\begin{aligned}
&((((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1)) \cdot 0, ((((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2)) \cdot 0)T,\\
&((((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3)) \cdot 0)I, ((((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4)) \cdot 0)F)\\
&= (((x_1 \cdot y_1) \cdot y_1) \cdot (0 \cdot y_1), (((x_2 \cdot y_2) \cdot y_2) \cdot (0 \cdot y_2))T,\\
&(((x_3 \cdot y_3) \cdot y_3) \cdot (0 \cdot y_3))I, (((x_4 \cdot y_4) \cdot y_4) \cdot (0 \cdot y_4))F)\\
&= (((x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot (y_1, y_2T, y_3I, y_4F)) \boxdot\\
&((0, 0T, 0I, 0F) \boxdot (y_1, y_2T, y_3I, y_4F))\\
&= ((\tilde{x} \boxdot \tilde{y}) \boxdot \tilde{y}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(K, J),
\end{aligned}$$

and so $(((x_i \cdot y_i) \cdot y_i) \cdot (0 \cdot y_i)) \cdot 0 \in K$ for $i = 1, 2$ and $(((x_j \cdot y_j) \cdot y_j) \cdot (0 \cdot y_j)) \cdot 0 \in J$ for $j = 3, 4$. Since $0 \in K \cap J$, and since K and J are BCI-implicative ideals of X, it follows that $x_i \cdot ((y_i \cdot (y_i \cdot x_i)) \cdot (0 \cdot (0 \cdot (x_i \cdot y_i)))) \in K$ for $i = 1, 2$, and $x_j \cdot ((y_j \cdot (y_j \cdot x_j)) \cdot (0 \cdot (0 \cdot (x_j \cdot y_j)))) \in J$ for $j = 3, 4$. Hence we have

$$\begin{aligned}
&\tilde{x} \boxdot ((\tilde{y} \boxdot (\tilde{y} \boxdot \tilde{x})) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{x} \boxdot \tilde{y}))))\\
&= (x_1 \cdot ((y_1 \cdot (y_1 \cdot x_1)) \cdot (0 \cdot (0 \cdot (x_1 \cdot y_1)))),\\
&(x_2 \cdot ((y_2 \cdot (y_2 \cdot x_2)) \cdot (0 \cdot (0 \cdot (x_2 \cdot y_2)))))T,\\
&(x_3 \cdot ((y_3 \cdot (y_3 \cdot x_3)) \cdot (0 \cdot (0 \cdot (x_3 \cdot y_3)))))I,\\
&(x_4 \cdot ((y_4 \cdot (y_4 \cdot x_4)) \cdot (0 \cdot (0 \cdot (x_4 \cdot y_4)))))F)\\
&\in N_q(K, J).
\end{aligned}$$

This completes the proof. □

We provide conditions for a neutrosophic quadruple set to be an NQ-BCI-implicative ideal.

Theorem 3.9. *Let K and J be ideals of X such that*

$$\begin{aligned}
&((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J)\\
&\Rightarrow\ x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J)
\end{aligned} \tag{3.2}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Assume that $(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z \in K$ (resp., J) for all $x, y \in X$ and $z \in K$ (resp., J). Then $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) since K and J are ideals of X. It follows from the condition (3.2) that $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J). Hence K and J are BCI-implicative ideals of X, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 3.10. *Let K be an ideal of X such that*

$$\begin{aligned}
&((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K\\
&\Rightarrow\ x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K
\end{aligned} \tag{3.3}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X, K).

Theorem 3.11. *Let K and J be ideals of X such that*

$$0 \cdot x \in K \text{ (resp., } J), \tag{3.4}$$

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J) \;\Rightarrow\; x \cdot (y \cdot (y \cdot x)) \in K \text{ (resp., } J) \tag{3.5}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Assume that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) for all $x, y \in X$. Then $x \cdot (y \cdot (y \cdot x)) \in K$ (resp., J) by (3.5). Using (I), (II), (2.3), (2.5), (2.6) and (3.4), we have

$$\begin{aligned}
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y))))) \cdot (x \cdot (y \cdot (y \cdot x)))\\
&\leq (y \cdot (y \cdot x)) \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y))))\\
&\leq 0 \cdot (0 \cdot (x \cdot y))\\
&= 0 \cdot ((0 \cdot x) \cdot (0 \cdot y))\\
&= 0 \cdot ((((0 \cdot y) \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y))\\
&= 0 \cdot ((((0 \cdot (0 \cdot (0 \cdot y))) \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y))\\
&= 0 \cdot ((((0 \cdot x) \cdot (0 \cdot y)) \cdot (0 \cdot y)) \cdot (0 \cdot (0 \cdot y)))\\
&= 0 \cdot (((0 \cdot (x \cdot y)) \cdot (0 \cdot y)) \cdot (0 \cdot (0 \cdot y)))\\
&= 0 \cdot (0 \cdot (((x \cdot y) \cdot y) \cdot (0 \cdot y)))\\
&\in K \text{ (resp., } J).
\end{aligned}$$

It follows that $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J). Hence K and J are BCI-implicative ideals of X by Lemma 2.1. Therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 3.12. *Let K be an ideal of X such that*

$$0 \cdot x \in K, \tag{3.6}$$

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \;\Rightarrow\; x \cdot (y \cdot (y \cdot x)) \in K \tag{3.7}$$

for all $x, y \in X$. Then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X, K).

Theorem 3.13. *Let X be a BCI-algebra satisfying the conditions:*

$$(\forall x, y \in X)(x \cdot y = ((x \cdot y) \cdot y) \cdot (0 \cdot y)), \tag{3.8}$$

$$(\forall x, y \in X)(x \cdot (x \cdot y) = y \cdot (y \cdot (x \cdot (x \cdot y)))). \tag{3.9}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J).

Proof. Let K and J be closed ideals of X. Assume that $((x\cdot y)\cdot y)\cdot(0\cdot y)\in K$ (resp., J). Then $0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\in K$ (resp., J). Using the conditions (3.8), (3.9), (2.3), (2.5), (I) and (III), we have

$$\begin{aligned}
&(x\cdot(y\cdot(y\cdot x)))\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&=(x\cdot(y\cdot(y\cdot x)))\cdot(x\cdot y)\\
&=(x\cdot(x\cdot y))\cdot(y\cdot(y\cdot x))\\
&=(y\cdot(y\cdot(x\cdot(x\cdot y))))\cdot(y\cdot(y\cdot x))\\
&=(y\cdot(y\cdot(y\cdot x)))\cdot(y\cdot(x\cdot(x\cdot y)))\\
&=(y\cdot x)\cdot(y\cdot(x\cdot(x\cdot y)))\\
&\leq(x\cdot(x\cdot y))\cdot x\\
&=0\cdot(x\cdot y)\\
&=0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&\in K\text{ (resp., }J).
\end{aligned}\tag{3.10}$$

It follows that $x\cdot(y\cdot(y\cdot x))\in K$ (resp., J), and so that

$$x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in K\text{ (resp., }J)$$

in the proof of Theorem 3.18. Thus K and J are BCI-implicative ideals of X by Lemma 2.1, and therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J) by Theorem 3.6. □

Corollary 3.14. *Let X be a BCI-algebra satisfying the conditions* (3.8) *and* (3.9). *If K is a closed ideal of X, then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X,K).*

Corollary 3.15. *Let X be a BCI-algebra satisfying the condition:*

$$(\forall x,y\in X)((x\cdot(x\cdot y))\cdot(y\cdot x)=y\cdot(y\cdot x)).\tag{3.11}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J).

Proof. If X satisfies the condition (3.11), then it satisfies two conditions (3.8) and (3.9) (see [?, ?]). Hence the result is induced from Theorem 3.13. □

Corollary 3.16. *Let X be a BCI-algebra satisfying the condition* (3.11). *If K is a closed ideal of X, then the neutrosophic quadruple set based on K is an NQ-BCI-implicative ideal over (X,K).*

Theorem 3.17. *Let X be a BCI-algebra satisfying the condition* (3.9) *and*

$$(\forall x,y\in X)((x\cdot(y\cdot x))\cdot(0\cdot(y\cdot x))=x).\tag{3.12}$$

If K and J are closed ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J).

Proof. Let K and J be closed ideals of X. The conditions (3.12) and (III) lead to the following fact.

$$(z\cdot y)\cdot(((z\cdot y)\cdot(z\cdot(z\cdot y)))\cdot(0\cdot(z\cdot(z\cdot y))))=0.\tag{3.13}$$

It follows from (2.1), (I), (2.2), (2.3) and (III) that

$$\begin{aligned}
&(z\cdot y)\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y))=((z\cdot y)\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y)))\cdot 0\\
&=((z\cdot y)\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y)))\cdot((z\cdot y)\cdot(((z\cdot y)\cdot(z\cdot(z\cdot y)))\cdot\\
&(0\cdot(z\cdot(z\cdot y)))))\\
&\leq(((z\cdot y)\cdot(z\cdot(z\cdot y)))\cdot(0\cdot(z\cdot(z\cdot y))))\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y))\\
&\leq(((z\cdot y)\cdot y)\cdot(0\cdot(z\cdot(z\cdot y))))\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y))\\
&\leq(0\cdot y)\cdot(0\cdot(z\cdot(z\cdot y)))\\
&\leq(z\cdot(z\cdot y))\cdot y\\
&=(z\cdot y)\cdot(z\cdot y)=0.
\end{aligned}\tag{3.14}$$

Hence $(z\cdot y)\cdot(((z\cdot y)\cdot y)\cdot(0\cdot y))=0$ since 0 is a minimal element of X, that is,

$$z\cdot y\leq((z\cdot y)\cdot y)\cdot(0\cdot y).\tag{3.15}$$

On the other hand, we get

$$\begin{aligned}
&(((z\cdot y)\cdot y)\cdot(0\cdot y))\cdot(z\cdot y)=(((z\cdot y)\cdot y)\cdot(z\cdot y))\cdot(0\cdot y)\\
&=(((z\cdot y)\cdot(z\cdot y))\cdot y)\cdot(0\cdot y)=(0\cdot y)\cdot(0\cdot y)=0
\end{aligned}$$

by (2.3) and (III), that is,

$$((z\cdot y)\cdot y)\cdot(0\cdot y)\leq z\cdot y.\tag{3.16}$$

Conditions (3.15) and (3.16) induce

$$z\cdot y=((z\cdot y)\cdot y)\cdot(0\cdot y).$$

Therefore the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X,K,J) by Theorem 3.13. □

We now consider extension property of NQ-BCI-implicative ideal.

Theorem 3.18. *For any nonempty subsets K and J of X, let A and B be closed ideals of X such that $K\subseteq A$ and $J\subseteq B$. If K and J are BCI-implicative ideals of X, then the neutrosophic quadruple set based on A and B is an NQ-BCI-implicative ideal over (X,A,B), which is larger than the NQ-BCI-implicative ideal over (X,K,J).*

Proof. Assume that K and J are BCI-implicative ideals of X. It is clear that $N_q(K,J)\subseteq N_q(A,B)$. Let $((x\cdot y)\cdot y)\cdot(0\cdot y)\in A$ (resp., B) for all $x,y\in X$. Then $0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\in A$ (resp., B) since A and B are closed ideals of X. Using (2.3) and (III) induce

$$\begin{aligned}
&(((x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot y)\cdot y)\cdot(0\cdot y)\\
&=(((x\cdot y)\cdot y)\cdot(0\cdot y))\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&=0\in K\text{ (resp., }J),
\end{aligned}\tag{3.17}$$

which implies from Lemma 2.1 that

$$\begin{aligned}
&(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot((y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\cdot\\
&(0\cdot(0\cdot((x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot y))))\\
&\in K\subseteq A\text{ (resp., }J\subseteq B).
\end{aligned}\tag{3.18}$$

Since

$$\begin{aligned}
&0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))=((0\cdot(x\cdot y))\cdot(0\cdot y))\cdot(0\cdot(0\cdot y))\\
&=(((0\cdot x)\cdot(0\cdot y))\cdot(0\cdot y))\cdot(0\cdot(0\cdot y))\\
&=(((0\cdot(0\cdot(0\cdot y)))\cdot x)\cdot(0\cdot y))\cdot(0\cdot y)\\
&=(((0\cdot y)\cdot x)\cdot(0\cdot y))\cdot(0\cdot y)\\
&=(0\cdot x)\cdot(0\cdot y)\\
&=0\cdot(x\cdot y)
\end{aligned}\tag{3.19}$$

by (2.6), (2.3), (2.5) and (III), we have

$$\begin{aligned}
&0\cdot(0\cdot((x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot y))\\
&=0\cdot(0\cdot((x\cdot y)\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))\\
&=0\cdot((0\cdot(x\cdot y))\cdot(0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))\\
&=0\cdot((0\cdot(x\cdot y))\cdot(0\cdot(x\cdot y)))\\
&=0.
\end{aligned}\tag{3.20}$$

Combining (3.18) and (3.20) implies that

$$\begin{aligned}
&(x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))))\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&=(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\\
&\in A\text{ (resp., }B).
\end{aligned}\tag{3.21}$$

Since A and B are ideals of X, it follows that

$$x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\in A\text{ (resp., }B).\tag{3.22}$$

On the other hand, we have

$$\begin{aligned}
&(x\cdot(y\cdot(y\cdot x)))\cdot(x\cdot(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))))\\
&\leq(y\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))))\cdot(y\cdot(y\cdot x))\\
&\leq(y\cdot x)\cdot(y\cdot(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))))\\
&\leq(x\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\cdot x\\
&=0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y))\\
&\in A\text{ (resp., }B).
\end{aligned}\tag{3.23}$$

By (3.22) and (3.23), we get $x\cdot(y\cdot(y\cdot x))\in A$ (resp., B). Using (3.19), (I), (II) we get

$$\begin{aligned}
&(x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y)))))\cdot(x\cdot(y\cdot(y\cdot x)))\\
&\leq(y\cdot(y\cdot x))\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\\
&\leq 0\cdot(0\cdot(x\cdot y))\\
&=0\cdot(0\cdot(((x\cdot y)\cdot y)\cdot(0\cdot y)))\in A\text{ (resp., }B).
\end{aligned}\tag{3.24}$$

It follows that $x\cdot((y\cdot(y\cdot x))\cdot(0\cdot(0\cdot(x\cdot y))))\in A$ (resp., B). Hence A and B are BCI-implicative ideals of X by Lemma 2.1. Therefore the neutrosophic quadruple set based on A and B is an NQ-BCI-implicative ideal over (X, A, B) by Theorem 3.6. □

Corollary 3.19. *For any nonempty subset K of X, let A be a closed ideal of X such that $K \subseteq A$. If K is a BCI-implicative ideals of X, then the neutrosophic quadruple set based on A is an NQ-BCI-implicative ideal over (X, A), which is larger than the NQ-BCI-implicative ideal over (X, K).*

4. RELATIONS BETWEEN NQ-BCI-COMMUTATIVE IDEAL, NQ-BCI-POSITIVE IMPLICATIVE IDEAL AND NQ-BCI-IMPLICATIVE IDEAL

Theorem 4.1. *For any nonempty subsets K and J of X, every NQ-BCI-implicative ideal over (X, K, J) is an NQ-BCI-commutative ideal over (X, K, J).*

Proof. Let K and J be nonempty subsets of X such that the neutrosophic quadruple set based on K and J is an NQ-BCI-implicative ideal over (X, K, J). Let $x, y, z \in X$ be such that $z \in K$ (resp., J) and $(((x\cdot y)\cdot y)\cdot(0\cdot y))\cdot z \in K$ (resp., J). Then $(z, zT, zI, zF) \in N_q(K, J)$ and

$$\begin{aligned}
&((((x, xT, xI, xF) \boxdot (y, yT, yI, yF)) \boxdot (y, yT, yI, yF))\boxdot \\
&((0, 0T, 0I, 0F) \boxdot (y, yT, yI, yF))) \boxdot (z, zT, zI, zF) \\
&= ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z, ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)T, \\
&((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)I, ((((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot z)F) \\
&\in N_q(K, J)
\end{aligned}$$

Since $N_q(K, J)$ is a BCI-implicative ideal of $N_q(X)$, it follows that

$$\begin{aligned}
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))), (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))T, \\
&(x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))I, (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))F) \\
&= (x, xT, xI, xF) \boxdot (((y, yT, yI, yF) \boxdot ((y, yT, yI, yF) \boxdot (x, xT, xI, xF)))\boxdot \\
&((0, 0T, 0I, 0F) \boxdot ((0, 0T, 0I, 0F) \boxdot ((x, xT, xI, xF) \boxdot (y, yT, yI, yF))))) \\
&\in N_q(K, J).
\end{aligned}$$

Hence $x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K$ (resp., J), and so K and J are BCI-implicative ideals of X. Thus K and J are ideals of X. Assume that $x \cdot y \in K$ (resp., J) for all $x, y \in X$. Then

$$(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot (x \cdot y) = (0 \cdot y) \cdot (0 \cdot y) = 0 \in K \text{ (resp., } J)$$

by using (2.3) and (III), which implies that

$$((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K \text{ (resp., } J).$$

Hence $(((x \cdot y) \cdot y) \cdot (0 \cdot y)) \cdot 0 \in K$ (resp., J) and $0 \in K$ (resp., J). Since K (resp., J) is a BCI-implicative ideal of X, it follows that

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J).$$

Therefore K (resp., J) is a BCI-commutative ideal of X by Lemma 2.2, and consequently the neutrosophic quadruple set based on K and J is an NQ-BCI-commutative ideal over (X, K, J). □

The converse of Theorem 4.1 is not true in general. In fact, $N_q(K)$ in Example 3.4 is not a BCI-implicative ideal of $N_q(X)$. But it is routine to verify that $N_q(K)$ is a BCI-commutative ideal of $N_q(X)$.

Lemma 4.2 ([6])**.** *If K and J are BCI-positive implicative ideals of X, then the neutrosophic quadruple set based on K and J is an NQ-BCI-positive implicative ideal over (X, K, J).*

Theorem 4.3. *For any nonempty subsets K and J of X, every NQ-BCI-implicative ideal over (X, K, J) is an NQ-BCI-positive implicative ideal over (X, K, J).*

Proof. Let K and J be nonempty subsets of X such that $N_q(K, J)$ is a BCI-implicative ideal of $N_q(X)$. Then K and J are ideals of X (see the proof of Theorem 4.1). Let $x, y \in X$ be such that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J). Then

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K \text{ (resp., } J)$$

by Lemma 2.1. Note that

$$\begin{aligned}
&(x \cdot y) \cdot (x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))))\\
&\leq ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \cdot y\\
&= (0 \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))\\
&= (0 \cdot (x \cdot y)) \cdot (y \cdot x)\\
&= ((0 \cdot x) \cdot (0 \cdot y)) \cdot (y \cdot x)\\
&= (0 \cdot (0 \cdot x)) \cdot x\\
&= 0 \in K \text{ (resp., } J).
\end{aligned}$$

It follows that $x \cdot y \in K$ (resp., J). Hence K and J are BCI-positive implicative ideals of X by Lemma 2.3, and therefore $N_q(K, J)$ is a BCI-positive implicative ideal of $N_q(X)$ by Lemma 4.2. □

In the following example, we can see that the converse of Theorem 4.3 is not true in general.

Example 4.4. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the binary operation "·", which is given in Table 3.

TABLE 3. Cayley table for the binary operation "·"

·	0	1	2	3	4
0	0	0	0	0	4
1	1	0	1	0	4
2	2	2	0	0	4
3	3	3	3	0	4
4	4	4	4	4	0

Then X is a BCI-algebra (see [8]), and the neutrosophic quadruple BCI-algebra $N_q(X)$ has 625 elements. If we take $K = \{0, 2\}$, then the neutrosophic quadruple set based on K has 16-elements, that is,

$$N_q(K) = \{\tilde{0}, \tilde{\rho}_i \mid i = 1, 2, \cdots, 15\},$$

where

$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{\rho}_1 = (0, 0T, 0I, 2F)$, $\tilde{\rho}_2 = (0, 0T, 2I, 0F)$,
$\tilde{\rho}_3 = (0, 0T, 2I, 1F)$, $\tilde{\rho}_4 = (0, 2T, 0I, 0F)$, $\tilde{\rho}_5 = (0, 2T, 0I, 2F)$,

$\tilde{\rho}_6 = (0, 2T, 2I, 0F)$, $\tilde{\rho}_7 = (0, 2T, 2I, 2F)$, $\tilde{\rho}_8 = (2, 0T, 0I, 0F)$,
$\tilde{\rho}_9 = (2, 0T, 0I, 2F)$, $\tilde{\rho}_{10} = (2, 0T, 2I, 0F)$, $\tilde{\rho}_{11} = (2, 0T, 2I, 2F)$,
$\tilde{\rho}_{12} = (2, 2T, 0I, 0F)$, $\tilde{\rho}_{13} = (2, 2T, 0I, 2F)$, $\tilde{\rho}_{14} = (2, 2T, 2I, 0F)$,
$\tilde{\rho}_{15} = (2, 2T, 2I, 2F)$.

It is routine to verify that $N_q(K)$ is an NQ-BCI-positive implicative ideal over (X, K). If we take $\tilde{\alpha}_1 = (1, 1T, 1I, 1F)$ and $\tilde{\alpha}_3 = (3, 3T, 3I, 3F)$ in $N_q(X)$, then $\tilde{0} \in N_q(K)$ and

$$(((\tilde{\alpha}_1 \boxdot \tilde{\alpha}_3) \boxdot \tilde{\alpha}_3) \boxdot (\tilde{0} \boxdot \tilde{\alpha}_3)) \boxdot \tilde{0} = \tilde{0} \in N_q(K).$$

But,

$$\tilde{\alpha}_1 \boxdot ((\tilde{\alpha}_3 \boxdot (\tilde{\alpha}_3 \boxdot \tilde{\alpha}_1)) \boxdot (\tilde{0} \boxdot (\tilde{0} \boxdot (\tilde{\alpha}_1 \boxdot \tilde{\alpha}_3)))) = \tilde{\alpha}_1 \boxdot (\tilde{0} \boxdot \tilde{0}) = \tilde{\alpha}_1 \notin N_q(K).$$

Hence $N_q(K)$ is not an NQ-BCI-implicative ideal over (X, K).

We display a characterization of an NQ-BCI-implicative ideal.

Theorem 4.5. *For any nonempty subsets K and J of X, the neutrosophic quadruple set based on K and J is both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K, J) if and only if it is an NQ-BCI-implicative ideal over (X, K, J).*

Proof. For the sufficiency, see Theorems 4.1 and 4.3. Let $N_q(K, J)$ be both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K, J). Then K and J are both a BCI-commutative ideal and a BCI-positive implicative ideal of X. Assume that $((x \cdot y) \cdot y) \cdot (0 \cdot y) \in K$ (resp., J) for all $x, y \in X$. Then $x \cdot y \in K$ (resp., J) by Lemma 2.3, and so

$$x \cdot ((y \cdot (y \cdot x)) \cdot (0 \cdot (0 \cdot (x \cdot y)))) \in K (\text{resp., } J)$$

by Lemma 2.2. It follows from Lemma 2.1 that K and J are BCI-implicative ideals of X. Therefore the neutrosophic quadruple set based on K and J is an NQ-implicative ideal over (X, K, J) by Theorem 3.6. □

Corollary 4.6. *For any nonempty subset K of X, the neutrosophic quadruple set based on K is both an NQ-BCI-commutative ideal and an NQ-BCI-positive implicative ideal over (X, K) if and only if it is an NQ-BCI-implicative ideal over (X, K).*

5. CONCLUSIONS

Smarandache introduced the notion of neutrosophic quadruple numbers by considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where a, b, c and d are real or complex numbers and T, I, F have their usual neutrosophic logic meanings. Jun et al. made up neutrosophic quadruple BCK/BCI-algebras and (positive) implicative neutrosophic quadruple BCK-algebras using neutrosophic quadruple numbers based on BCK/BCI-algebras (instead of real or complex numbers). In this article, we have studied BCI-implicative ideal in BCI-algebra using neutrosophic quadruple structure. We have introduced neutrosophic quadruple BCI-implicative ideal based on nonempty subsets in BCI-algebra, and have investigated their related properties. We have consulted relationship between neutrosophic quadruple ideal, neutrosophic quadruple BCI-implicative ideal, neutrosophic quadruple BCI-positive implicative ideal and neutrosophic quadruple BCI-commutative ideal. We have provided conditions for the neutrosophic quadruple set to be neutrosophic quadruple BCI-implicative ideal. We have discussed a characterization of an NQ-BCI-implicative ideal, and have established the extension property of neutrosophic quadruple BCI-implicative ideal. Based on the contents and ideas of this manuscript, we will study neutrosophic quadruple structure for various algebraic sub-structures in the future.

Acknowledgement The second author, Seok-Zun Song, was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812). The last author, G. Muhiuddin, is partially supported by the research grant S-0064-1439, Deanship of Scientific Research, University of Tabuk, Tabuk-71491, Saudi Arabia.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Isolation numbers of matrices over nonbinary Boolean semiring

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Abstract. Let $\mathbb{B}_k$ be the nonbinary Boolean semiring and A be a $m \times n$ Boolean matrix over $\mathbb{B}_k$. The Boolean rank of a Boolean matrix A is the smallest k such that A can be factored as an $m \times k$ Boolean matrix times a $k \times n$ Boolean matrix. The isolation number of A is the maximum number of nonzero entries in A such that no two are in any row or any column, and no two are in a 2×2 submatrix of all nonzero entries. We have that the isolation number of A is a lower bound on the Boolean rank of A. We also compare the isolation number with the binary Boolean rank of the support of A, and determine the equal cases of them.

1. Introduction

There are many papers on the study of rank of matrices over several semirings containing binary Boolean algebra, fuzzy semiring, semiring of nonegative integers, and so on ([2], [3], [6], and [7]). But there are few papers on isolation numbers of matrices. Gregory et al ([7]) introduced set of isolated entries and compared binary Boolean rank with biclique covering number. Recently Beasley ([2]) introduced isolation number of Boolean matrix and compare it with binary Boolean rank.

In this paper, we investigate the possible isolation number of Boolean matrix and compare it with Boolean rank of Boolean matrix and the binary Boolean rank of the support of the Boolean matrix.

2. Preliminaries

Definition 2.1. A *semiring* $\mathcal{S}$ consists of a set $\mathcal{S}$ with two binary operations, addition and multiplication, such that:

- $\mathcal{S}$ is an Abelian monoid under addition (the identity is denoted by 0);
- $\mathcal{S}$ is a monoid under multiplication (the identity is denoted by 1, $1 \neq 0$);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$ for all $s \in \mathcal{S}$.

Definition 2.2. A semiring $\mathcal{S}$ is called *antinegative* if the zero element is the only element with an additive inverse.

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[0] **2010 Mathematics Subject Classification**: 15A23; 15A03; 15B15.

[0] **Keywords**: Boolean rank; nonbinary Boolean semiring; binary Boolean algebra; isolation number.

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Definition 2.3. A semiring $\mathcal{S}$ is called a *Boolean semiring* if $\mathcal{S}$ is equivalent to a set of subsets of a given set $\mathbb{X}$, the sum of two subsets is their union, and the product is their intersection. The zero element 0 is the empty set and the identity element 1 is the whole set $\mathbb{X}$.

Let $S_k = \{a_1, a_2, \cdots, a_k\}$ be a set of k-elements, $\mathcal{P}(S_k)$ be the set of all subsets of S_k. Then $\mathcal{P}(S_k)$ is the Boolean semiring of all subsets of S_k with operations in above definition. Let $\mathbb{B}_k$ be a Boolean semiring of subsets of $S_k = \{a_1, a_2, \cdots, a_k\}$, that is a subset of $\mathcal{P}(S_k)$. It is straightforward to see that a Boolean semiring $\mathbb{B}_k$ is a commutative and antinegative semiring. Moreover, all of its elements, except 0 and 1, are zero-divisors. If $\mathbb{B}_k$ consists of only 0 (the empty subset) and 1 (the whole set S_k) then it is called a *binary Boolean semiring*, which is denoted as $\mathbb{B}_1$. If $\mathbb{B}_k$ is not a binary Boolean semiring then it is called a *nonbinary Boolean semiring.*

Throughout the paper, we assume that $m \leq n$ and $\mathbb{B}_k$ denotes a nonbinary Boolean semiring, which contains at least 3 elements. Let $\mathcal{M}_{m,n}(\mathbb{B}_k)$ denote the set of $m \times n$ matrices with entries from a Boolean semiring $\mathbb{B}_k$.

Let $\mathcal{M}_n(\mathbb{B}_k) = \mathcal{M}_{m,n}(\mathbb{B}_k)$ if $m = n$, let I_m denote the $m \times m$ identity matrix, $O_{m,n}$ denote the zero matrix in $\mathcal{M}_{m,n}(\mathbb{B}_k)$, $J_{m,n}$ denote the matrix of all ones in $\mathcal{M}_{m,n}(\mathbb{B}_k)$. The subscripts are usually omitted if the order is obvious, and we write I, O, J.

Definition 2.4. The matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ is said to be of *Boolean rank r* if there exist matrices $B \in \mathcal{M}_{m,r}(\mathbb{B}_k)$ and $C \in \mathcal{M}_{r,n}(\mathbb{B}_k)$ such that $A = BC$ and r is the smallest positive integer such that such a factorization exists. We denote $b(A) = r$.

By definition, the unique matrix with Boolean rank equal to 0 is the zero matrix O.

Now let $\mathcal{M}_{m,n}(\mathbb{B}_1)$ denote the set of all $m \times n$ binary Boolean matrices with entries in $\mathbb{B}_1$. The *binary Boolean rank* of $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ is the Boolean rank over $\mathbb{B}_1$ and denoted $b_1(A)$.

Definition 2.5. For two (binary) Boolean matrices A and B, A *dominates* B if $a_{i,j} = 0$ implies $b_{i,j} = 0$.

Given a matrix $X \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, we let $\mathbf{x}^{(j)}$ denote the j^{th} column of X and $\mathbf{x}_{(i)}$ denote the i^{th} row. Now if $b(A) = r$ and $A = BC$ is a factorization of $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}$. Since each of the terms $\mathbf{b}^{(i)}\mathbf{c}_{(i)}$ is a Boolean rank one matrix, the Boolean rank of A is also the minimum number of Boolean rank one matrices whose sum is A.

The binary Boolean rank has many applications in combinatorics, especially graph theory, for example, if $A \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ is the adjacency matrix of the bipartite graph G with bipartition (X, Y), the binary Boolean rank of A is the minimum number of bicliques that cover the edges of G, called the *biclique covering number.*

Definition 2.6. Given a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, a set of *isolated entries* ([7]) is a set of entries, usually written as $E = \{a_{i,j}\}$ such that $a_{i,j} \neq 0$, no two entries in E are in the same row, no two entries in E are in the same column, and, if $a_{i,j}, a_{k,l} \in E$ then, $a_{i,l} = 0$ or $a_{k,j} = 0$. That is, isolated entries are independent entries and any two isolated entries $a_{i,j}$ and $a_{k,l}$ do not lie in a submatrix of A of the form $\begin{bmatrix} a_{i,j} & a_{i,l} \\ a_{k,j} & a_{k,l} \end{bmatrix}$ with all entries nonzero. The *isolation number of* A, $\iota(A)$, is the maximum size of a set of isolated entries.

Note that $\iota(A) = 0$ if and only if $A = O$.

Example 2.7. Let $\sigma \in \mathbb{B}_k$ be neither 0 nor 1 and

$$A = \begin{bmatrix} 1 & 1 & \sigma & 0 & 0 \\ \sigma & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & \sigma \\ 0 & \sigma & 0 & 1 & 1 \\ 0 & 0 & 1 & \sigma & 1 \end{bmatrix}$$

be a Boolean matrix over $\mathbb{B}_k$ and E_1 is the set of $\sigma's$ which are located at the positions $\{a_{1,3}, a_{2,1}, a_{3,5}, a_{4,2}, a_{5,4}\}$ of A. The entries $\sigma's$ of A are isolated entries and hence $\iota(A) = 5$. But the entries of A in the position in $E_2 = \{a_{1,1}, a_{2,2}, a_{3,5}, a_{4,4}, a_{5,3}\}$ are not isolated.

Suppose that $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ and $b(A) = r$. Then there are r Boolean rank one matrices A_i such that

$$A = A_1 + A_2 + \cdots + A_r. \tag{2.1}$$

Because each Boolean rank one matrix can be permuted to a matrix of the form $\begin{bmatrix} N & O \\ O & O \end{bmatrix}$ with all nonzero entries in N, it is easily seen that the matrix consisting of two isolated entries of A cannot be dominated by any one A_i among the Boolean rank one summand of A in (2.1). Thus

$$i(A) \leq b(A). \tag{2.2}$$

Many functions, sets and relations concerning matrices do not depend upon the magnitude or nature of the individual entries of a matrix, but rather only on whether the entry is zero or nonzero. These combinatorially significant matrices have become increasingly important in recent years. Of primary interest is the binary Boolean rank. Finding the binary Boolean rank of a $(0,1)$-matrix is an NP-Complete problem ([8]), and consequently finding bounds on the binary Boolean rank of a matrix is of interest to those researchers that would use binary Boolean rank in their work. If the $(0,1)$-matrix is the reduced adjacency matrix of a bipartite graph, the isolation number of the matrix is the maximum size of a non-competitive matching in the bipartite graph. This is related to the study of such combinatorial problems as the patient hospital problem, the stable marriage problem, etc. An additional reason for studying the isolation number is that it is a lower bound on the Boolean rank of a Boolean matrix over $\mathbb{B}_k$. While finding the isolation number as well as finding the Boolean rank of a Boolean matrix is an NP-Complete problem ([1]), for some matrices finding the isolation number can be easier than finding the Boolean rank especially if the matrix is sparse:

Example 2.8. Let $\sigma \in \mathbb{B}_k$ and

$$F = \begin{bmatrix} 1 & 1 & 1 & \sigma & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & \sigma & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & \sigma & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \sigma & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be a Boolean matrix in $\mathcal{M}_9(\mathbb{B}_k)$.

Then we can easily see $b(F) \leqq 6$ from first 3 rows and columns, however to find that Boolean rank is not 5, requires much calculation if the isolation number is not considered. However, the isolation number is easily seen to be 6, both computationally and visually, the σ's in this matrix represent a set of isolated entries. Thus we have $b(F) = 6$ by (2.2).

Note that if any of the 1's in F are replaced by zeros, the resulting matrix still has Boolean rank 6 as well as isolation number 6.

Terms not specifically defined here can be found in Brualdi and Ryser [5] for matrix terms, or Bondy and Murty [4] for graph theoretic terms.

For our use in the next section, we define the support matrix of a Boolean matrix. If $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then the *support* of A is the binary Boolean matrix $\overline{A} = (\overline{a_{i,j}}) \in \mathcal{M}_{m,n}(\mathbb{B}_1)$ such that $\overline{a_{i,j}} = 1$ if $a_{i,j} \neq 0$ and $\overline{a_{i,j}} = 0$ if $a_{i,j} = 0$.

3. Comparisons between isolation numbers and Boolean ranks over $\mathcal{M}_{m,n}(\mathbb{B}_k)$

In this section, we compare the isolation number with Boolean rank of a Boolean matrix, and also we compare the isolation number with binary Boolean rank of the support of a Boolean matrix.

Lemma 3.1. *For $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $b(A+B) \leq b(A) + b(B)$. And for $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_1)$, $b_1(A+B) \leq b_1(A) + b_1(B)$.*

Proof. It follows from the definition of Boolean rank and equation (2.1). ■

Lemma 3.2. *For $A, B \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $\overline{A+B} = \overline{A} + \overline{B}$ in $\mathcal{M}_{m,n}(\mathbb{B}_1)$.*

Proof. It follows from the facts that $\mathbb{B}_k$ is an antinegative semiring and $1+1=1$ in $\mathbb{B}_1$. ■

Lemma 3.3. *For $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $b_1(\overline{A}) \leq b(A)$.*

Proof. If $b(A) = r$, then A has a Boolean rank one factorization such that $A = \mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}$ with $B = [\mathbf{b}^{(1)}\mathbf{b}^{(2)} \cdots \mathbf{b}^{(r)}] \in \mathcal{M}_{m,k}(\mathbb{B}_k)$ and $C = [\mathbf{c}_{(1)}\mathbf{c}_{(2)} \cdots \mathbf{c}_{(k)}]^t \in \mathcal{M}_{k,n}(\mathbb{B}_k)$ from (2.1). Therefore
$b_1(\overline{A}) = b_1(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)} + \mathbf{b}^{(2)}\mathbf{c}_{(2)} + \cdots + \mathbf{b}^{(r)}\mathbf{c}_{(r)}}) = b_1(\overline{\mathbf{b}^{(1)}\mathbf{c}_{(1)}} + \overline{\mathbf{b}^{(2)}\mathbf{c}_{(2)}} + \cdots + \overline{\mathbf{b}^{(r)}\mathbf{c}_{(r)}}) \leq r$, from Lemma 3.2. Hence $b_1(\overline{A}) \leq b(A)$. ■

We may have strict inequality in Lemma 3.3 as we see in the following example.

Example 3.4. Let $S_3 = \{x, y, z\}$ and $\mathbb{B}_3 = \{0, \{x\}, \{x, y\}, 1\}$ with $1 = \{x, y, z\}$. Consider $X = \begin{bmatrix} 1 & \{x\} \\ \{x,y\} & \{x,y\} \end{bmatrix}$ and $Y = \begin{bmatrix} 1 & \{x\} \\ \{x,y\} & \{x\} \end{bmatrix}$ in $\mathcal{M}_2(\mathbb{B}_3)$. Then $b(X) = 2$ but $b_1(\overline{X}) = b_1(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}) = 1$. Hence $b_1(\overline{X}) < b(X)$. But $b(Y) = b_1(\overline{Y}) = 1$ since $Y = \begin{bmatrix} 1 \\ \{x,y\} \end{bmatrix} \begin{bmatrix} 1 & \{x\} \end{bmatrix}$ over $\mathbb{B}_3$.

Lemma 3.5. *For $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, $\iota(A) = \iota(\overline{A})$.*

Proof. If $a_{i,j}$ and $a_{k,l}$ are any isolated entries in A, then $i \neq k$ and $j \neq l$, and that $a_{i,l} = 0$ or $a_{k,j} = 0$. Hence $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are isolated entries in $\overline{A}$, so we have $\iota(A) \leq \iota(\overline{A})$.

Conversely, if $\overline{a_{i,j}}$ and $\overline{a_{k,l}}$ are any isolated entries in $\overline{A}$, then $a_{i,j} \neq 0$ and $a_{k,l} \neq 0$ and that $a_{i,l} = \overline{a_{i,l}} = 0$ or $a_{k,j} = \overline{a_{k,j}} = 0$. Hence $a_{i,j}$ and $a_{k,l}$ are isolated entries in A, so we have $\iota(\overline{A}) \leq \iota(A)$. ■

Theorem 3.6. *If $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$, then $\iota(A) = 1$ if and only if $b_1(\overline{A}) = 1$.*

Proof. Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. If $b_1(\overline{A}) = 1$ then $A \neq O$ so that $\iota(A) \neq 0$ and since $\iota(A) = \iota(\overline{A}) \leq b_1(\overline{A})$ by (2.2), we have $\iota(A) = 1$.

Conversely, suppose on the contrary that there exists a matrix $A = [a_{i,j}] \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ such that $\iota(A) = 1$, $b_1(\overline{A}) > 1$. Then, there exists two non-equal and nonzero rows of $\overline{A}$, say ith and jth. Hence, without loss of generality, there exists a k such that $\overline{a_{i,k}} = 1$ and $\overline{a_{j,k}} = 0$. Then, $\overline{a_{i,k}}$ and any unit entry in jth row of $\overline{B}$ constitute a set of two isolated entries. Thus, $\iota(A) = \iota(\overline{A}) > 1$, a contradiction. ■

It follows that the subset of $\mathcal{M}_{m,n}(\mathbb{B}_k)$ of matrices with isolation number 1 is the same as the set of matrices whose support has Boolean rank 1.

For $A = A_1 + A_2 + \cdots + A_r$ with $b(A) = r$, let $\mathcal{R}_i$ denote the indices of the nonzero rows of A_i and $\mathcal{C}_j$ denote the indices of the nonzero columns of A_j, $i, j = 1.\cdots, k$. Let $r_i = |\mathcal{R}_i|$, the number of nonzero rows of A_i and $c_j = |\mathcal{C}_j|$, the number of nonzero columns of A_j.

Lemma 3.7. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then if $b(A) \geq b_1(\overline{A}) = 2$ then $\iota(A) = 2$, and if $\iota(A) = 2$ then $b_1(\overline{A}) \neq 3$.*

Proof. If $b_1(\overline{A}) = 2$, then $\iota(A) > 1$ by Theorem 3.6. Since $\iota(A) = \iota(\overline{A}) \leq b_1(\overline{A})$ from Lemma 3.5 and (2.2), we have that $\iota(A) = \iota(\overline{A}) = 2$.

Now, suppose that $\iota(A) = 2$ and that $b_1(\overline{A}) = 3$. Then, we have a factorization of $\overline{A}$ as $\overline{A} = C \times D$ with $C \in \mathcal{M}_{m,3}(\mathbb{B}_1)$ and $D \in \mathcal{M}_{3,n}(\mathbb{B}_1)$. Then, the three rows of D generate all the rows of $\overline{A}$. Since $b_1(\overline{A}) = 3$, D cannot have binary Boolean rank 2 or less. Thus, we have $b_1(D) = 3$. Therefore, we have a factorization of D as $D = E \times F$ with $E \in \mathcal{M}_{3,3}(\mathbb{B}_1)$ and $F \in \mathcal{M}_{3,n}(\mathbb{B}_1)$. Then, the three column of E generate all the columns of D and $b_1(E) = 3$. Therefore, it is sufficient to consider 3×3 matrices of binary Boolean rank 3. However, there are only 10 following 3×3 matrices of binary Boolean rank 3 up to permutations:

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_6 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, B_7 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, B_8 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

$$B_9 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_{10} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since B_5 can be permuted to B_2 and B_7 can be permuted to B_4, and B_9 can be permuted to B_6 with transposing. Therefore, there are only seven non-equivalent 3×3 matrices of binary Boolean rank 3. However, these matrices have three isolation entries on the main diagonal. Thus, we have a contradiction to the conditions that $\iota(B) = 2$ and $r_{\mathbb{B}_1}(\overline{B}) = 3$. Thus, if $\iota(A) = 2$ then $b_1(\overline{A}) \neq 3$. ■

Theorem 3.8. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then, $\iota(A) = 2$ if and only if $b_1(\overline{A}) = 2$.*

Proof. From Lemma 3.7, we have the sufficiency. So we only need show the necessity.

Suppose there exists $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ with $\iota(A) = \iota(\overline{A}) = 2$ and $b_1(\overline{A}) > 2$. By Lemma 3.7, $b_1(\overline{A}) \neq 3$, and hence $b_1(\overline{A}) \geq 4$. Thus we choose A such that if $b_1(\overline{A}) > b_1(\overline{C}) > 2$ then $\iota(C) > 2$. Suppose that $\overline{A} = \overline{A_1} + \overline{A_2} + \cdots + \overline{A_r}$ for $r = b_1(\overline{A})$ where each $\overline{A_i}$ is binary Boolean rank 1, i.e., r is the minimum r such that $b_1(\overline{A}) = r$ and $\iota(A) = 2$. Suppose that $\overline{A_1}$ has the fewest number of nonzero rows of the $\overline{A_i}$'s. As in the proof of the above lemma 3.7, permute the rows of $\overline{A}$ so that $\overline{A_1}$ has nonzero rows $1, 2, \cdots, r_1$. For $j = 1, \cdots, r_1$, let $\overline{B_j}$ be the matrix whose first j rows are the first j rows of $\overline{A}$ and whose last $m - j$ rows are all zero. Let $\overline{C_j}$ be the matrix whose first j rows are all zero and whose last $m - j$ rows are the last $m - j$ rows of $\overline{A}$. Then $\overline{A} = \overline{B_j} + \overline{C_j}$. Further any set of isolated entries of $\overline{C_j}$ is a set of isolated entries for $\overline{A}$. Now, from $b_1(\overline{A}) \leq b_1(\overline{B_j}) + b_1(\overline{C_j})$, and the fact that $b_1(\overline{C_j}) = b_1(\overline{C_{j-1}})$ or $b_1(\overline{C_j}) = b_1(\overline{C_{j-1}}) - 1$, there is some t such that $b_1(\overline{C_t}) = b_1(\overline{A}) - 1$. Since $b_1(\overline{C_t}) < r$ by the choice of $\overline{A}$, for this t, we have that $\iota(\overline{C_t}) > 2$ since $b_1(\overline{C_t}) \geq 3$. That is, $\iota(A) = \iota(\overline{A}) > 2$, which is impossible since $\iota(A) = 2$. Therefore $b_1(\overline{A}) = 2$. ■

Now, as we can see in the following example, there is a Boolean matrix $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$ such that $\iota(\overline{A}) = 3$ and $b_1(\overline{A})$ is relative large, depending on m and n.

Example 3.9. For $n \geq 3$, let $\overline{D_n} = J \setminus I \in \mathcal{M}_n(\mathbb{B}_1)$. Then, it is easily shown that $\iota(\overline{D_n}) = 3$ while $b_1(\overline{D_n}) = r$ where $r = min\left\{h : n \leq \left(\begin{array}{c} h \\ \frac{h}{2} \end{array} \right)\right\}$, see [6](Corollary 2). So, $\iota(\overline{D_{20}}) = 3$ while $b_1(\overline{D_{20}}) = 6$.

Definition 3.10. A *tournament matrix* $[T] \in \mathcal{M}_n(\mathbb{B}_k)$ is the adjacency matrix of a directed graph called a tournament, T. It is characterized by $[T] \circ [T]^t = O$ and $[T] + [T]^t = J - I$, where $\circ$ denotes entrywise multiplication of two matrices.

Now, for each $r = 1, 2, \cdots, \min\{m, n\}$, can we characterize the matrices in $\mathcal{M}_{m,n}(\mathbb{B}_k)$ for which $\iota(A) = b_1(\overline{A})$? Of course it is done if $r = 1$ or $r = 2$ in the above theorems, but only in those cases. For $r = m$ we can also find a characterization:

Theorem 3.11. *Let $1 \leq m \leq n$ and $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. Then, $\iota(A) = b_1(\overline{A}) = m$ if and only if there exist permutation matrices $P \in \mathcal{M}_m(\mathbb{B}_1)$ and $Q \in \mathcal{M}_n(\mathbb{B}_1)$ such that $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T} \in \mathcal{M}_m(\mathbb{B}_1)$ where $\overline{T} \in \mathcal{M}_m(\mathbb{B}_1)$ is dominated by a tournament matrix. (There are no restrictions on C.)*

Proof. Suppose that $\iota(A) = m$. Then we permute A by permutation matrices P and Q so that the set of isolated entries are in the (d, d) positions, $d = 1, \cdots, m$. That is, if $X = PAQ$ then $I = \{x_{1,1}, x_{2,2}, \cdots, x_{m,m}\}$ is the set of isolated entries in X. Therefore $X = [B|C]$, with $\overline{b_{i,i}} = \overline{x_{i,i}} = 1$ and $\overline{b_{i,j}} \cdot \overline{b_{j,i}} = 0$ for every i and $j \neq i$ from the definition of the isolated entries. Thus, $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix. Thus, $PAQ = [B|C]$ where $\overline{B} = I_m + \overline{T}$ and clearly there are no conditions on C.

Conversely, if $PAQ = [B|C]$ and $\overline{B} = I_m + \overline{T}$ where $\overline{T}$ is an m square matrix which is dominated by a tournament matrix, then the diagonal entries of B form a set of isolated entries for PAQ and hence A has a set of m isolated entries. Thus $\iota(A) = b_1(\overline{A}) = m$. ■

Corollary 3.12. *Let* $1 \leq m \leq n$ *and* $A \in \mathcal{M}_{m,n}(\mathbb{B}_k)$. *If there exist permutation matrices* $P \in \mathcal{M}_m(\mathbb{B}_1)$ *and* $Q \in \mathcal{M}_n(\mathbb{B}_1)$ *such that* $PAQ = [B|C]$ *where* $B \in \mathcal{M}_m(\mathbb{B}_k)$ *is a diagonal matrix or a triangular matrix with nonzero diagonal entries, then* $\iota(A) = b_1(\overline{A}) = m$.

4. Conclusions

In this paper, we investigated the nonbinary Boolean rank of a matrix A and the rank of its support for the given isolation number k over nonbinary Boolean semirings. Thus, we proved that the isolation number of A is the same as the Boolean rank of the support of it if the isolation numbers are 1 and 2. If the isolation number were greater than 2, then we showed by example that binary Boolean rank of the support of the given nonbinary Boolean matrix may be strictly greater than the isolation number of the matrix. In addition, in some special cases involving tournament matrices, we obtained that the isolation number of the given matrix and the Boolean rank of its support of the nonbinary Boolean matrix are the same.

Acknowledgement The third author, Seok-Zun Song, was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (No. 2016R1D1A1B02006812).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

ORTHOGONALLY EULER-LAGRANGE TYPE CUBIC FUNCTIONAL EQUATIONS IN ORTHOGONALITY NORMED SPACES

CHANG IL KIM AND GILJUN HAN*

ABSTRACT. In this paper, we investigate the orthogonally Euler-Lagrange type cubic functional equation

$$\begin{aligned} &f(ax+by)+f(ax-by)-ab^2[f(x+y)+f(x-y)]-2a(a^2-b^2)f(x) \\ &+c[f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y)]=0,\ \ x\perp y \end{aligned}$$

for fixed non-zero rational numbers a, b and a fixed non-zero real number c with $a^2 \neq b^2$ and $a \neq \pm 1$ and prove the generalized Hyers-Ulam stability for it by using the fixed point method,

1. INTRODUCTION

Assume that X is a real inner product space and $f : X \longrightarrow \mathbb{R}$ is a solution of the orthogonally Cauchy functional equation $f(x+y) = f(x)+f(y)$, $< x, y >= 0$. By the Pythagorean theorem, $f(x) = \|x\|^2$ is a solution of the conditional equation. Of course, this function does not satisfy the additivity equation everywhere. Thus, orthogonal Cauchy equation is not equivalent to the classic Cauchy equation on the whole inner product space.

The orthogonally Cauchy functional equation

$$f(x+y) = f(x)+f(y),\ \ x\perp y$$

in which $\perp$ is an abstract orthogonality relation, was first investigated by Gudder and Strawther [5]. Rätz [16] introduced a new definition of orthogonality by using more restrictive axioms than of Gudder and Strawther. Moreover, he investigated the structure of orthogonally additive mappings. Rätz and Szabó [17] investigated the problem in a rather more general framework.

Definition 1.1. [17] Let X be a real vector space with $\dim X \geq 2$ and $\perp$ a binary relation on X with the following properties:

(O1) totality for zero: $x\perp 0$ and $0\perp x$ for all $x \in X$;

(O2) independence: if $x, y \in X - \{0\}$, $x\perp y$, then x, y are linearly independent;

(O3) homogeneity: if $x, y \in X$, $x\perp y$, then $\alpha x\perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;

(O4) the Thalesian property: if P is a 2-dimensional subspace of X, $x \in P$ and a non-negative real number k, then there exists an $y \in P$ such that $x\perp y$ and $x + y\perp kx - y$.

The pair $(X, \perp)$ is called an orthogonality space. By an orthogonality normed space, we mean an orthogonality space having a normed structure.

2010 *Mathematics Subject Classification.* 39B55, 47H10, 39B52, 46H25.

Key words and phrases. Hyers-Ulam stability, fixed point theorem, orthogonally cubic functional equation, orthogonality space.

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Remark 1.2. (i) The trivial orthogonality on a vector space X defined by (O1) and for non-zero elements $x, y \in X$, $x \perp y$ if and only if x, y are linearly independent.

(ii) The ordinary orthogonality on an inner product space $(X, <\cdot,\cdot>)$ given by $x \perp y$ if and only if $<x, y>= 0$.

(iii) The Birkhoff-James orthogonality on a normed space $(X, \|\cdot\|)$ defined by $x \perp y$ if and only if $\|x + ky\| \ge \|x\|$ for all $k \in \mathbb{R}$.

The relation $\perp$ is called symmetric if $x \perp y$ implies that $y \perp x$ for all $x, y \in X$. Then clearly examples (i) and (ii) are symmetric but example (iii) is not. However, that a real normed space of dimension greater than 2 is an inner product space if and only if the Birkhoff-James orthogonality is symmetric.

In 1940, S. M. Ulam proposed the following stability problem (cf. [19]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [6] gave a partial solution of Ulam's problem for the case of approximate additive mappings. In 1978, Rassias [14] extended the theorem of Hyers by considering the unbounded Cauchy difference. The result of Rassias has provided a lot of influence in the development of what we now call the generalized Hyers-Ulam stability or Hyers-Ulam stability of functional equations. Ger and Sikorska [4] investigated the orthogonal stability of the Cauchy functional equation

$$f(x + y) = f(x) + f(y), \ \ x \perp y \tag{1.1}$$

and Vajzović [20] investigated the orthogonally additive-quadratic equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y), \ \ x \perp y \tag{1.2}$$

when X is a Hilbert space, Y is a scalar field, f is continuous and $\perp$ means the Hilbert space orthogonality. Later, many mathematicians have investigated the orthogonal stability of functional equations ([3], [9], [10], [11], [12], [13], and [18]).

In 2001, Rassias [15] introduced the following cubic functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) = 0 \tag{1.3}$$

and every solution of the cubic functional equation is called *a cubic mapping* and Jun, Kim, and Chang [8] introduced the Euler-Lagrange cubic functional equation.

In this paper, we consider the following orthogonally Euler-Lagrange type cubic functional equation

$$\begin{aligned} &f(ax + by) + f(ax - by) - ab^2[f(x + y) + f(x - y)] - 2a(a^2 - b^2)f(x) \\ &+ c[f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y)] = 0, \ \ x \perp y. \end{aligned} \tag{1.4}$$

for fixed non-zero rational numbers a, b and a fixed non-zero real numbers c with $a^2 \ne b^2$ and $a \ne \pm 1$ and prove the generalized Hyers-Ulam stability for it. Every solution of (1.4) is called *an orthogonally Euler-Lagrange type cubic mapping.*

Throughtout this paper, $(X, \perp)$ is an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ is a Banach space.

2. Solutions of (1.4)

In this section, we investigate solutiuons of (1.4). We will show that a mapping f satisfying (1.4) is an orthogonally cubic mapping.

Theorem 2.1. *Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. If f satisfies (1.4) and $c \neq 0$, then f is an orthogonally cubic mapping.*

Proof. Suppose that f satisfies (1.4). Setting $y = 0$ in (1.4), we have

$$f(ax) = a^3 f(x) \tag{2.1}$$

for all $x \in X$ and setting $x = 0$ and $y = x$ in (1.4), we have

$$f(bx) + f(-bx) = (ab^2 + 9c)f(x) + (ab^2 + c)f(-x) - cf(2x) \tag{2.2}$$

for all $x \in X$. Replacing x by $-x$ in (2.2), we have

$$f(bx) + f(-bx) = (ab^2 + 9c)f(-x) + (ab^2 + c)f(x) - cf(-2x) \tag{2.3}$$

for all $x \in X$. Since $c \neq 0$, by (2.2) and (2.3), we have

$$f(2x) - f(-2x) = 8[f(x) - f(-x)] \tag{2.4}$$

for all $x \in X$. Relpacing y by ay in (1.4), by (2.2), we have

$$\begin{aligned} &a^3[f(x + by) + f(x - by)] - (ab^2 + 3c)f(x + ay) - (ab^2 + c)f(x - ay) \\ &+ cf(x + 2ay) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(ay) = 0 \end{aligned} \tag{2.5}$$

for all $x, y \in X$ with $x \perp y$ and letting $y = \frac{y}{b}$ in (2.5), we have

$$\begin{aligned} &a^3[f(x + y) + f(x - y)] - (ab^2 + 3c)f(x + py) - (ab^2 + c)f(x - py) \\ &+ cf(x + 2py) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(py) = 0 \end{aligned} \tag{2.6}$$

for all $x, y \in X$ with $x \perp y$, where $p = \frac{a}{b}$. Letting $y = -y$ in (2.6), we have

$$\begin{aligned} &a^3[f(x - y) + f(x + y)] - (ab^2 + 3c)f(x - py) - (ab^2 + c)f(x + py) \\ &+ cf(x - 2py) - (2a^3 - 2ab^2 - 3c)f(x) - 6cf(-py) = 0 \end{aligned} \tag{2.7}$$

for all $x, y \in X$ with $x \perp y$. By (2.6) and (2.7), we have

$$\begin{aligned} &c[f(x + 2py) - f(x - 2py)] - 2c[f(x + py) - f(x - py)] \\ &- 6c[f(py) - f(-py)] = 0 \end{aligned} \tag{2.8}$$

for all $x, y \in X$ with $x \perp y$. Letting $y = \frac{1}{p}y$ in (2.8), we have

$$[f(x + 2y) - f(x - 2y)] - 2[f(x + y) - f(x - y)] - 6[f(y) - f(-y)] = 0 \tag{2.9}$$

for all $x, y \in X$ with $x \perp y$.

Let $f_o(x) = \frac{f(x)-f(-x)}{2}$. Then f_o satisfies (2.9). Letting $x = 0$ in (2.9), we have

$$f_o(2y) = 8f_o(y) \tag{2.10}$$

for all $y \in X$. Letting $x = 2x$ in (2.9), by (2.10), we have

$$4[f_o(x + y) - f_o(x - y)] = f_o(2x + y) - f_o(2x - y) + 6f_o(y) \tag{2.11}$$

for all $x, y \in X$ with $x \perp y$. Interchanging x and y in (2.11), we have

$$4[f_o(x + y) + f_o(x - y)] = f_o(x + 2y) + f_o(x - 2y) + 6f_o(x) \tag{2.12}$$

for all $x, y \in X$ with $x \perp y$. By (2.9) and (2.12), we have

$$f_o(x + 2y) - 3f_o(x + y) + 3f_o(x) - f_o(x - y) - 6f_o(y) = 0$$

for all $x, y \in X$ with $x \perp y$ and hence f_0 is an orthogonally cubic mapping.

Let $f_e(x) = \frac{f(x)+f(-x)}{2}$. Then f_e satisfies (2.9) and so we have

$$f_e(x+2y) - f_e(x-2y) - 2[f_e(x+y) - f_e(x-y)] = 0 \tag{2.13}$$

for all $x, y \in X$ with $x \perp y$. Letting $y = x$ in (2.13), we have

$$f_e(3x) = 2f_e(2x) + f_e(x)$$

for all $x \in X$ and letting $y = 2x$ in (2.13), we have

$$f_e(4x) = 2f_e(3x) - 2f_e(x)$$

for all $x \in X$. Hence we have $f_e(4x) = 4f_e(2x)$ for all $x \in X$ and so

$$f_e(2x) = 4f_e(x), \;\; f_e(3x) = 9f_e(x), \;\; f_e(4x) = 16f_e(x)$$

for all $x \in X$. By induction on n, we have

$$f_e(nx) = n^2 f_e(x)$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence

$$f_e(rx) = r^2 f_e(x)$$

for all $x \in X$ and all rational number r. By (2.1), since a is a non-zero rational number with $a \neq 1$, $f(x) = 0$ for all $x \in X$. Hence $f = f_o + f_e = f_o$ is an orthogonally cubic mapping. ☐

3. The Generalized Hyers-Ulam stability for (1.4)

In this section, we prove the generalized Hyers-Ulam stability for the orthogonally cubic functional equation (1.4) by using the fixed point method.

In 1996, Isac and Rassias [7] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 3.1. [1], [2] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all nonnegative integer n or there exists a positive integer n_0 such that*
(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$* and
(4) *$d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ for all $y \in Y$.*

For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$\begin{aligned} Df(x,y) = {} & f(ax+by) + f(ax-by) - ab^2[f(x+y) + f(x-y)] - 2a(a^2-b^2)f(x) \\ & + c[f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y)] \end{aligned}$$

for all $x, y \in X$.

Theorem 3.2. *Assume that* $\phi : X^2 \longrightarrow [0,\infty)$ *is a function such that*

$$\phi(x,y) \le \frac{L}{|a|^3}\phi(ax,ay) \tag{3.1}$$

for all $x, y \in X$ *and some real number* L *with* $0 < L < 1$. *Let* $f : X \longrightarrow Y$ *be a mapping such that* $f(0) = 0$ *and*

$$\|Df(x,y)\| \le \phi(x,y) \tag{3.2}$$

for all $x, y \in X$ *with* $x \perp y$. *Then there exists a unique orthogonally cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F(x) - f(x)\| \le \frac{L}{2|a|^3(1-L)}\phi(x,0) \tag{3.3}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and define the generalized metric d on S by

$$d(g,h) = \inf\{c \in [0,\infty) \mid \|g(x) - h(x)\| \le c\ \phi(x,0), \forall x \in X\}.$$

Then (S,d) is a complete metric space([9]). Define a mapping $T : S \longrightarrow S$ by $Tg(x) = a^3 g(\frac{x}{a})$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (3.1), we have

$$\|Tg(x) - Th(x)\| = |a|^3 \Big\| g\Big(\frac{x}{a}\Big) - h\Big(\frac{x}{a}\Big)\Big\| \le cL\phi(x,0)$$

for all $x \in X$. Hence we have $d(Tg,Th) \le Ld(g,h)$ for all $g, h \in S$ and so T is a strictly contractive mapping. Putting $y = 0$ in (3.2), we get

$$\|2f(ax) - 2a^3 f(x)\| \le \phi(x,0)$$

for all $x \in X$ and hence

$$\Big\| f(x) - a^3 f\Big(\frac{x}{a}\Big)\Big\| \le \frac{L}{2|a|^3}\phi(x,0)$$

for all $x \in X$ and hence $d(f,Tf) \le \frac{L}{2|a|^3} < \infty$. By Theorem 3.1, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of T such that $d(T^n f, F) \to 0$ as $n \to \infty$ and

$$\|F(x) - f(x)\| \le \frac{L}{2|a|^3(1-L)}\phi(x,0)$$

for all $x \in X$. Replacing x, y by $\frac{x}{a^n}$, $\frac{y}{a^n}$ in (3.2), respectively, and multiplying (3.2) by $|a|^{3n}$, by (O3), we have

$$\Big\| a^{3n} Df\Big(\frac{x}{a^n}, \frac{y}{a^n}\Big)\Big\| \le L^n \phi(x,y)$$

for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DF(x,y) = 0$$

for all $x, y \in X$ with $x \perp y$ and by Theorem 2.1, F is an orthogonally cubic mapping.

Now, we will show the uniqueness of F. Let $G : X \longrightarrow Y$ be another orthogonally cubic mapping with (3.3). Since F and G are fixed points of T, by (3.3), we get

$$\begin{aligned}\|G(x)-F(x)\| &= \|T^nG(x)-T^nF(x)\|\\ &\le \|T^nG(x)-T^nf(x)\|+\|T^nF(x)-T^nf(x)\|\\ &\le \frac{L^{n+1}}{|a|^3(1-L)}\phi(x,0)\end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Since $0 < L < 1$, letting $n \to \infty$ in the above inequality, we have $F = G$. ☐

Related with Theorem 3.2, we can also have the following theorem. And the proof is similar to that of Theorem 3.2.

Theorem 3.3. *Assume that $\phi : X^2 \longrightarrow [0,\infty)$ is a function such that*

$$\phi(ax,ay) \le |a|^3L\phi(x,y) \tag{3.4}$$

for all $x, y \in X$ and some real number L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping such that satisfying (3.2). Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x)-f(x)\| \le \frac{1}{2|a|^3(1-L)}\phi(x,0) \tag{3.5}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and define the generalized metric d on S by

$$d(g,h) = \inf\{c \in [0,\infty) \mid \|g(x)-h(x)\| \le c\ \phi(x,0), \forall x \in X\}.$$

Then (S,d) is a complete metric space([9]). Define a mapping $T : S \longrightarrow S$ by $Tg(x) = \frac{1}{a^3}g(ax)$ for all $x \in X$ and all $g \in S$.

Let $g, h \in S$ and $d(g,h) \le c$ for some $c \in [0,\infty)$. Then by (3.4), we have

$$\|Tg(x)-Th(x)\| = \frac{1}{|a|^3}\|g(ax)-h(ax)\| \le cL\phi(x,0)$$

for all $x \in X$. Hence we have $d(Tg,Th) \le Ld(g,h)$ for all $g, h \in S$ and so T is a strictly contractive mapping. Putting $y = 0$ in (3.2), we get

$$\|2f(ax)-2a^3f(x)\| \le \phi(x,0)$$

for all $x \in X$ and hence

$$\left\|f(x)-\frac{1}{a^3}f(ax)\right\| \le \frac{1}{2|a|^3}\phi(x,0)$$

for all $x \in X$ and hence $d(f,Tf) \le \frac{1}{2|a|^3} < \infty$. By Theorem 3.1, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of T such that $d(T^nf,F) \to 0$ as $n \to \infty$ and

$$\|F(x)-f(x)\| \le \frac{1}{2|a|^3(1-L)}\phi(x,0)$$

for all $x \in X$. Replacing x, y by a^nx, a^ny in (3.2), respectively, and multiplying (3.2) by $|a|^{-3n}$, by (O3), we have

$$\left\|a^{-3n}Df(a^nx,a^ny)\right\| \le L^n\phi(x,y)$$

for all $x, y \in X$ with $x \perp y$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in the last inequality, we get

$$DF(x, y) = 0$$

for all $x, y \in X$ with $x \perp y$ and by Theorem 2.1, F is an orthogonally cubic mapping.

Now, we will show the uniqueness of F. Let $G : X \longrightarrow Y$ be another orthogonally cubic mapping with (3.3). Since F and G are fixed points of T, by (3.3), we get

$$\begin{aligned}\|G(x) - F(x)\| &= \|T^nG(x) - T^nF(x)\|\\ &\le \|T^nG(x) - T^nf(x)\| + \|T^nF(x) - T^nf(x)\|\\ &\le \frac{L^n}{|a|^3(1-L)}\phi(x, 0)\end{aligned}$$

for all $x \in V$ and for all $n \in \mathbb{N}$. Since $0 < L < 1$, letting $n \to \infty$ in the above inequality, we have $F = G$. ☐

As an example of $\phi(x, y)$ in Theorem 3.2 and Theorem 3.3, we can take $\phi(x, y) = \epsilon(\|x\|_X^p\|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p})$ for some positive real numbers ϵ and p. Then we can formulate the following corollary :

Corollary 3.4. *Let $(X, \perp)$ be an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ a Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$\|Df(x, y)\| \le \epsilon(\|x\|_X^p\|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p}) \tag{3.6}$$

for all $x, y \in X$ with $x \perp y$ and a fixed positive number p with $p \neq \frac{3}{2}$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \frac{1}{2\Big||a|^{2p} - |a|^3\Big|}\|x\|^{2p}$$

for all $x \in X$.

By Theorem 2.1, if $c = -\frac{1}{3}ab^2$, then we have the following orthogonally Euler-Lagrange type cubic functional equation :

$$\begin{aligned}&f(ax + by) + f(ax - by) - \frac{2}{3}ab^2f(x - y) - \frac{1}{3}ab^2f(x + 2y)\\ &- a(2a^2 - b^2)f(x) + 2ab^2f(y) = 0\end{aligned}$$

for all $x, y \in X$ with $x \perp y$. By Corollary 3.6, we have the following exmaple.

Example 3.5. Let $(X, \perp)$ be an orthogonality normed space with the norm $\|\cdot\|_X$ and $(Y, \|\cdot\|)$ a Banach space. Let $f : X \longrightarrow Y$ be a mapping such that

$$\begin{aligned}&\|f(ax + by) + f(ax - by) - \frac{2}{3}ab^2f(x - y) - \frac{1}{3}ab^2f(x + 2y)\\ &- a(2a^2 - b^2)f(x) + 2ab^2f(y)\| \le \epsilon(\|x\|_X^p\|x\|_X^p + \|x\|_X^{2p} + \|y\|_X^{2p})\end{aligned}$$

for all $x, y \in X$ with $x \perp y$ and a fixed positive number p with $p \neq \frac{3}{2}$. Then there exists a unique orthogonally cubic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \le \frac{1}{2\Big||a|^{2p} - |a|^3\Big|}\|x\|^{2p}$$

for all $x \in X$.

It should be remarked that if a functional inequality can be deformed into the type of (3.2), then a solution of the original functional equation is cubic. In the following theorems, we give a simple example.

Theorem 3.6. *Let* $\phi : X^2 \longrightarrow [0, \infty)$ *be a function such that*

$$\phi(x, y) \le \frac{1}{8} L\phi(2x, 2y) \tag{3.7}$$

for all $x, y \in X$*, some real number* L *with* $0 < L < 1$ *and* $f : X \longrightarrow Y$ *a mapping such that* $f(0) = 0$ *and*

$$\|f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x)\| \le \phi(x, y) \tag{3.8}$$

for all $x, y \in X$ *with* $x\perp y$*. Then there exists a unique orthogonally cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F(x) - f(x)\| \le \frac{L}{16(1 - L)}[3\phi(x, 0) + 8\phi(0, x)]$$

for all $x \in X$*.*

Proof. Letting $x = 0$ in (3.8), we have

$$\|f(y) + f(-y)\| \le \phi(0, y) \tag{3.9}$$

for all $y \in X$ and letting $y = 0$ in (3.8), we have

$$\|f(2x) - 8f(x)\| \le \frac{1}{2}\phi(x, 0) \tag{3.10}$$

for all $y \in X$. Letting $y = 2y$ in (3.8), by (3.10), we have

$$\begin{aligned} &\|8f(x + y) + 8f(x - y) - 2f(x + 2y) - 2f(x - 2y) - 12f(x)\| \\ &\le \frac{1}{2}\phi(x + y, 0) + \frac{1}{2}\phi(x - y, 0) + \phi(x, 2y) \end{aligned} \tag{3.11}$$

for all $x, y \in X$ with $x\perp y$. Interchang x and y in (3.8), by (3.9), we get

$$\begin{aligned} &\|f(x + 2y) - f(x - 2y) - 2f(x + y) + 2f(x - y) - 12f(y)\| \\ &\le \phi(y, x) + \phi(0, x - 2y) + 2\phi(0, x - y) \end{aligned} \tag{3.12}$$

for all $x, y \in X$ with $x\perp y$. Putting $a = 2$, $b = 1$, and $c = -4$ in $Df(x, y)$, by (3.8), (3.11), and (3.12), we have

$$\|Df(x, y)\| \le \psi(x, y)$$

for all $x, y \in X$, where

$$\begin{aligned} \psi(x, y) = \phi(x, y) + 2\phi(y, x) + \frac{1}{2}\phi(x + y, 0) + \frac{1}{2}\phi(x - y, 0) + \phi(x, 2y) \\ + 2\phi(0, x - 2y) + 4\phi(0, x - y) \end{aligned}$$

Since ψ satisfies (3.1), by Theorem 3.2, we get the result. □

Similar to Theorem 3.6, we have the following theorem :

Theorem 3.7. *Let* $\phi : X^2 \longrightarrow [0, \infty)$ *be a function such that*

$$\phi(2x, 2y) \le 8L\phi(2x, 2y) \tag{3.13}$$

for all $x, y \in X$*, some real number* L *with* $0 < L < 1$ *and* $f : X \longrightarrow Y$ *a mapping satisfying* $f(0) = 0$ *(3.8). Then there exists a unique orthogonally cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F(x) - f(x)\| \leq \frac{1}{16(1-L)}[3\phi(x,0) + 8\phi(0,x)]$$

for all $x \in X$*.*

By Theorem 3.6 and Theorem 3.7, we have the following corollary :

Corollary 3.8. *Let* $f : X \longrightarrow Y$ *be a mapping such that* $f(0) = 0$ *and*

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \leq \|x\|^p\|y\|^p+\|x\|^{2p}+\|y\|^{2p}.$$

for all $x, y \in X$ *and a fixed positive real number* p *with* $p \neq \frac{3}{2}$*. Then there exists a unique orthogonally cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F(x) - f(x)\| \leq \frac{11}{2|8-2^{2p}|}\|x\|^{2p}$$

for all $x \in X$*.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

CERTAIN SUBCLASS OF HARMONIC MULTIVALENT FUNCTIONS DEFINED BY DERIVATIVE OPERATOR

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Abstract. In the present paper, we investigate new properties of a new subclass of multivalent harmonic functions in the open unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$, under certain conditions involving a new generalized differential operator. Furthermore, a representation theorem, an integral property and convolution conditions for the subclass denoted by $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ are also obtained. Finally, we will give an application of neighborhood.

Keywords: differential operator, harmonic function, extreme points, convolution, neighborhood.
2000 Mathematical Subject Classification: 30C45.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simple connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. (See also Clunie and Sheil-Small [5] for more details.)

Denote by $S_{\mathcal{H}}(p, n), (p, n \in \mathbb{N} = \{1, 2, \dots\})$ the class of functions $f = h + \bar{g}$ that are harmonic multivalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : \ |z| < 1\}$. Then for $f = h + \bar{g} \in S_{\mathcal{H}}(p, n)$ we may express the analytic functions h and g as

$$h(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad g(z) = \sum_{k=p+n-1}^{\infty} b_k z^k, \quad |b_{p+n-1}| < 1. \tag{1.1}$$

Let $\tilde{S}_{\mathcal{H}}(p,n,m), (p,n\in\mathbb{N}, m\in\mathbb{N}_0\cup\{0\})$ denote the family of functions $f_m = h+\bar{g}_m$ that are harmonic in D with the normalization

(1.2)

$$h(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k,\quad g_m(z)=(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|z^k,\quad |b_{p+n-1}|<1.$$

Definition 1.1. [4] Let $H(U)$ denote the class of analytic functions in the open unit disc $U=\{z\in\mathbb{C}:\ |z|<1\}$ and let $\mathcal{A}(p)$ be the subclass of the functions belonging to $H(U)$ of the form

$$h(z)=z^p+\sum_{k=p+n}^{\infty}a_kz^k.$$

For $m\in\mathbb{N}_0$, $\lambda\geq 0$, $\delta\in\mathbb{N}_0$, $l\geq 0$ we define the generalized differential operator $I^m_{\lambda,\delta}(p,l)$ on $\mathcal{A}(p)$ by the following infinite series

$$I^m_{\lambda,\delta}(p,l)h(z)=(p+l)^mz^p+\sum_{k=p+n}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)a_kz^k, \tag{1.3}$$

where

$$C(\delta,k)=\left(\begin{array}{c}k+\delta-1\\ \delta\end{array}\right)=\frac{\Gamma(k+\delta)}{\Gamma(k)\Gamma(\delta+1)}. \tag{1.4}$$

Remark 1.2. When $\lambda=1$, $p=1$, $l=0$, $\delta=0$ we get Sălăgean differential operator [13]; $p=1$, $m=0$ gives Ruscheweyh operator [12]; $p=1$, $l=0$, $\delta=0$ implies Al-Oboudi differential operator of order m (see [1]); $\lambda=1$, $p=1$, $l=0$ operator (1.3) reduces to Al-Shaqsi and Darus differential operator [2] and when $p=1$, $l=0$ we reobtain the operator introduced by Darus and Ibrahim in [6].

Definition 1.3. [4] Let $f\in S_{\mathcal{H}}(p,n)$, $p\in\mathbb{N}$. Using the operator (1.3) for $f=h+\bar{g}$ given by (1.1) we define the differential operator of f as

$$I^m_{\lambda,\delta}(p,l)f(z)=I^m_{\lambda,\delta}(p,l)h(z)+(-1)^m\overline{I^m_{\lambda,\delta}(p,l)g(z)} \tag{1.5}$$

where

$$I^m_{\lambda,\delta}(p,l)h(z)=(p+l)^mz^p+\sum_{k=p+n}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)a_kz^k \tag{1.6}$$

and

$$I^m_{\lambda,\delta}(p,l)g(z)=\sum_{k=p+n-1}^{\infty}[p+\lambda(k-p)+l]^mC(\delta,k)b_kz^k. \tag{1.7}$$

Remark 1.4. When $\lambda = 1$, $l = 0$, $\delta = 0$ the operator (1.5) reduces to the operator introduced earlier in [8] by Jahangiri et al.

Definition 1.5. [4] A function $f \in S_{\mathcal{H}}(p, n)$ is said to be in the class $AL_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ if

$$\frac{1}{p+l}\mathrm{Re}\left\{\frac{I^{m+1}_{\lambda,\delta}(p,l)f(z)}{I^{m}_{\lambda,\delta}(p,l)f(z)}\right\} \geq \alpha, \quad 0 \leq \alpha < 1, \tag{1.8}$$

where $I^m_{\lambda,\delta}f$ is defined by (1.5), for $m \in \mathbb{N}_0$.

Finally, we define the subclass

$$\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l) \equiv AL_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l) \cap \tilde{S}_{\mathcal{H}}(p, n, m). \tag{1.9}$$

Remark 1.6. The class $AL_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ includes a variety of well-known subclasses of $S_{\mathcal{H}}(p, n)$. For example, letting $n = 1$ we get $AL_{\mathcal{H}}(1, 1, 0, \alpha, 1, 0) \equiv HK(\alpha)$ in [7], for $n = 1$, $AL_{\mathcal{H}}(1, m-1, 0, \alpha, 1, 0) \equiv S_H(t, u, \alpha)$ in [14], $AL_{\mathcal{H}}(p, n+p, 0, \alpha, 1, 0) \equiv SH_p(n, \alpha)$ in [11] and $n = 1$, $AL_{\mathcal{H}}(1, m, \delta, \alpha, 1, 0) \equiv M_{\mathcal{H}}(m, \delta, \alpha)$ in [3].

Theorem 1.7. [4] *Let* $f_m = h + \bar{g_m}$ *be given by (1.2). Then* $f_m \in \widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ *if and only if*

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha) + \lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k| + \tag{1.10}$$

$$+ \sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha) + \lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k| \leq 1,$$

where $\lambda n \geq \alpha(p+l)$, $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda \geq 0$ *and*

$$d_{p,k}(m,\lambda,l) = [p + \lambda(k-p) + l]^m. \tag{1.11}$$

Remark 1.8. The harmonic function

$$f(z) = z^p + \sum_{k=p+n}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha) + \lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} x_k z^k + \tag{1.12}$$

$$+ \sum_{k=p+n-1}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha) + \lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} \overline{y_k z^k},$$

where $\sum_{k=p+n}^{\infty}|x_k| + \sum_{k=p+n-1}^{\infty}|y_k| = 1$, $0 \leq \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda n \geq \alpha(p+l)$, $\lambda \geq 0$ and $d_{p,k}(m,\lambda,l)$ is given in (1.11), show that the coefficient bound expressed by (1.10) is sharp.

2. Convex combination and extreme points

In this section, we show that the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is closed under convex combination of its members.

For $i=1,2,3,...$, let the functions $f_{m_i}(z)$ be

$$f_{m_i}(z)=z^p-\sum_{k=p+n}^{\infty}|a_{k,i}|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_{k,i}|\bar{z}^k. \tag{2.1}$$

Theorem 2.1. *The class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ is closed under convex combination.*

Proof. For $i=1,2,3,...$, let $f_{m_i}(z)\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, where the functions $f_{m_i}(z)$ are defined by (2.1). Then by (1.10) we have

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_{k,i}|+ \tag{2.2}$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_{k,i}|\le 1.$$

For $\sum\limits_{i=1}^{\infty}t_i=1$, $0\le t_i\le 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty}t_if_{m_i}(z)=z^p-\sum_{k=p+n}^{\infty}\left(\sum_{i=1}^{\infty}t_i|a_{k,i}|\right)z^k+(-1)^m\sum_{k=p+n-1}^{\infty}\left(\sum_{i=1}^{\infty}t_i|b_{k,i}|\right)\bar{z}^k.$$

Then by (2.2) one obtains

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\left(\sum_{i=1}^{\infty}t_i|a_{k,i}|\right)+$$
$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\left(\sum_{i=1}^{\infty}t_i|b_{k,i}|\right)=$$
$$\sum_{i=1}^{\infty}t_i\cdot\left\{\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_{k,i}|+\right.$$
$$\left.+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_{k,i}|\right\}\le\sum_{i=1}^{\infty}t_i=1,$$

and therefore $\sum_{i=1}^{\infty} t_i f_{m_i}(z) \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. □

Further, we will determine a representation theorem for functions in $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ from which we also establish the extreme points of closed convex hulls of $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ denoted by $clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Theorem 2.2. *Let $f_m(z)$ given by (1.2). Then $f_m(z) \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ if and only if*

$$f_m(z) = X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z), \tag{2.3}$$

where $h_p(z) = z^p$

$$h_k(z) = z^p - \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} z^k, \tag{2.4}$$
$$k = p+n, p+n+1, ...,$$

and

$$g_{m_k}(z) = z^p + (-1)^m \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} \bar{z}^k, \tag{2.5}$$
$$k = p+n-1, p+n, ...,$$

with $X_k \geq 0, Y_k \geq 0, X_p = 1 - \sum_{k=p+n}^{\infty} X_k - \sum_{k=p+n-1}^{\infty} Y_k$.

In particular, the extreme points of $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ are $\{h_k\}$ and $\{g_{m_k}\}$.

Proof. For the functions f_m of the form (2.3), we have

$$f_m(z) = X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z) =$$

$$= z^p - \sum_{k=p+n}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} X_k z^k +$$

$$+(-1)^m \sum_{k=p+n-1}^{\infty} \frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)} Y_k \bar{z}^k.$$

Consequently,

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)} a_k +$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}b_k=$$

$$=\sum_{k=p+n}^{\infty}X_k+\sum_{k=p+n-1}^{\infty}Y_k=1-X_p\leq 1,$$

where

$$a_k=\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}X_k$$

$$b_k=\frac{(p+l)^{m+1}(1-\alpha)}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}Y_k$$

and therefore $f_m\in clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Conversely, suppose that $f_m\in clco\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.
Setting

$$X_k=\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|, \tag{2.6}$$

$$k=p+n,p+n+1,...,$$

$$Y_k=\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|$$

$$k=p+n-1,p+n,...,$$

and $X_p=1-\sum_{k=p+n}^{\infty}X_k-\sum_{k=p+n-1}^{\infty}Y_k$. We note by Theorem 1.7 that $0\leq Y_k\leq 1, 0\leq X_k\leq 1,\ and\ X_p\geq 0$.

We obtain the required representation since f_m can be written as

$$f_m(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|\bar{z}^k=$$

$$=z^p-\sum_{k=p+n}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)X_k}{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}z^k+$$

$$+(-1)^m\sum_{k=p+n-1}^{\infty}\frac{(p+l)^{m+1}(1-\alpha)Y_k}{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}\bar{z}^k=$$

$$=z^p-\sum_{k=p+n}^{\infty}(z^p-h_k(z))X_k+\sum_{k=p+n-1}^{\infty}(g_{m_k}(z)-z^p)Y_k=$$

$$=\sum_{k=p+n}^{\infty}h_k(z)X_k+\sum_{k=p+n-1}^{\infty}g_{m_k}(z)Y_k+z^p\left(1-\sum_{k=p+n}^{\infty}X_k-\sum_{k=p+n-1}^{\infty}Y_k\right)=$$

$$= X_p h_p(z) + \sum_{k=p+n}^{\infty} X_k h_k(z) + \sum_{k=p+n-1}^{\infty} Y_k g_{m_k}(z),$$

as required. □

3. Integral property and convolution conditions

In this section we will examine the closure properties of the class $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$ under the generalized Bernardi-Libera-Livingston integral operator and also convolution properties of the same class.

Now, for $f = h + \bar{g}$ given by (1.1) we define the modified generalized Bernardi-Libera-Livingston integral operator of f as

$$\mathcal{L}_c(f(z)) = \mathcal{L}_c(h(z)) + \overline{\mathcal{L}_c(g(z))}, \quad c > -p, \tag{3.1}$$

where

$$\mathcal{L}_c(h(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} h(t)dt$$

and

$$\mathcal{L}_c(g(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1} g(t)dt.$$

Putting $g = 0$ in (3.1), we get the definition of the generalized Bernardi-Libera-Livingston integral operator on analytic functions, (see [9], [10]).

Theorem 3.1. *Let $f \in \widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$. Then $\mathcal{L}_c(f)$ belongs to the class $\widetilde{AL}_{\mathcal{H}}(p, m, \delta, \alpha, \lambda, l)$.*

Proof. From the representation of $\mathcal{L}_c(f)$, it follows that

$$\mathcal{L}_c(f(z)) = \frac{c+p}{z^c} \int_0^z t^{c-1}(h(t) + \bar{g}_m(t))dt =$$

$$= \frac{c+p}{z^c} \left[\int_0^z t^{c-1} \left(t^p - \sum_{k=p+n}^{\infty} |a_k| t^k \right) dt + (-1)^m \overline{\int_0^z t^{c-1} \left(\sum_{k=p+n-1}^{\infty} |b_k| t^k \right) dt} \right] =$$

$$= z^p - \sum_{k=p+n}^{\infty} |A_k| z^k + (-1)^m \sum_{k=p+n-1}^{\infty} |B_k| \bar{z}^k,$$

where

$$A_k = \frac{c+p}{c+k} a_k, \ B_k = \frac{c+p}{c+k} b_k.$$

Further, one obtains

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha) + \lambda(k-p)] d_{p,k}(m, \lambda, l) C(\delta, k)}{(p+l)^{m+1}(1-\alpha)} \cdot \frac{c+p}{c+k} |a_k| +$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}\cdot\frac{c+p}{c+k}|b_k|\le$$

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\le 1.$$

Since $f\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$, by Theorem 1.7 we have $\mathcal{L}_c(f)\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. □

For the harmonic functions

$$f_1(z)=z^p-\sum_{k=p+n}^{\infty}|a_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_k|\bar{z}^k,\ |b_{p+n-1}|<1, \tag{3.2}$$

and

$$f_2(z)=z^p-\sum_{k=p+n}^{\infty}|A_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|B_k|\bar{z}^k,\ |B_{p+n-1}|<1, \tag{3.3}$$

we define the convolution of f_1 and f_2 as

$$(f_1*f_2)(z)=f_1(z)*f_2(z)=z^p-\sum_{k=p+n}^{\infty}|a_kA_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|b_kB_k|\bar{z}^k.$$

In the following theorem, we examine the convolution properties of the class $\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Theorem 3.2. *For* $0\le\beta\le\alpha<1$ *let* $f_1\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ *and* $f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. *Then* $f_1*f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)\subset\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$.

Proof. Let $f_1\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$ and $f_2\in\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. Obviously, the coefficients of f_1 and f_2 must satisfy similar conditions to the inequality (1.10). Therefore, for the coefficients of f_1*f_2 we can write

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|a_kA_k|+$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|b_kB_k|\le$$

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|a_k|+$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\beta)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\beta)}|b_k|\leq$$

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+$$

$$+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)+}|b_k|\leq 1,$$

because $f_1 \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. In view of Theorem 1.7, it follows that $f_1 * f_2 \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l) \subset \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\beta,\lambda,l)$. □

4. AN APPLICATION OF NEIGHBORHOOD

Let us define a generalized (n,η)-neighborhood of a function f given in (1.2) to be the set

$$N_{n,\eta}(f)=\Big\{F_m(z)\in\tilde{S}_{\mathcal{H}}(p,n,m):$$

$$\sum_{k=p+n}^{\infty}\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k-A_k|+$$

$$\left.+\sum_{k=p+n-1}^{\infty}\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k-B_k|\leq\eta\right\}$$

where $F_m(z)=z^p-\sum_{k=p+n}^{\infty}|A_k|z^k+(-1)^m\sum_{k=p+n-1}^{\infty}|B_k|\bar{z}^k$.

Theorem 4.1. *Let $f_m=h+\bar{g}_m$ be given by (1.2). If the functions f_m satisfy the conditions*

$$\sum_{k=p+n}^{\infty}k\cdot\left[\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+\right.$$

$$\left.+\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\right]\leq 1-U_{p,\delta}^{\alpha}(m,\lambda,l) \tag{4.1}$$

and

$$\eta\leq\frac{p+n-\alpha-1}{p+n-\alpha}\left(1-U_{p,\delta}^{\alpha}(m,\lambda,l)\right), \tag{4.2}$$

$\lambda n\geq\alpha(p+l)$, where

$$U_{p,\delta}^{\alpha}(m,\lambda,l)=\frac{[(p+l)(1+\alpha)+\lambda(n-1)]d_{p,p+n-1}(m,\lambda,l)C(\delta,p+n-1)}{(p+l)^{m+1}(1-\alpha)}|b_{p+n-1}|$$

then $N_{n,\eta}(f)\subset\widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$.

Proof. Let f_m satisfy (4.1) and $F_m \in N_{n,\eta}(f)$. We have

$$\sum_{k=p+n}^{\infty} \frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|A_k|+$$

$$+\sum_{k=p+n-1}^{\infty} \frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|B_k| \leq$$

$$\leq \eta + \sum_{k=p+n}^{\infty} \left(\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+\right.$$

$$\left.\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\right) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq$$

$$\eta + \frac{1}{p+n-\alpha}\sum_{k=p+n}^{\infty} k \cdot \left(\frac{[(p+l)(1-\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|a_k|+\right.$$

$$\left.\frac{[(p+l)(1+\alpha)+\lambda(k-p)]d_{p,k}(m,\lambda,l)C(\delta,k)}{(p+l)^{m+1}(1-\alpha)}|b_k|\right) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq$$

$$\leq \eta + \frac{1}{p+n-\alpha}(1-U^{\alpha}_{p,\delta}(m,\lambda,l)) + U^{\alpha}_{p,\delta}(m,\lambda,l) \leq 1.$$

Hence, for $\eta \leq \frac{p+n-\alpha-1}{p+n-\alpha}\left(1-U^{\alpha}_{p,\delta}(m,\lambda,l)\right)$ we deduce that $f_m \in \widetilde{AL}_{\mathcal{H}}(p,m,\delta,\alpha,\lambda,l)$. ☐

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

ARGUMENT ESTIMATES FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. TThe purpose of the present paper is to investigate some argument properties for certain analytic functions in the open unit disk. The main results presented in here generalize some previous those concerning starlike function of reciprocal of order beta and strongly starlike functions.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C}\ ,\ |z| < 1\}). \tag{1.1}$$

A function $f(z) \in \mathcal{A}$ is said to be in the class $C(\alpha)$ of convex functions of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leq \alpha < 1) \tag{1.2}$$

and is said to be in the class $\mathbb{S}^*(\alpha)$ of starlike functions of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1). \tag{1.3}$$

We note that $\mathcal{C}(0) = \mathcal{C}$ and $\mathbb{S}^*(0) = \mathbb{S}^*$, where $\mathcal{C}$ and $\mathbb{S}^*$ are, respectively, the well-known classes of convex and starlike functions.

The classical result of Marx [5] and Strahhäcker [8] asserts that a convex function is starlike of order 1/2, that is,

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{U}) \Longrightarrow \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{U}). \tag{1.4}$$

If $f(z) \in \mathbb{S}^*$ satisfies the condition

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \beta \quad (0 \leq \beta < 1; z \in \mathbb{U}), \tag{1.5}$$

2010 *Mathematics Subject Classification.* 30C45.

Key words and phrases. univalent functions, starlike function of reciprocal of order β, strongly starlike functions, convex functions.

then $f(z)$ is said to be starlike of reciprocal of order β (see Nunokawa et al. [4]).

In [7] Sakaguchi proved that: If $f(z) \in \mathcal{A}$ and $g(z) \in \mathbb{S}^*$, then

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \quad (z \in \mathbb{U}) \Longrightarrow \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{1.6}$$

In [6] Pommerenke generalized Sakaguchi's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{C}$ and

$$\left|\arg\frac{f'(z)}{g'(z)}\right| \le \frac{\pi}{2}\alpha \quad (0 < \alpha \le 1; z \in \mathbb{U}), \tag{1.7}$$

then

$$\left|\arg\frac{f(z_2)-f(z_1)}{g(z_2)-g(z_1)}\right| \le \frac{\pi}{2}\alpha \quad (|z_1| < 1, |z_2| < 1). \tag{1.8}$$

Recently, Nunokawa et al. [4] generalized Pommerenke's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{C}$, then $g(z)$ is starlike of reciprocal of order β and

$$\left|\arg\frac{f'(z)}{g'(z)}\right| \le \frac{\pi}{2}\alpha + \tan^{-1}\frac{\alpha\beta}{1+\alpha} \quad (z \in \mathbb{U}; 0 < \alpha \le 1; 0 \le \beta < 1), \tag{1.9}$$

then

$$\left|\arg\frac{f(z)}{g(z)}\right| \le \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{1.10}$$

Also Kanas et al. [1] generalized Sakaguchi's result as follows.

If $f(z) \in \mathcal{A}$ and $g(z) \in \mathbb{S}^*$, then

$$\operatorname{Re}\left\{\left(\frac{f(z)}{g(z)}\right)^{1-\alpha}\left(\frac{f'(z)}{g'(z)}\right)^{\alpha}\right\} > 0 \ (z \in \mathbb{U}; 0 \le \alpha \le 1) \Longrightarrow \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\} > \alpha \ (z \in \mathbb{U}), \tag{1.11}$$

where the powers in (1.11) are meant as the principal values.

Also Kanas et al. [1] defined the class $\mathcal{H}(\alpha)$ as follows.

$$\mathcal{H}(\alpha) = \left\{f(z) \in \mathcal{A}, g(z) \in \mathbb{S}^* : \operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > 0 \ (0 \le \alpha \le 1)\right\}. \tag{1.12}$$

In the present paper, we extend some results obtained by Kanas et al. [1], Liu [2], Nunokawa et al. [4], Pommerenke [6] and Sakaguchi [7] by using Nunowawa's lemma [3].

2. Main results

To derive our results, we need the following lemma due to Nunokawa [3].

Lemma 2.1. *[3] Let a function $p(z)$ with $p(0) = 1$ and $p(z) \neq 0$ be analytic in $\mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|\,,\ \alpha > 0),$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha \text{ and } |\arg p(z_0)| = \frac{\pi}{2}\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \geq 1 \text{ when } \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \leq -1 \text{ when } \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where

$$p(z_0)^{\frac{1}{\alpha}} = \pm ia (a > 0).$$

Theorem 2.2. *Let $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{C}$ and $g(z)$ is starlike of reciprocal of order β. Suppose that*

$$\left|\arg\left[(1-\lambda)\frac{f(z)}{g(z)} + \lambda\frac{f'(z)}{g'(z)} - \gamma\right]\right| < \frac{\pi}{2}\rho \quad (0 \leq \lambda \leq 1;\ 0 \leq \gamma < 1;\ z \in \mathbb{U}), \tag{2.1}$$

where

$$\rho = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\beta\lambda}{1+\alpha\lambda}\right) (0 < \alpha \leq 1;\ 0 \leq \gamma < 1). \tag{2.2}$$

Then we have

$$\left|\left(\arg\frac{f(z)}{g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{2.3}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{f(z)}{g(z)} - \gamma\right). \tag{2.4}$$

Then $p(z)$ is analytic in $\mathbb{U}$, $p(0) = 1$ and $p(z) \neq 0$. It follows from (2.4) that

$$\frac{f'(z)}{g'(z)} = \gamma + (1-\gamma)p(z)\left[1 + \frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right]. \tag{2.5}$$

Also, from (2.4) and (2.5), we have

$$(1-\lambda)\frac{f(z)}{g(z)} + \lambda\frac{f'(z)}{g'(z)} - \gamma = (1-\gamma)p(z)\left[1 + \lambda\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right]. \tag{2.6}$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1).$$

Then from Lemma 1, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k,$$

where

$$k \geq \frac{1}{2}(a + a^{-1}) \geq 1 \text{ when } \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leq -\frac{1}{2}(a + a^{-1}) \leq -1 \text{ when} \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where $(p(z_0))^{1/\alpha} = \pm ia (a > 0)$. Since $g(z) \in \mathcal{C}$, from Marx-Strohhäcker's theorem [5,8], we have

$$\operatorname{Re}\left\{\frac{zg'(z)}{g(z)}\right\} > 1/2 \quad (z \in \mathbb{U}),$$

so that $g(z) \in \mathbb{S}^*(1/2)$. Putting $\frac{zg'(z)}{g(z)} = u + iv$, where $u > 1/2$. Then

$$\left|\frac{g(z)}{zg'(z)} - 1\right|^2 = \left|\frac{1 - u - iv}{u + iv}\right|^2 = \frac{1 - 2u + u^2 + v^2}{u^2 + v^2} < 1.$$

Therefore,

$$\left|\frac{g(z)}{zg'(z)} - 1\right| < 1 \quad (z \in \mathbb{U}), \tag{2.7}$$

which implies that

$$\left|\operatorname{Im}\left\{\frac{g(z)}{zg'(z)}\right\}\right| < 1 \quad (z \in \mathbb{U}), \tag{2.8}$$

and from the assumption of the theorem, we have

$$\operatorname{Re}\left\{\frac{g(z)}{zg'(z)}\right\} > \beta \quad (0 \leq \beta < 1; \; z \in \mathbb{U}). \tag{2.9}$$

For the case $|\arg p(z_0)| = \frac{\pi}{2}\alpha$, from (2.5), (2.6) and (2.8), we have

$$\begin{aligned}
&\arg\left\{(1-\lambda)\frac{f(z_0)}{g(z_0)} + \lambda\frac{f'(z_0)}{g'(z_0)} - \gamma\right\}\\
=\ &\arg p(z_0) + \arg\left\{1 + \lambda\frac{z_0 p'(z_0)}{p(z_0)}\left(\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\}\\
=\ &\frac{\pi}{2}\alpha + \arg\left\{1 + i\alpha k\lambda\left(\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)} + i\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right)\right\}\\
=\ &\frac{\pi}{2}\alpha + \arg\left\{1 - \alpha k\lambda\left(\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right) + ik\alpha\lambda\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)}\right\}\\
=\ &\frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha k\lambda\mathrm{Re}\frac{g(z_0)}{z_0 g'(z_0)}}{1 + \alpha k\lambda\left|\mathrm{Im}\frac{g(z_0)}{z_0 g'(z_0)}\right|}\right\}\\
\geq\ &\frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha k\lambda\beta}{1+\alpha k\lambda}\right\} \geq \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{\alpha\lambda\beta}{1+\alpha\lambda}\right\}.
\end{aligned}$$

This contradicts the assumption of the theorem, then

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

For the case $|\arg p(z_0)| = -\frac{\pi}{2}\alpha$, applying the same method above, we have a contradiction. This completes the proof of Theorem 2.2. ▌

Remark. Putting $\lambda = 1$ in Theorem 1, we get the result obtained by Liu [2, Theorem 2.1]. Also, from Theorem 1, we have the results obtained by Kanas [1], Nunokawa [4] and Sakaguchi [7].

Theorem 2.3. *Let* $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{C}$ *and* $g(z)$ *is starlike of reciprocal of order* β $(0 \leq \beta < 1)$. *Suppose that*

$$\left|\arg\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma}\right| < \frac{\pi}{2}\rho \quad (z \in \mathbb{U}), \tag{2.10}$$

where

$$\rho = (\mu+\gamma)\alpha + \frac{2\gamma}{\pi}\tan^{-1}\left(\frac{\alpha\beta}{1+\alpha}\right) \quad (z \in \mathbb{U}), \tag{2.11}$$

μ *and* γ *are fixed positive real numbers with* $0 < \mu+\gamma \leq 1$ *and* $0 < \alpha \leq 1$. *Then*

$$\left|\arg\frac{f(z)}{g(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{2.12}$$

Proof. Let us define the function $p(z)$ by (2.4). It follows from (2.4) and (2.5) that

$$\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma}=(p(z))^{\mu+\gamma}\left(1+\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right)^{\gamma}$$

and

$$\begin{aligned}\arg\left(\frac{f(z)}{g(z)}\right)^{\mu}\left(\frac{f'(z)}{g'(z)}\right)^{\gamma} &= \mu\arg\frac{f(z)}{g(z)}+\gamma\arg\frac{f'(z)}{g'(z)}\\ &= (\mu+\gamma)\arg p(z)+\gamma\arg\left(1+\frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right).\end{aligned} \tag{2.13}$$

Suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$|\arg p(z)|<\frac{\pi}{2}\alpha\ \ (|z|<|z_0|)\ \textit{and}\ |\arg p(z_0)|=\frac{\pi}{2}\alpha\ \ (0<\alpha\le 1).$$

Then, using Lemma 1, we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\beta.$$

For the case $\arg p(z)=\frac{\pi}{2}\alpha$, from (2.7), (2.8) and (2.13), we have

$$\begin{aligned}\arg\left(\frac{f(z_0)}{g(z_0)}\right)^{\mu}\left(\frac{f'(z_0)}{g'(z_0)}\right)^{\gamma} &= (\mu+\gamma)\arg p(z_0)+\gamma\arg\left(1+\frac{z_0p'(z_0)}{p(z_0)}\frac{g(z_0)}{z_0g'(z_0)}\right)\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\arg\left\{1+i\alpha k\left(\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}+i\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)\right\}\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\arg\left\{1-\alpha k\left(\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right)+i\alpha k\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}\right\}\\ &= (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha k\mathrm{Re}\frac{g(z_0)}{z_0g'(z_0)}}{1+\alpha k\left|\mathrm{Im}\frac{g(z_0)}{z_0g'(z_0)}\right|}\right\}\\ &\ge (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha\beta k}{1+\alpha k}\right\}\\ &\ge (\mu+\gamma)\frac{\pi}{2}\alpha+\gamma\tan^{-1}\left\{\frac{\alpha\beta}{1+\alpha}\right\}.\end{aligned}$$

This contradicts the assumption of the theorem, then we have

$$|\arg p(z)|<\frac{\pi}{2}\alpha\ \ (z\in\mathbb{U}).$$

For the case $|\arg p(z_0)|=-\frac{\pi}{2}\alpha$, applying the same method above, we have a contradiction. This completes the proof of Theorem 2.3.

Putting $\mu = 1-\gamma$ $(\gamma > 0)$ in Theorem 2, we obtain the following corollary.

Corollary 1. Let $f(z) \in \mathcal{A}, g(z) \in \mathcal{C}$ and $g(z)$ is starlike of reciprocal of order β $(0 < \beta \leq 1)$. Suppose that

$$\left| \arg \left(\frac{f(z)}{g(z)} \right)^{1-\gamma} \left(\frac{f'(z)}{g'(z)} \right)^{\gamma} \right| < \frac{\pi}{2}\rho \quad (\gamma > 0;\ z \in \mathbb{U}),$$

$$\rho = \alpha + \frac{2\gamma}{\pi} \tan^{-1}\left(\frac{\alpha\beta}{1+\alpha} \right) \quad (0 < \alpha \leq 1).$$

Then

$$\left| \arg \frac{f(z)}{g(z)} \right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

Remark. Putting $\gamma = 1$ in Corollary 1, we have the result obtained by Nunokawa et al. [[4], Theorem 2.3].

Acknowledgement

The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2019R1I1A3A01050861).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Some properties of the second kind degenerate q-Euler polynomials associated with the p-adic integral on $\mathbb{Z}_p$

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Abstract : In this paper, we introduce the second kind degenerate q-Euler numbers and polynomials associated with the p-adic integral on $\mathbb{Z}_p$. We also obtain some explicit formulas for the second kind degenerate q-Euler numbers and polynomials.

Key words : Euler numbers and polynomials, the second kind Euler numbers and polynomials, the second kind degenerate Euler numbers and polynomials, the second kind degenerate q-Euler numbers and polynomials, p-adic integral on $\mathbb{Z}_p$.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{C}$ denotes the complex number field, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ denotes the set of complex numbers. Let p be a fixed odd prime number. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

We say that f is uniformly differentiable function at a point $a \in \mathbb{Z}_p$ and denote this property by $g \in UD(\mathbb{Z}_p)$, if the difference quotients

$$F_g(x,y) = \frac{g(x)-g(y)}{x-y}$$

have a limit $l = g'(a)$ as $(x,y) \to (a,a)$. For $g \in UD(\mathbb{Z}_p)$, the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0 \leq x < p^N} g(x)(-1)^x, \text{ (see [3]).} \tag{1}$$

If we take $g_1(x) = g(x+1)$ in (1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{2}$$

We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and the second kind $S_2(n,k)$ are defined by the relations(see [6])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k) x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1} (x - \lambda k) \tag{3}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$. Note that $(x|\lambda)$ is a homogeneous polynomials in λ and x of degree n, so if $\lambda \neq 0$ then $(x|\lambda)_n = \lambda^n(\lambda^{-1}x|1)_n$. Clearly $(x|0)_n = x^n$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n \frac{t^n}{n!}. \tag{5}$$

For $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, if we take $g(x) = q^x e^{(2x+1)t}$ in (2), then we easily see that

$$I_{-1}(q^x e^{(2x+1)t}) = \int_{\mathbb{Z}_p} q^x e^{(2x+1)t} d\mu_{-1}(x) = \frac{2e^t}{qe^{2t}+1}.$$

Let us define the second kind q-Euler numbers $E_{n,q}$ and polynomials $E_{n,q}(x)$ as follows(see [5]):

$$\int_{\mathbb{Z}_p} q^y e^{(2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!}, \tag{6}$$

$$\int_{\mathbb{Z}_p} q^y e^{(x+2y+1)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}. \tag{7}$$

Recently, many mathematicians have studied in the area of the degenerate Bernoulli umbers and polynomials, degenerate Euler numbers and polynomials, degenerate tangent numbers and polynomials(see [1, 2, 3, 4, 6]). Our aim in this paper is to define the second kind degenerate q-Euler polynomials $\mathcal{E}_{n,q}(x,\lambda)$. We investigate some properties which are related to the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$.

2. Some properties of the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$

In this section, we introduce the second kind degenerate q-Euler numbers and polynomials, and we obtain explicit formulas for them. For $t, \lambda \in \mathbb{Z}_p$ such that $|\lambda t|_p < p^{-\frac{1}{p-1}}$, if we take $g(x) = q^x(1+\lambda t)^{(2x+1)/\lambda}$ in (2), then we easily see that

$$\int_{\mathbb{Z}_p} q^x(1+\lambda t)^{(2x+1)/\lambda} d\mu_{-1}(x) = \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}. \tag{8}$$

Let us define the second kind degenerate q-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$ as follows:

$$\int_{\mathbb{Z}_p} q^y(1+\lambda t)^{(2y+1)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!}, \tag{9}$$

$$\int_{\mathbb{Z}_p} q^y(1+\lambda t)^{(2y+1+x)/\lambda} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!}. \tag{10}$$

Note that $(1+\lambda t)^{1/\lambda}$ tends to e^t as $\lambda \to 0$. From (7) and (10), we note that

$$\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} = \lim_{\lambda\to 0} \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!}.$$

Thus, we have

$$\lim_{\lambda\to 0} \mathcal{E}_{n,q}(x,\lambda) = E_{n,q}(x), (n \geq 0).$$

From (5) and (9), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} &= \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} \\ &= \left(\sum_{m=0}^{\infty} \mathcal{E}_{m,q}(\lambda)\frac{t^m}{m!}\right)\left(\sum_{l=0}^{\infty}(x|\lambda)_l\frac{t^l}{l!}\right) = \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,q}(\lambda)(x|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{11}$$

Therefore, we obtain the following theorem.

Theorem 1. For $n \geq 0$, we have

$$\mathcal{E}_{n,q}(x,\lambda) = \sum_{l=0}^{n} \binom{n}{l} \mathcal{E}_{l,q}(\lambda)(x|\lambda)_{n-l}.$$

By (8), (9), and (10), we obtain the following Witt's formula.

Theorem 2. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} q^x (2x+1|\lambda)_n d\mu_{-1}(x) = \mathcal{E}_{n,q}(\lambda),\\
&\int_{\mathbb{Z}_p} q^y (x+2y+1|\lambda)_n d\mu_{-1}(y) = \mathcal{E}_{n,q}(x,\lambda).
\end{aligned}$$

By (5) and (9), we can derive the following recurrence relation:

$$\begin{aligned}
\sum_{n=0}^{\infty} 2(1|\lambda)_n \frac{t^n}{n!} &= 2(1+\lambda t)^{1/\lambda} = (q(1+\lambda t)^{2/\lambda}+1)\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!}\\
&= q(1+\lambda t)^{2/\lambda}\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!}\\
&= \left(\sum_{l=0}^{\infty} q(2|\lambda)_l \frac{t^l}{l!}\sum_{m=0}^{\infty}\mathcal{E}_{m,q}(\lambda)\frac{t^m}{m!}\right) + \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} q(2|\lambda)_l \mathcal{E}_{n-l,q}(\lambda) + \mathcal{E}_{n,q}(\lambda)\right)\frac{t^n}{n!}.
\end{aligned} \tag{12}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (12), we obtain the following theorem.

Theorem 3. For $n \in \mathbb{Z}_+$, we have

$$q\sum_{l=0}^{n}\binom{n}{l}(2|\lambda)_l\mathcal{E}_{n-l,q}(\lambda) + \mathcal{E}_{n,q}(\lambda) = 2(1|\lambda)_n.$$

By (5), (9), and (10), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} q\mathcal{E}_{n,q}(x+2,\lambda)\frac{t^n}{n!} + \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!}\\
&= \frac{2q(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+2)/\lambda} + \frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\
&= 2(1+\lambda t)^{(x+1)/\lambda} = 2\sum_{n=0}^{\infty}(x+1|\lambda)_n\frac{t^n}{n!}.
\end{aligned} \tag{13}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (13), we have the following theorem.

Theorem 4. For $h \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$, we have

$$q\mathcal{E}_{n,q}(x+2,\lambda) + \mathcal{E}_{n,q}(x,\lambda) = 2(x+1|\lambda)_n.$$

By (1) and (5), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\left(q^n\mathcal{E}_{m,q}(2n,\lambda) + \mathcal{E}_{m,q}(\lambda)\right)\frac{t^m}{m!}\\
&= \int_{\mathbb{Z}_p} q^{x+n}(1+\lambda t)^{(2x+2n+1)/\lambda}d\mu_{-1}(x) + (-1)^n\int_{\mathbb{Z}_p} q^x(1+\lambda t)^{(2x+1)/\lambda}d\mu_{-1}(x)\\
&= 2\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l(1+\lambda t)^{(2l+1)/\lambda} = \sum_{m=0}^{\infty}\left(2\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l(2l+1|\lambda)_m\right)\frac{t^m}{m!}.
\end{aligned} \tag{14}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (14), we have the following theorem.

Theorem 5. For $m \in \mathbb{Z}_+$, we have

$$q^n\mathcal{E}_{m,q}(2n,\lambda)+\mathcal{E}_{m,q}(\lambda)=2\sum_{l=0}^{n-1}(-1)^{n-1-l}q^l(2l+1|\lambda)_m.$$

By (10), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,q^{-1}}(-x,-\lambda)\frac{t^n}{n!}&=\frac{2(1-\lambda t)^{-1/\lambda}}{q^{-1}(1-\lambda t)^{-2/\lambda}+1}(1-\lambda t)^{x/\lambda}\\&=\frac{2q}{(1-\lambda t)^{2/\lambda}+1}(1-\lambda t)^{(x+1)/\lambda}=\sum_{n=0}^{\infty}(-1)^nq\mathcal{E}_{n,q}(x+1,\lambda)\frac{t^n}{n!}.\end{aligned}\tag{15}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (15), we have the following theorem.

Theorem 6. For $n \in \mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q^{-1}}(-x,-\lambda)=(-1)^nq\mathcal{E}_{n,q}(x+1,\lambda),\quad \mathcal{E}_{n,q^{-1}}(-\lambda)=(-1)^nq\mathcal{E}_{n,q}(1|\lambda).$$

For $d\in\mathbb{N}$ with $d\equiv 1(\text{mod } 2)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!}&=\frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}\\&=\frac{2(1+\lambda t)^{1/\lambda}}{q^d(1+\lambda t)^{2d/\lambda}+1}(1+\lambda t)^{x/\lambda}\sum_{l=0}^{d-1}(-1)^lq^l(1+\lambda t)^{2l/\lambda}\\&=\sum_{n=0}^{\infty}\left(d^n\sum_{l=0}^{d-1}(-1)^lq^l\mathcal{E}_{n,q^d}\left(\frac{2l+x+1-d}{d},\frac{\lambda}{d}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\dfrac{t^n}{n!}$ in the above equation, we have the following theorem:

Theorem 7. For $d\in\mathbb{N}$ with $d\equiv 1(\text{mod } 2)$ and $n\in\mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x,\lambda)=d^n\sum_{l=0}^{d-1}(-1)^lq^l\mathcal{E}_{n,q^d}\left(\frac{2l+x+1-d}{d},\frac{\lambda}{d}\right).$$

In particular,

$$\mathcal{E}_{n,q}(\lambda)=d^n\sum_{l=0}^{d-1}(-1)^lq^l\mathcal{E}_{n,q^d}\left(\frac{2l+1-d}{d},\frac{\lambda}{d}\right).$$

From (10), we derive

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x+y,\lambda)\frac{t^n}{n!}&=\frac{2(1+\lambda t)^{1/\lambda}}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{(x+y)/\lambda}\\&=\frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}(1+\lambda t)^{y/\lambda}\\&=\left(\sum_{n=0}^{\infty}\mathcal{E}_{m,q}(x,\lambda)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y|\lambda)_n\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,q}(x,\lambda)(y|\lambda)_{n-l}\right)\frac{t^n}{n!}.\end{aligned}\tag{16}$$

Therefore, by (16), we have the following theorem.

Theorem 8. For $n\in\mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x+y,\lambda)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{E}_{l,q}(x,\lambda)(y|\lambda)_{n-l}.$$

From Theorem 8, we note that $\mathcal{E}_{n,q}(x,\lambda)$ is a Sheffer sequence.

By replacing t by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (10), we obtain

$$\begin{aligned}\frac{2e^t}{qe^{2t}+1}e^{xt} &= \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n\frac{1}{n!} = \sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\lambda^{-n}\sum_{m=n}^{\infty}S_2(m,n)\lambda^m\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{E}_{n,q}(x,\lambda)\lambda^{m-n}S_2(m,n)\right)\frac{t^m}{m!}.\end{aligned}\tag{17}$$

Thus, by (17), we have the following theorem.

Theorem 9. For $n\in\mathbb{Z}_+$, we have

$$E_{m,q}(x)=\sum_{n=0}^{m}\lambda^{m-n}\mathcal{E}_{n,q}(x,\lambda)S_2(m,n).$$

By replacing t by $\log(1+\lambda t)^{1/\lambda}$ in (7), we have

$$\sum_{n=0}^{\infty}E_{n,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!}=\frac{2(1+\lambda t)^{1/\lambda}}{q(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda}=\sum_{m=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^m}{m!},\tag{18}$$

and

$$\sum_{n=0}^{\infty}E_{n,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n\frac{1}{n!}=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{E}_{n,q}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}.\tag{19}$$

Thus, by (18) and (19), we have the following theorem.

Theorem 10. For $n\in\mathbb{Z}_+$, we have

$$\mathcal{E}_{n,q}(x,\lambda)=\sum_{n=0}^{m}\lambda^{m-n}E_{n,q}(x)S_1(m,n).$$

Letting $q\to 1$ in Theorem 10 gives the theorem

$$\mathcal{E}_{n}(x,\lambda)=\sum_{n=0}^{m}\lambda^{m-n}E_{n}(x)S_1(m,n).$$

which was proved by Ryoo [4].

Acknowledgement: This work was supported by 2021 Hannam University Research Fund.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.4, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Some symmetric identities for twisted (p, q)-L-function

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Abstract : The main of this paper is to obtain some interesting symmetric identities for twisted (p, q)-L-function in complex field. We define the twisted (p, q)-L-function by generalizing the Carlitz's type twisted (p, q)-Euler numbers and polynomials. We give some new symmetric identities for twisted (p, q)-L-function. We also obtain symmetric identities for Carlitz's type twisted (p, q)-Euler numbers and polynomials by using symmetric property for twisted (p, q)-L-function.

Key words : Euler numbers and polynomials, q-Euler numbers and polynomials, twisted q-Euler numbers and polynomials, twisted (p, q)-Euler numbers and polynomials, q-L-function, twisted (p, q)-L-function, symmetric identities.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.

1. Introduction

Many (p, q)-extensions of some special numbers, polynomials, and functions have been studied(see [1, 2, 3, 4, 7]). Luo and Zhou [5] introduced the l-function and q-L-function. Ryoo [6] investigated some identities on the higher-order twisted q-Euler numbers and polynomials. In [8], Ryoo presented the multiple twisted (h, q)-l-function. In this paper, we construct twisted (p, q)-L-function in complex field and Carlitz's type twisted (p, q)-Euler numbers and polynomials. We obtain some new symmetric identities for twisted (p, q)-L-function. We also give symmetric identities for Carlitz's type twisted (p, q)-Euler numbers and polynomials of by using symmetric property for twisted (p, q)-L-function.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. The (p, q)-number is defined as

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + p^2q^{n-3} + pq^{n-2} + q^{n-1}.$$

Note that this number is q-number when $p = 1$. By substituting q by $\frac{q}{p}$ in the q-number, we can not obtain (p, q)-number. Therefore, much research has been developed in the area of special numbers and polynomials, and functions by using (p, q)-number(see [1, 2, 3, 4, 7]).

By using q-number, Luo and Zhou defined the q-L-function $L_q(s, a)$ and q-l-function $l_q(s)$ (see [5])

$$L_q(s, a) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+a}}{[n + a]_q^s}, \quad (Re(s) > 1; a \notin \mathbb{Z}_0^-), \text{ and } l_q(s) = \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{[n]_q^s}, \quad (Re(s) > 1).$$

Inspired by their work, the (p, q)-extension of the twisted q-L-function can be defined as follow: Let ζ be rth root of 1 and $\zeta \neq 1$. For $s, x \in \mathbb{C}$ with $Re(x) > 0$, the twisted (p, q)-L-function $L_{p,q,\zeta}(s, x)$ is define by

$$L_{p,q,\zeta}(s, x) = [2]_q \sum_{m=0}^{\infty} \frac{(-1)^m \zeta^m}{[m + x]_{p,q}^s}.$$

2. Twisted (p,q)-Euler numbers and polynomials

In this section, we define twisted (p,q)-Euler numbers and polynomials and provide some of their relevant properties. Let r be a positive integer, and let ζ be rth root of 1.

Definition 1. For $0<q<p\le 1$, the Carlitz's type twisted (p,q)-Euler numbers $E_{n,p,q,\zeta}$ and polynomials $E_{n,p,q,\zeta}(x)$ are defined by means of the generating functions

$$G_{p,q,\zeta}(t)=\sum_{n=0}^{\infty}E_{n,p,q,\zeta}\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m\zeta^m e^{[m]_{p,q}t}. \tag{2.1}$$

and

$$G_{p,q,\zeta}(t,x)=\sum_{n=0}^{\infty}E_{n,p,q,\zeta}(x)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m\zeta^m e^{[m+x]_{p,q}t}, \tag{2.2}$$

respectively.

Setting $p=1$ in (2.1) and (2.2), we can obtain the corresponding definitions for the Carlitz's type twisted q-Euler number $E_{n,q,\zeta}$ and q-Euler polynomials $E_{n,q,\zeta}(x)$, respectively.

By (2.1), we get

$$\sum_{n=0}^{\infty}E_{n,p,q,\zeta}\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m\zeta^m e^{[m]_{p,q}t}=\sum_{n=0}^{\infty}\left([2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+\zeta q^lp^{n-l}}\right)\frac{t^n}{n!}.$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

Theorem 2. For $n\in\mathbb{Z}_+$, we have

$$E_{n,p,q,\zeta}=[2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1+\zeta p^{n-l}q^l}.$$

By (2.2), we obtain

$$E_{n,p,q,\zeta}(x)=[2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^lq^{xl}p^{(n-l)x}\frac{1}{1+\zeta p^{n-l}q^l}. \tag{2.3}$$

Next, we introduce Carlitz's type twisted (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q,\zeta}(x)$.

Definition 3. The Carlitz's type twisted (h,p,q)-Euler polynomials $E^{(h)}_{n,p,q,\zeta}(x)$ are defined by

$$E^{(h)}_{n,p,q,\zeta}(x)=[2]_q\sum_{m=0}^{\infty}(-1)^mp^{hm}\zeta^m[m+x]^n_{p,q}. \tag{2.4}$$

When $x=0, E^{(h)}_{n,p,q,\zeta}=E^{(h)}_{n,p,q,\zeta}(0)$ are called the twisted (h,p,q)-Euler numbers $E^{(h)}_{n,p,q,\zeta}$.

By using (2.4) and (p,q)-number, we have the following theorem.

Theorem 4. For $n\in\mathbb{Z}_+$, we have

$$E^{(h)}_{n,p,q,\zeta}(x)=[2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^lq^{xl}p^{(n-l)x}\frac{1}{1+\zeta p^{n-l+h}q^l}.$$

By (2.4) and Theorem 2, we have

$$\begin{aligned}E_{n,p,q,\zeta}(x)&=\sum_{l=0}^{n}\binom{n}{l}q^{(n-l)x}E^{(l)}_{n-l,p,q,\zeta}[x]^l_{p,q}\\ E_{n,p,q,\zeta}(x+y)&=\sum_{l=0}^{n}\binom{n}{l}p^{xl}q^{(n-l)y}E^{(l)}_{n-l,p,q,\zeta}(x)[y]^l_{p,q}.\end{aligned} \tag{2.5}$$

By (2.1) and (2.2), we get

$$-[2]_q\sum_{l=0}^{\infty}(-1)^{l+n}\zeta^{l+n}e^{[l+n]_{p,q}t}+[2]_q\sum_{l=0}^{\infty}(-1)^l\zeta^l e^{[l]_{p,q}t}=[2]_q\sum_{l=0}^{n-1}(-1)^l\zeta^l e^{[l]_{p,q}t}.$$

Hence we have

$$(-1)^{n+1}\zeta^n\sum_{m=0}^{\infty}E_{m,p,q,\zeta}(n)\frac{t^m}{m!}+\sum_{m=0}^{\infty}E_{m,p,q,\zeta}\frac{t^m}{m!}=\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^l\zeta^l[l]_{p,q}^m\right)\frac{t^m}{m!}. \tag{2.6}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (2.6), we have the following theorem.

Theorem 5. For $m\in\mathbb{Z}_+$, we have

$$\sum_{l=0}^{n-1}(-1)^l\zeta^l[l]_{p,q}^m=\frac{(-1)^{n+1}\zeta^nE_{m,p,q,\zeta}(n)+E_{m,p,q,\zeta}}{[2]_q}.$$

3. Twisted (p,q)-l-function and twisted (p,q)-L-function

By using twisted (p,q)-Euler numbers and polynomials, twisted (p,q)-L-function is defined. These functions interpolate the twisted (p,q)-Euler numbers $E_{n,p,q,\zeta}$, and polynomials $E_{n,p,q,\zeta}(x)$, respectively. From (2.1), we note that

$$\frac{d^k}{dt^k}G_{p,q,\zeta}(t)\bigg|_{t=0}=[2]_q\sum_{m=0}^{\infty}(-1)^n\zeta^m[m]_{p,q}^k=E_{k,p,q,\zeta},\ (k\in\mathbb{N}).$$

By using the above equation, we are now ready to define twisted (p,q)-l-function.

Definition 6. Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$.

$$l_{p,q,\zeta}(s)=[2]_q\sum_{n=1}^{\infty}\frac{(-1)^n\zeta^n}{[n]_{p,q}^s}. \tag{3.1}$$

Relation between $l_{p,q,\zeta}(s)$ and $E_{k,p,q,\zeta}$ is given by the following theorem.

Theorem 7. For $k\in\mathbb{N}$, we have

$$l_{p,q,\zeta}(-k)=E_{k,p,q,\zeta}.$$

By using (2.2), we note that

$$\frac{d^k}{dt^k}G_{p,q,\zeta}(t,x)\bigg|_{t=0}=[2]_q\sum_{m=0}^{\infty}(-1)^m\zeta^m[m+x]_{p,q}^k \tag{3.2}$$

and

$$\left(\frac{d}{dt}\right)^k\left(\sum_{n=0}^{\infty}E_{n,p,q,\zeta}(x)\frac{t^n}{n!}\right)\bigg|_{t=0}=E_{k,p,q,\zeta}(x),\ \text{for } k\in\mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the twisted (p,q)-L-function.

Definition 8. Let $s\in\mathbb{C}$ with $\mathrm{Re}(s)>0$ and $x\notin\mathbb{Z}_0^-$.

$$L_{p,q,\zeta}(s,x)=[2]_q\sum_{n=0}^{\infty}\frac{(-1)^n\zeta^n}{[n+x]_{p,q}^s}. \tag{3.4}$$

Note that $L_{p,q,\zeta}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $L_{p,q,\zeta}(s,x)$ and $E_{k,p,q,\zeta}(x)$ is given by the following theorem.

Theorem 9. For $k\in\mathbb{N}$, we have $L_{p,q,\zeta}(-k,x)=E_{k,p,q,\zeta}(x)$.

Observe that $L_{p,q,\zeta}(-k,x)$ function interpolates $E_{k,p,q,\zeta}(x)$ numbers at non-negative integers.

3. Some symmetric identities for twisted (p, q)-L-function

Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain certain symmetric identities for twisted (p, q)-L-function.

Theorem 10. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. Then we obtain

$$\begin{aligned}
&[w_2]_{p,q}^s[2]_{q^{w_2}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}L_{p^{w_1},q^{w_1},\zeta^{w_1}}\left(s, w_2x+\frac{w_2}{w_1}j\right)\\
&=[w_1]_{p,q}^s[2]_{q^{w_1}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}L_{p^{w_2},q^{w_2},\zeta^{w_2}}\left(s, w_1x+\frac{w_1}{w_2}j\right).
\end{aligned}\tag{4.1}$$

Proof. Note that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$. In (3.4), by substitute $w_2x + \frac{w_2}{w_1}j$ for x in and replace q, p, and ζ by q^{w_1}, p^{w_1} and ζ^{w_1}, respectively, we derive next result

$$\begin{aligned}
\frac{1}{[2]_{q^{w_1}}}L_{p^{w_1},q^{w_1},\zeta^{w_1}}\left(s, w_2x+\frac{w_2}{w_1}j\right) &= \sum_{m=0}^{\infty}\frac{(-1)^m\zeta^{w_1m}}{\left[m+w_2x+\frac{w_2}{w_1}j\right]_{p^{w_1},q^{w_1}}^s}\\
&=\sum_{m=0}^{\infty}\frac{(-1)^m\zeta^{w_1m}}{\left[\frac{w_1m+w_1w_2x+w_2j}{w_1}\right]_{p^{w_1},q^{w_1}}^s}\\
&=[w_1]_{p,q}^s\sum_{m=0}^{\infty}\sum_{i=0}^{w_2-1}\frac{(-1)^{w_2m+i}\zeta^{w_1(w_2m+i)}}{[w_1(w_2m+i)+w_1w_2x+w_2j]_{p,q}^s}\\
&=[w_1]_{p,q}^s\sum_{m=0}^{\infty}\sum_{i=0}^{w_2-1}\frac{(-1)^m(-1)^i\zeta^{w_1w_2m}\zeta^{w_1i}}{[w_1w_2(x+m)+w_1i+w_2j]_{p,q}^s}.
\end{aligned}\tag{4.2}$$

Thus, from (4.2), we can derive the following equation.

$$\begin{aligned}
&\frac{[w_2]_{p,q}^s}{[2]_{q^{w_1}}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}L_{p^{w_1},q^{w_1},\zeta^{w_1}}\left(s, w_2x+\frac{w_2}{w_1}j\right)\\
&=[w_1]_{p,q}^s[w_2]_{p,q}^s\sum_{m=0}^{\infty}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\frac{(-1)^{j+i+m}\zeta^{w_1w_2m}\zeta^{w_1i}\zeta^{w_2j}}{[w_1w_2(x+m)+w_1i+w_2j]_{p,q}^s}
\end{aligned}\tag{4.3}$$

By using the same method as (4.3), we have

$$\begin{aligned}
&\frac{[w_1]_{p,q}^s}{[2]_{q^{w_2}}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}L_{p^{w_2},q^{w_2},\zeta^{w_2}}\left(s, w_1x+\frac{w_1}{w_2}j\right)\\
&=[w_1]_{p,q}^s[w_2]_{p,q}^s\sum_{m=0}^{\infty}\sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}\frac{(-1)^{j+i+m}\zeta^{w_1w_2m}\zeta^{w_2i}\zeta^{w_1j}}{[w_1w_2(x+m)+w_1j+w_2i]_{p,q}^s}
\end{aligned}\tag{4.4}$$

Therefore, by (4.3) and (4.4), we have the following theorem. □

Taking $w_2 = 1$ in Theorem 10, we obtain the following corollary.

Corollary 11. Let $w_1 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$L_{p,q,\zeta}(s, w_1x) = \frac{[2]_q}{[2]_{q^{w_1}}[w_1]_{p,q}^s}\sum_{j=0}^{w_1-1}(-1)^j\zeta^jL_{p^{w_1},q^{w_1},\zeta^{w_1}}\left(s, x+\frac{j}{w_1}\right).$$

Let us take $s=-n$ in Theorem 10. For $n \in \mathbb{Z}_+$, we obtain certain symmetry identities for twisted (p,q)-Euler polynomials.

Theorem 12. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[w_1]_{p,q}^n[2]_{q^{w_2}}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=[w_2]_{p,q}^n[2]_{q^{w_1}}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}E_{n,p^{w_2},q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}j\right).\end{aligned}$$

Taking $w_2=1$ in Theorem 12, we obtain the following distribution relation.

Corollary 13. Let $w_1 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$E_{n,p,q,\zeta}(w_1x)=\frac{[2]_q}{[2]_{q^{w_1}}}[w_1]_{p,q}^n\sum_{j=0}^{w_1-1}(-1)^j\zeta^jE_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(s,x+\frac{j}{w_1}\right).$$

By (2.5), we have

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}\sum_{i=0}^{n}\binom{n}{i}q^{w_2j(n-i)}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\left[\frac{w_2}{w_1}j\right]^i_{p^{w_1},q^{w_1}}\\&=\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}\sum_{i=0}^{n}\binom{n}{i}q^{w_2j(n-i)}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\left(\frac{[w_2]_{p,q}}{[w_1]_{p,q}}\right)^i[j]^i_{p^{w_2},q^{w_2}}\end{aligned}$$

Hence we have the following theorem.

Theorem 14. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=\sum_{i=0}^{n}\binom{n}{i}[w_2]^i_{p,q}[w_1]^{-i}_{p,q}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}q^{w_2(n-i)j}[j]^i_{p^{w_2},q^{w_2}}.\end{aligned}$$

For each integer $n \geq 0$, let $\mathcal{A}_{n,i,p,q,\zeta}(w)=\sum_{j=0}^{w-1}(-1)^j\zeta^jq^{j(n-i)}[j]^i_{p,q}$. The sum $\mathcal{A}_{n,i,p,q,\zeta}(w)$ is called the alternating twisted (p,q)-power sums.

By Theorem 14, we have

$$\begin{aligned}&[2]_{q^{w_2}}[w_1]^n_{p,q}\sum_{j=0}^{w_1-1}(-1)^j\zeta^{w_2j}E_{n,p^{w_1},q^{w_1},\zeta^{w_1}}\left(w_2x+\frac{w_2}{w_1}j\right)\\&=[2]_{q^{w_2}}\sum_{i=0}^{n}\binom{n}{i}[w_2]^i_{p,q}[w_1]^{n-i}_{p,q}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2x)\mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1)\end{aligned} \tag{4.5}$$

By using the same method as in (4.5), we have

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]^n_{p,q}\sum_{j=0}^{w_2-1}(-1)^j\zeta^{w_1j}E_{n,p^{w_2},q^{w_2},\zeta^{w_2}}\left(w_1x+\frac{w_1}{w_2}j\right)\\&=[2]_{q^{w_1}}\sum_{i=0}^{n}\binom{n}{i}[w_1]^i_{p,q}[w_2]^{n-i}_{p,q}p^{w_1w_2xi}E^{(i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1x)\mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2)\end{aligned} \tag{4.6}$$

Therefore, by (4.5) and (4.6) and Theorem 12, we have the following theorem.

Theorem 15. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]_{p,q}^{i} [w_2]_{p,q}^{n-i} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1 x) \mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2)\\ &= [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]_{p,q}^{i} [w_1]_{p,q}^{n-i} p^{w_1 w_2 x i} E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2 x) \mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1).\end{aligned}$$

By Theorem 15, we obtain the interesting symmetric identity for the twisted (h, p, q)-Euler numbers $E^{(h)}_{n,p,q,\zeta}$ in complex field.

Corollary 16. Let $w_1, w_2 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod 2$, $w_2 \equiv 1 \pmod 2$. For $n \in \mathbb{Z}_+$, we obtain

$$\begin{aligned}&[2]_{q^{w_1}} \sum_{i=0}^{n} \binom{n}{i} [w_1]_{p,q}^{i} [w_2]_{p,q}^{n-i} p^{w_1 w_2 x i} \mathcal{A}_{n,i,p^{w_1},q^{w_1},\zeta^{w_1}}(w_2) E^{(i)}_{n-i,p^{w_2},q^{w_2},\zeta^{w_2}}\\ &= [2]_{q^{w_2}} \sum_{i=0}^{n} \binom{n}{i} [w_2]_{p,q}^{i} [w_1]_{p,q}^{n-i} p^{w_1 w_2 x i} \mathcal{A}_{n,i,p^{w_2},q^{w_2},\zeta^{w_2}}(w_1) E^{(i)}_{n-i,p^{w_1},q^{w_1},\zeta^{w_1}}.\end{aligned}$$

Acknowledgement: This work was supported by 2020 Hannam University Research Fund.

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Volume 29, Number 5 October 2021
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS, LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)
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Journal of Computational Analysis and Applications(JoCAAA) is published by **EUDOXUS PRESS,LLC**,1424 Beaver Trail Drive,Cordova,TN38016,USA,anastassioug@yahoo.com http://www.eudoxuspress.com. **Annual Subscription Prices**:For USA and Canada,Institutional:Print $800, Electronic OPEN ACCESS. Individual:Print $400. For any other part of the world add $160 more(handling and postages) to the above prices for Print. No credit card payments.

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JoCAAA is reviewed and abstracted by AMS Mathematical Reviews,MATHSCI,and Zentralblaat MATH.

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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Differential equations and inclusions involving mixed fractional derivatives with four-point nonlocal fractional boundary conditions

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Abstract

We study a new class of boundary value problems of mixed fractional differential equations and inclusions involving both left Caputo and right Riemann-Liouville fractional derivatives, and nonlocal four-point fractional boundary conditions. We apply the standard tools of the fixed-point theory to obtain the sufficient criteria for the existence and uniqueness of solutions for the problems at hand. Illustrative examples for the obtained results are also presented.

Keywords: Fractional differential equations; fractional differential inclusions; fractional derivative; boundary value problem; existence; fixed point theorems.
MSC 2000: 34A08, 34B15, 34A60.

1 Introduction

Fractional calculus deals with the study of fractional order integrals and derivatives and their diverse applications [1, 2, 3]. Riemann-Liouville and Caputo are kinds of fractional derivatives. They all generalize the ordinary integral and differential operators. However, the fractional derivatives have fewer properties than the corresponding classical ones. As a result, it makes these derivatives very useful at describing the anomalous phenomena, see [4, 5, 6] and references cited therein.

Some solutions of equations containing left and right fractional derivatives were investigated [7, 8, 9]. The left and the right derivatives found interesting applications in fractional variational principles, fractional control theory as well as in fractional Lagrangian and Hamiltonian dynamics. In [10], the existence of an extremal solution to a nonlinear system with the right-handed Riemann-Liouville fractional derivative

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was discussed. In [11, 12], the authors studied the existence of solutions for fractional boundary value problems involving both the left Riemann-Liouville and the right Caputo fractional derivatives.

In this paper, we investigate the existence and uniqueness of solutions for a mixed fractional differential equation involving both left Caputo and right Riemann-Liouville types fractional derivatives associated with nonlocal four-point fractional boundary conditions. Precisely, we study the following problems:

$$\begin{cases} {}^cD^{\alpha}_{1-}D^{\beta}_{0+}y(t) = f(t, y(t)), \quad t \in J := [0, 1], \\ y(0) = 0, \ D^{\beta}_{0+}y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \eta < 1, \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}^cD^{\alpha}_{1-}D^{\beta}_{0+}y(t) \in F(t, y(t)), \quad t \in J := [0, 1], \\ y(0) = 0, \ D^{\beta}_{0+}y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \xi, \eta < 1, \end{cases} \tag{1.2}$$

where ${}^cD^{\alpha}_{1-}$ and D^{β}_{0+} denote the left Caputo fractional derivative of order $\alpha \in (1, 2]$ and the right Riemann-Liouville fractional derivative of order $\beta \in (0, 1]$ respectively, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given function, $F : [0, 1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\delta \in \mathbb{R}$ is an appropriate constant. Here we remark that the problem (1.1) with $y'(0) = 0$ in palce of $D^{\beta}_{0+}y(\xi) = 0$, was studied recently in [13].

The rest of the paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and prove a basic result that plays a key role in the forthcoming analysis. Section 3 contains the existence and uniqueness results for the problem (1.1), which rely on fixed point theorems due to Banach, Krasnoselskii and Leray-Schauder nonlinear alternative. In Section 4, we discuss existence results for the problem (1.2), which rely on nonlineqar alternative for Kakutani maps and Covitz and Nadler fixed point theorem. Finally in Section 5 we study illustrative examples for the obtained results.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts [14] that we need in the sequel.

Definition 2.1 *We define the left and right Riemann-Liouville fractional integrals of order* $\alpha > 0$ *of a function* $g : (0, \infty) \to \mathbb{R}$ *as*

$$I^{\alpha}_{0+}g(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds, \tag{2.1}$$

$$I^{\alpha}_{1-}g(t) = \int_t^1 \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} g(s)ds, \tag{2.2}$$

provided the right-hand sides are point-wise defined on $(0,\infty)$, *where* Γ *is the Gamma function.*

Definition 2.2 *The left Riemann-Liouville fractional derivative and the right Caputo fractional derivative of order* $\alpha > 0$ *of a continuous function* $g : (0,\infty) \to \mathbb{R}$ *such that* $g \in C^n((0,\infty),\mathbb{R})$ *are respectively given by*

$$D^{\alpha}_{0+}g(t) = \frac{d^n}{dt^n}(I^{n-\alpha}_{0+}g)(t),$$

$${}^cD^{\alpha}_{1-}g(t) = (-1)^n I^{n-\alpha}_{1-}g^{(n)}(t),$$

where $n-1 < \alpha < n$.

The following lemma, dealing with a linear variant of the problem (1.1), plays an important role in the forthcoming analysis.

Lemma 2.3 *Let* $h \in C(J,\mathbb{R})$ *and* $P = [(1-\delta\eta^{\beta+1}) - (\beta+1)\xi(1-\delta\eta^{\beta})] \neq 0$. *The function* y *is a solution of the problem*

$$\begin{cases} {}^cD^{\alpha}_{1-}D^{\beta}_{0+}y(t) = h(t), \quad t \in J := [0,1], \\ y(0) = 0, \ D^{\beta}_{0+}y(\xi) = 0, \quad y(1) = \delta y(\eta), \quad 0 < \xi, \eta < 1, \end{cases} \tag{2.3}$$

if and only if

$$\begin{aligned} y(t) = {} & I^{\beta}_{0+}I^{\alpha}_{1-}h(t) + \frac{[t^{\beta+1}(1-\delta\eta^{\beta}) - t^{\beta}(1-\delta\eta^{\beta+1})]}{P\Gamma(\beta+1)} I^{\alpha}_{1-}h(t)|_{t=\xi} \\ & + \frac{[t^{\beta+1} - \xi(\beta+1)t^{\beta}]}{P}\left(\delta I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=\eta} - I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=1}\right), \end{aligned} \tag{2.4}$$

where $I^{\alpha}_{1-}y(s)$ *is defined by (2.2).*

Proof. Applying the right fractional integral I^{α}_{1-} to both sides of the equation in the problem (2.3), we get

$$D^{\beta}_{0+}y(t) = I^{\alpha}_{1-}h(t) + c_0 + c_1 t. \tag{2.5}$$

Using the condition $D^{\beta}_{0+}y(\xi) = 0$ in (2.5), we obtain

$$c_0 + c_1\xi = -I^{\alpha}_{1-}h(t)|_{t=\xi}. \tag{2.6}$$

Next we apply the left fractional integral I^{β}_{0+} to the equation (2.5) to get

$$y(t) = I^{\beta}_{0+}I^{\alpha}_{1-}h(t) + c_0\frac{t^{\beta}}{\Gamma(\beta+1)} + c_1\frac{t^{\beta+1}}{\Gamma(\beta+2)} + c_2 t^{\beta-1}. \tag{2.7}$$

Making use of the conditions $y(0)=0$ and $y(1)=\delta y(\eta)$ in (2.7) yields $c_2=0$ and

$$\frac{(1-\delta\eta^{\beta})}{\Gamma(\beta+1)}c_0+\frac{(1-\delta\eta^{\beta+1})}{\Gamma(\beta+2)}c_1=\delta I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=\eta}-I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=1}. \tag{2.8}$$

Solving (2.7) and (2.8) for c_0 and c_1, we find that

$$c_0=-\frac{\Gamma(\beta+2)}{P}\Big[\frac{(1-\delta\eta^{\beta+1})}{\Gamma(\beta+2)}I^{\alpha}_{1-}h(t)|_{t=\xi}+\xi\left(\delta I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=\eta}-I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=1}\right)\Big],$$

$$c_1=\frac{\Gamma(\beta+2)}{P}\Big[\delta I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=\eta}-I^{\beta}_{0+}I^{\alpha}_{1-}h(t)|_{t=1}+\frac{(1-\delta\eta^{\beta})}{\Gamma(\beta+1)}I^{\alpha}_{1-}h(t)|_{t=\xi}\Big].$$

Substituting the values of c_0 and c_1 in (2.6), we get the solution (2.4). By direct computation, we can obtain the converse of this lemma. This completes the proof. □

Remark 2.4 *Let $\|h\|=\sup_{t\in[0,1]}|h(t)$. Then we have the following estimate:*

$$\|y\|\le\|h\|\max_{t\in[0,1]}\left\{\frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)|+\frac{[t^{\beta}+(1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\}, \tag{2.9}$$

where

$$\mu_1(t)=\frac{t^{\beta+1}(1-\delta\eta^{\beta})-t^{\beta}(1-\delta\eta^{\beta+1})}{P\Gamma(\beta+1)},\quad \mu_2(t)=\frac{t^{\beta+1}-\xi(\beta+1)t^{\beta}}{P}. \tag{2.10}$$

Indeed, we have

$$\begin{aligned}|y(t)| &\le \|h\|\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds+|\mu_1(t)|\|h\|\int_{\xi}^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\\ &\quad+|\mu_2(t)|\|h\|\Bigg[\delta\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\\ &\quad+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\\ &= \|h\|\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds+|\mu_1(t)|\|h\|\int_{\xi}^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\\ &\quad+|\mu_2(t)|\|h\|\left[\delta\int_0^{\eta}\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds\right]\\ &\le \|h\|\max_{t\in[0,1]}\left\{\frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)|+\frac{[t^{\beta}+(1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)}\right\},\end{aligned}$$

where we taken $(1-s)^{\alpha}\le 1$.

For computation convenience, we introduce the notation:

$$\Lambda=\max_{t\in[0,1]}\Big\{\frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}|\mu_1(t)|+\frac{[t^{\beta}+(1+\delta\eta^{\beta})|\mu_2(t)|]}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Big\}. \tag{2.11}$$

3 Existence and uniqueness results for the problem (1.1)

Let $\mathcal{X} = C([0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ equipped with the norm $\|y\| = \sup\{|y(t)| : t \in [0,1]\}$.

In view of Lemma 2.3, we transform the problem (1.1) into a fixed point problem as

$$y = \mathcal{G}y, \tag{3.1}$$

where the operator $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is defined by

$$\begin{aligned} \mathcal{G}y(t) \;&=\; \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds + \mu_1(t)\int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds \qquad (3.2)\\ &\quad +\mu_2(t)\Big[\delta\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} f(s,y(s))ds\Big], \end{aligned}$$

where μ_1, μ_2 are defined by (2.10).

Our first result deals with the existence and uniqueness of solutions for the problem (1.1).

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that:*

(H_1) $|f(t,y) - f(t,z)| \le L|y-z|$, *for all* $t \in [0,1]$, $y, z \in \mathbb{R}$, $L > 0$.

Then the problem (1.1) has a unique solution on $[0,1]$ *if*

$$L\Lambda < 1, \tag{3.3}$$

where Λ is defined by (2.11).

Proof. Let us define $\sup_{t\in[0,1]} |f(t,0)| = M$ and select $r \ge \dfrac{M\Lambda}{1-L\Lambda}$ to establish that $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$, where $\mathcal{B}_r = \{y \in \mathcal{X} : \|y\| \le r\}$ and $\mathcal{G}$ is defined by (3.2). Using the condition (H_1), we have

$$\begin{aligned} |f(t,y)| &= |f(t,y) - f(t,0) + f(t,0)| \le |f(t,y) - f(t,0)| + |f(t,0)| \\ &\le L\|y\| + M \le Lr + M. \end{aligned} \tag{3.4}$$

Then, for $y \in \mathcal{B}_r$, by using Remark 2.4, we obtain

$$\begin{aligned} \|\mathcal{G}y\| \;&\le\; (Lr+M)\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)|\int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\ &\quad +|\mu_2(t)|\Bigg[\delta\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \end{aligned}$$

$$
\begin{aligned}
& +\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\Bigg\} \\
= \;& (Lr+M)\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds \\
& +|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\frac{(1-s)^{\alpha}}{\Gamma(\alpha+1)}ds\Bigg]\Bigg\} \\
\leq \;& (Lr+M)\Lambda < r.
\end{aligned}
$$

This show that $\mathcal{G}y \in \mathcal{B}_r$, $y \in \mathcal{B}_r$. Thus $\mathcal{G}\mathcal{B}_r \subset \mathcal{B}_r$. Next we show that $\mathcal{G}$ is a contraction. For that, let $y, z \in \mathcal{X}$. Then, for each $t \in [0,1]$, we have

$$
\begin{aligned}
& \|(\mathcal{G}y)-(\mathcal{G}z)\| \\
\leq \;& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|f(u,y(u))-f(u,z(u))|duds \\
& +|\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s))-f(s,z(s))|ds \\
& +|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|f(u,y(u))-f(u,z(u))|duds \\
& +\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}|f(u,y(u))-f(u,z(u))|duds\Bigg] \\
\leq \;& L\Lambda\|y-z\|,
\end{aligned}
$$

which, in view of the given condition $L\Lambda < 1$, implies that $\mathcal{G}$ is a contraction. In consequence, it follow by the contraction mapping principle that there exists a unique solution for the problem (1.1) on $[0,1]$. This completes the proof. □

Our next existence result for the problem (1.1) relies on Krasnoselskii's fixed point theorem.

Lemma 3.2 *(Krasnoselskii's fixed point theorem) [15]. Let S be a closed, bounded, convex and nonempty subset of a Banach space X. Let $\mathcal{Y}_1, \mathcal{Y}_2$ be the operators mapping S into X such that (a) $\mathcal{Y}_1 s_1 + \mathcal{Y}_2 s_2 \in S$ whenever $s_1, s_2 \in S$; (b) $\mathcal{Y}_1$ is compact and continuous; (c) $\mathcal{Y}_2$ is a contraction mapping. Then there exists $s_3 \in S$ such that $s_3 = \mathcal{Y}_1 s_3 + \mathcal{Y}_2 s_3$.*

Theorem 3.3 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition (H_1). In addition we assume that:*

(H_2) $|f(t,y)| \leq m(t)$, for all $(t,y) \in [0,1] \times \mathbb{R}$ and $m \in C([0,1], \mathbb{R}^+)$.

Then there exists at least one solution for the problem (1.1) on $[0,1]$ *provided that*

$$L \sup_{t\in[0,1]} \left\{ \frac{t^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)| \frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)} \right\} < 1. \tag{3.5}$$

Proof. Setting $\sup_{t\in[0,1]} |m(t)| = \|m\|$, we fix

$$\varrho \geq \|m\|\Lambda, \tag{3.6}$$

where Λ is defined by (2.11), and consider $B_\varrho = \{y \in \mathcal{X} : \|y\| \leq \varrho\}$. Introduce the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on B_ϱ as follows:

$$\begin{aligned} \mathcal{G}_1 y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u,y(u))duds \\ &\quad + \mu_1(t) \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} f(s,y(s))ds, \end{aligned}$$

and

$$\begin{aligned} \mathcal{G}_2 y(t) &= \mu_2(t) \Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u,y(u))duds \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} f(u,y(u))duds \Bigg]. \end{aligned}$$

Observe that $\mathcal{G} = \mathcal{G}_1 + \mathcal{G}_2$. Now we verify the hypotheses of Krasnoselskii's fixed point theorem in the following steps.

(i) For $y, z \in B_\varrho$, we have

$$\begin{aligned} &\|\mathcal{G}_1 y + \mathcal{G}_2 z\| = \sup_{t\in[0,1]} |(\mathcal{G}_1 y)(t) + (\mathcal{G}_2 z)(t)| \\ \leq\ & \|m\| \sup_{t\in[0,1]} \Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds \\ & + |\mu_2(t)| \Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \\ & + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \Bigg] \Bigg\} \\ & \leq \|m\|\Lambda \leq \varrho, \end{aligned}$$

where we used (3.6). Thus $\mathcal{G}_1 y + \mathcal{G}_2 z \in B_\varrho$.

(ii) We show that $\mathcal{G}_1$ is a contraction. Indeed, by using the assumption (H_1) together with (3.5) and the fact that $(1-s)^{\alpha} < 1, (1 < \alpha \leq 2)$ we have

$$\begin{aligned} |\mathcal{G}_1 y(t) - \mathcal{G}_1 z(t)| &\leq L\|y-z\|\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \\ &\quad +|\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\} \\ &\leq L\|y-z\|\Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}ds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\} \\ &\leq L\sup_{t\in[0,1]}\left\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\right\}\|y-z\|, \end{aligned}$$

which implies that

$$\|\mathcal{G}_1 y - \mathcal{G}_1 z\| \leq L\sup_{t\in[0,1]}\left\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\right\}\|y-z\|.$$

Hence $\mathcal{G}_1$ is a contraction by (3.5).

(iii) Using the continuity of f, it is easy to show that the operator $\mathcal{G}_2$ is continuous. Further, $\mathcal{G}_2$ is uniformly bounded on B_ϱ as

$$\|\mathcal{G}_2 x\| = \sup_{t\in[0,1]}|(\mathcal{G}_2 y)(t)| \leq \frac{\|m\|M_2(\delta\eta^\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}, \quad M_2 = \sup_{t\in[0,1]}|\mu_2(t)|.$$

In order to establish that $\mathcal{G}_2$ is compact, we define $\sup_{(t,y)\in[0,1]\times B_\varrho}|f(t,y)| = \overline{f}$. Thus, for $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned} |(\mathcal{G}_2 y)(t_2) - (\mathcal{G}_2 y)(t_1)| &\leq |\mu_2(t_2)-\mu_2(t_1)|\overline{f}\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds \\ &\quad +\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg] \\ &\leq |\mu_2(t_2)-\mu_2(t_1)|\overline{f}\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)} \to 0 \text{ as } t_1 \to t_2, \end{aligned}$$

independent of y. This shows that $\mathcal{G}_2$ is relatively compact on B_ϱ. As all the conditions of the Arzelá-Ascoli theorem are satisfied, so $\mathcal{G}_2$ is compact on B_ϱ. In view of steps *(i)-(iii)*, the conclusion of Krasnoselskii's fixed point theorem applies and hence there exists at least one solution for the problem (1.1) on $[0,1]$. The proof is completed. □

Remark 3.4 *Interchanging the roles of the operators $\mathcal{G}_1$ and $\mathcal{G}_2$ in the foregoing result, we can obtain a second result by requiring the condition:*

$$LM_1\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}<1,\quad M_1=\sup_{t\in[0,1]}|\mu_1(t)|,$$

instead of (3.5).

The following existence result is based on Leray-Schauder nonlinear alternative.

Lemma 3.5 *(Nonlinear alternative for single valued maps)[16]. Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u\in\partial U$ (the boundary of U in C) and $\lambda\in(0,1)$ with $u=\lambda F(u)$.

Theorem 3.6 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that*

(H_3) *There exist a function $g\in C([0,1],\mathbb{R}^+)$, and a nondecreasing function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ such that $|f(t,y)|\le g(t)\psi(\|y\|)$, $\forall(t,y)\in[0,1]\times\mathbb{R}$.*

(H_4) *There exists a constant $K>0$ such that*

$$\frac{K}{\|g\|\psi(K)\Lambda}>1.$$

Then the problem (1.1) has at least one solution on $[0,1]$.

Proof. Consider the operator $\mathcal{G}:\mathcal{X}\to\mathcal{X}$ defined by (3.2). We show that $\mathcal{G}$ maps bounded sets into bounded sets in $\mathcal{X}=C([0,1],\mathbb{R})$. For a positive number r, let $\mathcal{B}_r=\{y\in C([0,1],\mathbb{R}):\|y\|\le r\}$ be a bounded set in $C([0,1],\mathbb{R})$. Then, by using the fact that $(1-s)^{\alpha-1}\le 1$ $(1<\alpha\le 2)$ we have

$$\begin{aligned}|\mathcal{G}y(t)| &\le \|g\|\psi(r)\Bigg\{\int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds+|\mu_1(t)|\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds\\ &\quad+|\mu_2(t)|\Bigg[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\\ &\quad+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\Bigg\}\\ &\le \|g\|\psi(r)\Lambda,\end{aligned}$$

which, on taking the norm for $t \in [0,1]$, yields

$$\|\mathcal{G}y\| \le \|g\|\psi(r)\Lambda.$$

Next we show that $\mathcal{G}$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{B}_r$, where $\mathcal{B}_r$ is a bounded set of $C([0,1],\mathbb{R})$. Then, using the fact that $(1-s)^{\alpha-1} \le 1$ $(1 < \alpha \le 2)$ and the computations for $\mathcal{G}_2$ in previous theorem, we obtain

$$\begin{aligned}
&|\mathcal{G}y(t_2) - \mathcal{G}y(t_1)| \\
\le\ & \|g\|\psi(r)\Bigg\{\Bigg|\int_0^{t_1} \frac{[(t_2-s)^{\beta-1} - (t_1-s)^{\beta-1}]}{\Gamma(\beta)}ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}ds\Bigg| \\
& + |\mu_1(t_2) - \mu_1(t_1)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds + |\mu_2(t_2) - \mu_2(t_1)| \frac{\delta\eta^\beta + 1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\} \\
\le\ & \|g\|\psi(r)\Bigg\{\frac{2(t_2-t_1)^\beta + t_2^\beta - t_1^\beta}{\Gamma(\beta+1)} + |\mu_1(t_2) - \mu_1(t_1)| \frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)} \\
& + |\mu_2(t_2) - \mu_2(t_1)| \frac{\delta\eta^\beta + 1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\},
\end{aligned}$$

which tends to zero independently of $y \in \mathcal{B}_r$ as $t_2 - t_1 \to 0$. As $\mathcal{G}$ satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\mathcal{G} : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative once it is shown that the set of all solutions to the equation $y = \lambda\mathcal{G}y$ is bounded for $\lambda \in [0,1]$. For that, let y be a solution of $y = \lambda\mathcal{G}y$ for $\lambda \in [0,1]$. Then, for $t \in [0,1]$, we have

$$\begin{aligned}
|y(t)| = |\lambda\mathcal{G}y(t)| \le\ & \Bigg\{\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds \\
& + |\mu_2(t)| \Bigg[\delta \int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds \\
& + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds\Bigg]\Bigg\} |g(t)|\psi(\|y\|) \\
\le\ & \|g\|\psi(\|y\|)\Lambda,
\end{aligned}$$

which implies that

$$\frac{\|y\|}{\|g\|\psi(\|y\|)\Lambda} \le 1.$$

In view of (H_4), there is no solution y such that $\|y\| \ne K$. Let us set

$$U = \{y \in \mathcal{X} : \|y\| < K\}.$$

The operator $\mathcal{G} : \overline{U} \to \mathcal{X}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \lambda \mathcal{G}(u)$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [16, Theorem 5.2], we deduce that $\mathcal{G}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

4 Existence results for the problem (1.2)

Before presenting the existence results for the problem (1.2), we outline the necessary concepts on multi-valued maps [17], [18].

For a normed space $(X, \|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex }\}$. A multi-valued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. The map G is bounded on bounded sets if $G(\mathbb{B}) = \cup_{x \in \mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x \in \mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$. G is said to be completely continuous if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$. If the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph, i.e., $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0,1] \to \mathcal{P}_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable.

For each $y \in \mathcal{X}$, define the set of selections of F by

$$S_{F,y} := \{v \in L^1([0,1], \mathbb{R}) : v(t) \in F(t, y(t)) \text{ for a.e. } t \in [0,1]\}.$$

Definition 4.1 *A function $y \in C([0,1], \mathbb{R})$ is said to be a solution of the boundary value problem (1.2) if $y(0) = 0$, $D^{\beta}_{0+} y(\xi) = 0$, $y(1) = \delta y(\eta)$, $0 < \xi, \eta < 1$, and there exists a function $v \in S_{F,y}$ such that $v(t) \in F(t, y(t))$ and*

$$\begin{aligned} y(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-} v(s) ds + \mu_1(t) \int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s) ds \\ &\quad + \mu_2(t) \Big[\delta \int_0^{\eta} \frac{(\eta - s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-} v(s) ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I^{\alpha}_{1-} v(s) ds \Big], \quad t \in [0,1]. \end{aligned}$$

4.1 The upper semicontinuous case

In the case when F has convex values we prove an existence result based on nonlinear alternative of Leray-Schauder type.

Definition 4.2 *A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,y)$ *is measurable for each* $y \in \mathbb{R}$;

(ii) $y \longmapsto F(t,y)$ *is upper semicontinuous for almost all* $t \in [0,1]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \leq \varphi_\rho(t)$$

for all $y \in \mathbb{R}$ with $\|y\| \leq \rho$ and for a.e. $t \in [0,1]$.

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y : y \in G(x)\}$ and recall two results for closed graphs and upper-semicontinuity.

Lemma 4.3 *([17, Proposition 1.2]) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4.4 *([19]) Let X be a separable Banach space. Let $F : [0,1] \times X \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,1],X)$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_{F,x} : C([0,1],X) \to \mathcal{P}_{cp,c}(C([0,1],X)), \quad y \mapsto (\Theta \circ S_{F,y})(y) = \Theta(S_{F,y})$$

is a closed graph operator in $C([0,1],X) \times C([0,1],X)$.

For the forthcoming analysis, we need the following lemma.

Lemma 4.5 *(Nonlinear alternative for Kakutani maps)[16]. Let E be a Banach space, C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{cp,c}(C)$ is a upper semicontinuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or

(ii) there is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda F(u)$.

Theorem 4.6 *Assume that:*

(B_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is L^1-Carathéodory and has nonempty compact and convex values;*

(B_2) *there exist a function* $\phi \in C([0,1],\mathbb{R}^+)$, *and a nondecreasing function* $\Omega : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

$$\|F(t,y)\|_{\mathcal{P}} := \sup\{|w| : w \in F(t,y)\} \le \phi(t)\Omega(\|y\|)$$

for each $(t,y) \in [0,1] \times \mathbb{R}$;

(B_3) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\|\phi\|\Lambda\Omega(M)} > 1,$$

where Λ *is defined by (2.11).*

Then the boundary value problem (1.2) has at least one solution on $[0,1]$.

Proof. Define an operator $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ by

$$\Omega_F(y) = \{h \in \mathcal{X} : h(t) = N(y)(t)\}$$

where

$$\begin{aligned} N(y)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds + \mu_1(t)\int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Big[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds\Big]. \end{aligned}$$

We will show that Ω_F satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof consists of several steps. As a first step, we show that Ω_F *is convex for each* $y \in \mathcal{X}$. This step is obvious since $S_{F,y}$ is convex (F has convex values), and therefore we omit the proof.

In the second step, we show that Ω_F *maps bounded sets (balls) into bounded sets in* $\mathcal{X}$. For a positive number ρ, let $B_\rho = \{y \in \mathcal{X} : \|y\| \le \rho\}$ be a bounded ball in $\mathcal{X}$. Then, for each $h \in \Omega_F(y), y \in B_\rho$, there exists $v \in S_{F,y}$ such that

$$\begin{aligned} h(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds + \mu_1(t)\int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Bigg[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds \\ &\quad + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} v(u)duds\Bigg]. \end{aligned}$$

Then, by using the fact that $(1-s)^{\alpha-1} \le 1 \ (1 < \alpha \le 2)$ we have

$$|h(t)| \le \|g\|\Omega(r)\Bigg\{ \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} duds + |\mu_1(t)| \int_{\xi}^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$
\begin{aligned}
&+|\mu_2(t)|\Bigg[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\\
&+\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\int_s^1\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}duds\Bigg]\Bigg\}\\
\leq\ & \|\phi\|\Omega(r)\Lambda,
\end{aligned}
$$

which, on taking the norm for $t\in[0,1]$. yields

$$\|h\|\leq\|\phi\|\Omega(r)\Lambda.$$

Now we show that Ω_F *maps bounded sets into equicontinuous sets of* $\mathcal{X}$. Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $y\in B_\rho$. For each $h\in\Omega_F(y)$, using the fact that $(1-s)^{\alpha-1}\leq 1$ $(1<\alpha\leq 2)$, we obtain

$$
\begin{aligned}
&|h(t_2)-h(t_1)|\\
\leq\ & \|\phi\|\Omega(r)\Bigg\{\Bigg|\int_0^{t_1}\frac{[(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}]}{\Gamma(\beta)\Gamma(\alpha+1)}ds+\int_{t_1}^{t_2}\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)\Gamma(\alpha+1)}ds\Bigg|\\
&+|\mu_1(t_2)-\mu_1(t_1)|\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}ds+|\mu_2(t_2)-\mu_2(t_1)|\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\}\\
\leq\ & \|\phi\|\Omega(r)\Bigg\{\frac{2(t_2-t_1)^\beta+t_2^\beta-t_1^\beta}{\Gamma(\beta+1)\Gamma(\alpha+1)}+|\mu_1(t_2)-\mu_1(t_1)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\\
&+|\mu_2(t_2)-\mu_2(t_1)|\frac{\delta\eta^\beta+1}{\Gamma(\alpha+1)\Gamma(\beta+1)}\Bigg\},
\end{aligned}
$$

which tends to zero independently of $y\in\mathcal{B}_r$ as $t_2-t_1\to 0$. As Ω_F satisfies the above assumptions, therefore it follows by the Arzelá-Ascoli theorem that $\Omega_F: C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$ is completely continuous.

In our next step, we show that Ω_F *is upper semicontinuous.* To this end it is sufficient to show that Ω_F *has a closed graph,* by Lemma 4.3. Let $y_n\to y_*$, $h_n\in\Omega_F(y_n)$ and $h_n\to h_*$. Then we need to show that $h_*\in\Omega_F(y_*)$. Associated with $h_n\in\Omega_F(y_n)$, there exists $v_n\in S_{F,y_n}$ such that for each $t\in[0,1]$,

$$
\begin{aligned}
h_n(t)\ =\ & \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha v_n(s)ds+\mu_1(t)\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}v_n(s)ds\\
&+\mu_2(t)\Big[\delta\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha v_n(s)ds-\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha v_n(s)ds\Big].
\end{aligned}
$$

Thus it suffices to show that there exists $v_*\in S_{F,y_*}$ such that for each $t\in[0,1]$,

$$h_*(t)\ =\ \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}I_{1-}^\alpha v_*(s)ds+\mu_1(t)\int_\xi^1\frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}v_*(s)ds$$

$$+\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds\Big].$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to \mathcal{X}$ given by

$$\begin{aligned} v \mapsto \Theta(v)(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v(s)ds\Big]. \end{aligned}$$

Observe that

$$\begin{aligned} &\|h_n(t)-h_*(t)\| \\ = &\Big\|\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha (v_n-v_*)(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} (v_n-v_*)(s)ds \\ &+\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha (v_n-v_*)(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha (v_n-v_*)(s)ds\Big]\Big\| \to 0, \end{aligned}$$

as $n \to \infty$. Thus, it follows by Lemma 4.4 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,y_n})$. Since $y_n \to y_*$, therefore, we have

$$\begin{aligned} h_*(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds + \mu_1(t)\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_*(s)ds \\ &\quad +\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_*(s)ds\Big]. \end{aligned}$$

Finally, we show there exists an open set $U \subseteq \mathcal{X}$ with $y \notin \theta\Omega_F(y)$ for any $\theta \in (0,1)$ and all $y \in \partial U$. Let $\theta \in (0,1)$ and $y \in \theta\Omega_F(y)$. Then there exists $v \in L^1([0,1],\mathbb{R})$ with $v \in S_{F,y}$ such that, for $t \in [0,1]$, we can obtain

$$\begin{aligned} &|y(t)| = |\theta\Omega_F(y)(t)| \\ \le &\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v(u)|duds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} |v(s)|ds \\ &+|\mu_2(t)|\Bigg[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v(u)|duds \\ &+\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \int_s^1 \frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} |v(u)|duds\Bigg] \\ \le &\|\phi\|\Omega(\|y\|)\Lambda, \end{aligned}$$

which implies that

$$\frac{\|y\|}{\|\phi\|\Omega(\|y\|)\Lambda} \le 1.$$

In view of (B_3), there exists M such that $\|y\| \neq M$. Let us set

$$U = \{y \in \mathcal{X} : \|y\| < M\}.$$

Note that the operator $\Omega_F : \overline{U} \to \mathcal{P}(\mathcal{X})$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \theta\Omega_F(y)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 4.5), we deduce that Ω_F has a fixed point $y \in \overline{U}$ which is a solution of the problem (1.2). This completes the proof. □

4.2 The Lipschitz case

We prove in this subsection the existence of solutions for the problem (1.2) with a nonconvex valued right-hand side by applying a fixed point theorem for multivalued maps due to Covitz and Nadler [21].

Let (X,d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ defined by $H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\}$, where $d(A,b) = \inf_{a\in A} d(a;b)$ and $d(a,B) = \inf_{b\in B} d(a;b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [20]).

Definition 4.7 *A multivalued operator $N : X \to \mathcal{P}_{cl}(X)$ is called* (**a**) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x), N(y)) \leq \gamma d(x,y)$ for each $x, y \in X$ and* (**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Lemma 4.8 *([21]) Let (X,d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

Theorem 4.9 *Assume that:*

(A_1) *$F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, y(t)) : [0,1] \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$;*

(A_2) *$H_d(F(t,y), F(t,\bar{y}) \leq q(t)|y - \bar{y}|$ for almost all $t \in [0,1]$ and $y, \bar{y} \in \mathbb{R}$ with $q \in C([0,1], \mathbb{R}^+)$ and $d(0, F(t,0)) \leq q(t)$ for almost all $t \in [0,1]$.*

Then the problem (1.2) has at least one solution on $[0,1]$ if

$$\|q\|\Lambda < 1, \tag{4.1}$$

where Λ is defined by (2.11).

Proof. Consider the operator $\Omega_F : \mathcal{X} \to \mathcal{P}(\mathcal{X})$ defined in the beginning of the proof of Theorem 4.6. Observe that the set $S_{F,y}$ is nonempty for each $y \in \mathcal{X}$ by the assumption (A_1), so F has a measurable selection (see Theorem III.6 [22]). Now we show that the operator Ω_F satisfies the assumptions of Lemma 4.8. To show that $\Omega_F(y) \in \mathcal{P}_{cl}(\mathcal{X})$

for each $y \in \mathcal{X}$, let $\{u_n\}_{n\geq 0} \in \Omega_F(y)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathcal{X}$. Then $u \in C([0,1],\mathbb{R})$ and there exists $v_n \in S_{F,y}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds + \mu_1(t)\int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_n(s)ds \\ &\quad + \mu_2(t)\Big[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_n(s)ds\Big]. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1([0,1],\mathbb{R})$. Thus, $v \in S_{F,y}$ and for each $t \in [0,1]$, we have

$$\begin{aligned} u_n(t) \to u(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds + \mu_1(t)\int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v(s)ds \\ &\quad + \mu_2(t)\Big[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v(s)ds\Big]. \end{aligned}$$

Hence, $u \in \Omega_F(y)$.

Next we show that there exists $\hat{\theta} := \|q\|\Lambda < 1$ such that

$$H_d(\Omega_F(y), \Omega_F(\bar{y})) \leq \hat{\theta}\|y-\bar{y}\| \text{ for each } y, \bar{y} \in \mathcal{X}.$$

Let $y, \bar{y} \in \mathcal{X}$ and $h_1 \in \Omega_F(y)$. Then there exists $v_1(t) \in F(t, y(t))$ such that, for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds + \mu_1(t)\int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_1(s)ds \\ &\quad + \mu_2(t)\Big[\delta \int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_1(s)ds\Big]. \end{aligned}$$

By (A_2), we have

$$H_d(F(t,y), F(t,\bar{y}) \leq q(t)|y-\bar{y}|.$$

So, there exists $w \in F(t,\bar{y})$ such that

$$|v_1(t) - w| \leq q(t)|y(t) - \bar{y}(t)|, \quad t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq q(t)|y(t) - \bar{y}(t)|\}.$$

Since the multivalued operator $U(t) \cap F(t,\bar{y})$ is measurable (Proposition III.4 [22]), there exists a function $v_2(t)$ which is a measurable selection for $U(t) \cap F(t,\bar{y})$. So $v_2(t) \in F(t,\bar{y})$ and for each $t \in [0,1]$, we have $|v_1(t) - v_2(t)| \leq q(t)|y(t) - \bar{y}(t)|$. For each $t \in [0,1]$, let us define

$$h_2(t) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^{\alpha} v_2(s)ds + \mu_1(t)\int_{\xi}^{1} \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)} v_2(s)ds$$

$$+\mu_2(t)\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_2(s)ds - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha v_2(s)ds\Big].$$

Thus

$$\begin{aligned}
&|h_1(t)-h_2(t)|\\
\le\;& \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha |v_1-v_2|(s)ds + |\mu_1(t)|\int_\xi^1 \frac{(s-\xi)^{\alpha-1}}{\Gamma(\alpha)}|v_1-v_2|(s)ds\\
&+|\mu_2(t)|\Big[\delta\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha |v_1-v_2|(s)ds + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} I_{1-}^\alpha |v_1-v_2|(s)ds\Big]\\
\le\;& \|q\|\Lambda\|y-\bar{y}\|,
\end{aligned}$$

which yields $\|h_1-h_2\| \le \|q\|\Lambda\|y-\bar{y}\|$.

Analogously, interchanging the roles of y and $\overline{y}$, we can obtain

$$H_d(\Omega_F(y),\Omega_F(\bar{y})) \le \|q\|\Lambda\|y-\bar{y}\|.$$

By the condition (4.1), it follows that Ω_F is a contraction and hence it has a fixed point y by Lemma 4.8, which is a solution of the problem (1.2). This completes the proof.□

5 Examples

(a) We construct examples for the illustration of the results obtained in Section 3. For that, we consider the following problem:

$$\begin{cases} D_{1-}^{7/4}D_{0+}^{3/4}y(t) = f(t,y(t)),\ t\in J:=[0,1],\\ y(0)=0,\ D_{0+}^{3/4}y(\xi)=0,\quad y(1)=(5/2)y(2/3), \end{cases} \tag{5.1}$$

Here $\alpha=7/4, \beta=3/4, \xi=1/3, \eta=2/3, \delta=5/2$, and

$$f(t,y) = \frac{1}{2\sqrt{t^2+81}}\Big(\cos y + \frac{|y|}{1+|y|}\Big) + \frac{e^{-2t}}{t+4}. \tag{5.2}$$

With the given data, it is found that

$$\begin{aligned}
&P = [(1-\delta\eta^{\beta+1}) - (\beta+1)\xi(1-\delta\eta^\beta)] \approx 0.262961 \ne 0,\\
&\sup_{t\in[0,1]}\Big\{\frac{t^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + |\mu_1(t)|\frac{(1-\xi)^\alpha}{\Gamma(\alpha+1)}\Big\} \approx 1.454491,
\end{aligned}$$

and $\Lambda \approx 4.503584$ (Λ is given by (2.11)). Furthermore, $|f(t,y_1)-f(t,y_2)| \le L|y_1-y_2|$ with $L=1/9$ so that $L\Lambda \approx 0.0.500398 < 1$. Clearly the hypothesis of Theorem 3.1 is satisfied and hence the problem (5.1) has a unique solution by the conclusion of Theorem 3.1.

In order to illustrate Theorem 3.3, we notice that (3.5) is satisfied as

$$L\Big\{\frac{t^{\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+|\mu_1(t)|\frac{(1-\xi)^{\alpha}}{\Gamma(\alpha+1)}\Big\}\approx 0.161610<1,$$

and

$$|f(t,y)|\leq m(t)=\frac{1}{\sqrt{t^2+81}}+\frac{e^{-2t}}{t+4}.$$

As all the assumptions of Theorem 3.3 hold true, we deduce from the conclusion of Theorem 3.3 that the problem (5.1) has at least one solution on $[0,1]$.

Now we demonstrate the application of Theorem 3.6 by considering the nonlinear function

$$f(t,y)=\frac{e^{-t}}{\sqrt{t+36}}\Big(y+\frac{2}{\pi}\tan^{-1}y+\frac{1}{10}\Big). \tag{5.3}$$

Clearly $|f(t,y)|\leq g(t)\psi(\|y\|)$, where $g(t)=\frac{e^{-t}}{\sqrt{t+36}}$, $\psi(\|y\|=(\frac{11}{10}+\|y\|)$. By the condition (H_4), we find that $K>3.310535$. Thus all the conditions of Theorem 3.6 are satisfied and consequently, the problem (5.1) with $f(t,y)$ given by (5.3) has has at least one solution on $[0,1]$.

(b) Here we illustrate the results obtained in Section 4. Let us consider the following fractional differential inclusion involving both left Caputo and right Riemann-Liouville types fractional derivatives equipped with fractional boundary conditions:

$$\begin{cases} D^{7/4}_{1-}D^{3/4}_{0+}y(t)\in F(t,y(t)),\ t\in J:=[0,1],\\ y(0)=0,\ D^{3/4}_{0+}y(\xi)=0,\quad y(1)=(5/2)y(2/3), \end{cases} \tag{5.4}$$

In order to illustrate Theorem 4.6, we take

$$F(t,y(t))=\left[\frac{1}{\sqrt{t^2+49}}\Big(\frac{|y(t)|}{2(1+|y(t)|)}+|y(t)|+\frac{1}{2}\Big),\ \frac{e^{-t}}{9+t}\Big(\sin y(t)+\frac{1}{80}\Big)\right]. \tag{5.5}$$

Clearly $|F(t,y(t))|\leq\phi(t)\Omega(\|y\|)$, where $\phi(t)=\frac{1}{\sqrt{t^2+49}}$ and $\Omega(\|y\|)=\|y\|+1$. Using the condition (B_3), we find that $M>1.804018$. As the hypothesis of Theorem 4.6 is satisfied, the problem (5.4) with $F(t,y(t))$ given by (5.5) has at least one solution on $[0,1]$.

Now we illustrate Theorem 4.9 by considering

$$F(t,x(t))=\left[\frac{1}{\sqrt{100+t^2}},\ \frac{\sin x(t)}{(6+t)}+\frac{1}{50}\right]. \tag{5.6}$$

Obviously $q(t)=1(6+t)$ with $\|q\|=1/6$ and $d(0,F(t,0))\leq q(t)$ for almost all $t\in[0,1]$. Moreover, $\|q\|\Lambda\approx 0.750597$. Thus all the assumptions of Theorem 4.9 hold true and consequently its conclusion applies to the problem (5.4) with $F(t,y(t))$ given by (5.6).

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces

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Abstract

In this paper, we study the existence of solutions and L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces. We no longer require the Azera-Ascoli theorem to prove the existence of continuous solutions of nonlinear differential systems, but instead we apply the regularity results of general linear differential equations to the case the L^2-primitive process for retarded stochastic neutral functional differential systems with unbounded principal operators, delay terms and local Lipschitz continuity of the nonlinear term. Finally, we give a simple example to which our main result can be applied.

Keywords: stochastic neutral differential equations, retarded system, L^2-primitive process, analytic semigroup, fractional power

AMS Classification 34K50, 93E03, 60H15

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2014R1A6A3A01009038).

1 Introduction

In this paper, we study the existence of solutions and L^2-primitive process for the following retarded stochastic neutral functional differential equations in Hilbert spaces:

$$\begin{cases} d[x(t)+g(t,x_t)] = [Ax(t)+\int_{-h}^{0} a_1(s)A_1x(t+s)ds + k(t)]dt + f(t,x_t)dW(t), \\ x(0)=\phi^0 \in L^2(\Omega,H), \quad x(s)=\phi^1(s),\ s\in[-h,0). \end{cases} \tag{1.1}$$

where $t>0$, $h>0$, $a_1(\cdot)$ is Hölder continuous, k is a forcing term, $W(t)$ stands for K-valued Brownian motion or Winner process with a finite trace nuclear covariance operator Q, and g, f, are given functions satisfying some assumptions. Moreover, $A : D(A) \subset H \to H$ is unbounded and $A_1 \in B(H)$, where $B(X,Y)$ is the collection of all bounded linear operators from X into Y, and $B(X,X)$ is simply written as $B(X)$.

This kind of systems arises in many practical mathematical models, such as, population dynamics, physical, biological and engineering problems, etc. (see [6, 11, 23]).

Many authors have studied for the theory of stochastic differential equations in a variety of ways (see [4] [7] and reference therein), impulsive stochastic neutral differential equations [14, 21], approximate controllability of stochastic equations [5, 27, 26].

As for the retarded differential equations, Jeong et al [17, 18], Wang [32], and Sukavanam et al. [28] have discussed the regularity of solutions and controllability of the semilinear retarded systems, and see [8, 15, 16, 24] and references therein for the linear retarded systems.

In [10, 12, 13], the authors have discussed the existence of solutions for mild solutions for the neutral differential systems with state-dependence delay. Most studies about the neutral initial value problems governed by retarded semilinear parabolic equation have been devoted to the control problems.

Recently, second order neutral impulsive integrodifferential systems have been studied in [2, 25], and Stochastic differential systems with impulsive conditions in [1, 3, 29]. Further, as for impulsive neutral stochastic differential inclusions with nonlocal initial conditions have been studied for the existence results by Lin and Hu [22], and controllability results by [19].

Let $(\Omega,\mathcal{F},P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t\in I\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. An H valued random variables is an $\mathcal{F}$ -measurable function $x(t) : \Omega \to H$. Usually we suppress the dependence on $w \in \Omega$ in the stochastic process $\mathcal{S} = \{x(t,w) : \Omega \to H :$

$t \in [0,T]\}$and write $x(t)$ instead of $x(t,w)$ and $x(t) : [0,T] \to H$ in the space of $\mathcal{S}$. Then we have to study on results in connection with solutions of random differential and integral equations in Hilbert spaces. It should be ensured that $x(t,w)$ is a H-valued random variable with finite second moments and L^2-primitive process of (1.1) for all $t \in T$ in order to study stationary random function, Brownian motion, Markov process, and etc. But the papers treating the regularity for second moments of the systems and L^2-primitive process for retarded stochastic neutral functional differential equations in Hilbert spaces are not many.

In this paper, we propose a different approach of the earlier works used Azera-Ascoli theorem to prove the existence of the mild solutions of functional differential systems in the Banach space of all continuous functions. Our approach is that regularity results of general differential equations results of the linear cases of Di Blasio et al. [8] and semilinear cases of [17] remain valid under the above formulation of the stochastic neutral differential system (1.1) even though the system (1.1) contains unbounded principal operators, delay term and local Lipschitz continuity of the nonlinear term.

The paper is organized as follows. In Section 2, we construct the strict solution of the semilinear functional differential equations and introduce basic properties. In Section 3, by using properties of the strict solutions in dealt in Section 2, we will obtain the L^2-primitive process of (1.1), and a variation of constant formula of L^2-primitive process of (1.1) on the solution space. Finally, we give a simple example to which our main result can be applied.

2 Preliminaries and Lemmas

The inner product and norm in H are denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. V is another Hilbert space densely and continuously embedded in H. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. For brevity we may regard that

$$||u||_* \le |u| \le ||u||, \quad u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \quad c_1 \ge 0. \tag{2.2}$$

Let A be the operator associated with the sesquilinear form $-a(\cdot,\cdot)$:

$$((c_1 - A)u, v) = -a(u,v), \quad u,\ v \in V.$$

It follows from (2.2) that for every $u \in V$

$$\text{Re}\,(Au,u) \ge c_0||u||^2.$$

Then A is a bounded linear operator from V to V^* according to the Lax-Milgram theorem, and its realization in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

is also denoted by A. Then A generates an analytic semigroup $S(t) = e^{tA}$ in both H and V^* as in Theorem 3.6.1 of [30]. Moreover, there exists a constant C_0 such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}, \tag{2.3}$$

for every $u \in D(A)$, where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*,$$

where each space is dense in the next one and continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [31]).

If X is a Banach space and $1 < p < \infty$, $L^p(0, T; X)$ is the collection of all strongly measurable functions from $(0, T)$ into X the p-th powers of norms are integrable.

For the sake of simplicity we assume that the semigroup $S(t)$ generated by A is uniformly bounded, that is, There exists a constant M_0 such that

$$||S(t)||_{B(H)} \leq M_0, \quad ||AS(t)||_{B(H)} \leq \frac{M_0}{t}. \tag{2.4}$$

The following lemma is from [30, Lemma 3.6.2].

Lemma 2.2. *There exists a constant M_0 such that the following inequalities hold:*

$$||S(t)||_{B(H,V)} \leq t^{-1/2} M_0, \tag{2.5}$$
$$||S(t)||_{B(V^*,V)} \leq t^{-1} M_0, \tag{2.6}$$
$$||AS(t)||_{B(H,V)} \leq t^{-3/2} M_0. \tag{2.7}$$

First, consider the following initial value problem for the abstract linear parabolic equation

$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + \int_{-h}^{0} a_1(s)A_1x(t+s)ds + k(t), \quad 0 < t \le T, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \ s \in [-h, 0). \end{cases} \tag{2.8}$$

By virtue of Theorem 2.1 of [15] or [8], we have the following result on the corresponding linear equation of (2.8).

Lemma 2.3. *1) For $(\phi^0, \phi^1) \in V \times L^2(-h, 0; D(A))$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (2.8) belonging to*

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; V)$$

and satisfying

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||\phi^0|| + ||\phi^1||_{L^2(-h,0;D(A))} + ||k||_{L^2(0,T;H)}), \tag{2.9}$$

where C_1 is a constant depending on T and

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} = \max\{||x||_{L^2(0,T;D(A))}, ||x||_{W^{1,2}(0,T;H)}\}$$

(2) Let $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (2.8) belonging to

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}), \tag{2.10}$$

where C_1 is a constant depending on T.

Let the solution spaces $\mathcal{W}_0(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}_0(T) = L^2(0, T; D(A)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}_0(T) \subset C([0, T]; V), \quad \mathcal{W}_1(T) \subset C([0, T]; H).$$

Thus, there exists a constant $c_1 > 0$ such that

$$||x||_{C([0,T];V)} \le c_1||x||_{\mathcal{W}_0(T)}, \quad ||x||_{C([0,T];H)} \le c_1||x||_{\mathcal{W}_1(T)}. \tag{2.11}$$

Lemma 2.4. *Suppose that* $k \in L^2(0,T;H)$ *and* $x(t) = \int_0^t S(t-s)k(s)ds$ *for* $0 \le t \le T$. *Then there exists a constant* C_2 *such that*

$$\begin{aligned} &||x||_{L^2(0,T;D(A))} \le C_1||k||_{L^2(0,T;H)}, \\ &||x||_{L^2(0,T;H)} \le C_2T||k||_{L^2(0,T;H)}, \end{aligned} \tag{2.12}$$

and

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.13}$$

Proof. The first assertion is immediately obtained by (2.9). Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t S(t-s)k(s)ds|^2dt \\ &\le M_0\int_0^T (\int_0^t |k(s)|ds)^2dt \\ &\le M_0\int_0^T t\int_0^t |k(s)|^2dsdt \\ &\le M_0\frac{T^2}{2}\int_0^T |k(s)|^2ds, \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le T\sqrt{M_0/2}||k||_{L^2(0,T;H)}. \tag{2.14}$$

From (2.3), (2.12), and (2.14) it holds that

$$||x||_{L^2(0,T;V)} \le C_0\sqrt{C_1T}(M_0/2)^{1/4}||k||_{L^2(0,T;H)}.$$

So, if we take a constant $C_2 > 0$ such that

$$C_2 = \max\{\sqrt{M_0/2}, C_0\sqrt{C_1}(M_0/2)^{1/4}\},$$

the proof is complete. □

In what follows in this section, we assume $c_1 = 0$ in (2.2) without any loss of generality. So we have that $0 \in \rho(A)$ and the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \ge 0\}$ is contained in the resolvent set of A. In this case, it is possible to define the fractional power A^α for $\alpha > 0$. The subspace $D(A^\alpha)$ is dense in H and the expression

$$||x||_\alpha = ||A^\alpha x||, \quad x \in D(A^\alpha)$$

defines a norm on $D(A^\alpha)$. It is also well known that A^α is a closed operator with its domain dense and $D(A^\alpha) \supset D(A^\beta)$ for $0 < \alpha < \beta$. Due to the well known fact that $A^{-\alpha}$ is a bounded operator, we can assume that there is a constant $C_{-\alpha} > 0$ such that

$$||A^{-\alpha}||_{\mathcal{L}(H)} \le C_{-\alpha}, \quad ||A^{-\alpha}||_{\mathcal{L}(V^*,V)} \le C_{-\alpha}. \tag{2.15}$$

Lemma 2.5. *For any $T > 0$, there exists a positive constant C_α such that the following inequalities hold for all $t > 0$:*

$$||A^\alpha S(t)||_{\mathcal{L}(H)} \le \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(H,V)} \le \frac{C_\alpha}{t^{3\alpha/2}}. \tag{2.16}$$

Proof. The relation is from the inequalities (2.6) and (2.7) by properties of fractional power of A and the definition of $S(t)$. □

3 Existence of solutions

In this paper $(H, |\cdot|)$ and $(K, |\cdot|_K)$ denote real separable Hilbert spaces. Consider the following retarded semilinear impulsive neutral differential system in Hilbert space H:

$$\begin{cases} d[x(t) + g(t, x_t)] = [Ax(t) + \int_{-h}^0 a_1(s)A_1x(t+s)ds + k(t)]dt + f(t, x_t)dW(t), \\ x(0) = \phi^0 \in L^2(\Omega, H), \quad x(s) = \phi^1(s), \ s \in [-h, 0). \end{cases} \tag{3.1}$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space furnished with complete family of right continuous increasing sub σ-algebras $\{\mathcal{F}_t, t \in I\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$.

An H valued random variables is an $\mathcal{F}$ -measurable function $x(t) : \Omega \to H$ and the collection of random variables $\mathcal{S} = \{x(t, w) : \Omega \to H \ : \ t \in [0, T], \ w \in \Omega\}$ is a stochastic process. Generally, we just write $x(t)$ instead of $x(t, w)$ and $x(t) : [0, T] \to H$ in the space of $\mathcal{S}$

Let $\{e_n\}_{n=1}^\infty$ be a complete orthonormal basis of K, and let $Q \in B(K, K)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite $\mathrm{Tr}(Q) = \sum_{n=1}^\infty \sqrt{\lambda_n} = \lambda < \infty$ (Tr denotes the trace of the operator), where $\lambda_n \ge 0 (n = 1, 2, \cdots)$, and $B(K, K)$ denotes the space of all bounded linea operators from K into K.

$\{W(t) : t \ge 0\}$ be a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator Q over $(\Omega, \mathcal{F}, P)$, which satisfies that

$$W(t) = \sum_{n=1}^\infty \sqrt{\lambda_n} w_i(t) e_n, \quad t \ge 0,$$

where $\{w_i(t)\}_{i=1}^\infty$ be mutually independent one dimensional standard Wiener processes over $(\Omega, \mathcal{F}, P)$. Then the above K-valued stochastic process $W(t)$ is called a Q-Wiener process.

We assume that $\mathcal{F}_t = \sigma\{W(s) : 0 \le s \le t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. Let $\psi \in B(K, H)$ and define

$$|\psi|_Q^2 = \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} |\sqrt{\lambda_n} \psi e_n|^2.$$

If $|\psi|_Q^2 < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. $B_Q(K, H)$ stands for the space of all Q-Hilbert-Schmidt operators. The completion $B_Q(K, H)$ of $B(K, H)$ with respect to the topology induced by the norm $|\psi|_Q$, where $|\psi|_Q^2 = (\psi, \psi)$ is a Hilbert space with the above norm topology.

Let V be a dense subspace of H as mentioned in Section 2. For $T > 0$ we define

$$M^2(-h, T; V) = \{x : [-h, T] \to V : E(\int_{-h}^{T} ||x(s)||^2 ds) < \infty\}$$

with norm defined by

$$||x||_{M^2(0,T;V)} = \big[E(\int_{-h}^{T} ||x(s)||^2 ds)\big]^{1/2}.$$

The spaces $M^2(-h, 0; V)$, $M^2(0, T; V)$, and $M^2(0, T; V^*)$ are also defined as the same way and the basic theory of the class of all nonanticipative functions can be founded in [9]. For $h > 0$, we assume that $\phi^1 : [-h, 0) \to V$ is a given initial value satisfying

$$E(\int_{-h}^{0} ||\phi^1(s)||^2 ds) < \infty,$$

that is, $\phi^1 \in M^2(-h, 0; V)$. In this note, a random variable $x(t) : \Omega \to H$ will be called an L^2-primitive process if $x \in M^2(-h, T; V)$.

For each $s \in [0, T]$, we define $x_s : [-h, 0] \to H$ as

$$x_s(r) = x(s + r), \quad -h \le r \le 0.$$

We will set

$$\Pi = M^2(-h, 0; V).$$

Definition 3.1. *A stochastic process $x : [-h, T] \times \Omega \to H$ is called a solution of (3.1) if*

(i) *$x(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \ge 0$.*

(ii) *$x(t) \in H$ has cádlág paths on $t \in (0, T)$ such that*

$$\begin{aligned} x(t) =& S(t)[\phi^0 + g(0, x_0)] - g(t, x_t) + \int_0^t AS(t-s)g(s, x_s)ds \\ &+ \int_0^t S(t-s)\{ \int_{-h}^0 a_1(\tau)A_1 x(s+\tau)d\tau ds + f(s, x_s)dW(s)\} \\ &+ \int_0^t S(t-s)k(s)ds, \\ x(0) =& \phi^0, \quad x(s) = \phi^1(s), \quad s \in [-h, 0). \end{aligned} \tag{3.2}$$

(iii) *$x \in M^2(0, T; V)$ i.e., $E(\int_0^T ||x(s)||^2 ds) < \infty$ and $C([0, T]; H)$.*

To establish our results, we introduce the following assumptions on system (3.1).
Assumption (A). We assume that $a_1(\cdot)$ is Hölder continuous of order ρ:

$$|a_1(0)| \leq H_1, \quad |a_1(s) - a_1(\tau)| \leq H_1(s-\tau)^\rho.$$

Assumption (G). Let $g : [0, T] \times \Pi \to H$ be a nonlinear mapping satisfying the following conditions hold:

(i) For any $x \in \Pi$, the mapping $g(\cdot, x)$ is strongly measurable.

(ii) There exist positive constants L_g and $\beta > 2/3$ such that

$$\begin{aligned} &E|A^\beta g(t, x)|^2 \leq L_g(||x||_\Pi + 1)^2, \\ &E|A^\beta g(t, x) - A^\beta g(t, \hat{x})|^2 \leq L_g ||x - \hat{x}||_\Pi^2, \end{aligned}$$

for all $t \in [0, T]$, and $x, \hat{x} \in \Pi$.

Assumption (F). Let $f : \mathbb{R} \times \Pi \to B(K, H)$ be a nonlinear mapping satisfying the following:

(i) For any $x \in \Pi$, the mapping $f(\cdot, x)$ is strongly measurable.

(ii) There exists a function $L_f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$E|f(t, x) - f(t, y)|^2 \leq L_f(r)||x - y||_\Pi^2, \quad t \in [0, T]$$

hold for $||x||_\Pi \leq r$ and $||y||_\Pi \leq r$.

(iii) The inequality

$$E|f(t,x)|^2 \le L_f(r)(||x||_\Pi + 1)^2$$

holds for every $t \in [0,T]$ and $||x||_\Pi \le r$.

Lemma 3.1. *Let $x \in M^2(-h,T;V)$. Then the mapping $s \mapsto x_s$ belongs to $C([0,T];\Pi)$, and for each $0 < t \le T$*

$$\begin{aligned} &||x_t||_\Pi \le ||x||_{M^2(-h,t;V)} = ||\phi^1||_\Pi + ||x||_{M^2(0,t;V)}, \\ &E(||x||^2_{L^2(0,t;V)}) = ||x||^2_{M^2(0,t;V)}, \\ &||x.||_{L^2(0,t;\Pi)} \le \sqrt{t}||x||_{M^2(-h,t;V)}. \end{aligned} \tag{3.3}$$

Proof. The first paragraph is easy to verify. In fact, it is from the following inequality;

$$||x_t||^2_\Pi = E\big(\int_{-h}^0 ||x(t+\tau)||^2 d\tau\big) \le E\big[\int_{-h}^t ||x(\tau)||^2 d\tau\big] \le ||x||^2_{M^2(-h,t;V)}, \ t > 0.$$

The second paragraph is immediately obtained by definition. From the inequality (3.3), we have

$$\begin{aligned} \int_0^t ||x_s||^2_\Pi ds &= \int_0^t \big[E\big(\int_{-h}^0 ||x(s+\tau)||^2 d\tau\big)\big]^2 ds \\ &= \int_0^t \big[E\big(\int_{s-h}^s ||x(\tau)||^2 d\tau\big)\big]^2 ds \le t||x||^2_{M^2(-h,t;V)}, \end{aligned}$$

which completes the last paragraph. □

One of the main useful tools in the proof of existence theorems for nonlinear functional equations is the following Sadvoskii's fixed point theorem.

Lemma 3.2. *(Krasnoselski [20]) Suppose that Σ is a closed convex subset of a Banach space X. Assume that K_1 and K_2 are mappings from Σ into X such that the following conditions are satisfied:*

(i) $(K_1 + K_2)(\Sigma) \subset \Sigma$,

(ii) K_1 is a completely continuous mapping,

(iii) K_2 is a contraction mapping.

Then the operator $K_1 + K_2$ has a fixed point in Σ.

From now on, we establish the following results on the solvability of the equation (3.1).

Theorem 3.1. *Let Assumptions (A), (G) and (F) be satisfied. Assume that* $(\phi^0, \phi^1) \in L^2(\Omega, H) \times \Pi$ *and* $k \in M^2(0, T; V^*)$ *for* $T > 0$*. Then, there exists a solution* x *of the system (3.1) such that*

$$x \in M^2(0, T; V) \cap C([0, T]; H).$$

Moreover, there is a constant C_3 *independent of the initial data* (ϕ^0, ϕ^1) *and the forcing term* k *such that*

$$||x||_{M^2(-h,T;V)} \le C_3(1 + E(|\phi^0|^2) + ||\phi^1||_\Pi + ||k||_{M^2(0,T;V^*)}). \tag{3.4}$$

Proof. Let

$$r := 2\big[C_1 C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1) + \sqrt{3}C_1\big(E(|\phi^0|^2) + ||\phi^1||_\Pi^2 + ||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\big],$$

and

$$\begin{aligned} N :=& \sqrt{3}C_{-\alpha}\sqrt{L_g}\big(||\phi^1||_\Pi + r + 1\big) \\ &+ (3\beta - 2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}(||\phi^1||_\Pi + r + 1) \\ &+ C_2\mathrm{Tr}(Q)\sqrt{L_f(r)}(||\phi^1||_\Pi + r + 1), \end{aligned}$$

where $\beta > 2/3$, C_1 and C_2 is constants in Lemma 2.3 and Lemma 2.4, respectively. Let

$$T_1^\gamma := \max\{T_1^{1/2}, T_1^{3\beta/2}\}$$

and choose $0 < T_1 < T$ such that

$$T_1^\gamma N \le \frac{r}{2} = C_1 C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi + 1) + \sqrt{3}C_1\big(\sqrt{E(|\phi^0|^2)} + ||\phi^1||_\Pi + ||k||_{M^2(0,T_1;V^*)}, \tag{3.5}$$

and

$$\hat{N} := T_1^\gamma\big\{\sqrt{3}C_{-\alpha}\sqrt{L_g} + (3\beta - 1)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g} + C_2\mathrm{Tr}(Q)\sqrt{L_f(r)}\big\} < 1. \tag{3.6}$$

Let J be the operator on $M^2(0,T_1;V)$ defined by

$$\begin{aligned}(Jx)(t) =&S(t)[\phi^0+g(0,\phi^1)]-g(t,x_t)+\int_0^t AS(t-s)g(s,x_s)ds\\&+\int_0^t S(t-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau ds+f(s,x_s)dW(s)\big\}\\&+\int_0^t S(t-s)k(s)ds.\end{aligned}$$

It is easily seen that J is continuous from $C([0,T_1];H)$ into itself. Let

$$\Sigma=\{x\in M^2(-h,T;V): x(0)=\phi^0,\ \text{and } x(s)=\phi^1(s)(s\in[-h,0))\}.$$

and

$$\Sigma_r=\{x\in\Sigma: ||x||_{M^2(0,T_1;V)}\le r\},$$

which is a bounded closed subset of $M^2(0,T_1;V)$. Now, we give the proof of Theorem 3.1 in the following several steps:

Now, in order to show that the operator J has a fixed point in $\Sigma_r\subset M^2(0,T_1;V)$, we take the following steps.

Step 1. J maps Σ_r into Σ_r.

By (2.10), (2.15) and Assumption (G), and noting $x_0=\phi^1$, we know

$$\begin{aligned}E\big[\int_0^{T_1}\|S(t)g(0,x_0)\|^2dt\big]&=E\big[C_1^2|g(0,\phi^1)|^2\big]\\&=E\big[C_1^2\big(||A^{-\beta}||_{B(H)}|A^\beta g(0,\phi^1)|\big)^2\big]\\&\le(C_1C_{-\alpha})^2L_g(||\phi^1||_\Pi+1)^2.\end{aligned}\tag{3.7}$$

From (2.10) of Lemma 2.3 it follows

$$\begin{aligned}&E\big[\int_0^{T_1}\big\|S(t)\phi^0+\int_0^t S(t-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau+k(s)\big\}ds\big\|^2dt\big]\\&\le E\big[C_1^2\{|\phi^0|+||\phi^1||_{L^2(-h,0;V)}+||k||_{L^2(0,T_1;V^*)}\}^2\big]\\&\le 3C_1^2\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big).\end{aligned}\tag{3.8}$$

By using Assumption (G) and Lemma 3.1, we have

$$\begin{aligned}||g(\cdot,x.)||^2_{M^2(0,T_1;V)}&=E\big(\int_0^{T_1}\big\|A^{-\beta}A^\beta g(t,x_t)\big\|^2dt\big)\\&\le C_{-\alpha}^2E\big(\int_0^{T_1}\big|A^\beta g(t,x_t)\big|^2dt\big)\le C_{-\alpha}^2L_gT_1\big(||x_t||_\Pi+1\big)^2\\&\le 3C_{-\alpha}^2L_gT_1\big(||\phi^1||_\Pi^2+||x||_{M^2(0,T_1;V)}+1\big)\end{aligned}\tag{3.9}$$

Define $H_1 : M^2(0, T_1; V) \to M^2(0, T_1; V)$ by

$$(H_1x)(t) = \int_0^t AS(t-s)g(s, x_s)ds.$$

Then from Lemma 2.5 and Assumption (G) we have

$$\begin{aligned}||AS(t-s)g(s, x_s)|| &= ||A^{1-\beta}S(t-s)||_{B(H,V)}|A^\beta g(s, x_s)|\\ &\le \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^\beta g(s, x_s)|,\end{aligned}$$

and hence, by using Hólder inequality and Assumption (G),

$$\begin{aligned}&||H_1x||^2_{M^2(0,T_1;V)} = E\big[\int_0^{T_1} \big\|\int_0^t AS(t-s)g(s, x_s)ds\big\|^2 dt\big] \qquad (3.10)\\ &\le E\big[\int_0^{T_1} \big(\int_0^t \frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^\beta g(s, x_s)|ds\big)^2 dt\big]\\ &\le E\big[\int_0^{T_1} C^2_{1-\beta}(3\beta-2)^{-1}t^{3\beta-2}\int_0^t |A^\beta g(s, x_s)|^2 dsdt\big]\\ &\le (3\beta-2)^{-1}C^2_{1-\beta}L_g(||x_s||_\Pi + 1)^2 \int_0^{T_1} t^{3\beta-1}dt\\ &\le (3\beta-2)^{-1}(3\beta)^{-1}C^2_{1-\beta}L_gT_1^{3\beta}(||x||_{M^2(-h,T_1;V)} + 1)^2\\ &= (3\beta-2)^{-1}(3\beta)^{-1}C^2_{1-\beta}L_gT_1^{3\beta}(||\phi^1||_\Pi + ||x||_{M^2(0,T_1;V)} + 1)^2\end{aligned}$$

Let

$$(H_2x)(t) = \int_0^t S(t-s)f(s, x_s)dW(s).$$

For (2.13) of Lemma 2.4 it follows

$$\begin{aligned}||H_2x||^2_{M^2(0,T_1;V)} &= E\big[\int_0^{T_1} \big|\int_0^t S(t-s)f(s, x_s)dW(s)\big|^2 dt\big] \qquad (3.11)\\ &\le E[C_2^2\mathrm{Tr}(Q)^2T_1||f(s, x_s)||^2_{L^2(0,T;V^*)}]\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1||f(s, x_s)||^2_{M^2(0,T;V^*)}\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1L_f(r)(||x_s||_\Pi + 1)^2\\ &\le C_2^2\mathrm{Tr}(Q)^2T_1L_f(r)(||\phi^1||_\Pi + ||x||_{M^2(0,T_1;V)} + 1)^2\end{aligned}$$

Therefore, from (3.7)-(3.11) it follows that

$$\begin{aligned}||Jx||_{M^2(0,T_1;V)} \leq& C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi+1)\\
&+\sqrt{3}C_1\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\\
&+\sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1\big)\\
&+(3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)\\
&+C_2\mathrm{Tr}(Q)\sqrt{T_1L_f(r)}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)\\
\leq& C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi+1)\\
&+\sqrt{3}C_1\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}+T_1^\gamma N\leq r,\end{aligned}$$

and so, J maps Σ_r into Σ_r.

Define mapping K_1+K_2 on $L^2(0,T_1;V)$ by the formula

$$\begin{aligned}&(Jx)(t)=(K_1x)(t)+(K_2x)(t),\\
&(K_1x)(t)=\int_0^t S(t-s)\int_0^s a_1(\tau-s)A_1x(\tau)d\tau ds,\end{aligned}$$

and

$$\begin{aligned}(K_2x)(t)=&S(t)[\phi^0+g(0,x_0)]-g(t,x_t)+\int_0^t AS(t-s)g(s,x_s)ds\\
&+\int_0^t S(t-s)\big\{\int_{s-h}^0 a_1(\tau-s)A_1\phi^1(\tau)d\tau ds+f(s,x_s)dW(s)\big\}\\
&+\int_0^t S(t-s)k(s)ds.\end{aligned}$$

Step 2. K_1 is a completely continuous mapping.

We can now employ Lemma 3.2 with Σ_r. Assume that a sequence $\{x_n\}$ of $M^2(0,T_1;V)$ converges weakly to an element $x_\infty\in M^2(0,T_1;V)$, i.e., $w-\lim_{n\to\infty}x_n=x_\infty$. Then we will show that

$$\lim_{n\to\infty}||K_1x_n-K_1x_\infty||_{M^2(0,T_1;V)}=0, \tag{3.12}$$

which is equivalent to the completely continuity of K_1 since $M^2(0,T_1;V)$ is reflexive. For a fixed $t\in[0,T_1]$, let $x_t^*(x)=(K_1x)(t)$ for every $x\in M^2(0,T_1;V)$. Then $x_t^*\in M^2(0,T_1;V^*)$ and we have $\lim_{n\to\infty}x_t^*(x_n)=x_t^*(x_\infty)$ since $w-\lim_{n\to\infty}x_n=x_\infty$. Hence,

$$\lim_{n\to\infty}(K_1x_n)(t)=(K_1x_\infty)(t),\quad t\in[0,T_1].$$

Set

$$h(s)=\int_0^s a_1(\tau-s)A_1x(\tau)d\tau.$$

Then by using Hólder inequality we obtain the following inequality

$$\begin{aligned}|h(s)| &\le \Big|\int_0^s (a_1(\tau-s)-a_1(0))A_1x(\tau)d\tau\Big| \\ &\quad+\Big|\int_0^s a_1(0)A_1x(\tau)d\tau\Big| \\ &\le \big\{\big((2\rho+1)^{-1}s^{2\rho+1}\big)^{1/2}+\sqrt{s}\big\}H_1||A_1||_{B(H)}\big(\int_0^s ||x(\tau)||^2d\tau\big)^{1/2}.\end{aligned} \tag{3.13}$$

Thus, by (2.5) and (3.13) it holds

$$\begin{aligned}&||(K_1x)(t)||^2=\Big\|\int_0^t S(t-s)h(s)ds\Big\|^2 \\ &\le (H_1||A_1||_{B(H)})^2\big(\int_0^t ||x(\tau)||^2d\tau\big)\Big\|\int_0^t \frac{1}{(t-s)^{1/2}}\{((2\rho+1)^{-1}s^{(2\rho+1)/2}+\sqrt{s}\}ds\Big\|^2 \\ &\le (H_1||A_1||_{B(H)})^2||x||^2_{L^2(0,t;V)}\big\{(2\rho+1)^{-1}B(1/2,(2\rho+3)/2)t^{\rho+1}+B(1/2,3/2)t\big\}^2. \\ &:= c_2||x||^2_{L^2(0,t;V)},\end{aligned}$$

where c_2 is a constant and $B(\cdot,\cdot)$ is the Beta function. Here we used

$$B(1/2,(2\rho+3)/2)t^{\rho+1}=\int_0^t (t-s)^{-1/2}s^{(2\rho+1)/2}ds.$$

And we know

$$\sup_{0\le t\le T_1}||E[(K_1x)(t)]^2||\le c_2||x||^2_{M^2(0,T_1;V)}\le\infty.$$

Therefore, by Lebesgue's dominated convergence theorem it holds

$$\lim_{n\to\infty}E\big(\int_0^{T_1}||(K_1x_n)(t)||^2dt\big)=E\big(\int_0^{T_1}||(K_1x_\infty)(t)||^2dt\big),$$

i.e., $\lim_{n\to\infty}||K_1x_n||_{M^2(0,T_1;V)}=||K_1x_\infty||_{M^2(0,T_1;V)}$. Since $M^2(0,T_1;V)$ is a reflexive space, it holds (3.12).

Step 3. K_2 is a contraction mapping.

For every x_1 and $x_2 \in \Sigma_r$, we have

$$\begin{aligned}(K_2x_1)(t)-(K_2x_2)(t) =&g(t,x_{2_t})-g(t,x_{1_t})\\&+\int_0^t AS(t-s)\big(g(t,x_{1_s})-g(t,x_{2_s})\big)ds\\&+\int_0^t S(t-s)\{f(s,x_1(s))-f(s,x_2(s))\}dW(s).\end{aligned}$$

In a similar way to (3.9)-(3.11) and Proposition 2.3, we have

$$\begin{aligned}||K_2x_1-K_2x_2||_{M^2(0,T_1;V)} \le&\big\{C_{-\alpha}\sqrt{T_1L_g}+(3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}T_1^{3\beta/2}\sqrt{L_g}\\&+C_2\sqrt{T_1L_f(r)}\mathrm{Tr}(Q)\big\}||x_1-x_2||_{M^2(0,T_1;V)}\\\le&\hat{N}||x_1-x_2||_{M^2(0,T_1;V)}.\end{aligned}$$

So by virtue of the condition (3.6) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $M^2(0,T_1;V)$. This has proved the local existence and uniqueness of the solution of (3.1).

Step 4. We drive a priori estimate of the solution.

To prove the global existence, we establish a variation of constant formula (3.4) of solution of (3.1). Let x be a solution of (3.1) and $\phi^0 \in H$. Then we have that from (3.7)-(3.11) it follows that

$$\begin{aligned}||x||_{M^2(0,T_1;V)} \le&C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi+1)\\&+\sqrt{3}C_1\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\\&+\sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1\big)\\&+(3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)\\&+C_2\mathrm{Tr}(Q)T_1\sqrt{T_1L_f(r)}(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)\\\le&\hat{N}||x||_{L^2(0,T_1;V)}+\hat{N}_1,\end{aligned}$$

where

$$\begin{aligned}\hat{N}_1 =&C_1C_{-\alpha}\sqrt{L_g}(||\phi^1||_\Pi+1)\\&+\sqrt{3}C_1\big(E[|\phi^0|^2]+||\phi^1||_\Pi^2+||k||^2_{M^2(0,T_1;V^*)}\big)^{1/2}\\&+\sqrt{3}C_{-\alpha}\sqrt{T_1L_g}\big(||\phi^1||_\Pi+1\big)\\&+(3\beta-2)^{-1/2}(3\beta)^{-1/2}C_{1-\beta}\sqrt{L_g}T_1^{3\beta/2}(||\phi^1||_\Pi+1)\\&+C_2\mathrm{Tr}(Q)\sqrt{T_1L_f(r)}(||\phi^1||_\Pi+1).\end{aligned}$$

Taking into account (3.6) there exists a constant C_3 such that

$$\begin{aligned}||x||_{L^2(0,T_1;V)} \leq &(1-\hat{N})^{-1}\hat{N}_1\\ \leq &C_3(1+E(|\phi^0|^2)+||\phi^1||_\Pi+||k||_{M^2(0,T_1;V^*)}),\end{aligned}$$

which obtain the inequality (3.4).

Now we will prove that $E[x(T_1)^2]<\infty$ in order that the solution can be extended to the interval $[T_1,2T_1]$.

Define a mapping $H_3: L^2(0,T_1;V)\to L^2(0,T_1;V)$ as

$$(H_3x)(t)=S(t)[\phi^0+g(0,x_0)]+\int_0^t S(t-s)\Big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau+k(s)\Big\}ds$$

The from (2.11) and Lemma 2.3 it follows that

$$\begin{aligned}E|(H_3x)(T_1)|^2 &\leq c_1E||H_3x||^2_{\mathcal{W}_1} \qquad (3.14)\\ &\leq 3c_1C_1E\big\{|\phi^0+g(0,\phi^1)|+||\phi^1||_{L^2(-h,0;V)}+||k||_{L^2(0,T_1;V^*)}\big\}^2\\ &\leq c_1C_1\big\{E|\phi^0+g(0,\phi^1)|^2+||\phi^1||^2_{M^2(-h,0;V)}+||k||^2_{M^2(0,T_1;V^*)}\big\}:=I,\end{aligned}$$

and from (2.4) and Assumption (F),

$$\begin{aligned}E|(H_2x)(T_1)|^2 =&E\Big|\int_0^{T_1}S(T_1-s)f(s,x_s)dW(s)\Big|^2 \qquad (3.15)\\ \leq&M_0^2\mathrm{Tr}(Q)^2T_1L_f(r)(||x_s||_\Pi+1)^2\\ \leq&M_0^2\mathrm{Tr}(Q)^2T_1L_f(r)(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1)^2:=II.\end{aligned}$$

Moreover, by using Assumption (G) we have

$$\begin{aligned}E|g(T_1,x_{T_1})|^2 &\leq E\big\|A^{-\beta}A^\beta g(t,x_{T_1})\big\|^2, \qquad (3.16)\\ &\leq C\_ALPHA^2L_g\big(||x_{T_1}||_\Pi+1\big)^2\\ &\leq C\_ALPHA^2L_g\big(||\phi^1||_\Pi+||x||_{M^2(0,T_1;V)}+1\big)^2:=III,\end{aligned}$$

and

$$\begin{aligned}E|(H_1x)(T_1)|^2 &= E\Big|\int_0^{T_1}AS(T_1-s)g(s,x_s)ds\Big|^2 \qquad (3.17)\\ &= E\Big|\int_0^{T_1}A^{1-\beta}W(T_1-s)A^\beta g(s,x_s)ds\Big|^2\\ &\leq E\Big[\int_0^{T_1}\frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}|A^\beta(g(s,x_s)|ds\Big]^2\\ &\leq E\big[C^2_{1-\beta}(3\beta-2)^{-1}T_1^{3\beta-2}\int_0^t|A^\beta(g(s,x_s)|^2ds\big]^2\\ &= C^2_{1-\beta}(3\beta-2)^{-1}T_1^{3\beta-1}L_g(||x||_{M^2(0,T_1;V)}+||\phi^1||_{M^2(-h,0;V)}+1)^2:=IV.\end{aligned}$$

Thus,by (3.14)-(3.17) we have

$$\begin{aligned}E|x(T_1)|^2 &= E\big|(H_3x)(T_1) - g(T_1, x_{T_1}) + \int_0^{T_1} AS(T_1 - s)g(s, x_s)ds \\ &\quad + \int_0^{T_1} S(T_1 - s)f(s, x_s)dW(s)\big| \\ &\le I + II + III + IV < \infty.\end{aligned}$$

Hence we can solve the equation in $[T_1, 2T_1]$ with the initial $(x(T_1), x_{T_1})$ and an analogous estimate to (3.4). Since the condition (3.6) is independent of initial values, the solution can be extended to the interval $[0, nT_1]$ for any natural number n, and so the proof is complete. □

Remark 3.1. *Thanks for Lemma 2.3, we note that the solution of (3.1) with the conditions of Theorem 3.1 satisfies also that*

$$E(\int_{-h}^{T} ||x'(s)||_*^2 ds) < \infty.$$

Here we note that by a simple calculation using the properties of analytic semigroup, it is immediately seen that $x \in M^2(-h, T; H)$.

Now, we obtain that the solution mapping is continuous in the following result, which is useful for the control problem and physical applications of the given equation.

Theorem 3.2. *Let Assumptions (A), (G) and (F) be satisfied. Assuming that the initial data* $(\phi^0, \phi^1) \in L^2(\Omega, H) \times \Pi$ *and the forcing term* $k \in M^2(0, T; V^*)$. *Then the solution* x *of the equation (3.1) belongs to* $x \in M^2(0, T; V)$ *and the mapping*

$$L^2(\Omega, H) \times \Pi \times M^2(0, T; V^*) \ni (\phi^0, \phi^1, k) \mapsto x \in M^2(0, T; V) \tag{3.18}$$

is continuous.

Proof. From Theorem 3.1, it follows that if $(\phi^0, \phi^1, k) \in L^2(\Omega, H) \times \Pi \times M^2(0, T; V^*)$ then x belongs to $M^2(0, T; V)$. Let $(\phi_i^0, \phi_i^1, k_i)$ and x^i be the solution of (3.1) with

$(\phi_i^0, \phi_i^1, k_i)$ in place of (ϕ^0, ϕ^1, k) for $i = 1,\ 2$. Let $x_i (i = 1, 2) \in \Sigma_r$. Then it holds

$$\begin{aligned}
&x^1(t) - x^2(t) = S(t)[(\phi_1^0 - \phi_2^0) + (g(0, x_0^1) - g(0, x_0^2))] \\
&- (g(t, x_t^1) - g(t, x_t^2)) + \int_0^t AS(t-s)(g(s, x_s^1) - g(t, x_s^2))ds \\
&+ \int_0^t S(t-s)\{ \int_{-h}^0 a_1(\tau)A_1(x^1(s+\tau) - x^2(s+\tau))d\tau ds \\
&+ \int_0^t S(t-s)\{((Fx^1)(s) - (Fx^2)(s)) + (k_1(s) - k_2(s))\}ds. \\
&+ \int_0^t S(t-s)(k_1(s) - k_2(s))ds
\end{aligned}$$

Hence, by applying the same argument as in the proof of Theorem 3.1, we have

$$||x_1 - x_2||_{M^2(0,T_1;V)} \leq \hat{N} ||x_1 - x_2||_{L^2(0,T_1;V)} + \hat{N}_2,$$

where

$$\begin{aligned}
\hat{N}_2 =& C_1 C\_ALPHA \sqrt{L_g}(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ \sqrt{3} C_1 \big(E[|\phi_1^0 - \phi_1^0|^2] + ||\phi_1^1 - \phi_2^1||_\Pi^2 + ||k_1 - k_2||^2_{M^2(0,T_1;V^*)}\big)^{1/2} \\
&+ \sqrt{3} C\_ALPHA \sqrt{T_1 L_g}\big(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ (3\beta - 2)^{-1/2}(3\beta)^{-1/2} C_{1-\beta} \sqrt{L_g} T_1^{(3\beta+1)/2}(||\phi_1^1 - \phi_2^1||_\Pi) \\
&+ C_2 \mathrm{Tr}(Q) \sqrt{T_1 L_f(r)}(||\phi_1^1 - \phi_2^1||_\Pi),
\end{aligned}$$

which implies

$$||x||_{M^2(0,T_1;V)} \leq \hat{N}_2 (1 - \hat{N})^{-1}.$$

Therefore, it implies the inequality (3.18). □

4 Example

Let

$$H = L^2(0, \pi),\ V = H_0^1(0, \pi),\ V^* = H^{-1}(0, \pi).$$

Consider the following retarded neutral stochastic differential system in Hilbert space H:

$$\begin{cases} d[x(t,y)+g(t,x_t(t,y))] = [Ax(t,y)+\int_{-h}^{0} a_1(s)A_1x(t+s,y)ds+k(t,y)]dt \\ \qquad\qquad +F(t,x(t,y))dW(t), \quad (t,y)\in[0,T]\times[0,\pi], \\ x(0,y)=\phi^0(y)\in L^2(\Omega,H), \quad x(s,y)=\phi^1(s,y), \quad (s,y)\in[-h,0)\times[0,\pi], \end{cases} \tag{3.19}$$

where $h>0$, $a_1(\cdot)$ is Hölder continuous, $A_1\in B(H)$, and $W(t)$ stands for a standard cylindrical Winner process in H defined on a stochastic basis $(\Omega,\mathcal{F},P)$. Let

$$a(u,v)=\int_0^\pi \frac{du(y)}{dy}\overline{\frac{dv(y)}{dy}}dy.$$

Then

$$A=\partial^2/\partial y^2 \quad \text{with} \quad D(A)=\{x\in H^2(0,\pi): x(0)=x(\pi)=0\}.$$

The eigenvalue and the eigenfunction of A are $\lambda_n=-n^2$ and $z_n(y)=(2/\pi)^{1/2}\sin ny$, respectively. Moreover,

(a1) $\{z_n: n\in N\}$ is an orthogonal basis of H and

$$S(t)x=\sum_{n=1}^{\infty} e^{n^2t}(x,z_n)z_n, \quad \forall x\in H, \ t>0.$$

Moreover, there exists a constant M_0 such that $||S(t)||_{B(H)}\le M_0$.

(a2) Let $0<\alpha<1$. Then the fractional power $A^\alpha: D(A^\alpha)\subset H\to H$ of A is given by

$$A^\alpha x=\sum_{n=1}^{\infty} n^{2\alpha}(x,z_n)z_n, \ D(A^\alpha):=\{x: A^\alpha x\in H\}.$$

In particular,

$$A^{-1/2}x=\sum_{n=1}^{\infty}\frac{1}{n}(x,z_n)z_n, \ \text{and} \ ||A^{-1/2}||=1.$$

The nonlinear mapping f is a real valued function belong to $C^2([0,\infty))$ which satisfies the conditions

(f1) $f(0)=0$, $f(r)\ge 0$ for $r>0$,

(f2) $|f^{'}(r) \leq c(r+1)$ and $|f''(r)| \leq c$ for $r \geq 0$ and $c > 0$.

If we present

$$F(t, x(t,y)) = f^{'}(|x(t,y)|^2)x(t,y),$$

Then it is well known that F is a locally Lipschitz continuous mapping from the whole V into H by Sobolev's imbedding theorem (see [30, Theorem 6.1.6]). As an example of q in the above, we can choose $q(r) = \mu^2 r + \eta^2 r^2/2$ (μ and η is constants).

Define $g : [0,T] \times \Pi \to H$ as

$$g(t, x_t) = \sum_{n=1}^{\infty} \int_0^t e^{n^2 t} (\int_{-h}^0 a_2(s)x(t+s)ds, z_n)z_n, \quad , t > 0.$$

Then it can be checked that Assumption (G) is satisfied. Indeed, for $x \in \Pi$, we know

$$Ag(t, x_t) = (S(t) - I) \int_{-h}^0 a_2(s)x(t+s)ds,$$

where I is the identity operator form H to itself and

$$|a_2(0)| \leq H_2, \quad |a_2(s) - a_2(\tau)| \leq H_2(s-\tau)^\kappa, \quad s, \ \tau \in [-h, 0]$$

for a constant $\kappa > 0$. Hence we have

$$\begin{aligned} E|Ag(t,x_t)|^2 \leq &(M_0+1)^2 \big\{ \big| \int_{-h}^0 (a_2(s) - a_2(0))x(t+s)d\tau \big|^2 \\ &+ \big| \int_{-h}^0 a_2(0)x(t+s)d\tau \big|^2 \big\} \\ \leq &(M_0+1)^2 H_2^2 \big\{ (2\kappa+1)^{-1} h^{2\rho+1} + h \big\} ||x_t||_\Pi^2. \end{aligned}$$

It is immediately seen that Assumption (G) has been satisfied. Thus, all the conditions stated in Theorem 3.1 have been satisfied for the equation (3.19), and so there exists a solution x of the equation (3.19) such that

$$E(\int_{-h}^T ||x(s)||^2 ds) < \infty, \text{ and } E(\int_{-h}^T ||x'(s)||_*^2 ds) < \infty.$$

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

FUZZY STABILITY OF CUBIC FUNCTIONAL EQUATIONS WITH EXTRA TERMS

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ABSTRACT. In this paper, we consider the generalized Hyers-Ulam stability for the following cubic functional equation

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)-6f(y)+G_f(x,y)=0.$$

with an extra term G_f which is a functional operator of f.

1. INTRODUCTION AND PRELIMINARIES

In 1940, Ulam proposed the following stability problem (cf. [20]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists an unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [8] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings and by Rassias [18] for linear mappings to consider the stability problem with unbounded Cauchy differences. During the last decades, the stability problem of functional equations have been extensively investigated by a number of mathematicians ([3], [4], [5], [7], and [16]).

Katsaras [11] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a vector space in different points of view. In particular, Bag and Samanta [2] gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. In this paper, we use the definition of fuzzy normed spaces given in [2],[14], [15].

Definition 1.1. Let X be a real vector space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ is called *a fuzzy norm on X* if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,

(N1) $N(x, t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
(N5) $N(x, \cdot)$ is a nondecreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$;
(N6) for any $x \neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X, N) is called *a fuzzy normed space*.

2010 *Mathematics Subject Classification.* 39B52, 46S40.

Key words and phrases. fuzzy normed space, cubic functional equation.

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Let (X, N) be a fuzzy normed space. (i) A sequence $\{x_n\}$ in X is said to be *convergent in* (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$. (ii) A sequence $\{x_n\}$ in X is said to be *Cauchy in* (X, N) if for any $\epsilon > 0$ and any $t > 0$, there exists an $m \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $n \geq m$ and all positive integer p.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space.*

For example, it is well known that for any normed space $(X, ||\cdot||)$, the mapping $N_X : X \times \mathbb{R} \longrightarrow [0, 1]$, defined by

$$N_X(x,t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+||x||}, & \text{if } t > 0 \end{cases}$$

is a fuzzy norm on X.

In 1996, Isac and Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Theorem 1.2. [6] *Let* (X, d) *be a complete generalized metric space and let* $J : X \longrightarrow X$ *be a strictly contractive mapping with some Lipschitz constant* L *with* $0 < L < 1$. *Then for each given element* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integer* n *or there exists a positive integer* n_0 *such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \geq n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J ;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ and
(4) $d(y, y^*) \leq \dfrac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

In 2001, Rassias [19] introduced the following cubic functional equation

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) = 0 \tag{1.1}$$

and the following cubic functional equations were investigated

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.2}$$

in ([10]). Every solution of a cubic functional equation is called *a cubic mapping* and Kim and Han [12] investigated the following cubic functional equation

$$\begin{aligned} &f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) \\ &+ k[f(mx+y) + f(mx-y) - m[f(x+y) + f(x-y)] - 2(m^3-m)f(x)] = 0 \end{aligned}$$

for some rational number m and some real number k and proved the stability for it in fuzzy normed spaces.

In this paper, we investigate the following functional equation which is added a term by G_f to (1.1)

$$f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) + G_f(x,y) = 0,$$

where G_f is a functional operator depending on functions f. The definition of G_f is given in section 2 and prove the stability for it in fuzzy normed spaces.

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and (Z, N') is a fuzzy normed space.

2. Cubic functional equations with extra terms

For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \cdots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. It is clear that $\sigma_i(0, 0) = 0$.

Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))).$$

Now consider the functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + G_f(x, y) = 0 \tag{2.1}$$

with the functional operator G_f.

Theorem 2.1. *Suppose that the mapping $f : X \longrightarrow Y$ is a solution of (2.1) with $f(0) = 0$. Then f is cubic if and only if $f(2x) = 8f(x)$ and $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$.*

Proof. Suppose that $f(2x) = 8f(x)$ and $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$. Interchanging x and y in (2.1) , we have

$$f(2x + y) - 3f(x + y) + 3f(y) - f(y - x) - 6f(x) + G_f(y, x) = 0. \tag{2.2}$$

for all $x, y \in X$ and letting $x = -x$ in (2.2) , we have

$$f(-2x + y) - 3f(-x + y) + 3f(y) - f(x + y) - 6f(-x) + G_f(y, -x) = 0. \tag{2.3}$$

for all $x, y \in X$. By (2.2) and (2.3), we have

$$f(2x + y) - f(-2x + y) - 2f(x + y) + 2f(y - x) - 6f(x) + 6f(-x) = 0. \tag{2.4}$$

for all $x, y \in X$, because $G_f(y, x) = G_f(y, -x)$. Letting $y = x$ in (2.4), we have

$$f(3x) - 22f(x) + 5f(-x) = 0. \tag{2.5}$$

for all $x \in X$ and letting $y = 2x$ in (2.4), by (2.5), we have

$$f(4x) - 2f(3x) - 4f(x) + 6(-x) = 16f(x) + 16f(-x) = 0.$$

for all $x \in X$, because $f(2x) = 8f(x)$. Hence f is odd and by (2.2) and (2.3), f satisfies (1.2). Thus f is a cubic mapping. The converse is trivial. □

3. The Generalized Hyers-Ulam stability for (2.1)

In this section, we prove the generalized Hyers-Ulam stability of (2.1) in fuzzy normed spaces. For any mapping $f : X \longrightarrow Y$, we define the difference operator $Df : X^2 \longrightarrow Y$ by

$$Df(x, y) = f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) + G_f(x, y)$$

for all $x, y \in X$.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow Z$ be a function such that there is a real number L satisfying $0 < L < 1$ and*

$$N'(\phi(2x, 2y), t) \geq N'(8L\phi(x, y), t) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$N(Df(x, y), t) \geq N'(\phi(x, y), t) \tag{3.2}$$

for all $x, y \in X$ and all $t > 0$ and

$$N(f(2x) - 8f(x), t) \geq \min\{N'(a\phi(x, 0), t), N'(b\phi(0, x), t), N'(c\phi(x, -x), t)\} \tag{3.3}$$

for all $x \in X$, all $t > 0$ and some nonnegative real numbers a, b, c. Further, assume that if g satisfies (2.1), then g is a cubic mapping. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$\begin{aligned} &N\Big(f(x) - C(x), \frac{1}{8(1-L)}t\Big) \\ &\geq \min\{N'(a\phi(x, 0), t), N'(b\phi(0, x), t), N'(c\phi(x, -x), t)\} \end{aligned} \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Let $\psi(x, t) = \min\{N'(a\phi(x, 0), t), N'(b\phi(0, x), t), N'(c\phi(x, -x), t)\}$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g, h) = \inf\{c \in [0, \infty) \mid N(g(x) - h(x), ct) \ \geq \ \psi(x, t), \ \forall x \in X, \ \forall t > 0\}.$$

Then (S, d) is a complete metric space(see [17]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 2^{-3}g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (3.1), we have

$$N(Jg(x) - Jh(x), cLt) \geq N(2^{-3}(g(2x) - h(2x)), cLt) \geq \psi(x, t)$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (3.3), $d(f, Jf) \leq \frac{1}{8} < \infty$ and by Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. Moreover, $C(x) = N - \lim_{n\to\infty} 2^{-3n} f(2^n x)$ for all $x \in X$ and $d(f, C) \leq \frac{1}{8(1-L)}$ and hence we have (3.4).

Replacing x, y, and t by $2^n x$, $2^n y$, and $2^{3n} t$ in (3.2), respectively, we have

$$N(D_f(2^n x, 2^n y), 2^{3n} t) \geq N'(\phi(2^n x, 2^n y), 2^{3n} t) \geq N'(L^n \phi(x, y), t)$$

for all $x, y \in X$ and all $t > 0$. Letting $n \longrightarrow \infty$ in the last inequality, we have

$$C(x + 2y) - 3C(x + y) + 3C(x) - C(x - y) - 6C(y) + G_C(x, y) = 0$$

for all $x, y \in X$ and thus C is a cubic mapping.

Now, we show the uniqueness of C. Let $C_0 : X \longrightarrow Y$ be another cubic mapping with (3.4). Then C_0 is a fixed ponit of J in S and by (3.4), we get

$$d(Jf, C_0) \leq d(Jf, JC) \leq Ld(f, C_0) \leq \frac{L}{8(1-L)} < \infty$$

and by (3) of Theorem 1.2, we have $C = C_0$. □

Similar to Theorem 3.1, we can also have the following theorem.

Theorem 3.2. *Let $\phi : X^2 \longrightarrow Z$ be a function such that there is a real number L satisfying $0 < L < 1$ and*

$$N'(\phi(x, y), t) \geq N'\Big(\frac{L}{8}\phi(2x, 2y), t\Big) \tag{3.5}$$

for all $x, y \in X$ and all $t > 0$. Let $f : X \longrightarrow Y$ be a mapping satisfying $f(0) = 0$, (3.2), and (3.3). Further, assume that if g satisfies (2.1), then g is a cubic mapping.

Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that the inequality

$$\begin{aligned} &N\Big(f(x) - C(x), \frac{L}{8(1-L)}t\Big) \\ &\geq \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\} \end{aligned} \tag{3.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let $\psi(x,t) = \min\{N'(a\phi(x,0),t), N'(b\phi(0,x),t), N'(c\phi(x,-x),t)\}$. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \ \geq \ \psi(x,t), \ \forall x \in X, \ \forall t > 0\}.$$

Then (S,d) is a complete metric space(see [17]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 8g(2^{-1}x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (3.2) and (3.5), we have

$$N(Jg(x) - Jh(x), cLt) \geq N(8(g(2^{-1}x) - h(2^{-1}x)), cLt) \geq \psi(x,t)$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g,h)$ for any $g, h \in S$ and so J is a strictly contractive mapping. By (3.3), we get

$$N\Big(f(x) - 8f(2^{-1}x), \frac{L}{8}t\Big) \geq \psi\Big(2^{-1}x, \frac{L}{8}t\Big) \geq \psi(x,t) \tag{3.7}$$

for all $x \in X$ and all $t > 0$. Hence $d(f, Jf) \leq \frac{L}{8} < \infty$ and by Theorem 1.2, there exists a mapping $C : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, C) \to 0$ as $n \to \infty$. Moreover, $C(x) = N - \lim_{n\to\infty} 2^{3n} f(2^{-n}x)$ for all $x \in X$ and $d(f,C) \leq \frac{L}{8(1-L)}$ and hence we have (3.6). The rest of the proof is similar to that of Theorem 3.1. □

Using Theorem 3.1 and Theorem 3.2, we have the following corollaries.

Corollary 3.3. *Let $\phi : X^2 \longrightarrow Z$ be a function with (3.1). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and (3.2). Further, assume that if g satisfies (2.1), then g is a cubic mapping and that*

$$\begin{aligned} &N(G_f(0,x),t) \geq \min\{N'(a_1\phi(x,0),t), N'(a_2\phi(0,x),t), N'(a_3\phi(x,-x),t)\}, \\ &N(G_f(x,-x),t) \geq \min\{N'(b_1\phi(x,0),t), N'(b_2\phi(0,x),t), N'(b_3\phi(x,-x),t)\} \end{aligned} \tag{3.8}$$

for all $x \in X$, all $t > 0$ and for some nonnegative real numbers $a_i, b_i (i = 1,2,3)$. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$\begin{aligned} &N\Big(f(x) - C(x), \frac{7}{24(1-L)}t\Big) \\ &\geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\} \end{aligned} \tag{3.9}$$

for all $x \in X$ and all $t > 0$, where $c_1 = \max\{a_1, b_1\}$, $c_2 = \max\{1, a_2, b_2\}$, and $c_3 = \max\{1, a_3, b_3\}$.

Proof. Setting $x = 0$ and $y = x$ in (3.2), we have

$$N(f(2x) - 9f(x) - f(-x) + G_f(0,x), t) \geq N'(\phi(0,x),t) \tag{3.10}$$

for all $x \in X$ and all $t > 0$. Setting $y = -x$ in (3.2), we have

$$N(3f(x) - 5f(-x) - f(2x) + G_f(x,-x), t) \geq N'(\phi(x,-x),t) \tag{3.11}$$

for all $x \in X$ and all $t > 0$. Hence by (3.10) and (3.11), we get

$$\begin{aligned} &N(6f(x) + 6f(-x) - G_f(0,x) - G_f(x,-x), 2t) \\ &\geq \min\{N'(\phi(0,x),t), N'(\phi(x,-x),t)\} \end{aligned} \tag{3.12}$$

for all $x \in X$ and all $t > 0$. Thus by (3.8), (3.10), and (3.12), we get

$$\begin{aligned} &N\Big(f(2x) - 8f(x), \frac{7}{3}t\Big) \\ &= \min\Big\{N(f(2x) - 9f(x) - f(-x) + G_f(0,x), t), N\Big(\frac{5}{6}G_f(0,x), \frac{5}{6}t\Big), \\ &\quad N\Big(f(x) + f(-x) - \frac{1}{6}G_f(0,x) - \frac{1}{6}G_f(x,-x), \frac{1}{3}t\Big), N\Big(\frac{1}{6}G_f(x,-x), \frac{1}{6}t\Big)\Big\} \\ &\geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. By Theorem 3.1, there exists an unique cubic mapping $C : X \longrightarrow Y$ with (3.9). ☐

Corollary 3.4. *Let $\phi : X^2 \longrightarrow Z$ be a function with (3.5). Let $f : X \longrightarrow Y$ be a mapping satisfying $f(0) = 0$ and (3.2). Further, assume that if g satisfies (2.1), then g is a cubic mapping and that (3.8) hold. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that the inequality*

$$\begin{aligned} &N\Big(f(x) - C(x), \frac{L}{8(1-L)}t\Big) \\ &\geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\} \end{aligned} \tag{3.13}$$

holds for all $x \in X$ and all $t > 0$, where $c_1 = \max\{a_1, b_1\}$, $c_2 = \max\{1, a_2, b_2\}$, and $c_3 = \max\{1, a_3, b_3\}$.

Proof. By (**??**), we get

$$\begin{aligned} &N\Big(f(x) - 8f(2^{-1}x), \frac{7L}{24}t\Big) \geq \psi\Big(2^{-1}x, \frac{L}{8}t\Big) \\ &\geq \min\{N'(c_1\phi(x,0),t), N'(c_2\phi(0,x),t), N'(c_3\phi(x,-x),t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. By Theorem 3.2, there exists an unique cubic mapping $C : X \longrightarrow Y$ with (3.13). ☐

From now on, we consider the following functional equation

$$\begin{aligned} &f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) \\ &+ k[f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)] = 0 \end{aligned} \tag{3.14}$$

for some positive real number k.

Lemma 3.5. [12] *A mapping $f : X \longrightarrow Y$ satisfies (3.14) if and only if f is a cubic mapping.*

Using Theorem 2.1, Theorem 3.1, and Theorem 3.2, we have the following example.

Example 3.6. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\begin{aligned} &N(f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y) + k[f(2x+y) + f(2x-y) \\ &- 2f(x+y) - 2f(x-y) - 12f(x)], t) \geq \frac{t}{t + \|x\|^{2p} + \|y\|^{2p} + \|x\|^p\|y\|^p} \end{aligned} \tag{3.15}$$

for all $x, y \in X$, all $t > 0$ and some positive real numbers k, p with $p \neq \frac{3}{2}$. Then there exists an unique cubic mapping $C : X \longrightarrow Y$ such that

$$N(f(x) - C(x), t) \geq \frac{2k|8 - 2^{2p}|t}{2k|8 - 2^{2p}|t + \|x\|^{2p}} \tag{3.16}$$

for all $x \in X$.

Proof. Let $G_f(x, y) = k[f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)]$ and $\phi(x, y) = \|x\|^{2p} + \|y\|^{2p} + \|x\|^{2p}\|y\|^p$. Then $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$ and f satisfies (3.2). Letting $y = 0$ in (3.15), we have

$$N(f(2x) - 8f(x), t) \geq N'\Big(\frac{1}{2k}\phi(x, 0), t\Big)$$

for all $x \in X$ and all $t > 0$, where

$$N'(r, t) = \begin{cases} 0, & \text{if } t \leq 0 \\ \frac{t}{t+|r|}, & \text{if } t > 0 \end{cases}$$

for all $r \in \mathbb{R}$. By Theorem 3.1, and Theorem 3.2, there exists an unique mapping $C : X \longrightarrow Y$ with (2.1) and (3.16). Since $G_f(y, x) = G_f(y, -x)$ for all $x, y \in X$, $G_C(y, x) = G_C(y, -x)$ for all $x, y \in X$ and letting $y = 0$ in $D_C(x, y) = 0$, we have $C(2x) = 8C(x)$ for all $x \in X$. By Theorem 2.1, we have the result. □

We can use Corollary 3.3 and Corollary 3.4 to get a classical result in the framework of normed spaces. As an example of $\phi(x, y)$ in Corollary 3.3 and Corollary 3.4, we can take $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then we can formulate the following example.

Example 3.7. Let X be a normed space and Y a Banach space. Suppose that if g satisfies (2.1), then g is a cubic mapping. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\|Df(x, y)\| \leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \tag{3.17}$$

for all $x, y \in X$ and a fixed positive real numbers p, ϵ with $p \neq \frac{3}{2}$. Suppose that

$$\|G_f(0, x)\| \leq \epsilon \max\{a_1, a_2, a_3\}\|x\|^{2p}, \ \ \|G_f(x, -x)\| \leq \epsilon \max\{b_1, b_2, b_3\}\|x\|^{2p}$$

for all $x \in X$, all $t > 0$ and for some nonnegative real numbers $a_i, b_i (i = 1, 2, 3)$. Then there is an unique cubic mapping $C : X \longrightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{7\epsilon}{3|8 - 2^{2p}|} \max\{3, a_1, a_2, 3a_3, b_1, b_2, 3b_3\}\|x\|^{2p}$$

for all $x \in X$.

Proof. Define a fuzzy norm N' on $\mathbb{R}$ by

$$N_{\mathbb{R}}(x, t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

for all $x \in \mathbb{R}$ and all $t > 0$. Similary we can define a fuzzy norm N_Y on Y. Then (Y, N_Y) is a fuzzy Banach space. Let $\phi(x, y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then by definitions N_Y and N', the following inequality holds :

$$N_Y(Df(x, y), t) \geq N_{\mathbb{R}}(\phi(x, y), t)$$

for all $x, y \in X$ and all $t > 0$. By Corollary 3.3 and Corollary 3.4, we have the result. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

DRYGAS FUNCTIONAL EQUATIONS WITH EXTRA TERMS AND ITS STABILITY

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ABSTRACT. In this paper, we consider the generalized Hyers-Ulam stability for the following functional equation with an extra term G_f

$$f(x+y)+f(x-y)+G_f(x,y)=2f(x)+f(y)+f(-y),$$

where G_f is a functional operator of f.

1. INTRODUCTION AND PRELIMINARIES

In 1940, Ulam [12] proposed the following stability problem :

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists an unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In 1941, Hyers [6] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [11] generalized the result of Hyers. Rassias [11] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p+\|y\|^p)$$

for some $\epsilon \ge 0$ and p with $p < 1$ and for all $x, y \in X$, where $f : X \longrightarrow Y$ is a function between Banach spaces. The paper of Rassias [11] has provided a lot of influence in the development of what we call *the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassis approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [10] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability for the quadratic functional equation and Park [9] proved the generalized Hyers-Ulam stability of the quadratic functional eqution in Banach modules over a C^*-algebra.

2010 *Mathematics Subject Classification.* 39B52, 39B82.

Key words and phrases. Hyers-Ulam Stability, Banach Space.

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In this paper, we are interested in what kind of terms can be added to the Drygas functional equation [4]

$$f(x+y)+f(x-y)=2f(x)+f(y)+f(-y)$$

while the generalized Hyers-Ulam stability still holds for the new functional equation. We denote the added term by $G_f(x,y)$ which can be regarded as a functional operator depending on the variables x, y, and functions f. Then the new functional equation can be written as

$$f(x+y)+f(x-y)+G_f(x,y)=2f(x)+f(y)+f(-y). \tag{1.2}$$

In fact, the functional operator $G_f(x,y)$ was introduced and considered in the cases of additive, quadratic functional equations with somewhat different point of view by the authors([7], [8]).

2. Solutions of 1.2 as additive-quadratic mappings

Let X and Y be normed spacese. For given $l\in\mathbb{N}$ and any $i\in\{1,2,\cdots,l\}$, let $\sigma_i: X\times X\longrightarrow X$ be a binary operation such that

$$\sigma_i(rx,ry)=r\sigma_i(x,y)$$

for all $x,y\in X$ and all $r\in\mathbb{R}$. It is clear that $\sigma_i(0,0)=0$. Also let $F: Y^l\longrightarrow Y$ be a linear, continuous function. For a map $f: X\longrightarrow Y$, define

$$G_f(x,y)=F(f(\sigma_1(x,y)),f(\sigma_2(x,y)),\cdots,f(\sigma_l(x,y))).$$

From now on, for any mapping $f: X\longrightarrow Y$, we deonte

$$f_o(x)=\frac{f(x)-f(-x)}{2},\ \ f_e(x)=\frac{f(x)+f(-x)}{2}$$

First, we consider the following functional equation

$$\begin{aligned}&af(x+y)+bf(x-y)-cf(y-x)\\&=(a+b)f(x)-cf(-x)+(a-c)f(y)+bf(-y)\end{aligned} \tag{2.1}$$

for fixed real numbers a,b,c with $a=b-c$ and $a\neq 0$. We can easily show the following lemma.

Lemma 2.1. *Let $f: X\longrightarrow Y$ be a mapping. Then f satisfies (2.1) if and only if f is an additive-quadratic mapping.*

Definition 2.2. The functional operator G is called *additive-quadratic* if whenever $G_h(x,y)=0$ for all $x,y\in X$, h is an additive-quadratic mapping.

Lemma 2.3. *Let $f: X\longrightarrow Y$ be a mapping satisfying (1.2) and G additve-quadratic. Then the following are equivalent :*
(1) f is additive-quadratic,
(2) the following equality

$$G_f(x,y)=-G_f(y,x) \tag{2.2}$$

holds for all x, $y\in X$, and
(3) there exist real numbers b,c such that $b\neq c$ and

$$bG_f(x,y)=cG_f(y,x) \tag{2.3}$$

holds for all x, $y\in X$.

Proof. (1) (⇒) (2) (⇒) (3) are trivial.

(3) (⇒) (1) By (2.2), we have $f(0) = 0$ and by (1.2), we have

$$\begin{aligned}G_f(x, y) &= 2f(x) + f(y) + f(-y) - f(x + y) - f(x - y), \text{ and}\\ G_f(y, x) &= 2f(y) + f(x) + f(-x) - f(x + y) - f(y - x)\end{aligned}$$

for all x, $y \in X$. Hence by (2.3), we have

$$(b+c)f(x+y)+bf(x-y)-cf(y-x) = (2b+c)f(x)+cf(-x)+(b+2c)f(y)+bf(-y)$$

for all x, $y \in X$ and by Lemma 2.1, we have that f is additive-quadratic. ☐

3. The generalized Hyers-Ulam stability of (1.2)

In this section, we deal with the generalized Hyers-Ulam stability of (1.2). Throughout this paper, assume that G is additive-quadratic and the following inequalities hold

$$\begin{aligned}&\|G_h(x,x)\| \le \|G_h(0,x)\| + \sum_{i=1}^{t} |b_i|\|G_h(\delta_i x, 0)\| \quad if \;\; h : odd,\\ &\|G_h(x,x)\| \le \sum_{i=1}^{r} |p_i|\|G_h(0,\alpha_i x)\| + \sum_{i=1}^{s} |a_i|\|G_h(\lambda_i x, 0)\| \quad if \;\; h : even\end{aligned} \tag{3.1}$$

for some $r, s, t \in \mathbb{N} \cup \{0\}$, some real numbers $p_i, a_i, b_i, \alpha_i, \lambda_i$, and δ_i and for all $x \in X$.

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty \tag{3.2}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an odd mapping such that

$$\|f(x + y) + f(x - y) + G_f(x, y) - 2f(x)\| \le \phi(x, y). \tag{3.3}$$

for all $x, y \in X$. Then there exists an odd mapping $A : X \longrightarrow X$ such that A satisfies (1.2) and

$$\|A(x) - f(x)\| \le \sum_{n=0}^{\infty} 2^{-n-1}\Big[\phi(2^n x, 2^n x) + \phi(0, 2^n x) + \sum_{i=1}^{t} |b_i|\phi(2^n\delta_i x, 0)\Big]. \tag{3.4}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $A : X \longrightarrow X$ is an unique additive mapping with (3.4).

Proof. By (3.3), we have

$$\|G_f(x, 0)\| \le \phi(x, 0), \;\; \|G_f(0, x)\| \le \phi(0, x)$$

for all $x, y \in X$. Setting $y = x$ in (3.3), we have

$$\|f(2x) + G_f(x, x) - 2f(x)\| \le \phi(x, x) \tag{3.5}$$

for all $x \in X$. Hence by (3.1) and (3.5), we have

$$\|f(x) - 2^{-1}f(2x)\| \le 2^{-1}\Big[\phi(x, x) + \phi(0, x) + \sum_{i=1}^{t} |b_i|\phi(\delta_i x, 0)\Big] \tag{3.6}$$

for all $x \in X$. By (3.6), we have

$$\begin{aligned}&\|f(x) - 2^{-n}f(2^nx)\|\\ &\leq \sum_{k=0}^{n-1} 2^{-k-1}\Big[\phi(2^kx, 2^kx) + \phi(0, 2^kx) + \sum_{i=1}^{t}|b_i|\phi(2^k\delta_ix, 0)\Big]\end{aligned}$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N} \cup \{0\}$ with $0 \leq m < n$,

$$\begin{aligned}&\|2^{-m}f(2^mx) - 2^{-n}f(2^nx)\|\\ &= 2^{-m}\|f(2^mx) - 2^{-(n-m)}f(2^{n-m}(2^mx))\|\\ &\leq \sum_{k=m}^{n-1} 2^{-k-1}\Big[\phi(2^kx, 2^kx) + \phi(0, 2^kx) + \sum_{i=1}^{t}|b_i|\phi(2^k\delta_ix, 0)\Big]\end{aligned} \tag{3.7}$$

for all $x \in X$. By (3.2) and (3.7), $\{2^{-n}f(2^nx)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $A : X \longrightarrow Y$ such that $A(x) = \lim_{n\to\infty} 2^{-n}f(2^nx)$ for all $x \in X$. By (3.7), we have (3.4).

Replacing x and y by 2^nx and 2^ny in (3.3), respectively and deviding (3.3) by 2^n, we have

$$\|2^{-n}[f(2^n(x + y)) + f(2^n(x - y)) + G_f(2^nx, 2^ny) - 2f(2^nx)]\| \leq 2^{-n}\phi(2^nx, 2^ny)$$

for all $x, y \in X$ and letting $n \to \infty$, we can show that A satisfies (1.2). Since f is odd, A is odd.

Suppose that G_f satisfies (2.2). Then clearly, we can show that G_A satisfies (2.2) and hence by Lemma 2.3, A is an additive-quadratic mapping. Since A is odd, A is an additive mapping.

Now, we show the uniqueness of A. Let $E : X \longrightarrow Y$ be an additive mapping with (3.4). Since A and E are additive,

$$\begin{aligned}\|A(x) - E(x)\| &= \|A(2^nx) - E(2^nx)\|\\ &\leq 2^{-k}\sum_{n=0}^{\infty} 2^{-n}\Big[\phi(2^nx, 2^nx) + \phi(0, 2^nx) + \sum_{i=1}^{t}|b_i|\phi(2^n\delta_ix, 0)\Big]\end{aligned}$$

for all $x \in X$ and all $k \in \mathbb{N}$. Hence, letting $k \to \infty$, by (3.2), we have $A = E$. ☐

Similar to Theorem 3.1, we have the following theorem.

Theorem 3.2. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^n\phi(2^{-n}x, 2^{-n}y) < \infty \tag{3.8}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an odd mapping satisfying (3.3). Then there exists an odd mapping $A : X \longrightarrow X$ such that A satisfies (1.2) and

$$\|A(x)-f(x)\| \leq \sum_{n=0}^{\infty} 2^{n-1}\Big[\phi(2^{-n}x, 2^{-n}x)+\phi(0, 2^{-n}x)+\sum_{i=1}^{t}|b_i|\phi(2^{-n}\delta_ix, 0)\Big] \tag{3.9}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $A : X \longrightarrow X$ is an unique additive mapping with (3.9)

Proof. By (3.3), we have

$$\|G_f(x,0)\| \le \phi(x,0), \ \ \|G_f(0,x)\| \le \phi(0,x)$$

for all $x, y \in X$. Setting $y = x = \frac{x}{2}$ in (3.5), we have

$$\left\| f(x) + G_f\Big(\frac{x}{2}, \frac{x}{2}\Big) - 2f\Big(\frac{x}{2}\Big) \right\| \le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) \tag{3.10}$$

for all $x \in X$. Hence by (3.1), (3.3), and (3.10), we have

$$\left\| f(x) - 2f\Big(\frac{x}{2}\Big) \right\| \le \phi(x,x) + \phi(0,x) + \sum_{i=1}^{t} |b_i| \phi(\delta_i x, 0) \tag{3.11}$$

for all $x \in X$. By (3.11), we have

$$\|f(x) - 2^n f(2^{-n}x)\| \le \sum_{k=0}^{n-1} 2^k \Big[\phi(2^{-k}x, 2^{-k}x) + \phi(0, 2^{-k}x) + \sum_{i=1}^{t} |b_i| \phi(2^{-k}\delta_i x, 0)\Big]$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N} \cup \{0\}$ with $0 \le m < n$,

$$\begin{aligned} &\|2^m f(2^{-m}x) - 2^n f(2^{-n}x)\| \\ &= 2^m \|f(2^{-m}x) - 2^{(n-m)} f(2^{-(n-m)}(2^{-m}x))\| \\ &\le \sum_{k=m}^{n-1} 2^k \Big[\phi(2^{-k}x, 2^{-k}x) + \phi(0, 2^{-k}x) + \sum_{i=1}^{t} |b_i| \phi(2^{-k}\delta_i x, 0)\Big] \end{aligned} \tag{3.12}$$

for all $x \in X$. By (3.12), $\{2^n f(2^{-n}x)\}$ is a Cauchy sequence in Y. The rest of proof is similar to Theorem 3.1. ☐

Theorem 3.3. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{-2n} \phi(2^n x, 2^n y) < \infty \tag{3.13}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an even mapping such that

$$\|f(x+y) + f(x-y) + G_f(x,y) - 2f(x) - 2f(y)\| \le \phi(x,y). \tag{3.14}$$

for all $x, y \in X$. Then there exists an even mapping $Q : X \longrightarrow X$ such that

$$\|Q(x) - f(x)\| \le \sum_{n=0}^{\infty} 2^{-2n-2} \Big[\phi(2^n x, 2^n x) + \sum_{i=1}^{r} |p_i| \phi(0, 2^n a_i x) + \sum_{i=1}^{s} |a_i| \phi(2^n \lambda_i x, 0)\Big] \tag{3.15}$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $Q : X \longrightarrow Y$ is an unique quadratic mapping with (3.15)

Proof. Setting $y = x$ in (3.14), we have

$$\|2^2 f(x) - f(2x) + G_f(x,x)\| \le \phi(x,x)$$

for all $x \in X$ and by (3.14), we have

$$\|G_f(x,0)\| \le \phi(x,0), \ \ \|G_f(0,x)\| \le \phi(0,x)$$

for all $x \in X$. Since f is even, letting $y = x$ in (3.14), by (3.1), we have

$$\begin{aligned}
&\|f(x) - 2^{-2}f(2x)\| \\
&\le 2^{-2}\Big[\phi(x,x) + \|G_f(x,x)\|\Big] \\
&\le 2^{-2}\Big[\phi(x,x) + \sum_{i=1}^{r}|p_i|\|G_f(0,\alpha_i x)\| + \sum_{i=1}^{s}|a_i|\|G_f(\lambda_i x,0)\|\Big] \\
&\le 2^{-2}\Big[\phi(x,x) + \sum_{i=1}^{r}|p_i|\phi(0,\alpha_i x) + \sum_{i=1}^{s}|a_i|\phi(\lambda_i x,0)\Big]
\end{aligned}$$

for all $x \in X$. Hence we have

$$\begin{aligned}
&\|f(x) - 2^{-2n}f(2^n x)\| \\
&\le \sum_{k=0}^{n-1} 2^{-2k-2}\Big[\phi(2^k x, 2^k x) + \sum_{i=1}^{r}|p_i|\phi(0,2^k a_i x) + \sum_{i=1}^{s}|a_i|\phi(2^k\lambda_i x,0)\Big]
\end{aligned} \tag{3.16}$$

for all $x \in X$ and all $n \in N$. For $m, n \in \mathbb{N}\cup\{0\}$ with $0 \le m < n$, by (3.16)

$$\begin{aligned}
&\|2^{-2m}f(2^m x) - 2^{-2n}f(2^n x)\| \\
&= 2^{-2m}\|f(2^m x) - 2^{-2(n-m)}f(2^{n-m}(2^m x))\| \\
&\le \sum_{k=m}^{n-1} 2^{-2k-2}\Big[\phi(2^k x, 2^k x) + \sum_{i=1}^{r}|p_i|\phi(0,2^k a_i x) + \sum_{i=1}^{s}|a_i|\phi(2^k\lambda_i x,0)\Big]
\end{aligned} \tag{3.17}$$

for all $x \in X$. By (3.17), $\{2^{-2n}f(2^n x)\}$ is a Cauchy sequence in Y. The rest of proof is similar to Theorem 3.1. ☐

Theorem 3.4. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^{2n}\phi(2^{-n}x, 2^{-n}y) < \infty \tag{3.18}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be an even mapping satisfying (3.14). Then there exists an even mapping $Q : X \longrightarrow X$ such that

(3.19)

$$\|Q(x)-f(x)\| \le \sum_{n=0}^{\infty} 2^{2n}\Big[\phi(2^{-n}x, 2^{-n}x)+\sum_{i=1}^{r}|p_i|\phi(0,2^{-n}a_i x)+\sum_{i=1}^{s}|a_i|\phi(2^{-n}\lambda_i x,0)\Big]$$

for all $x \in X$. Further, if G_f satisfies (2.2), then $Q : X \longrightarrow Y$ is an unique quadratic mapping with (3.19)

Proof. Setting $y = x = \frac{x}{2}$ in (3.14), we have

$$\Big\|2^2 f\Big(\frac{x}{2}\Big) - f(x) + G_f\Big(\frac{x}{2},\frac{x}{2}\Big)\Big\| \le \phi\Big(\frac{x}{2},\frac{x}{2}\Big)$$

for all $x \in X$. By (3.14), we have

$$\|G_f(x,0)\| \le \phi(x,0), \;\; \|G_f(0,x)\| \le \phi(0,x)$$

for all $x \in X$ and so, we have

$$\begin{aligned}\Big\|2^2 f\Big(\frac{x}{2}\Big) - f(x)\Big\| &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \Big\|G_f\Big(\frac{x}{2}, \frac{x}{2}\Big)\Big\| \\ &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \sum_{i=1}^{r} |p_i| \Big\|G_f\Big(0, \alpha_i \frac{x}{2}\Big)\Big\| + \sum_{i=1}^{s} |a_i| \Big\|G_f\Big(\lambda_i \frac{x}{2}, 0\Big)\Big\| \\ &\le \phi\Big(\frac{x}{2}, \frac{x}{2}\Big) + \sum_{i=1}^{r} |p_i| \phi\Big(0, \alpha_i \frac{x}{2}\Big) + \sum_{i=1}^{s} |a_i| \phi\Big(\lambda_i \frac{x}{2}, 0\Big)\end{aligned}$$

for all $x \in X$. Similar to Theorem 3.1, we have the result. ☐

Theorem 3.5. *Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function with (3.2). Let $f : X \longrightarrow Y$ be a mapping with (3.3). Then there exists a mapping $F : X \longrightarrow X$ such that F satisfies (1.2) and*

$$\begin{aligned}&\|F(x) - f(x)\| \\ &\le \sum_{n=0}^{\infty} 2^{-2n-2} \Big[\phi_1(2^n x, 2^n x) + \sum_{i=1}^{r} |p_i| \phi_1(0, 2^n x) + \sum_{i=1}^{s} |a_i| \phi_1(\lambda_i 2^n x, 0)\Big] \\ &\quad + \sum_{n=0}^{\infty} 2^{-n-1} \Big[\phi_1(2^n x, 2^n x) + \phi_1(0, 2^n x) + \sum_{i=1}^{t} |b_i| \phi_1(\delta_i 2^n x, 0)\Big]\end{aligned} \tag{3.20}$$

for all $x \in X$, where $\phi_1(x, y) = \frac{1}{2}\Big[\phi(x, y) + \phi(-x, -y)\Big]$. Further, if G_f satisfies (2.2), then $F : X \longrightarrow X$ is an unique additive-quadratic mapping with (3.20)

Proof. By (3.3), we have

$$\|f_e(x+y) + f_e(x-y) + G_{f_e}(x, y) - 2f_e(x) - 2f_e(y)\| \le \phi_1(x, y) \tag{3.21}$$

for all $x, y \in X$. By Theorem 3.3, there exists an even mapping $Q : X \longrightarrow Y$ such that $Q(x) = \lim_{n \longrightarrow \infty} 2^{-2n} f_e(2^n x)$ for all $x \in X$,

$$Q(x+y) + Q(x-y) + G_Q(x, y) = 2Q(x) + 2Q(y) \tag{3.22}$$

for all $x, y \in X$, and

$$\begin{aligned}&\|Q(x) - f_e(x)\| \\ &\le \sum_{n=0}^{\infty} 2^{-2n-2} \Big[\phi_1(2^n x, 2^n x) + \sum_{i=1}^{r} |p_i| \phi_1(0, 2^n a_i x) + \sum_{i=1}^{s} |a_i| \phi_1(2^n \lambda_i x, 0)\Big]\end{aligned} \tag{3.23}$$

for all $x \in X$. Similarly, there exists an odd mapping $A : X \longrightarrow Y$ such that $A(x) = \lim_{n \longrightarrow \infty} 2^{-n} f_o(2^n x)$ for all $x \in X$,

$$A(x+y) + A(x-y) + G_A(x, y) - 2A(x) = 0 \tag{3.24}$$

for all $x, y \in X$, and

$$\|A(x) - f_o(x)\| \le \sum_{n=0}^{\infty} 2^{-n-1} \Big[\phi_1(2^n x, 2^n x) + \phi_1(0, 2^n x) + \sum_{i=1}^{t} |b_i| \phi_1(2^n \delta_i x, 0)\Big] \tag{3.25}$$

for all $x \in X$.

Let $F = Q + A$. Since Q is even and A is odd, $2Q(y) = F(y) + F(-y)$ and by (3.22) and (3.24), F satisfies (1.2). Since $\|F(x) - f(x)\| \le \|Q(x) - f_e(x)\| + \|A(x) - f_o(x)\|$, by (3.23) and (3.25), we have (3.20).

Suppose that G_f satisfies (2.2). Then clearly, we can show that G_F satisfies (2.2) and hence by Lemma 2.3, F is an additive-quadratic mapping. The proof of the uniqueness of F is similar to Theorem 3.1. $\square$

Theorem 3.6. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function such that*

$$\sum_{n=0}^{\infty} 2^n \phi(2^{-n}x, 2^{-n}y) < \infty$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping with (3.3). Then there exist a mapping $F : X \longrightarrow X$ such that

(3.26)

$$\begin{aligned} &\|F(x) - f(x)\| \\ &\leq \sum_{n=0}^{\infty} 2^{2n-2}\Big[\phi_1(2^{-n}x, 2^{-n}x) + \sum_{i=1}^{r}|p_i|\phi_1(0, 2^{-n}x) + \sum_{i=1}^{s}|a_i|\phi_1(\lambda_i 2^{-n}x, 0)\Big] \\ &\quad + \sum_{n=0}^{\infty} 2^{n-1}\Big[\phi_1(2^{-n}x, 2^{-n}x) + \phi_1(0, 2^{-n}x) + \sum_{i=1}^{t}|b_i|\phi_1(\delta_i 2^{-n}x, 0)\Big] \end{aligned}$$

for all $x \in X$, where $\phi_1(x,y) = \frac{1}{2}\Big[\phi(x,y) + \phi(-x,-y)\Big]$. Further, if G_f satisfies (2.2), then $F : X \longrightarrow X$ is an unique additive-quadratic mapping with (3.26).

4. Applicaions

In this section, we illustrate how the theorems in section 3 work well for the generalized Hyers-Ulam stability of various additive-quadratic functional equations.

As examples of $\phi(x,y)$ in Theorem 3.5 and Theorem 3.6, we can take $\phi(x,y) = \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|y\|^{2p})$. Then we can formulate the following theorem :

Theorem 4.1. *Assume that all of the conditions in Theorem 3.1 hold and G_f satisfies (2.2). Let p be a real number with $0 < p < \frac{1}{2}, 1 < p$. Let $f : X \longrightarrow Y$ be a mapping such that*

(4.1)

$$\|f(x+y)+f(x-y)-2f(x)-f(y)-f(-y)+G_f(x,y)\| \leq \epsilon(\|x\|^p\|y\|^p+\|x\|^{2p}+\|x\|^{2p})$$

for all $x, y \in X$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \begin{cases} \Psi_1(x), & \textit{if } 0 < p < \frac{1}{2} \\ \Psi_2(x), & \textit{if } 1 < p \end{cases}$$

for all $x \in X$, where

$$\Psi_1(x) = \Big[3 + \sum_{i=1}^{r}|p_i| + \sum_{i=1}^{s}|a_i||\lambda_i|^{2p}\Big]\frac{\epsilon}{4-4^p}\|x\|^{2p} + \Big[4 + \sum_{i=1}^{t}|b_i||\delta_i|^{2p}\Big]\frac{\epsilon}{2-4^p}\|x\|^{2p}$$

and

$$\Psi_2(x) = \Big[3 + \sum_{i=1}^{r}|p_i| + \sum_{i=1}^{s}|a_i||\lambda_i|^{2p}\Big]\frac{4^{p-1}\epsilon}{4^p-4}\|x\|^{2p} + \Big[4 + \sum_{i=1}^{t}|b_i||\delta_i|^{2p}\Big]\frac{2^{2p-1}\epsilon}{4^p-2}\|x\|^{2p}$$

Lemma 4.2. *Let G be the operator defined by*

$$G_f(x,y) = f(2x+y) - f(x+2y) + f(x-y) - f(y-x) - 3f(x) + 3f(y)$$

for all mapping $f : X \longrightarrow Y$. Then G is additive-quadratic.

Proof. Suppose that $G_f(x,y) = 0$ for all $x, y \in X$. Then we have

$$f(2x+y) - f(x+2y) + f(x-y) - f(y-x) - 3f(x) + 3f(y) = 0. \tag{4.2}$$

and so we have

$$f_e(2x+y) - f_e(x+2y) - 3f_e(x) + 3f_e(y) = 0 \tag{4.3}$$

for all $x, y \in X$ and letting $y = y - x$ in (4.3), we have

$$f_e(x+y) - f_e(x-2y) - 3f_e(x) + 3f_e(x-y) = 0 \tag{4.4}$$

for all $x, y \in X$. Letting $y = -y$ in (4.4), we have

$$f_e(x-y) - f_e(x+2y) - 3f_e(x) + 3f_e(x+y) = 0 \tag{4.5}$$

for all $x, y \in X$. By (4.4) and (4.5), we have

$$f_e(x+2y)+f_e(x-2y)-2f_e(x)-8f_e(y)-4[f_e(x+y)+f_e(x-y)-2f_e(x)-2f_e(y)] = 0$$

for all $x, y \in X$ and so f_e is quadratic.

Since f_o is an odd mapping, by (4.2), we have

$$f_o(2x+y) - f_o(x+2y) + 2f_o(x-y) - 3f_o(x) + 3f_o(y) = 0 \tag{4.6}$$

for all $x, y \in X$ and letting $y = -x - y$ in (4.6), we have

$$f_o(x-y) + f_o(x+2y) + 2f_o(2x+y) - 3f_o(x) - 3f_o(x+y) = 0 \tag{4.7}$$

for all $x, y \in X$. By (4.6) and (4.7), we have

$$f_o(2x+y) + f_o(x-y) - 2f_o(x) + f_o(y) - f_o(x+y) = 0 \tag{4.8}$$

for all $x, y \in X$ and letting $y = -y$ in (4.10), we have

$$f_o(2x-y) + f_o(x+y) - 2f_o(x) - f_o(y) - f_o(x-y) = 0 \tag{4.9}$$

for all $x, y \in X$. By (4.10) and (4.9), we have

$$f_o(2x+y) + f_o(2x-y) - 4f_o(x) = 0 \tag{4.10}$$

for all $x, y \in X$ and hence f_o is additive. Thus f is an additive-quadratic mapping. □

By Lemma 2.3, Theorem 4.1, and Lemma 4.2, we have the following theorem :

Theorem 4.3. *Let $f : X \longrightarrow Y$ be a mapping such that*

$$\begin{aligned}&\|f(x+2y) - f(2x+y) + f(x+y) + f(y-x) + f(x) - 4f(y) - f(-y)\| \\ &\leq \epsilon(\|x\|^p\|y\|^p + \|x\|^{2p} + \|x\|^{2p})\end{aligned}$$

for all $x, y \in X$ and some a real number p with $0 < p < \frac{1}{2}, 1 < p$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$\|F(x) - f(x)\| \leq \begin{cases} \left[\frac{3}{4-4^p} + \frac{4}{2-4^p}\right]\epsilon\|x\|^{2p}, & \text{if } 0 < p < \frac{1}{2} \\ \\ \left[\frac{3\times 4^{p-1}}{4^p-4} + \frac{2\times 4^p}{4^p-2}\right]\epsilon\|x\|^{2p}, & \text{if } 1 < p \end{cases}$$

for all $x \in X$.

Proof. For a mapping $h : X \longrightarrow Y$, let $G_h(x,y) = h(2x+y) - h(x+2y) + h(x-y) - h(y-x) - 3h(x) + 3h(y)$. By Lemma 4.2, G is additive-quadratic and f, G satisfiy (4.1). Since G_f satisfies (2.2) in Lemma 2.3, by Theorem 4.1, we have the result. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Theoretical and Numerical Discussion for the Mixed Integro–Differential Equations

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Abstract

In this paper, we tend to apply the proposed modified Laplace Adomian decomposition method that is the coupling of Laplace transform and Adomian decomposition method. The modified Laplace Adomian decomposition method is applied to solve the Fredholm–Volterra integro–differential equations of the second kind in the space $L_2[a, b]$. The nonlinear term will simply be handled with the help of Adomian polynomials. The Laplace decomposition technique is found to be fast and correct. Several examples are tested and also the results of the study are discussed. The obtained results expressly reveal the complete reliability, efficiency, and accuracy of the proposed algorithmic rule for solving the Fredholm–Volterra integro–differential equations and therefore will be extended to other problems of numerous nature.

Mathematics Subject Classification: 41A10, 45J05, 65R20.

Key-Words: Fredholm-Volterra Integro-Differential Equations; Adomian Decomposition Method; Laplace Transform Method; Laplace Adomian Decomposition Method.

1. Introduction

Mathematical modeling of real-life problems usually results in functional equations, such as differential, integral, and integro-differential equations. Many mathematical formulations of physical phenomena reduced to integro-differential equations, like fluid dynamics, biological models, chemical mechanics and contact problems, see [6, 14, 19].

Many problems from physics and engineering and alternative disciplines cause linear and nonlinear integral equations. Now, for the solution of those equations several analytical and numerical methods are introduced, however numerical methods are easier than analytical methods and most of the time numerical methods are used to solve these equations we refer to [1, 2, 18].

Laplace Adomians decomposition method was first introduced by Suheil A. Khuri [16,17] and has been with successfully used to find the solution of differential equations [20]. This method generates a solution in the form of a series whose terms are determined by a recursive relation using the Adomian polynomials.

Most of the nonlinear integro-differential equations don't have an exact analytic solution, therefore approximation and numerical technique should be used, there are only a number of techniques for the solution of integro-differential equations, since it's relatively a new subject in arithmetic.

The modified laplace decomposition technique has applied for solving some nonlinear ordinary, partial differential equations. Recently, the authors have used many methods for the numerical or the analytical solution of linear and nonlinear Fredholm and Volterra integral and integrodifferential equations of the second kind [8, 9, 11, 12, 21].

In this paper, we consider the Fredholm–Volterra integro–differential equations of the second kind with continuous kernels with respect to position. We applied Laplace transform and Adomian polynomials to solve nonlinear Fredholm–Volterra integro–differential equations. Al–Towaiq and Kasasbeh [7] have applied the modification of Laplace decomposition method to solve linear interval Fredholm integro–differential equations of the form :

$$u'(x) = f(x) + \int_a^b k(x,t)u(t)dt; \qquad u(a) = \alpha.$$

But in this paper, we will study the modification of Laplace Adomian decomposition method to solve the nonlinear interval Fredholm–Volterra integro–differential equation of the form:

$$\phi(u+q) = p(u) + \lambda \int_a^b k(u,v)\mu(v,\phi(v))dv + \lambda \int_0^u \psi(u,v)\nu(v,\phi(v))dv; \quad (q << 1), \tag{1}$$

where q is the Phase-Lag is positive, very small and assumed to be intrinsic properties of the medium. The constant parameter λ may be complex and has many physical meanings, the function $\phi(u)$ is unknown in the Banach space and continuous with their derivative with respect to time in the space $L_2[a,b]$, where $[a,b]$ is the domain of integration with respect to the position and it's called the potential function of the mixed integral equation. The kernels $k(u,v), \psi(u,v)$ are positive and continuous in $L_2[a,b]$ and the known function $p(u)$ is continuous and its derivatives with respect to position.

Using Taylor Expansion after neglecting the second derivative in the equation (1) we get,

$$\phi(u) + q\frac{d\phi(u)}{du} = p(u) + \lambda \int_a^b k(u,v)\mu(v,\phi(v))dv + \lambda \int_0^u \psi(u,v)\nu(v,\phi(v))dv; \quad (q << 1), \tag{2}$$

with initial condition,

$$\phi(a) = \alpha. \tag{3}$$

The equation (2) with initial condition (3) is called Integro-Differential Equation for the Phase-Lag. The Integro-Differential Equation is a kind of functional equation that has associate integral and derivatives of unknown function. These equations were named after the leading mathematicians who have first studied them such as Fredholm, Volterra. Fredholm and Volterra equations are the most encountered types, see [10]. There is, formally only one difference between them, in the Fredholm equation the region of integration is fixed where in the Volterra equation the region is variable. Integro-Differential Equations (IDEs) are given as a combination of differential and integral equations.

2. Preliminaries

In this section, we give some definitions and properties of the Adomian polynomials and Laplace transform.

2.1 Laplace transform

Definition 1. The Laplace transform of a function $\phi(u); u > 0$ is defined as

$$L[\phi(u)] = \Phi(s) = \int_0^{+\infty} e^{-su}\phi(u)du, \tag{4}$$

where s can be either real or complex.

Definition 2. Given two functions ϕ and ψ, we define, for any $u > 0$,

$$(\phi * \psi)(u) = \int_0^u \phi(v)\psi(u-v)dv, \tag{5}$$

the function $\phi * \psi$ is called the convolution of ϕ and ψ.

Theorem 1. *The convolution theorem*

$$L[\phi * \psi](u) = L[\phi(u)] * L[\psi(u)]. \tag{6}$$

Lemma 1. *Laplace Transform of an Integral: If $\Phi(s) = L[\phi(u)]$ then*

$$L\left[\int_0^u \phi(v)dv\right] = \frac{\Phi(s)}{s}. \tag{7}$$

Theorem 2. *The Laplace transform $L[\phi(u)]$ of the derivatives are defined by*

$$L[\phi^{(n)}(u)] = s^n L[\phi(u)] - s^{n-1}\phi(0) - s^{n-2}\phi'(0) - \cdots - \phi^{(n-1)}(0). \tag{8}$$

2.2 Adomians Decomposition method

Consider the general functional equation:

$$\phi = p + N_1\phi + N_2\phi, \tag{9}$$

where N_1, N_2 are a nonlinear operators, p is a known function, and we are seeking the solution ϕ satisfying (9). We assume that for every p , Eq. (9) has one and only one solution.

The Adomians technique consists of approximating the solution of (9) as an infinite series

$$\phi = \sum_{n=0}^{\infty} \phi_n, \tag{10}$$

and decomposing the nonlinear operators N_1, N_2 as respectively

$$N_1\phi = \sum_{n=0}^{\infty} A_n, \qquad N_2\phi = \sum_{n=0}^{\infty} B_n, \tag{11}$$

where A_n, B_n are polynomials (called Adomian polynomials)of $\{\phi_0, \phi_1, \ldots, \phi_n\}$ [4,5] given by

$$\begin{aligned} A_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_1\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0} ; \quad n = 0, 1, 2, \ldots \\ B_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N_2\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0} ; \quad n = 0, 1, 2, \ldots \end{aligned}$$

The proofs of the convergence of the series $\sum_{n=0}^{\infty}\phi_n, \sum_{n=0}^{\infty} A_n$ and $\sum_{n=0}^{\infty} B_n$ are given in [3,13]. Substituting (10) and (11) into (9) yields, we get

$$\sum_{n=0}^{\infty}\phi_n = p + \sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} B_n.$$

Thus, we can identify

$$\begin{aligned} \phi_0 =& p, \\ \phi_{n+1} =& A_n(\phi_0, \phi_1, \ldots, \phi_n) + B_n(\phi_0, \phi_1, \ldots, \phi_n); \quad n = 0, 1, 2, \ldots \end{aligned}$$

Thus all components of ϕ can be calculated once the A_n, B_n are given. We then define the n-terms approximate to the solution ϕ by

$$\Psi_n[\phi] = \sum_{i=0}^{n-1}\phi_i \quad , with \quad \lim_{n\to\infty}\Psi_n[\phi] = \phi.$$

3. Description of the Method

The purpose of this section is to discuss the use of modified Laplace decomposition algorithm for the Fredholm–Volterra integro–differential equation. Applying the Laplace transform (denoted by L) on the both sides of the equation yield (2), we have

$$\begin{aligned} L[\phi(u)] + qL\left[\frac{d\phi(u)}{du}\right] =& L[p(u)] + \lambda L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ &+ \lambda L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right], \end{aligned} \tag{12}$$

using the differentiation property of Laplace transform (8) we get

$$\begin{aligned} L[\phi(u)] + qsL[\phi(u)] - q\phi(0) =& L[p(u)] + \lambda L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ &+ \lambda L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right]. \end{aligned} \tag{13}$$

Thus, the given equation is equivalent to

$$\begin{aligned} L[\phi(u)] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)\mu(v,\phi(v))dv\right] \\ + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)\nu(v,\phi(v))dv\right]. \end{aligned} \tag{14}$$

The Adomian decomposition method and the Adomian polynomials can be used to handle (14) and to address the nonlinear terms $\mu(v,\phi(v)), \nu(v,\phi(v))$. We first represent the linear term $\phi(u)$ at the left side by an infinite series of components given by

$$\phi(u) = \sum_{n=0}^{\infty} \phi_n(u), \tag{15}$$

where the components $\phi_n; n \geq 0$ will be determined recursively. However, the nonlinear terms $\mu(v,\phi(v)), \nu(v,\phi(v))$ at the right side of Eq. (14) will be represented by an infinite series of the Adomian polynomials A_n, B_n respectively in the form

$$\mu(v,\phi(v)) = \sum_{n=0}^{\infty} A_n(v), \qquad \nu(v,\phi(v)) = \sum_{n=0}^{\infty} B_n(v), \tag{16}$$

where $A_n, B_n; n \geq 0$ are defined by

$$\begin{aligned} A_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mu\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0,1,2,\ldots \\ B_n =& \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\nu\left(\sum_{i=0}^{\infty}\lambda^i\phi_i\right)\right]_{\lambda=0}; \quad n = 0,1,2,\ldots \end{aligned}$$

where the so-called Adomian polynomials A_n, B_n can be evaluated for all forms of nonlinearity [22]. In other words, assuming that the nonlinear function is $\mu(v,\phi(v)), \nu(v,\phi(v))$, therefore the Adomian polynomials are given by

$$\begin{aligned} A_0 &= \mu(\phi_0), & B_0 &= \nu(\phi_0), \\ A_1 &= \phi_1\mu'(\phi_0), & B_1 &= \phi_1\nu'(\phi_0), \\ A_2 &= \phi_2\mu'(\phi_0) + \frac{1}{2}\phi_1^2\mu''(\phi_0), & B_2 &= \phi_2\nu'(\phi_0) + \frac{1}{2}\phi_1^2\nu''(\phi_0). \end{aligned}$$

Substituting (15) and (16) into (14), we will get

$$\begin{aligned} L\left[\sum_{0}^{\infty}\phi_n(u)\right] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)\sum_{0}^{\infty}A_n(v)dv\right] \\ + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)\sum_{0}^{\infty}B_n(v)dv\right]. \end{aligned} \tag{17}$$

The Adomian decomposition method presents the recursive relation

$$L[\phi_0(u)] = \frac{q\phi(0)}{(1+qs)} + \frac{L[p(u)]}{(1+qs)} + \frac{\lambda}{(1+qs)}, \tag{18}$$

$$L[\phi_1(u)] = \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_0(v)dv\right] + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_0(v)dv\right], \tag{19}$$

$$L[\phi_2(u)] = \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_1(v)dv\right] + \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_1(v)dv\right]. \tag{20}$$

In general, the recursive relation is given by

$$\begin{aligned} L[\phi_{n+1}(u)] =& \frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_n(v)dv\right] \\ &+ \frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_n(v)dv\right], \quad n = 0,1,2,\ldots \end{aligned} \tag{21}$$

A necessary condition for Eq. (21) to work is that

$$\lim_{s\to\infty}\frac{\lambda}{(1+qs)} = 0.$$

Applying inverse Laplace transform to Eqs. (18)–(21), so our required recursive relation

$$\phi_0(u) = G(u), \tag{22}$$

and

$$\begin{aligned} \phi_{n+1}(u) =& L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_n(v)dv\right]\right] \\ &+ L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_n(v)dv\right]\right], \end{aligned} \tag{23}$$

where $G(u)$ may be a function that arises from the source term and also the prescribed initial conditions, the initial solution is very important, the choice of (22) as the initial solution always leads to noise oscillation during the iteration procedure, the modified laplace decomposition method [15] suggests that the operate $G(u)$ defined above in (18) be rotten into two parts:

$$G(u) = G_1(u) + G_2(u).$$

Instead of iteration procedure (22) and (23), we suggest the following modification

$$\begin{aligned}
\phi_0(u) &= G_1(u),\\
\phi_1(u) &= G_2(u) + L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_0(v)dv\right]\right]\\
&\quad + L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_0(v)dv\right]\right],\\
\phi_{n+1}(u) &= L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_a^b k(u,v)A_n(v)dv\right]\right]\\
&\quad + L^{-1}\left[\frac{\lambda}{(1+qs)}L\left[\int_0^u \psi(u,v)B_n(v)dv\right]\right], \quad n = 0,1,2,\ldots
\end{aligned}$$

We then define the n-terms approximate to the solution $\phi(u)$ by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1}\phi_i(u), \quad with \quad \lim_{n\to\infty}\Psi_n[\phi(u)] = \phi(u).$$

In this paper, the obtained series solution converges to the exact solution.

3.1 A Test of Convergence

In fact, on every interval the inequality $\|\phi_{i+1}\|_2 < \beta\|\phi_i\|_2$ is required to hold for $i = 0,1,\ldots,n$, wherever $0 < \beta < 1$ may be a constant and n is that the maximum order of the approximate used in the computation. Of course, this is often only a necessary condition for convergence, as a result of it might be necessary to compute $\|\phi_i\|_2$ for each $i = 0,1,\ldots,n$ so as to conclude that the series is convergent.

4. Application of the Laplace transform–Adomian decomposition method

In this section, the Laplace transform–Adomian decomposition method for solving Fredholm–Volterra integro–differential equation is illustrated in the two examples given below. To show the high accuracy of the solution results from applying the present method to our problem (2) compared with the exact solution, the maximum error is defined as:

$$R_n = \|\phi_{Exact}(u) - \Psi_n[\phi(u)]\|_\infty,$$

where $n = 1,2,\ldots$ represents the number of iterations.

Example 1

Consider the nonlinear Fredholm–Volterra integro–differential equation

$$\phi(u+0.2) = p(u) + \frac{1}{4}\int_0^1 cos(u)\phi^2(v)dv + \int_0^u \phi^3(v)dv, \tag{24}$$

where

$$p(u) = \frac{1}{12}(-3 - u^3 cos(u)).$$

Using Taylor Expansion after neglecting the second derivative in the equation (24) we get,

$$\phi(u) + 0.2\frac{d\phi(u)}{du} = p(u) + \frac{1}{4}\int_0^1 cos(u)\phi^2(v)dv + \int_0^u \phi^3(v)dv; \quad \phi(0) = 0. \tag{25}$$

The exact solution for this problem is

$$\phi(u) = cos(u) - sin(u).$$

First, we apply the Laplace transform to both sides of (25)

$$L[\phi(u)] + 0.2L\left[\frac{d\phi(u)}{du}\right] = L[p(u)] + \frac{1}{4}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + L\left[\int_0^u \phi^3(v)dv\right], \tag{26}$$

Using the property of Laplace transform and the initial conditions, we get

$$L[\phi(u)] + 0.2sL[\phi(u)] = L[p(u)] + \frac{1}{4}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + L\left[\int_0^u \phi^3(v)dv\right], \tag{27}$$

or equivalently

$$L[\phi(u)] = \frac{L[p(u)]}{1+0.2s} + \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)\phi^2(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u \phi^3(v)dv\right]. \tag{28}$$

Substituting the series assumption for $\phi(u)$ and the Adomian polynomials for $\phi^2(u), \phi^3(u)$ as given above in (15) and (16) respectively into Eq. (28) we obtain

$$\begin{aligned} L\left[\sum_{n=0}^{\infty}\phi_n(u)\right] = \frac{L[p(u)]}{1+0.2s} &+ \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)\sum_{n=0}^{\infty}A_n(v)dv\right] \\ &+ \frac{1}{1+0.2s}L\left[\int_0^u \sum_{n=0}^{\infty}B_n(v)dv\right]. \end{aligned} \tag{29}$$

The recursive relation is given below

$$\begin{aligned} L[\phi_0(u)] &= \frac{L[p(u)]}{1+0.2s}, \\ L[\phi_1(u)] &= \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)A_0(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u B_0(v)dv\right], \\ L[\phi_{n+1}(u)] &= \frac{1}{4+0.8s}L\left[\int_0^1 cos(u)A_n(v)dv\right] + \frac{1}{1+0.2s}L\left[\int_0^u B_n(v)dv\right], \end{aligned} \tag{30}$$

where A_n, B_n are the Adomian polynomials for the nonlinear terms $\phi^2(u), \phi^3(u)$ respectively. The Adomian polynomials for $\mu(v,\phi(v)) = \phi^2(u), \nu(v,\phi(v)) = \phi^3(u)$ are given by

$$\begin{aligned}
A_0 &= \phi_0^2, & B_0 &= \phi_0^3,\\
A_1 &= 2\phi_0\phi_1, & B_1 &= 3\phi_0^2\phi_1,\\
A_2 &= 2\phi_0\phi_2 + \phi_1^2, & B_2 &= 3\phi_0^2\phi_2 + 3\phi_0\phi_1^2,\\
A_3 &= 2\phi_0\phi_3 + 2\phi_1\phi_2, & B_3 &= 3\phi_0^2\phi_3 + 6\phi_0\phi_1\phi_2 + \phi_1^3.
\end{aligned}$$

Taking the inverse Laplace transform of both sides of the first part of (30), and using the recursive relation (30) gives

$$\begin{aligned}
\phi_0(u) &= 1 - u - u^2 + \frac{1}{2}u^3 + \frac{1}{12}u^4 - \dots\\
\phi_1(u) &= \frac{1}{2}u^2 - \frac{1}{3}u^3 - \frac{1}{8}u^4 + \frac{1}{6}u^5 + \dots\\
\phi_2(u) &= \frac{1}{12}u^4 - \frac{1}{12}u^5 + \dots
\end{aligned} \tag{31}$$

Thus the series solution is given by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1}\phi_i(u) = \left(1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \dots\right) - \left(u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \dots\right) \quad n = 1, 2, \dots$$

$$\phi(u) = \lim_{n\to\infty}\Psi_n[\phi(u)] = \lim_{n\to\infty}\left[\left(1 - \frac{1}{2!}u^2 + \frac{1}{4!}u^4 + \dots\right) - \left(u - \frac{1}{3!}u^3 + \frac{1}{5!}u^5 + \dots\right)\right],$$

that converges to the exact solution

$$\phi(u) = cos(u) - sin(u).$$

Example 2

Consider the nonlinear Fredholm–Volterra integro–differential equation

$$\phi(u + 0.01) = p(u) + \int_0^1 \phi(v)dv + \int_0^u e^{-u}\phi^2(v)dv, \tag{32}$$

where

$$p(u) = 1 - \frac{1}{4}e^{-u} + 0.0100502e^u.$$

Using Taylor Expansion after neglecting the second derivative in the equation (32) we get,

$$\phi(u) + 0.01\frac{d\phi(u)}{du} = p(u) + \int_0^1 \phi(v)dv + \int_0^u e^{-u}\phi^2(v)dv; \quad \phi(0) = 1. \tag{33}$$

The exact solution for this problem is

$$\phi(u) = e^u.$$

First, we apply the Laplace transform to both sides of (33)

$$L[\phi(u)] + 0.01L\left[\frac{d\phi(u)}{du}\right] = L[p(u)] + L\left[\int_0^1 \phi(v)dv\right] + L\left[\int_0^u e^{-u}\phi^2(v)dv\right], \tag{34}$$

using the property of Laplace transform and the initial conditions, we get

$$L[\phi(u)] + 0.01sL[\phi(u)] - 0.01 = L[p(u)] + L\left[\int_0^1 \phi(v)dv\right] + L\left[\int_0^u e^{-u}\phi^2(v)dv\right], \tag{35}$$

or equivalently

$$\begin{aligned} L[\phi(u)] = \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s} + \frac{1}{1+0.01s}L\left[\int_0^1 \phi(v)dv\right] \\ + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}\phi^2(v)dv\right]. \end{aligned} \tag{36}$$

Substituting the series assumption for $\phi(u)$ and the Adomian polynomials for $\phi^2(u)$ as given above in (15) and (16) respectively into above equation, we obtain

$$\begin{aligned} L\left[\sum_{n=0}^{\infty}\phi_n(u)\right] = \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s} + \frac{1}{1+0.01s}L\left[\int_0^1 \sum_{n=0}^{\infty}\phi_n(v)dv\right] \\ + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}\sum_{n=0}^{\infty}A_n(v)dv\right], \end{aligned} \tag{37}$$

the recursive relation is given below

$$\begin{aligned} L[\phi_0(u)] &= \frac{0.01}{1+0.01s} + \frac{L[p(u)]}{1+0.01s}, \\ L[\phi_1(u)] &= \frac{1}{1+0.01s}L\left[\int_0^1 \phi_0(v)dv\right] + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}A_0(v)dv\right], \\ L[\phi_{n+1}(u)] &= \frac{1}{1+0.01s}L\left[\int_0^1 \phi_n(v)dv\right] + \frac{1}{1+0.01s}L\left[\int_0^u e^{-u}A_n(v)dv\right], \end{aligned} \tag{38}$$

where A_n are the Adomian polynomials for the nonlinear terms $\phi^2(u)$. The Adomian polynomials for $\mu(v, \phi(v)) = \phi^2(u)$ is given by

$$\begin{aligned} A_0 &= \phi_0^2, \\ A_1 &= 2\phi_0\phi_1, \\ A_2 &= 2\phi_0\phi_2 + \phi_1^2, \\ A_3 &= 2\phi_0\phi_3 + 2\phi_1\phi_2, \end{aligned}$$

Taking the inverse Laplace transform of both sides of the first part of (38), and using the recursive relation (38) gives

$$\begin{aligned}\phi_0(u) &= 1+u-\frac{1}{2}u^3-\frac{1}{2}u^4-\frac{13}{40}u^5+\dots\\ \phi_1(u) &= \frac{1}{2}u^2+\frac{2}{3}u^3+\frac{5}{12}u^4+\frac{7}{120}u^5+\dots\\ \phi_2(u) &= \frac{1}{8}u^4+\frac{11}{40}u^5+\dots\end{aligned}\tag{39}$$

Thus the series solution is given by

$$\Psi_n[\phi(u)] = \sum_{i=0}^{n-1}\phi_i(u) = \left(1+u+\frac{1}{2!}u^2+\frac{1}{3!}u^3+\frac{1}{4!}u^4+\frac{1}{5!}u^5+\dots\right) \quad n=1,2,\dots$$

$$\phi(u) = \lim_{n\to\infty}\Psi_n[\phi(u)] = \lim_{n\to\infty}\left[\left(1+u+\frac{1}{2!}u^2+\frac{1}{3!}u^3+\frac{1}{4!}u^4+\frac{1}{5!}u^5+\dots\right)\right],$$

that converges to the exact solution

$$\phi(u) = e^u.$$

5. Conclusions

In this work, the Laplace decomposition technique has been successfully applied to finding the approximate solution of the nonlinear Fredholm–Volterra integro–differential equation. The method is extremely powerful and efficient find analytical moreover as numerical solutions for wide classes of nonlinear Fredholm–Volterra integro–differential equations. It provides a lot of realistic series solutions that converge very rapidly in real physical issues.

The main advantage of this technique is that the fact that it provides the analytical solution. Some examples are given and therefore the results reveal that the method is extremely effective. some of the nonlinear equations are examined by the modified technique to Illustrate the effectiveness and convenience of this technique, and in all cases, the modified technique performed excellently. The results reveal that the proposed technique is extremely effective and easy.

Acknowledgements

The authors are very grateful to Prof. Dr. M. A. Abdou, (Dep. of Maths. Faculty of Education, Alexandria University) for their help and suggestions in the process of numerical calculation. We are also very grateful to the reviewers for their constructive suggestions towards upgrading the quality of the manuscript.

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Representation of the Matrix for Conversion between Triangular Bézier Patches and Rectangular Bézier Patches

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Abstract

In this paper we studied Bézier surfaces that are very famous techniques and widely used in Computer Aided Geometric Design. Mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches. In this paper we will give a representation for the conversion matrix which converts one type to another.

1 Introduction

The theory of Bézier curves has an important role and they are numerically the most stable among all polynomial bases currently used in CAD systems. On the other hand in these days Bézier surfaces are very famous techniques and widely used in Computer Aided Geometric Design [1]-[13]. Mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches and they are defined in terms of the univariate Bernstein polynomials $B_i^n(s) = \binom{n}{i} s^i (1-s)^{n-i}$ and the bivariate Bernstein polynomial $B_{i,j,k}^n(u,v,w) = \binom{n}{i,j,k} u^i v^j w^k$ where $u+v+w=1$. A triangular Bézier patch of degree n with control points $T_{i,j,k}$ is defined by

$$T\left(u,v,w\right) = \sum_{i+j+k=n} T_{i,j,k} B_{i,j,k}^n(u,v,w),\ \ u,v,w \geq 0,\ \ u+v+w=1.$$

and a rectangular Bézier patch of degree $n \times m$ with control points $P_{i,j}$ is represented by

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{ij} B_i^n(s) B_j^n(t) \quad 0 \leq s,t \leq 1, (\text{see [3]})$$

Since the two patches have different geometric properties it is not easy to use both of them in the same CAD system and conversion of one type to another is needed.

2 Construction of the Conversion Matrices

The following theorem gives the conversion of degree n triangular Bézier patch to degenerate rectangular Bézier patch of degree $n \times n$.

Definition 1 *For all nonnegative integers x the falling factorial is defined by*

$$(x)_n = x(x-1)...(x-n+1) = \prod_{k=1}^{n} (x-(k-1))$$

Theorem 2 *A degree n triangular Bézier patch $T(u,v,w)$ can be represented as a degenerate Bézier patch of degree $n \times n$:*

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n} P_{ij} B_i^n(s) B_j^n(t), \qquad 0 \le s,t \le 1$$

where the control points P_{ij} are determined by

$$\begin{pmatrix} P_{i0} \\ P_{i1} \\ \vdots \\ P_{in} \end{pmatrix} = A_1 A_2 ... A_i \begin{pmatrix} T_{i0} \\ T_{i1} \\ \vdots \\ T_{i,n-i} \end{pmatrix}, \qquad i = 0,1,2,...,n.$$

and $A_i (i = 0,1,...,n)$ are degree elevation operators in the form

$$A_k = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{n+1-k} & \frac{n-k}{n+1-k} & 0 & \dots & 0 & 0 \\ 0 & \frac{2}{n+1-k} & \frac{n-k-1}{n+1-k} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \frac{n-k}{n+1-k} & \frac{1}{n+1-k} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(n-k+2)\times(n-k+1)}$$

Until now no one has studied the generalization of the product $A_1 A_2 ... A_k$ mentioned in the above theorem and indeed the product of these matrices is not easy to calculate for different values of n and k. Here we will give the generalization of this product which will make all the computations easier.

Theorem 3 *The following formula is true*

$$A_1 A_2 ... A_k = \bar{A}_k = \left[\bar{a}_{i,j}^{(k)}\right]_{(n+1)\times(n-k+1)},$$

where

$$\bar{a}_{i,j}^{(k)} = \frac{\binom{i-1}{j-1} (k)_{i-j} (n-k)_{j-1}}{(n)_{i-1}},$$

$$(k)_n = k(k-1)...(k-n+1) = \prod_{j=1}^{n} (k-(j-1)) \text{ and}$$

$$A_k = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{n+1-k} & \frac{n-k}{n+1-k} & 0 & \dots & 0 & 0 \\ 0 & \frac{2}{n+1-k} & \frac{n-k-1}{n+1-k} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \frac{n-k}{n+1-k} & \frac{1}{n+1-k} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{(n-k+2)\times(n-k+1)}.$$

Proof. For $k = 1$,

$$\bar{A}_1 = A_1.$$

Suppose it is true for k, that is

$$A_1 A_2 ... A_k = \bar{A}_k.$$

We will show that it also true for $k+1$, i.e

$$\bar{A}_k A_{k+1} = \bar{A}_{k+1}.$$

Let $c_{i,j}$ be the element at the i^{th} row, j^{th} column of the matrix $\bar{A}_k A_{k+1}$:

$$\begin{aligned} c_{i,j} &= \sum_{m=1}^{n-k+1} \bar{a}_{i,m}^{(k)} a_{m,j}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{i-1}{m-1} (k)_{i-m} (n-k)_{m-1}}{(n)_{i-1}} a_{m,j}^{(k+1)}, \end{aligned}$$

where $i = \{1, 2, ..., n+1\}$ and $j = \{1, 2, ..., n-k\}$.

For $j = 1$ (first column)

$$\begin{aligned} c_{i,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{i,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{i-1}{m-1} (k)_{i-m} (n-k)_{m-1}}{(n)_{i-1}} a_{m,1}^{(k+1)}. \end{aligned}$$

For $i = 1$ and $j = 1$

$$\begin{aligned} c_{1,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{1,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1} (k)_{1-m} (n-k)_{m-1}}{(n)_0} a_{m,1}^{(k+1)} \\ &= a_{1,1}^{(k+1)} = 1 = \bar{a}_{1,1}^{(k+1)}. \end{aligned}$$

For $i = 2$ and $j = 1$,

$$\begin{aligned} c_{2,1} &= \sum_{m=1}^{n-k+1} \bar{a}_{2,m}^{(k)} a_{m,1}^{(k+1)} \\ &= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1} (k)_{2-m} (n-k)_{m-1}}{(n)_1} a_{m,1}^{(k+1)} \\ &= \frac{k+1}{n} = \frac{(k+1)_1}{(n)_1}. \end{aligned}$$

For $i = n+1$ and $j = 1$,

$$\begin{aligned} c_{n+1,1} &= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1} (k)_{n+1-m} (n-k)_{m-1}}{(n)_n} a_{m,1}^{(k+1)} \\ &= \frac{(k+1)k(k-1)...(k-n+2)}{n(n-1)(n-2)...1} \\ &= \frac{(k+1)_n}{(n)_n}. \end{aligned}$$

For $j = 2$ (second solumn), for $i = 1$ and $j = 2$

$$\begin{aligned} c_{1,2} &= \sum_{m=1}^{n-k+1} \bar{a}_{1,m}^{(k)} a_{m,2}^{(k+1)} \\ c_{1,2} &= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1} (k)_{1-m} (n-k)_0}{(n)_0} a_{m,2}^{(k+1)} \\ c_{1,2} &= a_{1,2}^{(k+1)} = 0 = \bar{a}_{1,2}^{(k+1)}. \end{aligned}$$

For $i=2$ and $j=2$

$$\begin{aligned}
c_{2,2} &= \sum_{m=1}^{n-k+1} \bar{a}_{i,m}^{(k)} a_{m,2}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1}(k)_{2-m}(n-k)_{m-1}}{(n)_1} a_{m,2}^{(k+1)} \\
&= \frac{n-k-1}{n} = \bar{a}_{2,2}^{(k+1)}.
\end{aligned}$$

For $i=n+1$ and $j=2$

$$\begin{aligned}
c_{n+1,2} &= \sum_{m=1}^{n-k+1} \bar{a}_{n+1,m}^{(k)} a_{m,n+1}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1}(k)_{n+1-m}(n-k)_{m-1}}{(n)_n} a_{m,2}^{(k+1)} \\
&= \frac{n(k+1)k(k-1)...(k-n+3)(n-k-1)}{n(n-1)(n-2)...1} \\
&= \frac{n(k+1)_{n-1}(n-k-1)_1}{(n)_n} = \bar{a}_{n+1,2}^{(k+1)}.
\end{aligned}$$

For $j=n-k$ (last column), for $i=1$ and $j=n-k$

$$\begin{aligned}
c_{1,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}_{1,m}^{(k)} a_{m,n-k}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{0}{m-1}(k)_{1-m}(n-k)_{m-1}}{(n)_0} a_{m,n-k}^{(k+1)} \\
&= a_{1,n-k}^{(k+1)} = 0 = \bar{a}_{1,n-k}^{(k+1)}.
\end{aligned}$$

For $i=2$ and $j=n-k$

$$\begin{aligned}
c_{2,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}_{2,m}^{(k)} a_{m,j}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{1}{m-1}(k)_{2-m}(n-k)_{m-1}}{(n)_1} a_{m,n-k}^{(k+1)} \\
&= 0 = \bar{a}_{2,n-k}^{(k+1)}.
\end{aligned}$$

For $i=n+1$ and $j=n-k$

$$\begin{aligned}
c_{n+1,n-k} &= \sum_{m=1}^{n-k+1} \bar{a}_{n+1,m}^{(k)} a_{m,n-k}^{(k+1)} \\
&= \sum_{m=1}^{n-k+1} \frac{\binom{n}{m-1}(k)_{n+1-m}(n-k)_{m-1}}{(n)_n} a_{m,n-k}^{(k+1)} \\
&= 1 = \bar{a}_{n+1,n-k}^{(k+1)}.
\end{aligned}$$

Hence, $\bar{A}_k A_{k+1} = [c_{i,j}]_{(n+1)\times(n-k)} = \left[\bar{a}_{i,j}^{(k+1)}\right]_{(n+1)\times(n-k)}$, where $\bar{a}_{i,j}^{(k+1)} = \frac{\binom{i-1}{j-1}(k+1)_{i-j}(n-k-1)_{j-1}}{(n)_{i-1}}$. ■

Remark 4 *Sum of the elements in each row of the matrix $\bar{A}_k$ is equal to* 1.

Now in the following theorem we consider the inverse process

Theorem 5 *A rectangular Bézier patch $P(s,t)$ of degree $n \times n$ can be represented as a Triangular Bézier patch $T(u,v,w)$ of degree n :*

$$T(u,v,w) = \sum_{i+j+k=n} T_{i,j,k}\ B^n_{i,j,k}(u,v,w), \qquad u,v,w \geq 0, u+v+w=1.$$

where the control points $T_{i,j,k}$ are determined by

$$\begin{pmatrix} T_{i0} \\ T_{i1} \\ \vdots \\ T_{i,n-i} \end{pmatrix} = B_i B_{i-1} ... B_1 \begin{pmatrix} P_{i0} \\ P_{i1} \\ \vdots \\ P_{in} \end{pmatrix} \qquad i = 0,1,2,...,n.$$

and $B_i (i = 0,1,...,n)$ are degree elevation operators in the form

$$B_k = \begin{bmatrix} 1-t & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-t & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1-t & t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1-t & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-t & t \end{bmatrix}_{(n-k+1)\times(n-k+2)}$$

Proof. Indeed

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j} B_i^n(s) B_j^n(t) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n} P_{i,j} B_i^n(s) \left\{ t B_{j-1}^{n-1}(t) + (1-t) B_j^{n-1}(t) \right\} \\ &= \sum_{i=0}^{n} B_i^n(s) \left\{ t \sum_{j=0}^{n} P_{i,j} B_{j-1}^{n-1}(t) + (1-t) \sum_{j=0}^{n} P_{i,j} B_j^{n-1}(t) \right\} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-r} P_{i,j}^r B_i^n(s) B_j^{n-r}(t), \end{aligned}$$

where

$$\begin{aligned} P^0_{i,j}(t) &\equiv P^0_{i,j} = P_{i,j} \\ P^r_{i,j}(t) &= t P^{r-1}_{i,j+1} + (1-t) P^{r-1}_{i,j}. \end{aligned}$$

Let $r = i$,

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^i B_i^n(s) B_j^{n-i}(t) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^i \binom{n}{i} \binom{n-i}{j} s^i (1-s)^{n-i} t^j (1-t)^{n-i-j} \end{aligned}$$

if we use the following reparametrization

$$\left\{\begin{array}{c} s = u \\ t = \frac{v}{1-u} = \frac{v}{v+w} \end{array}\right.$$

we get

$$P(s,t) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i}\binom{n}{i}\binom{n-i}{j}u^{i}(1-u)^{n-i}\left(\frac{v}{v+w}\right)^{j}(1-\frac{v}{v+w})^{n-i-j}.$$

Now if $i+j+k=n$

$$\begin{aligned} P(s,t) &= \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i}\binom{n}{i}\binom{n-i}{j}u^{i}(1-u)^{n-i}\left(\frac{v}{v+w}\right)^{j}(1-\frac{v}{v+w})^{k} \\ &= \sum_{i+j+k=n} T_{i,j,k}B_{i,j,k}^{n}(u,v,w) \end{aligned}$$

$$T_{i,j,k} = \sum_{i=0}^{n}\sum_{j=0}^{n-i} P_{i,j}^{i}(t).$$

For each value of i, we obtain $(n-i+1) \times (n-i+2)$ matrix B_i. ■

Theorem 6 *The product of the matrices in the above theorem $B_k B_{k-1} ... B_1$ can be generalized as follows*

$$Z^{k} = B_k B_{k-1} ... B_1 = \begin{bmatrix} b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & 0 & \cdots & 0 \\ 0 & 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{k,0} & b_{k,1} & b_{k,2} & \cdots & b_{k,k-1} & b_{k,k} \end{bmatrix}$$

where $b_{k,j} = \binom{k}{j}t^{j}(1-t)^{k-j}$ and Z^{k} is $(n-k+1) \times (n+1)$ matrix.

Proof. For $k=1$,

$$Z^{1} = B_1,$$

suppose it is true for k, that is

$$B_k B_{k-1} ... B_1 = Z^{k},$$

we will show that it also true for $k+1$, i.e

$$B_{k+1} B_k B_{k-1} ... B_1 = Z^{k+1}.$$

Let $z_{i,j}$ be the element at the i^{th} row, j^{th} column of the matrix $B_{k+1} B_k B_{k-1} ... B_1$,

$$z_{i,j} = \sum_{m=1}^{n-k+1} b_{i,m}^{(k+1)} b_{m,j}^{(k)}$$

where $b_{i,m}^{(k+1)}$ is the element at the i^{th} row, m^{th} column of the matrix B_{k+1}, $b_{m,j}^{(k)}$ is the element at the m^{th} row, j^{th} column of the matrix $B_k B_{k-1} ... B_1$, $i = \{1, 2, ..., n-k\}$ and $j = \{1, 2, ..., n+1\}$.

For $i = 1$(first row)

$$z_{1,j} = \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,j}^{(k)}.$$

For $i = 1$ and $j = 1$

$$\begin{aligned} z_{1,1} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,1}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,1}^{(k)} + b_{1,2}^{(k+1)} b_{2,1}^{(k)} + \ldots + b_{1,n-k+1}^{(k+1)} b_{n-k+1,1}^{(k)} \\ &= (1-t)\, b_{k,0} = (1-t)^{k+1} = Z_{(1,1)}^{k+1}. \end{aligned}$$

For $i = 1$ and $j = 2$,

$$\begin{aligned} z_{1,2} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,2}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,2}^{(k)} + b_{1,2}^{(k+1)} b_{2,2}^{(k)} + \ldots + b_{1,n-k+1}^{(k+1)} b_{n-k+1,2}^{(k)} \\ &= (1-t)\, b_{k,1} + t b_{k,0} \\ &= b_{k+1,1} = Z_{(1,2)}^{k+1}. \end{aligned}$$

For $i = 1$ and $j = n+1$,

$$\begin{aligned} z_{1,n+1} &= \sum_{m=1}^{n-k+1} b_{1,m}^{(k+1)} b_{m,n+1}^{(k)} \\ &= b_{1,1}^{(k+1)} b_{1,n+1}^{(k)} + b_{1,2}^{(k+1)} b_{2,n+1}^{(k)} + \ldots + b_{1,n-k+1}^{(k+1)} b_{n-k+1,n+1}^{(k)} \\ &= 0 = Z_{(1,n+1)}^{k+1}. \end{aligned}$$

For $i = 2$ (second row)

$$z_{2,j} = \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,j}^{(k)}.$$

For $i = 2$ and $j = 1$,

$$\begin{aligned} z_{2,1} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,1}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,1}^{(k)} + b_{2,2}^{(k+1)} b_{2,1}^{(k)} + \ldots + b_{2,n-k+1}^{(k+1)} b_{n-k+1,1}^{(k)} \\ &= 0 = Z_{(2,1)}^{k+1}. \end{aligned}$$

For $i = 2$ and $j = 2$

$$\begin{aligned} z_{2,2} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,2}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,2}^{(k)} + b_{2,2}^{(k+1)} b_{2,2}^{(k)} + \ldots + b_{2,n-k+1}^{(k+1)} b_{n-k+1,2}^{(k)} \\ &= (1-t) b_{k,0} \\ &= (1-t)^{k+1} = Z_{2,2}^{k+1}. \end{aligned}$$

For $i = 2$ and $j = n+1$,

$$\begin{aligned} z_{2,n+1} &= \sum_{m=1}^{n-k+1} b_{2,m}^{(k+1)} b_{m,n+1}^{(k)} \\ &= b_{2,1}^{(k+1)} b_{1,n+1}^{(k)} + b_{2,2}^{(k+1)} b_{2,n+1}^{(k)} + \ldots + b_{2,n-k+1}^{(k+1)} b_{n-k+1,n+1}^{(k)} \\ &= t b_{k,k} = Z_{2,n+1}^{k+1}. \end{aligned}$$

For $i = n - k + 1$

$$z_{n-k+1,j} = \sum_{m=1}^{n-k+1} b_{n-k+1,m}^{(k+1)} b_{m,j}^{(k)}.$$

For $i = n - k$ and $j = 1$

$$\begin{aligned} z_{n-k,1} &= \sum_{m=1}^{n-k+1} b_{n-k,m}^{(k+1)} b_{m,1}^{(k)} \\ &= b_{n-k,1}^{(k+1)} b_{1,1}^{(k)} + b_{n-k,2}^{(k+1)} b_{2,1}^{(k)} + \ldots + b_{n-k,n-k+1}^{(k+1)} b_{n-k+1,1}^{(k)} \\ &= 0 = Z_{n-k,1}^{k+1}. \end{aligned}$$

For $i = n - k$ and $j = 2$

$$\begin{aligned} z_{n-k,2} &= \sum_{m=1}^{n-k+1} b_{n-k,m}^{(k+1)} b_{m,2}^{(k)} \\ &= b_{n-k,1}^{(k+1)} b_{1,2}^{(k)} + b_{n-k,2}^{(k+1)} b_{2,2}^{(k)} + \ldots + b_{n-k,n-k+1}^{(k+1)} b_{n-k+1,2}^{(k)} \\ &= 0 = Z_{n-k,2}^{k+1}. \end{aligned}$$

For $i = n - k$ and $j = n + 1$

$$\begin{aligned} z_{n-k,n+1} &= \sum_{m=1}^{n-k+1} b_{n-k,m}^{(k+1)} b_{m,n+1}^{(k)} \\ &= b_{n-k,1}^{(k+1)} b_{1,n+1}^{(k)} + b_{n-k,2}^{(k+1)} b_{2,n+1}^{(k)} + \ldots + b_{n-k,n-k+1}^{(k+1)} b_{n-k+1,n+1}^{(k)} \\ &= tb_{k,k} = Z_{n-k,n+1}^{k+1}. \end{aligned}$$

$$\bar{A}_k A_{k+1} = \bar{A}_{k+1} = \left[\begin{array}{cc} 1 & 0 \\ \frac{k+1}{n} & \frac{n-k-1}{n} \\ \frac{(k+1)k}{n(n-1)} & \frac{2(k+1)(n-k-1)}{n(n-1)} \\ \frac{(k+1)(k)(k-1)}{n(n-1)(n-2)} & \frac{3(k+1)k(n-k-1)}{n(n-1)(n-2)} \\ \frac{(k+1)k(k-1)(k-2)}{n(n-1)(n-2)(n-3)} & \frac{4(k+1)k(k-1)(n-k-1)}{n(n-1)(n-2)(n-3)} \\ \frac{(k+1)k(k-1)(k-2)(k-3)}{n(n-1)(n-2)(n-3)} & \frac{5(k+1)k(k-1)(k-2)(n-k-1)}{n(n-1)(n-2)(n-3)(n-4)} \\ \vdots & \vdots \\ \frac{(k+1)k\ldots(k-n+3)}{n(n-1)(n-2)\ldots2} & \frac{(n-1)(k+1)k\ldots(k-n+4)(n-k-1)}{n(n-1)(n-2)\ldots2} \\ \frac{(k+1)k\ldots(k-n+2)}{n(n-1)(n-2)\ldots1} & \frac{n(k+1)k\ldots(k-n+3)(n-k-1)}{n(n-1)(n-2)\ldots1} \end{array} \right.$$

$$\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \frac{(n-k-1)(n-k-2)}{n(n-1)} & 0 \\ \frac{3(k+1)(n-k-1)(n-k-2)}{n(n-1)(n-2)} & \frac{(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)} \\ \frac{6(k+1)k(n-k-1)(n-k-2)}{n(n-1)(n-2)(n-3)} & \frac{4(k+1)(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)(n-3)} \\ \frac{10(k+1)k(k-1)(n-k-1)(n-k-2)}{n(n-1)(n-2)(n-3)(n-4)} & \frac{10(k+1)k(n-k-1)(n-k-2)(n-k-3)}{n(n-1)(n-2)(n-3)(n-4)} \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \end{array}$$

$$\left.\begin{matrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 \\
\frac{(n-k-1)(n-k-2)(n-k-3)(n-k-4)}{n(n-1)(n-2)(n-3)} & 0 & 0 & \ldots & 0 & 0 \\
\frac{5(k+1)(n-k-1)(n-k-2)(n-k-3)(n-k-4)}{n(n-1)(n-2)(n-3)(n-4)} & \frac{(n-k-1)(n-k-2)(n-k-3)(n-k-4)(n-k-5)}{n(n-1)(n-2)(n-3)(n-4)} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 & 0 \\
\vdots & \vdots & & & \frac{n-k-1}{n} & \frac{k+1}{n} \\
\vdots & \vdots & & & 0 & 1
\end{matrix}\right] \blacksquare$$

Conclusion 7 *As we mentioned before, mainly there are two types of Bézier surfaces which are rectangular and triangular Bézier patches. These two types of patches have different geometric properties so it is difficult to use both of them in the same CAD system. One may need to convert one type to another and here in this paper we studied on the conversion matrix to convert triangular Bézier patch to a rectangular Bézier patch and a rectangular Bézier patch to a triangular Bézier patch. We found simple representations for these two matrices which will allow the conversion in one step.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Permeable values with applications in BE-algebras

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Abstract. The notions of energetic subsets and (anti) permeable values are introduced, and related properties are investigated. These notions are applied to the theory of BE-algebras. Regarding (anti) fuzzy subalgebras/filters and energetic subsets are investigated.

1. INTRODUCTION

As a generalization of a BCK-algebra, the notion of BE-algebras has been introduced by H. S. Kim and Y. H. Kim in [5]. The study of BE-algebras has been continued in papers [1], [2], and [6]. Jun et al. [3] introduced the notions of S-energetic subsets and I-energetic subsets in BCK/BCI-algebras, and investigated several properties. Jun et.al [4] defined the notions of a C-energetic subset and (anti) permeable C-value in BCK-algebras and studied some related properties of them.

In this paper, we introduce the notions of energetic subsets and (anti) permeable values, and investigate some related properties. These notions are applied to the theory of BE-algebras. Regarding (anti) fuzzy subalgebras/filters and energetic subsets are investigated.

2. PRELIMINARIES

We display basic notions on BE-algebras. We refer the reader to the papers [2, 5] for further information regarding BE-algebras.

By a *BE-algebra* [5] we mean a system $(X;*,1)$ of type $(2,0)$ which the following axioms hold:
(BE1) $(\forall x\in X)\,(x*x=1)$,
(BE2) $(\forall x\in X)\,(x*1=1)$,
(BE3) $(\forall x\in X)\,(1*x=x)$,
(BE4) $(\forall x,y,z\in X)\,(x*(y*z)=y*(x*z))$ (exchange).

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[0] **2010 Mathematics Subject Classification**: 06F35; 03G25; 08A72.

[0] **Keywords**: S-energetic subset, F-energetic subset, (anti) fuzzy filter ideal, (anti) permeable S-value, (anti) permeable F-value.

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We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

A BE-algebra $(X; *, 1)$ is said to be *transitive* if it satisfies: for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$. A BE-algebra $(X; *, 1)$ is said to be *self distributive* if it satisfies: for any $x, y, z \in X$, $x * (y * z) = (x * y) * (x * z)$. Note that every self distributive BE-algebra is transitive, but the converse is not true in general [5].

Every self distributive BE-algebra $(X; *, 1)$ satisfies the following properties:

(2.1) $(\forall x, y, z \in X)\,(x \leq y \Rightarrow z * x \leq z * y$ and $y * z \leq x * z)$,
(2.2) $(\forall x, y \in X)\,(x * (x * y) = x * y)$,
(2.3) $(\forall x, y, z \in X)\,(x * y \leq (z * x) * (z * y))$.

Definition 2.1. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is a *filter* [5] of X if

(i) $1 \in F$;
(ii) $(\forall x, y \in X)(x * y, x \in F \Rightarrow y \in F)$.

The concept of fuzzy sets was introduced by Zadeh [7]. Let X be a set. The mapping $f : X \to [0, 1]$ is called a *fuzzy set* in X. A fuzzy set f in a BE-algebra X is called a *fuzzy subalgebra* of X if it satisfies

(F_0) $(\forall x, y \in X)(f(x * y) \geq \min\{f(x), f(y)\})$.

A fuzzy set f in a BE-algebra X is called a fuzzy filter of X if it satisfies

(F_1) $(\forall x \in X)(f(1) \geq f(x))$;
(F_2) $(\forall x, y \in X)(f(y) \geq \min\{f(x * y), f(x)\})$.

Note that every fuzzy filter f of a BE-algebra X satisfies

$$(\forall x, y \in X)(x \leq y \Rightarrow f(y) \geq f(x)).$$

For a fuzzy set f in X and $t \in [0, 1]$, the (strong) upper (resp. lower) t-level sets are defined as follows:

$$U(f; t) := \{x \in X | f(x) \geq t\}, \quad U^*(f; t) := \{x \in X | f(x) > t\},$$
$$L(f; t) := \{x \in X | f(x) \leq t\}, \quad L^*(f; t) := \{x \in X | f(x) < t\}.$$

3. Energetic subsets

In what follows, let X denote a BE-algebra unless otherwise specified.

Definition 3.1. A nonempty subset A of a BE-algebra X is said to be *S-energetic* if it satisfies

(S) $(\forall a, b \in X)(a * b \in A \Rightarrow \{a, b\} \cap A \neq \emptyset)$.

Definition 3.2. A nonempty subset A of a BE-algebra X is said to be *F-energetic* if it satisfies

(F) $(\forall x, y \in X)(y \in A \Rightarrow \{x * y, x\} \cap A \neq \emptyset)$.

Example 3.3. (1) Let $X := \{1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

It is easy to show that $A := \{b, c\}$ is a S-energetic subset of X. But $B := \{a\}$ is not an S-energetic subset of X since $c * b = a \in B$ and $\{c, b\} \cap B = \emptyset$. It is routine to verify that $C := \{c\}$ is an S-energetic subset of X. But it is not an F-energetic subset of X, since $c \in C$ and $\{b * c, b\} \cap C = \emptyset$.

(2) Let $X := \{1, a, b, c\}$ be a BE-algebra with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

It is easy to show that $A := \{a, b\}$ is an F-energetic subset of X.

Theorem 3.4. *For any nonempty subset A of X, if A is a subalgebra of a BE-algebra X, then $X \setminus A$ is an S-energetic a subset of X.*

Proof. Let $a, b \in X$ be such that $a * b \in X \setminus A$. If $\{a, b\} \cap (X \setminus A) = \emptyset$, then $a, b \in A$ and so $a * b \in A$ since A is a subalgebra of X. This is a contradiction. Thus $\{a, b\} \cap (X \setminus A) \neq \emptyset$. Therefore $X \setminus A$ is an S-energetic subset of X. ☐

Theorem 3.5. *For any nonempty subset A of X, if A is a filter of a BE-algebra X, then $X \setminus A$ is an F-energetic a subset of X.*

Proof. Let $x, y \in X$ be such that $y \in X \setminus A$. If $\{x * y, x\} \cap X \setminus A = \emptyset$, then $x * y, x \in A$ and so $y \in A$, since A is a filter of X. This is a contradiction. Therefore $\{x * y, x\} \cap X \setminus A \neq \emptyset$. Thus $X \setminus A$ is an F-energetic subset of X. ☐

Theorem 3.6. *Let A be a nonempty subset of a BE-algebra X with $1 \notin A$. If A is F-energetic, then $X \setminus A$ is a filter of X.*

Proof. Obviously, $1 \in X \setminus A$. Let $x, y \in X$ be such that $x * y, x \in X \setminus A$. Assume that $y \in A$. Then $\{x * y, x\} \cap A \neq \emptyset$ by (F). Hence $x * y \in A$ or $x \in A$, which is a contradiction. Therefore $y \in X \setminus A$. This completes the proof. ☐

Theorem 3.7. *If f is a fuzzy filter of a BE-algebra X, then the nonempty lower t-level set $L(f; t)$ is an F-energetic subset of X for all $t \in [0, 1]$.*

Proof. Assume that $L(f;t) \neq \emptyset$ for $t \in [0,1]$ and let $x, y \in X$ be such that $y \in L(f;t)$. Then $t \geq f(y) \geq \min\{f(x*y), f(x)\}$. Hence $f(x*y) \leq t$ or $f(x) \leq t$, i.e., $x*y \in L(f;t)$ or $x \in L(f;t)$. Thus $\{x*y, x\} \cap L(f;t) \neq \emptyset$. Therefore $L(f;t)$ is an F-energetic subset of X. □

Corollary 3.8. *If f is a fuzzy filter of a BE-algebra X, then the nonempty stronger lower t-level set $L^*(f;t)$ is an F-energetic subset of X.*

Since $L(f;t) \cup U^*(f;t) = X$ and $L(f;t) \cap U^*(f;t) = \emptyset$ for all $t \in [0,1]$, we have the following corollary.

Corollary 3.9. *If f is a fuzzy filter of a BE-algebra X, then $U^*(f;t)$ is empty set or a filter of X for all $t \in [0,1]$.*

For any $a, b \in X$, we consider sets

$$X_a^b := \{x \in X | a * (b * x) = 1\} \text{ and } A_a^b := X \setminus X_a^b.$$

Obviously, $a, b \notin A_a^b, A_a^b = A_b^a$ and $1 \notin A_a^b$. In the following example, we know that there exist $a, b \in X$ such that A_a^b may not be F-energetic.

Example 3.10. Let $X := \{1, a, b, c, d, 0\}$ be a BE-algebra [2] with the following Cayley table

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $A_c^d = \{0, b\}$ and it is not F-energetic since $b \in A_c^d$ but $\{a*b, a\} \cap A_c^d = \emptyset$.

We consider conditions for the set A_a^b to be F-energetic.

Theorem 3.11. *If X is a self distributive BE-algebra X, then A_a^b is F-energetic for all $a, b \in X$.*

Proof. Let $y \in A_a^b$ for any $a, b, y \in X$. Assume that $\{x*y, x\} \cap A_a^b = \emptyset$ for any $x \in X$. Then $x*y \notin A_a^b$ and $x \notin A_a^b$ and so $a*(b*(x*y)) = 1$ and $a*(b*x) = 1$. Using (BE3) and the self distributivity of X, we have

$$\begin{aligned} 1 =& a*(b*(x*y)) = a*((b*x)*(b*y)) \\ =& (a*(b*x))*(a*(b*y)) = 1*(a*(b*y)) = a*(b*y) \end{aligned}$$

and so $y \notin A_a^b$. This is a contradiction, and therefore $\{a*b, a\} \cap A_a^b \neq \emptyset$. Hence A_a^b is an F-energetic subset of X for all $a, b \in X$. □

Definition 3.12. A fuzzy set f in a BE-algebra X is called an *anti fuzzy subalgebra* of X if $f(x*y) \leq \max\{f(x), f(y)\}$ for all $x, y \in X$. A fuzzy set f in a BE-algebra X is called an *anti fuzzy filter* of X if it satisfies

(AF_1) $(\forall x \in X)(f(1) \leq f(x))$;
(AF_2) $(\forall x, y \in X)(f(y) \leq \max\{f(x*y), f(x)\})$.

Proposition 3.13. *For any anti fuzzy filter of a BE-algebra X, then following are valid.*

(i) $(\forall x, y \in X)(x \leq y \Rightarrow f(y) \leq f(x))$;
(ii) $(\forall x, y, z \in X)(f(x*z) \leq \max\{f(x*(y*z)), f(y)\})$;
(iii) $(\forall a, x \in X)(f((a*x)*x) \leq f(a))$.

Proof. (i) Let $x, y \in X$ be such that $x \leq y$. Then $x*y = 1$. It follows from Definition 3.12 that $f(y) \leq \max\{f(x*y), f(x)\} = \max\{f(1), f(x)\} = f(x)$.
(ii) Using (AF_2) and (BE4), we have $f(x*z) \leq \max\{f(y*(x*z)), f(y)\} = \max\{f(x*(y*z)), f(y)\}$ for any $x, y, z \in X$.
(iii) Taking $y := (a*x)*x$ and $x := a$ in (AF_2), we have $f((a*x)*x) \leq \max\{f(a*((a*x)*x)), f(a)\} = \max\{f((a*x)*(a*x)), f(a)\} = \max\{f(1), f(a)\} = f(a)$ for any $a, x \in X$. □

Theorem 3.14. *Any fuzzy set of a BE-algebra X satisfying (AF_1) and Proposition 3.13 (ii) is an anti fuzzy filter of X.*

Proof. Taking $x := 1$ in Proposition 3.13 (ii) and (BE3), we have $f(z) = f(1*z) \leq \max\{f(1*(y*z)), f(y)\} = \max\{f(y*z), f(y)\}$ for all $y, z \in X$. Hence f is an anti fuzzy filter of X. □

Corollary 3.15. *For any fuzzy set f of a BE-algebra X, f is an anti fuzzy filter of X if and only if it satisfies (AF_1) and Proposition 3.13 (ii).*

Theorem 3.16. *Any fuzzy set f of a BE-algebra X is an anti fuzzy filter of X if and only if it satisfies the following conditions:*

(i) $(\forall x, y \in X)(f(y*x) \leq f(x))$;
(ii) $(\forall x, a, b \in X)(f((a*(b*x))*x) \leq \max\{f(a), f(b)\})$.

Proof. Assume that f is an anti fuzzy filter of X. It follows from Definition 3.12 that $f(y*x) \leq \max\{f(x*(y*x)), f(x)\} = \max\{f(1), f(x)\} = f(x)$ for all $x, y \in X$. Using Proposition 3.13, we have $f((a*(b*x))*x) \leq \max\{f((a*(b*x))*(b*x)), f(b)\} \leq \max\{f(a), f(b)\}$ for any $a, b, x \in X$.

Conversely, let f be a fuzzy set satisfying conditions (i) and (ii). Setting $y := x$ in (i), we have $f(x*x) = f(1) \leq f(x)$ for all $x \in X$. Using (ii), we obtain $f(y) = f(1*y) = f((x*y)*(x*y))*y) \leq \max\{f(x*y), f(y)\}$ for all $x, y \in X$. Hence f is an anti fuzzy filter of X. □

Proposition 3.17. *For any fuzzy set of a BE-algebra X, then f is an anti fuzzy filter of X if and only if*

$(*)$ $(\forall x, y, z \in X)(z \leq x*y \Rightarrow f(y) \leq \max\{f(x), f(z)\})$.

Proof. Assume that f is an anti fuzzy filter of X. Let $x, y, z \in X$ be such that $z \leq x * y$. By Proposition 3.13, we have $f(y) \leq \max\{f(x * y), f(x)\} \leq \max\{f(z), f(x)\}$.

Conversely, suppose that f satisfies $(*)$. By (BE2), we have $x \leq x * 1 = 1$. Using $(*)$, we have $f(1) \leq f(x)$ for all $x \in X$. It follows from (BE1) and (BE4) that $x \leq (x * y) * y$ for all $x, y \in X$. Using $(*)$, we have $f(y) \leq \max\{f(x * y), f(x)\}$. Therefore f is an anti fuzzy filter of X. ☐

4. Permeable values in BE-algebras

Definition 4.1. Let f be a fuzzy set in a BE-algebra X. A number $t \in [0, 1]$ is called a *permeable S-value* for f if $U(f; t) \neq \emptyset$ and the following assertion is valid.

(4.1) $(\forall a, b \in X)(f(a * b) \geq t \Rightarrow \max\{f(a), f(b)\} \geq t)$.

Example 4.2. Consider a BE-algebra $X = \{1, a, b, c\}$ as in Example 3.3 (1). Let f be a fuzzy set of X defined by $f(1) = 0.2, f(a) = 0.3$, and $f(b) = f(c) = 0.6$. Take $t \in (0.3, 0.6]$. Then $U(f; t) = \{b, c\}$. It is easy to check that t is a permeable S-value for f.

Theorem 4.3. *Let f be a fuzzy subalgebra of a BE-algebra X. If $t \in [0, 1]$ is a permeable S-value for f, then the nonempty upper t-level set $U(f; t)$ is an S-energetic subset of X.*

Proof. Let $a, b \in X$ be such that $a * b \in U(f; t)$. Then $f(a * b) \geq t$ and so $\max\{f(a), f(b)\} \geq t$. Therefore $f(a) \geq t$ or $f(b) \geq t$, i.e., $a \in U(f; t)$ or $b \in U(f; t)$. Hence $\{a, b\} \cap U(f; t) \neq \emptyset$. Thus $U(f; t)$ is an S-energetic subset of X. ☐

Since $U(f; t) \cup L^*(f; t) = X$ and $U(f; t) \cap L^*(f; t) = \emptyset$ for all $t \in [0, 1]$, we have the following corollary.

Corollary 4.4. *Let f be a fuzzy subalgebra of a BE-algebra X. If $t \in [0, 1]$ is a permeable S-value for f, then $L^*(f; t)$ is empty or a subalgebra of X.*

Definition 4.5. Let f be a fuzzy set in a BE-algebra X. A number $t \in [0, 1]$ is called an *anti permeable S-value* for f if $L(f; t) \neq \emptyset$ and the following assertion is valid.

(4.2) $(\forall a, b \in X)(f(a * b) \leq t \Rightarrow \min\{f(a), f(b)\} \leq t)$.

Example 4.6. Consider a BE-algebra $X = \{1, a, b, c\}$ as in Example 3.3 (1). Let f be a fuzzy set of X defined by $f(1) = 0.4, f(a) = f(b) = 0.5$, and $f(c) = 0.3$. Take $t \in [0.3, 0.4)$. Then $L(f; t) = \{c\}$. It is easy to check that t is an anti permeable S-value for f.

Theorem 4.7. *Let f be an anti fuzzy subalgebra of a BE-algebra X. For any anti permeable S-value $t \in [0, 1]$ for f, we have $L(f; t) \neq \emptyset \Rightarrow L(f; t)$ is an S-energetic subset of X.*

Proof. Let $a, b \in X$ be such that $a * b \in L(f; t)$. Then $f(a * b) \leq t$ and so $\min\{f(a), f(b)\} \leq t$. Thus $f(a) \leq t$ or $f(b) \leq t$, i.e., $a \in L(f; t)$ or $b \in L(f; t)$. Hence $\{a, b\} \cap L(f; t) \neq \emptyset$. Therefore $L(f; t)$ is an S-energetic subset of X. ☐

Theorem 4.8. *Let f be a fuzzy subalgebra of a BE-algebra X and let $t \in [0,1]$ be such that $L(f;t) \neq \emptyset$. Then t is an anti permeable S-value for f.*

Proof. Let $a, b \in X$ be such that $f(a*b) \leq t$ for all $t \in [0,1]$. Then $\min\{f(a), f(b)\} \leq f(a*b) \leq t$. Therefore t is an anti permeable S-value for f. $\square$

Definition 4.9. Let f be a fuzzy set in a BE-algebra X. A number $t \in [0,1]$ is called a *permeable F-value* for f if $U(f;t) \neq \emptyset$ and the following assertion is valid.

(4.3) $(\forall x, y \in X)(f(y) \geq t \Rightarrow \max\{f(x*y), f(x)\} \geq t)$.

Example 4.10. Consider the BE-algebra $X = \{1, a, b, c, d, 0\}$ as in Example 3.10. Let f be a fuzzy set in X defined by $f(1) = 0.2, f(a) = f(b) = 0.4$, and $f(c) = f(d) = f(0) = 0.7$. If $t \in (0.4, 0.7]$, then $U(f;t) = \{0, c, d\}$ and it is easy to check that t is a permeable F-value for f.

Theorem 4.11. *Let f be a fuzzy filter of a BE-algebra X. If $t \in [0,1]$ is a permeable F-value for f, then the nonempty upper t-level set $U(f;t)$ is an F-energetic subset of X.*

Proof. Assume that $U(f;t) \neq \emptyset$ for $t \in [0,1]$. Let $y \in X$ be such that $y \in U(f;t)$. Then $t \leq f(y)$. It follows from (4.3) that $t \leq \max\{f(x*y), f(x)\}$ for all $x \in X$. Hence $f(x*y) \geq t$ or $f(x) \geq t$, i.e., $x*y \in U(f;t)$ or $x \in U(f;t)$. Hence $\{x*y, x\} \cap U(f;t) \neq \emptyset$. Therefore $U(f;t)$ is an F-energetic subset of X. $\square$

Since $U(f;t) \cup L^*(f;t) = X$ and $U(f;t) \cap L^*(f;t) = \emptyset$ for all $t \in [0,1]$, we have the following corollary.

Corollary 4.12. *Let f be a fuzzy filter of a BE-algebra X. If $t \in [0,1]$ is a permeable F-value for f, then $L^*(f;t)$ is empty or a filter of X.*

Theorem 4.13. *For a fuzzy set f in a BE-algebra X, if there exists a subset K of $[0,1]$ such that $\{U(f;t), L^*(f;t)\}$ is a partition of X and $L^*(f;t)$ is a filter of X for all $t \in K$, then t is a permeable F-value for f.*

Proof. Assume that $f(y) \geq t$ for any $y \in X$. Then $y \in U(f;t)$ and so $\{x*y, x\} \cap U(f;t) \neq \emptyset$ for any $x \in X$, since $U(f;t)$ is an F-energetic subset of X. Hence $x*y \in U(f;t)$ or $x \in U(f;t)$ and so $\max\{f(x*y), f(x)\} \geq t$. Therefore t is a permeable F-value for f. $\square$

Theorem 4.14. *Let f be a fuzzy set in a BE-algebra X with $U(f;t) \neq \emptyset$ for $t \in [0,1]$. If f is an anti fuzzy filter of X, then t is a permeable F-value for f.*

Proof. Let $y \in X$ be such that $f(y) \geq t$. Then $t \leq f(y) \leq \max\{f(x*y), f(x)\}$ for all $x \in X$. Hence t is a permeable F-value for f. $\square$

Theorem 4.15. *Let f be an anti fuzzy filter of a BE-algebra X. Then the following assertion is valid.*

$$(\forall t \in [0,1])(U(f;t) \neq \emptyset \Rightarrow U(f;t) \text{ is an } F\text{-energetic subset of } X).$$

Proof. Let $y \in X$ be such that $y \in U(f;t)$. Then $f(y) \geq t$. By (AF_2), we have $t \leq f(y) \leq \max\{f(x*y), f(x)\}$ for all $x \in X$. Hence $f(x*y) \geq t$ or $f(x) \geq t$, i.e., $x*y \in U(f;t)$ or $x \in U(f;t)$. Therefore $\{x*y, x\} \cap U(f;t) \neq \emptyset$. Thus $U(f;t)$ is an F-energetic subset of X. $\square$

Definition 4.16. Let f be a fuzzy set in a BE-algebra X. A number $t \in [0,1]$ is called an *anti permeable F-value* for f if $L(f;t) \neq \emptyset$ and the following assertion is valid.

(4.4) $(\forall x, y \in X)(f(y) \leq t \Rightarrow \min\{f(x*y), f(x)\} \leq t)$.

Theorem 4.17. *Let f be a fuzzy set in a BE-algebra X with $L(f;t) \neq \emptyset$ for $t \in [0,1]$. If f is a fuzzy filter of X, then t is an anti permeable F-value for f.*

Proof. Let $y \in X$ be such that $f(y) \leq t$. Then $\min\{f(x*y), f(x)\} \leq f(y) \leq t$ for all $x \in X$. Hence t is an anti permeable F-value for f. $\square$

Theorem 4.18. *Let f be an anti fuzzy filter of a BE-algebra X. If $t \in [0,1]$ is an anti permeable F-value for f, then the lower t-level set $L(f;t)$ is an F-energetic subset of X.*

Proof. Let $y \in X$ be such that $y \in L(f;t)$. Then $f(y) \leq t$. It follows from (4.4) that $\min\{f(x*y), f(x)\} \leq t$ for all $x \in X$. Hence $x*y \in L(f;t)$ or $x \in L(f;t)$ and so $\{x*y, x\} \cap L(f;t) \neq \emptyset$. Therefore $L(f;t)$ is an F-energetic subset of X. $\square$

Corollary 4.19. *Let f be an anti fuzzy filter of a BE-algebra X. If $t \in [0,1]$ is an anti permeable F-value for f, then $U^*(f;t)$ is empty or a filter of X.*

Theorem 4.20. *For a fuzzy set f in a BE-algebra X, if there exists a subset K of $[0,1]$ such that $\{U^*(f;t), L(f;t)\}$ is a partition of X and $U^*(f;t)$ is a filter of X for all $t \in K$, then t is an anti permeable F-value for f.*

Proof. Assume that $f(y) \leq t$ for any $y \in X$. Then $y \in L(f;t)$ and so $\{x*y, x\} \cap L(f;t)\} \neq \emptyset$ for all $x \in X$, since $L(f;t)$ is an F-energetic subset of X. Hence $f(x*y) \leq t$ or $f(x) \leq t$ and so $\min\{f(x*y), f(x)\} \leq t$. Therefore t is anti permeable F-value for f. $\square$

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Rapid gradient penalty schemes and convergence for solving constrained convex optimization problem in Hilbert spaces

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Abstract

The purposes of this paper are to establish and study the convergence of a new gradient scheme with penalization terms called rapid gradient penalty algorithm (**RGPA**) for minimizing a convex differentiable function over the set of minimizers of a convex differentiable constrained function. Under the observation of some appropriate choices for the available properties of the considered functions and scalars, we can generate a suitable algorithm that weakly converges to a minimal solution of the considered constraint minimization problem. Further, we also provide a numerical example to compare the rapid gradient penalty algorithm (**RGPA**) and the algorithm introduced by Peypouquet [20].

Keywords: Rapid gradient penalty algorithm, penalization, constraint minimization, fenchel conjugate

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2010 *Mathematics Subject Classification.* 65K05, 65K10, 90C25.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with the norm and inner product given by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. Let $f : \mathcal{H} \to \mathbb{R}$ and $g : \mathcal{H} \to \mathbb{R}$ be convex and (Fréchet) differentiable functions on the space $\mathcal{H}$ and the gradients ∇f and ∇g are Lipschitz continuous operators with constants L_f and L_g, respectively. We consider the following constrained convex optimization problem

$$\min_{x \in \arg\min g} f(x). \tag{1.1}$$

Throughout the paper, we also assume that the solution set $\mathcal{S} := \arg\min\{f(x) : x \in \arg\min g\}$ is a nonempty set. Further, without loss of generality, we may assume that $\min g = 0$.

Due to the interesting applications of (1.1) in many branches of mathematics and sciences, many researchers have paid attention to solve the problem (1.1) which can be mentioned briefly as follows: In 2010, Attouch and Czarnecki [1] initially presented and studied a numerical algorithm called the *multiscale asymptotic gradient* (**MAG**) for solving general constrained convex optimization problem. They proved that every sequence generated by (**MAG**) converges weakly to a solution of their

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[1] Supported by The Royal Golden Jubilee Project Grant no. PHD/0158/2557, Thailand.

considered problem. It seems that their representation is the starting point for the development of numerical algorithms in the context of solving type of constrained convex optimization problem (see, for instance [2–4, 7–10, 18]) and the references therein. Inspired by Attouch and Czarnecki [1], in 2012 Peypouquet [20] proposed and analyzed an algorithm called *diagonal gradient scheme* (**DGS**) via gradient method and exterior penalization scheme for constrained minimization of convex functions. He also provided a weak convergence to find a solution of the considered constrained minimization of convex functions. Several applications are provided such as relaxed feasibility, mathematical programming with convex inequality constraints, and Stokes equation and signal reconstruction, etc. In 2013, Shehu et al. [21] studied the problem (1.1) in the case when the constrained set is simple enough and also proposed an algorithm for solving (1.1). In the last two decades, intensive research efforts dedicated to algorithms of inertial type and their convergence behavior can be noticed (see [6, 11, 13–17, 19]). In 2017, Bot et al. [9] considered the problem of minimizing a smooth convex objective function subject to the set of minima of another differentiable convex function. They proposed a new algorithm called *gradient-type penalty with inertial effects method* (**GPIM**) for solving the problem (1.1). They also illustrated the usability of their method via a numerical experiment for image classification via support vector machines.

In the remaining part of this section, we recall some elements of convex analysis. For a function $h : \mathcal{H} \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ we denote by dom $h = \{x \in \mathcal{H} : h(x) < +\infty\}$ its effective domain and say that h is proper, if dom $h \neq \emptyset$ and $h(x) \neq -\infty$ for all $x \in \mathcal{H}$. The *Fenchel conjugate* of h is $h^* : \mathcal{H} \to \overline{\mathbb{R}}$, which is defined by

$$h^*(z) = \sup_{x\in\mathcal{H}}\{\langle z, x\rangle - h(x)\} \quad \text{for all } z \in \mathcal{H}.$$

The subdifferential of h at $x \in \mathcal{H}$, with $h(x) \in \mathbb{R}$, is the set

$$\partial h(x) := \{v \in \mathcal{H} : h(y) - h(x) \geq \langle v, y - x\rangle \ \forall y \in \mathcal{H}\}.$$

We take by convention $\partial h(x) := \emptyset$, if $h(x) \in \{\pm\infty\}$.

The convex and differentiable function $T : \mathcal{H} \to \mathbb{R}$ has a Lipschitz continuous gradient with Lipschitz constant $L_T > 0$, if $\|\nabla T(x) - \nabla T(y)\| \leq L_T\|x - y\|$ for all $x, y \in \mathcal{H}$.

Let $\mathcal{C} \subset \mathcal{H}$ be a nonempty closed convex set. The *indicator function* is defined as:

$$\delta_{\mathcal{C}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{C} \\ +\infty & \text{otherwise.} \end{cases}$$

The *support function* of $\mathcal{C}$ is defined as: $\sigma_{\mathcal{C}}(x) := \sup_{c\in\mathcal{C}}\langle x, c\rangle$ for all $x \in \mathcal{H}$. The *normal cone* $\mathcal{C}$ at a point x is

$$N_{\mathcal{C}}(x) := \begin{cases} \{\overline{x} \in \mathcal{H} : \langle \overline{x}, c - x\rangle \leq 0 \ \text{ for all } c \in \mathcal{C}\}, & \text{if } x \in \mathcal{C} \\ \emptyset, \ \text{ otherwise.} \end{cases}$$

We denote by $\mathbf{Ran}(N_{\mathcal{C}})$ for the range of $N_{\mathcal{C}}$. Notice that $\delta^*_{\mathcal{C}} = \sigma_{\mathcal{C}}$. Moreover, it holds that $\overline{x} \in N_{\mathcal{C}}(x)$ if and only if $\sigma_{\mathcal{C}}(\overline{x}) = \langle \overline{x}, x\rangle$.

Inspired by the research works in this direction, we are interested in the development and improvement of the method for finding solutions of the considered problem, that is, we wish to establish the algorithm called rapid gradient penalty algorithm (**RGPA**) for solving (1.1) which is generated by a controlling sequence of scalars together with the gradient of objective and feasibility gap functions as follows:

$$(\mathbf{RGPA}) \quad \begin{cases} x_1 \in \mathcal{H}; \\ y_n = x_n - \lambda_n \nabla f(x_n) - \lambda_n\beta_n\nabla g(x_n); \\ x_{n+1} = y_n + \alpha_n(y_n - x_n) \qquad \text{for all } n \geq 1, \end{cases}$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ are sequences of positive parameters and $\{\alpha_n\} \subseteq (0, 1)$.

For $n \geq 1$, we write $\Omega_n := f + \beta_n g$, which is also (Fréchet) differentiable function. Therefore, $\nabla\Omega_n$ is Lipschitz continuous with constant $L_n := L_f + \beta_n L_g$. In particular, if we setting $\alpha_n = 0$ for all $n \geq 1$, the algorithm (**RGPA**) can be reduced to (**DGS**) in Peypouquet [20].

In order to support our ideas, we also provide a numerical example to simulate an event for solving problem (1.1). We also compare the time and the iteration between two algorithms including (**RGPA**) and (**DGS**).

2. The Hypotheses

In this section, we will carry out the main assumptions to prove the convergence results for rapid gradient penalty algorithm (**RGPA**). In order to prove the convergence results, the following assumptions will be proposed.

Assumption A

- (I) The function f is bounded from below;
- (II) There exists a positive $K > 0$ such that $\beta_{n+1} - \beta_n \leq K\lambda_{n+1}\beta_{n+1}$, $\frac{L_n}{2} - \frac{1}{2\lambda_n} \leq -K$ and $\frac{\alpha_n^2 - 1}{2\lambda_n} + (1 + \alpha_n)^2 K < 0$ for all $n \geq 1$;
- (III) $\{\alpha_n\} \in l^2 \setminus l^1$, $\displaystyle\sum_{n=1}^{\infty} \lambda_n = +\infty$ and $\displaystyle\liminf_{n\to\infty} \lambda_n\beta_n > 0$;
- (IV) For each $p \in \mathbf{Ran}(N_{\arg\min g})$, we have $\displaystyle\sum_{n=1}^{\infty} \lambda_n\beta_n \left[g^*\left(\frac{p}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{p}{\beta_n}\right)\right] < +\infty$.

Remark 2.1. The conditions in **Assumption A** sparsely extend the hypotheses in [20]. The differences are given by the second and third inequality in (II), which here involves a sequence $\{\alpha_n\}$ which controls the inertial terms, and by $\{\alpha_n\} \in l^2 \setminus l^1$.

In the following remark, we present some situations where **Assumption A** is verified.

Remark 2.2. Let $K > 0$, $q \in (0, 1)$, $\delta > 0$ and $\gamma \in (0, \frac{1}{3L_g})$ be any given. Then we set $\alpha_n := \frac{1}{n+1}$ for all $n \geq 1$, which implies that $\displaystyle\lim_{n\to\infty} \alpha_n = 0$, $\displaystyle\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$ and $\alpha_n \leq \frac{1}{2}$ for all $n \geq 1$. We also set

$$\beta_n := \frac{3\gamma[L_f + 2(K+\delta)]}{1 - 3\gamma L_g} + \gamma K n^q \quad \text{and} \quad \lambda_n := \frac{\gamma}{\beta_n} \quad \text{for all } n \geq 1.$$

Since $\beta_n \geq \frac{3\gamma[L_f+2(K+\delta)]}{1-3\gamma L_g}$, we have for each $n \geq 1$

$$\beta_n(1 - 3\gamma L_g) \geq 3\gamma[L_f + 2(K + \delta)].$$

It follows that

$$\frac{1}{3\lambda_n} - \beta_n L_g \geq L_f + 2(K_+\delta) \quad \text{for all } n \geq 1,$$

which implies that

$$-(K + \delta) \geq \frac{L_n}{2} - \frac{1}{6\lambda_n} \quad \text{for all } n \geq 1. \tag{2.1}$$

According to (2.1), we obtain that

$$-K \geq \frac{L_n}{2} - \frac{1}{2\lambda_n} \quad \text{and} \quad \frac{1}{3} > 2\lambda_n K \quad \text{for all } n \geq 1.$$

Let us consider, for each $n \geq 1$

$$\frac{\alpha_n^2 - 1}{2\lambda_n} + (1+\alpha_n)^2 K \leq \frac{-\frac{3}{4} + \frac{9}{4} 2\lambda_n K}{2\lambda_n} < \frac{-\frac{3}{4} + \frac{3}{4}}{2\lambda_n} = 0.$$

On the other hand,

$$\beta_{n+1} - \beta_n = \gamma K[(n+1)^q - n^q] \leq \gamma K = K\lambda_{n+1}\beta_{n+1}.$$

Hence, we can conclude that **Assumption A** (II) holds.

Since $q \in (0,1)$, we obtain that $\sum_{n=1}^{\infty} \frac{1}{\beta_n} = +\infty$, so $\sum_{n=1}^{\infty} \lambda_n = +\infty$. Notice that $\lambda_n \beta_n = \gamma$ for all $n \geq 1$. It follows that $\liminf_{n\to\infty} \lambda_n\beta_n = \liminf_{n\to\infty} \gamma > 0$. Thus **Assumption A** (III) holds.

Finally, since $g^* - \sigma_{\arg\min g} \geq 0$. If $g(x) \geq \frac{k}{2}\text{dist}^2(x, \arg\min g)$ where $k > 0$, then $g^*(x) - \sigma_{\arg\min g}(x) \leq \frac{1}{2k}\|x\|^2$ for all $x \in \mathcal{H}$.

Therefore, for each $p \in \mathbf{Ran}(N_{\arg\min g})$, we obtain that

$$\lambda_n\beta_n \left[g^*\left(\frac{p}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{p}{\beta_n}\right)\right] \leq \frac{\lambda_n}{2k\beta_n}\|p\|^2.$$

Thus, $\sum_{n=1}^{\infty} \lambda_n\beta_n \left[g^*\left(\frac{p}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{p}{\beta_n}\right)\right]$ converges, if $\sum_{n=1}^{\infty} \frac{\lambda_n}{\beta_n}$ converges, which is equivalently to $\sum_{n=1}^{\infty} \frac{1}{\beta_n^2}$ converges. This holds for the above choices of $\{\beta_n\}$ and $\{\lambda_n\}$ when $q \in (\frac{1}{2}, 1)$.

3. Convergence analysis for convexity

In this section, we will prove the convergence of the sequence of $\{x_n\}$ generated by (**RGPA**) and of the sequence of objective values $\{f(x_n)\}$.

We start the convergence analysis of this section with three technical lemmas.

Lemma 3.1. *Let x^* be an arbitrary element in $\mathcal{S}$ and set $p^* := -\nabla f(x^*)$. Then for each $n \geq 1$*

$$\begin{aligned}\|x_{n+1} - x^*\|^2 - \|x_n - x^*\|^2 + (1+\alpha_n)\lambda_n\beta_n g(x_n) &\leq (1+\alpha_n)^2\|x_n - y_n\|^2 \\ &+ (1+\alpha_n)\lambda_n\beta_n\left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right]. \end{aligned} \tag{3.1}$$

Proof. Applying to the first-order optimality condition, we have

$$0 \in \nabla f(x^*) + N_{\arg\min g}(x^*).$$

It follows that

$$p^* = -\nabla f(x^*) \in N_{\arg\min g}(x^*).$$

Note that for each $n \geq 1$, $\frac{x_n - y_n}{\lambda_n} - \beta_n \nabla g(x_n) = \nabla f(x_n)$.

By monotonicity of ∇f, we obtain that

$$\begin{aligned}\left\langle \frac{x_n - y_n}{\lambda_n} - \beta_n\nabla g(x_n) + p^*, x_n - x^*\right\rangle &= \langle \nabla f(x_n) - \nabla f(x^*), x_n - x^*\rangle \\ &\geq 0 \quad ,\forall n \geq 1,\end{aligned}$$

and hence, for each $n \geq 1$

$$2\langle x_n - y_n, x_n - x^*\rangle \geq 2\lambda_n\beta_n \langle \nabla g(x_n), x_n - x^*\rangle - 2\lambda_n\langle p^*, x_n - x^*\rangle. \tag{3.2}$$

Since g is convex and differentiable, we have for each $n \geq 1$

$$\langle \nabla g(x_n), x^* - x_n \rangle + g(x_n) \leq g(x^*) = 0,$$

whence

$$2\lambda_n \beta_n g(x_n) \leq 2\lambda_n \beta_n \langle \nabla g(x_n), x_n - x^* \rangle. \tag{3.3}$$

On the other hand,

$$2\langle x_n - y_n, x_n - x^* \rangle = \|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \|y_n - x^*\|^2. \tag{3.4}$$

Combining (3.2), (3.3) and (3.4), we get that

$$\|y_n - x^*\|^2 \leq \|x_n - y_n\|^2 + \|x_n - x^*\|^2 - 2\lambda_n \beta_n g(x_n) + 2\lambda_n \langle p^*, x_n - x^* \rangle. \tag{3.5}$$

Since $x^* \in \mathcal{S}$ and $p^* \in N_{\arg\min g}(x^*)$, we have

$$\sigma_{\arg\min g}(p^*) = \langle p^*, x^* \rangle.$$

In (3.5), we observe that

$$\begin{aligned} 2\lambda_n \langle p^*, x_n - x^* \rangle - \lambda_n \beta_n g(x_n) &= 2\lambda_n \langle p^*, x_n \rangle - \lambda_n \beta_n g(x_n) - 2\lambda_n \langle p^*, x^* \rangle \\ &= \lambda_n \beta_n \left[\left\langle \frac{2p^*}{\beta_n}, x_n \right\rangle - g(x_n) - \left\langle \frac{2p^*}{\beta_n}, x^* \right\rangle \right] \\ &\leq \lambda_n \beta_n \left[g^* \left(\frac{2p^*}{\beta_n} \right) - \sigma_{\arg\min g} \left(\frac{2p^*}{\beta_n} \right) \right]. \end{aligned} \tag{3.6}$$

Combining (3.6) and (3.5), we obtain that

$$\|y_n - x^*\|^2 \leq \|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \lambda_n \beta_n g(x_n) + \lambda_n \beta_n \left[g^* \left(\frac{2p^*}{\beta_n} \right) - \sigma_{\arg\min g} \left(\frac{2p^*}{\beta_n} \right) \right]. \tag{3.7}$$

On the other hand, we observe that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|y_n + \alpha_n(y_n - x_n) - x^*\|^2 = \|(1+\alpha_n)(y_n - x^*) + \alpha_n(x^* - x_n)\|^2 \\ &= (1+\alpha_n)\|y_n - x^*\|^2 - \alpha_n \|x_n - x^*\|^2 + \alpha_n(1+\alpha_n)\|x_n - y_n\|^2. \end{aligned} \tag{3.8}$$

By (3.7) and (3.8), we obtain the desired result. □

Lemma 3.2. *For all $n \geq 1$, we have*

$$\begin{aligned} \Omega_{n+1}(x_{n+1}) \leq \Omega_n(x_n) + (\beta_{n+1} - \beta_n) g(x_{n+1}) + \frac{\alpha_n^2 - 1}{2\lambda_n} \|y_n - x_n\|^2 \\ + \left[\frac{L_n}{2} - \frac{1}{2\lambda_n} \right] \|x_{n+1} - x_n\|^2. \end{aligned}$$

Proof. Since $\nabla \Omega$ is L_n-Lipschitz continuous and by Descent Lemma (see [5, Theorem 18.15]), we obtain that

$$\Omega_n(x_{n+1}) \leq \Omega_n(x_n) + \langle \nabla \Omega_n(x_n), x_{n+1} - x_n \rangle + \frac{L_n}{2} \|x_{n+1} - x_n\|^2.$$

Recall that $-\frac{y_n - x_n}{\lambda_n} = \nabla \Omega_n(x_n)$.

It follows that

$$\begin{aligned}
&f(x_{n+1}) + \beta_n g(x_{n+1}) \\
&\leq f(x_n) + \beta_n g(x_n) - \left\langle \frac{y_n - x_n}{\lambda_n}, x_{n+1} - x_n \right\rangle + \frac{L_n}{2}\|x_{n+1} - x_n\|^2 \\
&= f(x_n) + \beta_n g(x_n) - \frac{1}{2\lambda_n}\|y_n - x_n\|^2 - \frac{1}{2\lambda_n}\|x_{n+1} - x_n\|^2 + \frac{1}{2\lambda_n}\|y_n - x_{n+1}\|^2 + \frac{L_n}{2}\|x_{n+1} - x_n\|^2 \\
&= f(x_n) + \beta_n g(x_n) + \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2 + \left[\frac{L_n}{2} - \frac{1}{2\lambda_n}\right]\|x_{n+1} - x_n\|^2.
\end{aligned}$$

Adding $\beta_{n+1}g(x_{n+1})$ to both sides, we have

$$\begin{aligned}
f(x_{n+1}) + \beta_{n+1} g(x_{n+1}) \leq{}& f(x_n) + \beta_n g(x_n) + (\beta_{n+1} - \beta_n) g(x_{n+1}) \\
&+ \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2 + \left[\frac{L_n}{2} - \frac{1}{2\lambda_n}\right]\|x_{n+1} - x_n\|^2,
\end{aligned}$$

which means that

$$\Omega_{n+1}(x_{n+1}) \leq \Omega_n(x_n) + (\beta_{n+1} - \beta_n) g(x_{n+1}) + \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2 + \left[\frac{L_n}{2} - \frac{1}{2\lambda_n}\right]\|x_{n+1} - x_n\|^2.$$

□

For $n \geq 1$ and $x^* \in \mathcal{S}$, we denote by

$$\begin{aligned}
\Lambda_n &:= f(x_n) + (1 - (1 + \alpha_n)K\lambda_n)\beta_n g(x_n) + K\|x_n - x^*\|^2 \\
&= \Omega_n(x_n) - (1 + \alpha_n)K\lambda_n\beta_n g(x_n) + K\|x_n - x^*\|^2.
\end{aligned}$$

Lemma 3.3. *Let $x^* \in \mathcal{S}$ and set $p^* := -\nabla f(x^*)$. Then there exists $\theta > 0$ such that for each $n \geq 1$*

$$\Lambda_{n+1} - \Lambda_n + \theta\|y_n - x_n\|^2 \leq (1 + \alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

Proof. From Lemma 3.2 and **Assumption A** (II), we obtain that

$$\begin{aligned}
\Omega_{n+1}(x_{n+1}) - \Omega_n(x_n) &\leq K\lambda_{n+1}\beta_{n+1} g(x_{n+1}) + \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2 \\
&\leq (1 + \alpha_{n+1})K\lambda_{n+1}\beta_{n+1} g(x_{n+1}) + \frac{\alpha_n^2 - 1}{2\lambda_n}\|y_n - x_n\|^2. \qquad (3.9)
\end{aligned}$$

On the other hand, multiplying (3.1) by K, we have

$$\begin{aligned}
&K\|x_{n+1} - x^*\|^2 - K\|x_n - x^*\|^2 + (1 + \alpha_n)K\lambda_n\beta_n g(x_n) \\
&\qquad \leq (1 + \alpha_n)^2 K\|x_n - y_n\|^2 + (1 + \alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right]. \qquad (3.10)
\end{aligned}$$

Combining (3.9) and (3.10), we have

$$\Lambda_{n+1} - \Lambda_n \leq \left[\frac{\alpha_n^2 - 1}{2\lambda_n} + (1 + \alpha_n)^2 K\right]\|y_n - x_n\|^2 + (1 + \alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right]. \qquad (3.11)$$

For each $n \geq 1$, $\frac{\alpha_n^2 - 1}{2\lambda_n} + (1 + \alpha_n)^2 K < 0$, we have there exists $\theta > 0$ such that

$$\frac{\alpha_n^2 - 1}{2\lambda_n} + (1 + \alpha_n)^2 K < -\theta.$$

From (3.11), we have

$$\Lambda_{n+1} - \Lambda_n + \theta\|y_n - x_n\|^2 \leq (1+\alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

This completes the proof. □

The next lemma is an important role in convergence analysis (see in [3, Lemma 2] or [12, Lemma 3.1]).

Lemma 3.4. *Let $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\varepsilon_n\}$ be real sequences. Assume that $\{\gamma_n\}$ is bounded from below, $\{\delta_n\}$ is non-negative and $\sum_{n=1}^{\infty}\varepsilon_n < +\infty$ such that*

$$\gamma_{n+1} - \gamma_n + \delta_n \leq \varepsilon_n \quad \text{for all} \quad n \geq 1.$$

Then $\lim_{n\to\infty}\gamma_n$ exists and $\sum_{n=1}^{\infty}\delta_n < +\infty$.

Lemma 3.5. *Let $x^* \in \mathcal{S}$. Then the following statements hold:*

- (i) *The sequence $\{\Lambda_n\}$ is bounded from below and $\lim_{n\to\infty}\Lambda_n$ exists;*
- (ii) $\sum_{n=1}^{\infty}\|y_n - x_n\|^2 < +\infty$;
- (iii) $\lim_{n\to\infty}\|x_n - x^*\|^2$ *exists and* $\sum_{n=1}^{\infty}\lambda_n\beta_n g(x_n) < +\infty$;
- (iv) $\lim_{n\to\infty}\Omega_n(x_n)$ *exists;*
- (v) $\lim_{n\to\infty} g(x_n) = 0$ *and every weak cluster point of the sequence $\{x_n\}$ lies in* $\arg\min g$.

Proof. We set $p^* := -\nabla f(x^*)$.

(i). From **Assumption A** (II) implies $1-(1+\alpha_n)K\lambda_n \geq 0$. Since f is convex and differentiable, we have for each $n \geq 1$

$$\begin{aligned}\Lambda_n &= f(x_n) + (1-(1+\alpha_n)K\lambda_n)\beta_n g(x_n) + K\|x_n - x^*\|^2 \geq f(x_n) + K\|x_n - x^*\|^2 \\ &\geq f(x^*) + \langle \nabla f(x^*), x_n - x^*\rangle + K\|x_n - x^*\|^2 = f(x^*) - \left\langle \frac{p^*}{\sqrt{2K}}, \sqrt{2K}(x_n - x^*)\right\rangle + K\|x_n - x^*\|^2 \\ &\geq f(x^*) - \frac{\|p^*\|^2}{4K} - K\|x_n - x^*\|^2 + K\|x_n - x^*\|^2 = f(x^*) - \frac{\|p^*\|^2}{4K}.\end{aligned}$$

Therefore, $\{\Lambda_n\}$ is bounded from below.

Next, we set $\gamma_n = \Lambda_n$, $\delta_n = \theta\|y_n - x_n\|^2$ and

$$\varepsilon_n = (1+\alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

Recall that $\min g = 0$. Thus $g \leq \delta_{\arg\min g}$. Therefore $\sigma_{\arg\min g} = (\delta_{\arg\min g})^* \leq g^*$ and hence, $g^* - \sigma_{\arg\min g} \geq 0$. It follows that

$$\varepsilon_n = (1+\alpha_n)K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right] \leq 2K\lambda_n\beta_n \left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

By using **Assumption A** (IV) and $p^* \in N_{\arg\min g}(x^*)$, we have $\sum_{n=1}^{\infty}\varepsilon_n < +\infty$. Applying Lemma 3.3 and Lemma 3.4, we obtain that $\lim_{n\to\infty}\Lambda_n$ exists.

(ii). Follows immediately from Lemmas 3.3 and 3.4.

(iii). We set $\gamma_n = \|x_n - x^*\|^2$, $\delta_n = (1+\alpha_n)\lambda_n\beta_n g(x_n)$ and

$$\varepsilon_n = (1+\alpha_n)^2\|y_n - x_n\|^2 + (1+\alpha_n)\lambda_n\beta_n\left[g^*\left(\frac{2p^*}{\beta_n}\right) - \sigma_{\arg\min g}\left(\frac{2p^*}{\beta_n}\right)\right].$$

From statement (ii), Lemma 3.4 and Lemma 3.1, we get that

$$\lim_{n\to\infty}\|x_n - x^*\| \text{ exists and } \sum_{n=1}^{\infty}\lambda_n\beta_n g(x_n) < +\infty.$$

For (iv) since for each $n \geq 1$ $\Omega_n(x_n) = \Lambda_n + (1+\alpha_n)K\lambda_n\beta_n g(x_n) - K\|x_n - x^*\|^2$, by using (i), (iii) and $\lim_{n\to\infty}\alpha_n = 0$, we have $\lim_{n\to\infty}\Omega_n(x_n)$ exists.

(v). It follows from **Assumption A** (III) that $\liminf_{n\to\infty}\lambda_n\beta_n > 0$. According to statement (iii) implies $\lim_{n\to\infty} g(x_n) = 0$. Let $\overline{x}$ be any weak cluster point of the sequence $\{x_n\}$. Therefore, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges weakly to $\overline{x}$ as $k \to \infty$. By the weak lower semicontinuity of g, we get that

$$g(\overline{x}) \leq \liminf_{k\to\infty} g(x_{n_k}) \leq \lim_{n\to\infty} g(x_n) = 0,$$

which means that $\overline{x} \in \arg\min g$. This completes the proof. □

Lemma 3.6. *Let $x^* \in \mathcal{S}$. Then*

$$\sum_{n=1}^{\infty}\lambda_n\left[\Omega_n(x_n) - f(x^*)\right] < +\infty.$$

Proof. Since f is differentiable and convex function, we obtain that for each $n \geq 1$

$$f(x^*) \geq f(x_n) + \langle\nabla f(x_n), x^* - x_n\rangle.$$

Since g is differentiable, convex function and $\min g = 0$, we obtain that for each $n \geq 1$

$$0 = g(x^*) \geq g(x_n) + \langle\nabla g(x_n), x^* - x_n\rangle,$$

which implies that

$$0 \geq \beta_n g(x_n) + \langle\beta_n\nabla g(x_n), x^* - x_n\rangle, \quad \text{for all } n \geq 1.$$

Therefore, we can conclude that

$$f(x^*) \geq \Omega_n(x_n) + \langle\nabla\Omega_n(x_n), x^* - x_n\rangle = \Omega_n(x_n) + \left\langle\frac{x_n - y_n}{\lambda_n}, x^* - x_n\right\rangle. \tag{3.12}$$

From (3.12), we obtain that

$$2\lambda_n\left[\Omega_n(x_n) - f(x^*)\right] \leq 2\langle x_n - y_n, x_n - x^*\rangle = \|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \|y_n - x^*\|^2. \tag{3.13}$$

On the other hand, for each $n \geq 1$

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \|x_{n+1} - \alpha_n(y_n - x_n) - x^*\|^2 = \|x_{n+1} - x^*\|^2 + \alpha_n^2\|y_n - x_n\|^2 - 2\langle\alpha_n(x_{n+1} - x^*), y_n - x_n\rangle \\
&= \|x_{n+1} - x^*\|^2 + \alpha_n^2\|y_n - x_n\|^2 - \alpha_n^2\|x_{n+1} - x^*\|^2 - \|y_n - x_n\|^2 + \|\alpha_n(x_{n+1} - x^*) - (y_n - x_n)\|^2 \\
&\geq \|x_{n+1} - x^*\|^2 + \alpha_n^2\|y_n - x_n\|^2 - \alpha_n^2\|x_{n+1} - x^*\|^2 - \|y_n - x_n\|^2,
\end{aligned}$$

which implies that

$$-\|y_n - x^*\|^2 \leq -\|x_{n+1} - x^*\|^2 - \alpha_n^2\|y_n - x_n\|^2 + \alpha_n^2\|x_{n+1} - x^*\|^2 + \|y_n - x_n\|^2. \tag{3.14}$$

Combining (3.13) and (3.14), we have for all $n \geq 1$

$$\begin{aligned} 2\lambda_n \left[\Omega_n(x_n) - f(x^*)\right] &\leq (2-\alpha_n^2)\|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n^2\|x_{n+1} - x^*\|^2 \\ &\leq 2\|x_n - y_n\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n^2\|x_{n+1} - x^*\|^2. \end{aligned}$$

Therefore, according to Lemma 3.5 (iii), we get that the sequence $\{\|x_n - x^*\|\}$ is bounded, which means that there exists $M > 0$ such that $\|x_n - x^*\| \leq M$ for all $n \geq 1$.

By **Assumption A** (III) and Lemma 3.5, we obtain that

$$2\sum_{n=1}^{\infty} \lambda_n[\Omega_n(x_n) - f(x^*)] \leq 2\sum_{n=1}^{\infty} \|y_n - x_n\|^2 + \|x_1 - x^*\|^2 + M^2 \sum_{n=1}^{\infty} \alpha_n^2 < +\infty.$$

□

The following proposition will play an important role in convergence analysis, which is the main result of this paper.

Proposition 3.7 ([5, Opial Lemma]). *Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{C} \subseteq \mathcal{H}$ be nonempty set and $\{x_n\}$ be any given sequence such that:*

(i) *For every $z \in \mathcal{C}$, $\lim_{n\to\infty} \|x_n - z\|$ exists;*

(ii) *Every weak cluster point of the sequence $\{x_n\}$ lies in $\mathcal{C}$.*

Then the sequence $\{x_n\}$ converges weakly to a point in $\mathcal{C}$.

Let $\{x_n\}$ be define by (**RGPA**). Then $\{x_n\}$ converges weakly to a point in $\mathcal{S}$. Moreover, the sequence $\{f(x_n)\}$ converges to the optimal objective value of the optimization problem (1.1).

Proof. From Lemma 3.5 (iii), $\lim_{n\to\infty} \|x_n - x^*\|$ exists for all $x^* \in \mathcal{S}$. Let $\overline{x}$ be any weak cluster point of $\{x_n\}$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\overline{x}$ as $k \to \infty$. According to Lemma 3.5 (v) implies $\overline{x} \in \arg\min g$. It suffices to show that $f(\overline{x}) \leq f(x)$ for all $x \in \arg\min g$. Since $\sum_{n=1}^{\infty} \lambda_n = +\infty$, and by Lemma 3.6 and Lemma 3.5 (iv), we have

$$\lim_{n\to\infty} \Omega_n(x_n) - f(x^*) \leq 0 \;\text{ for all } x^* \in \mathcal{S}.$$

Therefore, $f(\overline{x}) \leq \liminf_{k\to\infty} f(x_{n_k}) \leq \lim_{n\to\infty} \Omega_n(x_n) \leq f(x^*), \quad \forall x^* \in \mathcal{S}$, which implies that $\overline{x} \in \mathcal{S}$. Applying Opial Lemma, we obtain that $\{x_n\}$ converges weakly to a point in $\mathcal{S}$. The last statement follows immediately from the above. □

4. Numerical experiments

In this section, we present the convergence of the algorithm proposed (**RGPA**) in this paper by the minimization problem with linear equality constraints. Firstly, we are given a linear system of the form

$$\mathbf{A}x = \mathbf{b},$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$. In addition, we assume that $n > m$. In this section, the system has many solutions. This leads us to the question of which solution should be considered. As a result, we may consider the following problem, say, the minimization problem with linear equality constraints.

$$\begin{aligned} &\text{minimize } \frac{1}{2}\|x\|^2 \\ &\text{subject to } \mathbf{A}x = \mathbf{b}, \end{aligned}$$

Table 1: Comparison of the convergence of (**RGPA**) and (**DGS**) for the parameters $K = 0.001$ and $q \in (\frac{1}{2}, 1)$.

q	(**RGPA**)		(**DGS**)	
	Time (sec)	#(Iters)	Time (sec)	#(Iters)
0.6	2.38	566	10.23	2221
0.7	2.31	568	107.78	25336
0.8	2.46	581	384.00	90636
0.9	44.96	11458	447.11	103487

or , equivalently,

$$\begin{aligned}&\text{minimize } \frac{1}{2}\|x\|^2\\&\text{subject to } x \in \arg\min \frac{1}{2}\|\mathbf{A}(\cdot) - \mathbf{b}\|^2.\end{aligned}$$

The above problem can be written in the form of the problem (1.1) as

$$\begin{aligned}&\text{minimize } f(x) := \frac{1}{2}\|x\|^2\\&\text{subject to } g(x) := \frac{1}{2}\|\mathbf{A}(x) - \mathbf{b}\|^2.\end{aligned}$$

In this setting, we have $\nabla f(x) = x$ and notice that ∇f is 1-Lipschitz continuous.

Furthermore, we get that $\nabla g(x) = \mathbf{A}^\top(\mathbf{A}x - \mathbf{b})$ and notice that ∇g is $\|\mathbf{A}\|^2$-Lipschitz continuous.

All the numerical experiments were performed under MATLAB (R2015b). We begin with the problem by random matrix $\mathbf{A}$ in $\mathbb{R}^{m\times n}$, vector $x_1 \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$ with $m = 1000$ and $n = 4000$ generated by using MATLAB command **randi**, which the entries of $\mathbf{A}$, x_1 and $\mathbf{b}$ are integer in [-10,10]. Next, we are going to compare a performance of the algorithms (**RGPA**) and (**DGS**). The choice of the parameters for the computational experiment is based on Remark 2.2. We chooses $\gamma = \frac{1}{4\|\mathbf{A}\|^2}$ and $\delta = 1$. We consider different choices of the parameters $K \in (0, 1]$ and $q \in (\frac{1}{2}, 1)$. We will terminate the algorithms (**RGPA**) and (**DGS**) when the errors become small, i.e.,

$$\|x_n - x^*\| \le 10^{-6},$$

where $x^* = \mathbf{A}^\top(\mathbf{A}\mathbf{A}^\top)^{-1}\mathbf{b}$.

In Table 1 we present a comparison of the convergence between two algorithms including (**RGPA**) and (**DGS**) for the parameters $K = 0.001$ and different choices for the parameters $q \in (\frac{1}{2}, 1)$. We observe that when $q = 0.6$ leads to lowest computation time for (**RGPA**) and (**DGS**) with 2.38 second and 10.23 second, respectively. Furthermore, we also observe that (**DGS**) hit a big number of iterations than (**RGPA**) for all choices of parameter q.

In Table 2 we present a comparison of the convergence of (**RGPA**) and (**DGS**) for the parameters $q = 0.6$ and $K \in (0, 1]$. We observe that the number of iterations and computation time for (**RGPA**) smaller than the number of iterations for (**DGS**) for each choice of parameters K. Furthermore, (**RGPA**) needs tiny computation time to reach the optimality tolerance than (**DGS**) for each choice of parameter K.

We observe that our algorithm (**RGPA**) performs an advantage behavior when comparing with algorithm (**DGS**) for all different choices of parameters. Note that the number of iterations for (**RGPA**) smaller than the number of iterations for (**DGS**). Furthermore, (**RGPA**) needs tiny computation time to reach optimality tolerance than (**DGS**) for each different choice of parameters.

5. Conclusions

We have presented a new gradient penalty scheme, say, *rapid gradient penalty algorithm* **(RGPA)**. We provide sufficient conditions to guarantee the convergences of **(RGPA)** for the considered con-

Table 2: Comparison of the convergence of (**RGPA**) and (**DGS**) for the parameters $q = 0.6$ and $K \in (0, 1]$.

K	(**RGPA**)		(**DGS**)	
	Time (sec)	#(Iters)	Time (sec)	#(Iters)
0.001	2.38	566	10.23	2221
0.005	2.40	585	171.46	40888
0.01	6.63	1612	254.93	64469
0.05	83.22	20480	288.39	65722
0.1	107.41	26257	212.02	52464
0.5	79.95	18606	100.33	24419
1	51.46	13414	67.20	16616

strained convex optimization problem (1.1). We also provide a numerical example to compare the performance of the algorithms (**RGPA**) and (**DGS**). As a result, we observe that our algorithm (**RGPA**) performs an advantage behavior when comparing with (**DGS**) for all different choices of parameters.

Acknowledgements

The second author would like to thank Naresuan University and The Thailand Research Fund for financial support. Moreover, N. Artsawang is also supported by The Royal Golden Jubilee Program under Grant PHD/0158/2557, Thailand.

Disclosure statement

The authors declare that there is no conflict of interests regarding the publication of this paper.

Funding

N. Artsawang was supported by the Thailand Research Fund through the Royal Golden Jubilee PhD Program under Grant PHD/0158/2557, Thailand.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Approximation by modified Lupaş operators based on (p, q)–integers

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Abstract

The purpose of this paper is to introduce a new modification of Lupaş operators in the frame of post quantum setting and to investigate their approximation properties. First using the relations between q-calculus and post quantum calculus, the post quantum analogue of operators constructed will be linear and positive but will not follow Korovkin's theorem. Hence a new modification of q-Lupaş operators is constructed which will preserve test functions. Based on these modification of operators, approximation properties have been investigated. Further, the rate of convergence of operators by mean of modulus of continuity and functions belonging to the Lipschitz class as well as Peetre's K-functional are studied.

Keywords and phrases: Lupaş operators; Post quantum analogue; q analogue; Peetre's K-functional; Korovkin's type theorem; Convergence theorems.

AMS Subject Classification (2010): 41A10, 41A25, 41A36.

1. Introduction and preliminaries

A. Lupaş [17] introduced the linear positive operators at the International Dortmund Meeting held in Witten (Germany, March, 1995) as follows:

$$L_m(f;u) = (1-a)^{-mu}\sum_{l=0}^{\infty}\frac{(mu)_l a^l}{l!} f\left(\frac{l}{m}\right),\ u \geq 0, \tag{1.1}$$

with $f : [0, \infty) \to \mathbb{R}$. If we impose $L_m(u) = u$, we get $a = \frac{1}{2}$. Thus operators (1.1) becomes

$$L_m(f;u) = 2^{-mu}\sum_{l=0}^{\infty}\frac{(mu)_l}{l!2^l} f\left(\frac{l}{m}\right),\ u \geq 0, \tag{1.2}$$

where $(mu)_l$ is the rising factorial defined as:

$$(mu)_0 = 1,\ (mu)_l = mu(mu+1)(mu+2)\cdots(mu+l-1),\ l \geq 0.$$

The q-analogue of Lupaş operators (1.2) is defined in [26] as:

$$L_m^{p,q}(f;u) = 2^{-[m]_q u}\sum_{l=0}^{\infty}\frac{([m]_q u)_l}{[l]!2^l} f\left(\frac{[l]_q}{[m]_q}\right),\ u \geq 0. \tag{1.3}$$

2. CONSTRUCTION OF NEW OPERATORS AND AUXILIARY RESULTS

Let us recall certain notations and definitions of (p,q)-calculus. Let $p>0,\ q>0$. For each non negative integer $l,\ m,\ m\geq l\geq 0$, the (p,q)-integer, (p,q)-binomial are defined, as

$$[j]_{p,q}=p^{j-1}+p^{j-2}q+p^{j-3}q^2+...+pq^{j-2}+q^{j-1}=\begin{cases}\frac{p^j-q^j}{p-q}, & \text{when } p\neq q\neq 1,\\ j\ p^{j-1}, & \text{when } p=q\neq 1,\\ [j]_q, & \text{when } p=1,\\ j, & \text{when } p=q=1.\end{cases}$$

where $[j]_q$ denotes the q-integers and $m=0,1,2,\cdots$.

The formula for (p,q)-binomial expansion is as follows:

$$(au+bv)^m_{p,q}:=\sum_{l=0}^{m}p^{\frac{(m-l)(m-l-1)}{2}}q^{\frac{l(l-1)}{2}}\left[\begin{array}{c}m\\ l\end{array}\right]_{p,q}a^{m-l}b^lu^{m-l}v^l,$$

$$(u+v)^m_{p,q}=(u+v)(pu+qv)(p^2u+q^2v)\cdots(p^{m-1}u+q^{m-1}v),$$

$$(1-u)^m_{p,q}=(1-u)(p-qu)(p^2-q^2u)\cdots(p^{m-1}-q^{m-1}u),$$

where (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c}m\\ l\end{array}\right]_{p,q}=\frac{[m]_{p,q}!}{[l]_{p,q}![m-l]_{p,q}!}.$$

Details on (p,q)-calculus can be found in [9, 11, 21].

In the case of $p=1$, the above notations reduce to q-analogues and one can easily see that $[m]_{p,q}=p^{m-1}[m]_{q/p}$. Mursaleen et al. [21] introduced (p,q)-calculus in approximation theory and constructed post quantum analogue of Bernstein operators. On the other hand Khalid and Lobiyal defined the (p,q)- analogue of Lupaş Bernstein operators in [12] and have shown its application in computer aided geometric design for construction of Beizer curves and surfaces. For another applications of extra parameters p in the field of approximation on compact disk, one can refer [4]. For related literature, one can refer [1, 2, 9, 3, 13, 14, 18, 19, 20, 22, 23, 25, 24] papers based on q and (p,q) integers in approximation theory and CAGD. Motivated by the above mentioned work, we introduce a new analogue of Lupaş operators. The post quantum analogue of (1.3) are as follows:

Definition 2.1. Let $f:[0,\infty)\to\mathbb{R}$, $0<q<p\leq 1$ and for any $m\in\mathbb{N}$. we define the (p,q)-analogue of Lupaş operators as

$$L^{p,q}_m(f;u)=2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}f\left(\frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}}\right),\ u\geq 0. \tag{2.1}$$

The operators (2.1) are linear and positive. For $p=1$, the operators (2.1) turn out to be q-Lupaş operators defined in (1.3). Next, we prove some auxiliary results for (2.1).

Lemma 2.2. *Let* $0<q<p\leq 1$ *and* $m\in\mathbb{N}$. *We have*

(i) $L^{p,q}_m(1;u)=1$,
(ii) $L^{p,q}_m(t;u)=\frac{u}{p^{m-1}(2-p)^{[m]_{p,q}u+1}}$,
(iii) $L^{p,q}_m(t^2;u)=\frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}}+\frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}}+\frac{qu}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}$.

Proof. we have

(i)

$$L_m^{p,q}(1;u) = 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} = 1.$$

(ii)

$$\begin{aligned}
L_m^{p,q}(t;u) &= 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} \frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}} \\
&= 2^{-[m]_{p,q}u} \sum_{l=1}^{\infty} \frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l} \frac{p^{l-m}[l]_{p,q}}{[m]_{p,q}} \\
&= 2^{-[m]_{p,q}u} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}} p^{l+1-m} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{m-1}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l p^l}{[l]_{p,q}!2^l} \\
&= \frac{u}{(p^{m-1})(2-p)^{([m]_{p,q}u+1)}}.
\end{aligned}$$

(iii)

$$\begin{aligned}
L_m^{p,q}(t^2;u) &= 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} \frac{p^{2l-2m}[l]_{p,q}^2}{[m]_{p,q}^2} \\
&= 2^{-[m]_{p,q}u} \sum_{l=1}^{\infty} \frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!a^l} \frac{p^{2l-2m}[l]_{p,q}^2}{[m]_{p,q}^2} \\
&= 2^{-[m]_{p,q}u} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}} \frac{p^{2l+2-2m}[l+1]_{p,q}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{2l}[l+1]_{p,q}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{2l}[p^l+q[l]_{p,q}]}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{3l}}{[m]_{p,q}} \\
&= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}} u \sum_{l=0}^{\infty} \frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l} \frac{p^{2l}q[l]_{p,q}}{[m]_{p,q}} \\
&= I_1 + I_2(say),
\end{aligned}$$

□

we find that I_1 and I_2 are

$$\begin{aligned}
I_1 &= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{p^{3l}}{[m]_{p,q}}\\
&= \frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}}.\\
I_2 &= \frac{2^{-[m]_{p,q}u-1}}{p^{2m-2}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{p^{2l}q[l]_{p,q}}{[m]_{p,q}}\\
&= \frac{2^{-[m]_{p,q}u-1}([m]_{p,q}u+1)}{p^{2m-2}}qu\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+2)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l}\frac{p^{2l}[l]_{p,q}}{[m]_{p,q}}\\
&= \frac{2^{-[m]_{p,q}u-1}([m]_{p,q}u+1)}{[m]_{p,q}p^{2m-2}}qu\sum_{l=1}^{\infty}\frac{([m]_{p,q}u+2)_{l-1}p^{2l}}{[l-1]_{p,q}!2^l}\\
&= \frac{2^{-[m]_{p,q}u-2}([m]_{p,q}u+1)}{[m]_{p,q}p^{2m-4}}qu\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+2)_l p^{2l}}{[l]_{p,q}!2^l}\\
&= \frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}}+\frac{qu}{p^{2n-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}.
\end{aligned}$$

On adding $I_1$and I_2, we get

$$L_m^{p,q}(t^2;u) = \frac{u}{[m]_{p,q}p^{2m-2}(2-p^3)^{[m]_{p,q}u+1}}+\frac{qu^2}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}}+\frac{qu}{p^{2m-4}(2-p^2)^{[m]_{p,q}u+2}[m]_{p,q}}.$$

The sequence of (p,q)-Lupaş operators constructed in (2.1) however do not preserve the test functions t and t^2. Hence one can not guarantee approximation via these operators. Therefore, we construct the modified (p,q)- Lupaş operators as follows:

Lemma 2.3. *Let $0<q<p\leq 1$ and $m\in\mathbb{N}$. For $f:[0,\infty)\to\mathbb{R}$, we define the (p,q)-analogue of Lupaş operators as:*

$$\widetilde{L}_m^{p,q}(f;u) = 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!a^l}f\left(\frac{[l]_{p,q}}{[m]_{p,q}}\right),\ u\geq 0. \tag{2.2}$$

The operators (2.2) are linear and positive. For $p=1$, the operators (2.2) turn out to be q-Lupaş operator defined in (1.3).
We shall investigate approximation properties of the operators (2.2). We obtain rate of convergence of the operators via modulus of continuities. We also obtain approximation behaviors of the operators for functions belonging to Lipschitz spaces.

Lemma 2.4. *Let $0<q<p\leq 1$ and $m\in\mathbb{N}$. We have*

(i) $\widetilde{L}_m^{p,q}(1;u)=1$,
(ii) $\widetilde{L}_m^{p,q}(t;u)=u$,
(iii) $\widetilde{L}_m^{p,q}(t^2;u)=\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}+\frac{qu}{[m]_{p,q}}+qu^2$.

Proof. We have

(i)

$$\widetilde{L}_m^{p,q}(1;u) = 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} = 1.$$

(ii)

$$\begin{aligned}
\widetilde{L}_m^{p,q}(t;u) &= 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\frac{[l]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u}\sum_{l=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l}\frac{[l]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u-1}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\\
&= u.
\end{aligned}$$

(iii)

$$\begin{aligned}
\widetilde{L}_m^{p,q}(t^2;u) &= 2^{-[m]_{p,q}u}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l}\frac{[l]_{p,q}^2}{[m]_{p,q}^2}\\
&= 2^{-[m]_{p,q}u}\sum_{l=1}^{\infty}\frac{([m]_{p,q}u)([m]_{p,q}u+1)_{l-1}}{[l]_{p,q}[l-1]_{p,q}!2^l}\frac{[l]_{p,q}^2}{[m]_{p,q}^2}\\
&= 2^{-[m]_{p,q}u}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^{l+1}}\frac{[l+1]_{p,q}}{[m]_{p,q}}\\
&= 2^{-[m]_{p,q}u-1}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}\frac{[p^l+q[l]_{p,q}]}{[m]_{p,q}}\\
&= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}p^l\\
&+ \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l[l]_{p,q}}{[l]_{p,q}!2^l}\\
&= I_1+I_2(Say).
\end{aligned}$$

After solving I_1 and I_2, we get

$$\begin{aligned}
I_1 &= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}u\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l}{[l]_{p,q}!2^l}p^l\\
&= \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}.\\
I_2 &= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+1)_l[l]_{p,q}}{[l]_{p,q}!2^l}\\
&= \frac{2^{-[m]_{p,q}u-1}}{[m]_{p,q}}qu\sum_{l=1}^{\infty}\frac{([m]_{p,q}u+1)([m]_{p,q}u+2)_{l-1}[l]_{p,q}}{[l]_{p,q}[l-1]_{p,q}!2^l}\\
&= \frac{2^{-[m]_{p,q}u-2}([m]_{p,q}u+1)qu}{[n]_{p,q}}\sum_{l=0}^{\infty}\frac{([m]_{p,q}u+2)_l}{[l]_{p,q}!2^l}\\
&= \frac{qu}{[m]_{p,q}}+qu^2.
\end{aligned}$$

On adding $I_1$and I_2, we get

$$\widetilde{L}_m^{p,q}(t^2;u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}+\frac{qu}{[m]_{p,q}}+qu^2.$$

□

Corollary 2.5. *Using Lemma 2.4, we get the following central moments.*

$\widetilde{L}_n^{p,q}(t-u;u)=0$
$\widetilde{L}_n^{p,q}((t-u)^2;u)=\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}}+\frac{qu}{[m]_{p,q}}+qu^2-u^2=\delta_m(u)$ *(say).*

Remark 2.6. One can observe that

$$\lim_{m\to\infty}[m]_{p,q}=\begin{cases}0, & \text{p,q}\in(0,1),\\ \frac{1}{1-q}, & p=1 \text{ and } q\in(0,1).\end{cases}$$

Thus for approximation processes, one need to choose convergent sequences (p_m) and (q_m) such that for each n, $0<q_m<p_m\leq 1$ and $p_m,q_m\to 1$ so that $[m]_{p_m,q_m}\to\infty$ as $m\to\infty$.

Theorem 2.7. *Let $f\in C_B[0,\infty)$ and $q_m\in(0,1)$, $p_m\in(q_m,1]$ such that $q_m\to 1$, $p_m\to 1$, as $m\to\infty$. Then for each $u\in[0,\infty)$ we have*

$$\lim_{n\to\infty}\widetilde{L}_m^{p_m,q_m}(f;u)=f(u).$$

Proof. By Korovkin's theorem it is enough to show that

$$\lim_{m\to\infty}\widetilde{L}_m^{p_m,q_m}(t^m;u)=u^m,\ m=0,1,2.$$

By Lemma 2.4, it is clear that

$$\lim_{m\to\infty}\widetilde{L}_m^{p_m,q_m}(1;u)=1$$

$$\lim_{m\to\infty}\widetilde{L}_m^{p_m,q_m}(1;u)=u$$

and

$$\lim_{m\to\infty}\widetilde{L}_m^{p_m,q_m}(t^2;u)=\lim_{m\to\infty}\left[\frac{u}{(2-p_m)^{([m]_{p_m,q_m}u+1)}[m]_{p_m,q_m}}+\frac{q_m u}{[m]_{p_m,q_m}}+q_m u^2\right]$$

$$=u^2.$$

This completes the proof. □

3. Direct results

Let $C_B[0,\infty)$ be the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \|=\sup_{0\leq x<\infty}\mid f(x)\mid.$$

Let us consider the K-functional as:

$$K_2(f,\delta)=\inf_{s\in W^2}\{\| f-s\|+\delta\| s''\|\},$$

where $\delta>0$ and $W^2=\{s\in C_B[0,\infty):s',s''\in C_B[0,\infty)\}$.
Then as in ([4], p. 177, Theorem 2.4), there euists an absolute constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}). \tag{3.1}$$

Second order modulus of smoothness of $f \in C_B[0,\infty)$ is as follows

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{u\in[0,\infty)} \mid f(u+2h) - 2f(u+h) + f(u) \mid .$$

The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{u\in[0,\infty)} \mid f(u+h) - f(u) \mid .$$

Theorem 3.1. *Let $f \in C_B[0,\infty)$, $p,q \in (0,1)$ such that $0 < q < p \leq 1$. Then for every $u \in [0,\infty)$ we have*

$$\mid \widetilde{L}_m^{p,q}(f;u) - f(u) \mid \leq C\omega_2(f;\delta_m(u)),$$

where

$$\delta_m^2(u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2.$$

Proof. Let $s \in \mathcal{W}^2$. Then from Taylor's expansion, we get

$$s(t) = s(u) + s'(u)(t-u) + \int_u^t (t-u)s''(u)\mathrm{d}u,\ t \in [0,\mathcal{A}],\ \mathcal{A} > 0.$$

Now by Corollary 2.5, we have

$$\widetilde{L}_m^{p,q}(s;u) = s(u) + \widetilde{L}_m^{p,q}\left(\int_u^t (t-u)s''(u)\mathrm{d}u; u\right).$$

$$\begin{aligned} |\widetilde{L}_m^{p,q}(s;u)(s;u) - s(u)| &\leq \widetilde{L}_m^{p,q}\left(\left|\int_u^t \mid (t-u) \mid \mid s''(u) \mid \mathrm{d}u; u\right|\right) \\ &\leq \widetilde{L}_m^{p,q}\left((t-u)^2; u\right) \parallel s'' \parallel, \end{aligned}$$

hence we get

$$|\widetilde{L}_m^{p,q}(s;u(s;u) - s(u)| \leq \parallel s'' \parallel \left(\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2\right).$$

By Lemma 2.3, we have

$$|\widetilde{L}_m^{p,q}(f;u)| \leq 2^{-[m]_{p,q}u} \sum_{l=0}^{\infty} \frac{([m]_{p,q}u)_l}{[l]_{p,q}!2^l} \left| f\left(\frac{[l]_{p,q}}{[m]_{p,q}}\right)\right| \leq \parallel f \parallel .$$

Thus, we have

$$|\widetilde{L}_m^{p,q}(f;u)| \ \leq \mid \widetilde{L}_m^{p,q}\left((f-s);u\right) - (f-s)(u) \mid + |\widetilde{L}_m^{p,q}(s;u) - s(u)|.$$

After substituting all values, we get

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \ \leq \parallel f - s \parallel + \parallel s'' \parallel \left(\frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2\right).$$

By taking the infimum on the right hand side over all $s \in \mathcal{W}^2$, we get

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \ \leq \ \mathcal{C}K_2\left(f, \delta_m^2(u)\right).$$

By using the property of K-functional, we have

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \ \leq \ \mathcal{C}\omega_2\left(f, \delta_m(u)\right).$$

This completes the proof. □

4. Pointwise estimates

Theorem 4.1. *Let $0 < \alpha \leq 1$ and Ȩ be any bounded subset of the interval $[0, \infty)$. If $f \in C_B[0, \infty)$, is locally Lip(α), i.e., the condition*

$$|f(v) - f(u)| \ \leq \text{Ł}|v - u|^{\alpha}, \ v \in \text{Ȩ} \text{ and } u \in [0, \infty) \tag{4.1}$$

holds, then, for each $u \in [0, \infty)$, we have

$$|\widetilde{L}_m^{p,q}(f; u) - f(u)| \ \leq \text{Ł}\left\{\delta_m(u)^{\frac{\alpha}{2}} + 2(d(u, \text{Ȩ}))^{\alpha}\right\}, \ u \in [0, \infty)$$

where L is a constant depending on α and f and $d(u; \text{Ȩ})$ is the distance between u and Ȩ defined by

$$d(u, \text{Ȩ}) = inf\left\{|t - u|; t \in E\right\} \text{and } \delta_m(u) = \widetilde{L}_m^{p,q}((t - u)^2; u).$$

Proof. Let $\overline{\text{Ȩ}}$ be the closure of Ȩ in $[0, 1)$. Then, there exists a point $t_0 \in \overline{\text{Ȩ}}$ such that $d(u, \text{Ȩ}) = |u - t_0|$.
Using the triangle inequality, we have

$$|f(t) - f(u)| \ \leq |f(t) - f(t_0)| + |f(t_0) - f(u)|.$$

By using (4.1) we get,

$$\begin{aligned}
|\widetilde{L}_m^{p,q}(f; u) - f(u)| &\leq \widetilde{L}_m^{p,q}\left(|f(t) - f(t_0)|; u\right) + \widetilde{L}_m^{p,q}\left(|f(u) - f(t_0)|; u\right) \\
&\leq \text{Ł}\left\{\widetilde{L}_m^{p,q}\left(|t - t_0|^{\alpha}; u\right) + (|u - t_0|^{\alpha}; u) + |u - t_0|^{\alpha}\right\} \\
&\leq \text{Ł}\left\{\widetilde{L}_m^{p,q}\left(|t - u|^{\alpha}; u\right) + 2|u - t_0|^{\alpha}\right\}.
\end{aligned}$$

By choosing $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we get $\frac{1}{p} + \frac{1}{q} = 1$. Then by using Hölder's inequality we get

$$\begin{aligned}
|\widetilde{L}_m^{p,q}(f; u) - f(u)| &\leq \text{Ł}\left\{\widetilde{L}_m^{p,q}\left(|t - u|^{\alpha p}; u\right)^{\frac{1}{p}}\left[\widetilde{L}_m^{p,q}(1^q; u)\right]^{\frac{1}{q}} + 2(d(u, \text{Ȩ}))^{\alpha}\right\} \\
&\leq \text{Ł}\left\{\widetilde{L}_m^{p,q}\left(((t - u)^2; u)\right)^{\frac{\alpha}{2}} + 2(d(u, \text{Ȩ}))^{\alpha}\right\} \\
&\leq \text{Ł}\left\{\delta_m(u)^{\frac{\alpha}{2}} + 2(d(u, \text{Ȩ}))^{\alpha}\right\}.
\end{aligned}$$

Hence the proof is completed. □

Now, we recall local approximation in terms of α order Lipschitz-type maximal function given by Lenze [16] as

$$\widetilde{\omega}_{\alpha}(f; u) = \sup_{t \neq u, t \in (0, \infty)} \frac{|f(t) - f(u)|}{|t - u|^{\alpha}}, \ u \in [0, \infty) \text{ and } \alpha \in (0, 1]. \tag{4.2}$$

Then we get the next result

Theorem 4.2. *Let $f \in C_B[0, \infty)$ and $\alpha \in (0, 1]$. Then, for all $u \in [0, \infty)$, we have*

$$|\widetilde{L}_m^{p_m, q_m}(f; u) - f(u)| \ \leq \ \widetilde{\omega}_{\alpha}(f; u)\Big(\delta_m(u)\Big)^{\frac{\alpha}{2}},$$

where $\delta_m(u)$ is defined in Corollary 2.5.

Proof. We know that

$$|\widetilde{L}_m^{p_m, q_m}(f; u) - f(u)| \ \leq \ \widetilde{L}_m^{p_m, q_m}(|f(t) - f(u)|; u).$$

From equation (4.2), we have

$$|\widetilde{L}_m^{p_m, q_m}(f; u) - f(u)| \ \leq \ \widetilde{\omega}_{\alpha}(f; u)\widetilde{L}_m^{p_m, q_m}(|t - u|^{\alpha}; u).$$

From Hölder's inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we have

$$|\widetilde{L}_m^{p_m, q_m}(f; u) - f(u)| \ \leq \ \widetilde{\omega}_{\alpha}(f; u)\big(\widetilde{L}_m^{p_m, q_m}(|t - u|^2; u)\big)^{\frac{\alpha}{2}},$$

which proves the desired result. ☐

5. Weighted approximation by $\widetilde{L}_m^{p,q}$

In this section we shall discuss weighted approximation theorems for the operators $\widetilde{L}_m^{p,q}$ on the interval $[0,\infty)$.

Theorem 5.1 (cf. [5, 15])**.** *Let (T_m) be the sequence of linear positive operators from $C_{u^2}[0,\infty)$ to $B_{u^2}[0,\infty)$ satisfy*

$$\lim_{m\to\infty} \| T_m\kappa_i - \kappa_i \|_{u^2} = 0, \qquad\qquad i = 0,1,2.$$

Then for any function $f \in C^*_{u^2}[0,\infty)$

$$\lim_{m\to\infty} \| T_m f - f \|_{u^2} = 0.$$

Theorem 5.2 (cf. [6, 7])**.** *Let (q_m) and (p_m) be two sequences such that $0 < q_m < p_m \le 1$, for all n and both converge to 1. Then for each function $f \in C^*_{u^2}[0,\infty)$, we get*

$$\lim_{m\to\infty} \| L_m^{p_m,q_m} f - f \|_{u^2} = 0.$$

Proof. By Theorem 5.1, it is enough to show

$$\lim_{m\to\infty} \| \widetilde{L}_m^{p_m,q_m}\kappa_i - \kappa_i \|_{u^2} \quad = \quad 0, \qquad i = 0,1,2. \tag{5.1}$$

By Lemma 2.4 (i) and (ii), it is clear that

$$\begin{aligned} \| L_m^{p_m,q_m}(1;u) - 1 \|_{u^2} &= 0 \\ \| \widetilde{L}_m^{p_m,q_m}(t;u) - u \|_{u^2} &= 0 \end{aligned}$$

and by Lemma 2.4 (iii), we have

$$\begin{aligned} \| \widetilde{L}_m^{p_m,q_m}(t^2;u) - u^2 \|_2 &= \sup_{u\in[0,\infty)} \frac{\left(\frac{1}{(2-p_m)^{([m]_{p_m,q_m}u+1)}[m]_{p_m,q_m}} + \frac{q_m}{[m]_{p_m,q_m}}\right)u + (q_m-1)u^2}{1+u^2} \\ &\le \left(\frac{1}{(2-p_m)[m]_{p_m,q_m}} + \frac{q_m}{[m]_{p_m,q_m}}\right) + (q_m-1). \end{aligned}$$

Last inequality means that (5.1) holds for $i = 2$. By Theorem 5.1, the proof is completed. ☐

Theorem 5.3. *Let $q_m \in (0,1)$, $p_m \in (q,1]$ such that $q_m \to 1$, $p_m \to 1$ as $m \to \infty$. Let $f \in C^*_{u^2}[0,\infty)$, and its modulus of continuity $\omega_{d+1}(f;\delta)$ be defined on $[0,d+1] \subset [0,\infty)$. Then, we have*

$$|\widetilde{L}_m^{p_m,q_m}(f;u) - f(u)\|_{C[0,d]} \le 6M_f(1+d^2)\delta_m(d) + 2\omega_{d+1}(f;\sqrt{\delta_m(d)}),$$

where $\delta_m(d) = \widetilde{L}_m^{p,q}((t-u)^2;u) = \frac{u}{(2-p)^{([m]_{p,q}u+1)}[m]_{p,q}} + \frac{qu}{[m]_{p,q}} + qu^2 - u^2.$

Proof. From ([10] p. 378), for $u \in [0,d]$ and $t \in [0,\infty)$, we have

$$|f(t) - f(u)| \le 6M_f(1+d^2)(t-u)^2 + \left(1 + \frac{|t-u|}{\delta}\right)\omega_{d+1}(f;\delta).$$

Applying $\widetilde{L}_m^{p,q}$ both the sides, we have

$$|\widetilde{L}_m^{p,q}(f;u) - f(u)| \le 6M_f(1+d^2)\widetilde{L}_m^{p,q}((t-u)^2;u) + \left(1 + \frac{\widetilde{L}_m^{p,q}(|t-u|;u)}{\delta}\right)\omega_{d+1}(f;\delta).$$

Applying Cauchy-Schwarz inequality,for $u \in [0,d]$ and $t \ge 0$, we get

$$\begin{aligned} |\widetilde{L}_m^{p,q}(f;u) - f(u)| &\le \widetilde{L}_m^{p,q}\left(|(f;u) - f(u)|;u\right) \\ &\le 6M_f(1+d^2)\widetilde{L}_m^{p,q}((t-u)^2;u) \\ &+ \omega_{d+1}(f;\delta)\left(1 + \frac{1}{\delta}\widetilde{L}_m^{p,q}\left((t-u)^2;u\right)^{\frac{1}{2}}\right). \end{aligned}$$

Thus, from Lemma 2.4, for $u \in [0,d]$, we get

$$|\widetilde{L}_m^{p;q}(f;u)-f(u)| \leq 6M_f(1+d^2)\delta_m(d)+\omega_{d+1}(f;\delta)\left(1+\frac{\sqrt{\delta_m(d)}}{\delta}\right).$$

By Choosing $\delta=\sqrt{\delta_m(d)}$, we get the required result. □

Now, we prove a theorem to approximate all functions in $C^*_{u^2}$ Such type of results are given in [8] for locally integrable functions.

Theorem 5.4. *Let* $0<q_m<p_m\leq 1$ *such that* $q_m \to 1$, $p_m \to 1$ *as* $m\to\infty$. *Then for each function* $f\in C^*_{u^2}[0,\infty)$, *and* $\alpha>1$

$$\lim_{m\to\infty}\sup_{u\in[0,\infty)}\frac{|\widetilde{L}_m^{p_m,q_m}(f;u)-f(u)|}{(1+u^2)^\alpha}=0.$$

Proof. Let for any fixed $u_0>0$,

$$\begin{aligned}\sup_{u\in[0,\infty)}\frac{|\widetilde{L}_m^{p_m,q_m}(f;u)-f(u)|}{(1+u^2)^\alpha} &\leq \sup_{u\leq u_0}\frac{|\widetilde{L}_m^{p_m,q_m}(f;u)-f(u)|}{(1+u^2)^\alpha}+\sup_{u\geq u_0}\frac{|\widetilde{L}_m^{p_m,q_m}(f;u)-f(u)|}{(1+u^2)^\alpha}\\ &\leq \| L_m^{p_m,q_m}(f)-f\|_{[c_0,u_0]}+\|f\|_{u^2}\sup_{u\leq u_0}\frac{|\widetilde{L}_m^{p_m,q_m}(1+t^2;u)|}{(1+u^2)^\alpha}\\ &+\sup_{u\geq u_0}\frac{|f(u)|}{(1+u^2)^\alpha}. \end{aligned}\tag{5.2}$$

Since, $|f(u)|\leq M_f(1+u^2)$ we have,

$$\sup_{u\geq u_0}\frac{|f(u)|}{(1+u^2)^\alpha}\leq\sup_{u\geq u_0}\frac{M_f}{(1+u^2)^{\alpha-1}}\leq\frac{M_f}{(1+u^2)^{\alpha-1}}.$$

Let $\epsilon>0$, and let us choose u_0 large then we have

$$\frac{M_f}{(1+{u_0}^2)^{\alpha-1}}<\frac{\epsilon}{3}\tag{5.3}$$

and in view of (2.4), we get,

$$\begin{aligned}\|f\|_{u^2}\lim_{m\to\infty}\frac{|\widetilde{L}_m^{p_m,q_m}(1+t^2;u)|}{(1+u^2)^\alpha} &= \|f\|_{u^2}\frac{1+u^2}{(1+u^2)^\alpha}\\ &\leq \frac{\|f\|_{u^2}}{(1+u^2)^{\alpha-1}}\\ &\leq \frac{\|f\|_{u^2}}{(1+{u_0}^2)^{\alpha-1}}\\ &\leq \frac{\epsilon}{3}.\end{aligned}\tag{5.4}$$

By using Theorem 5.3, the first term of inequality (5.2) becomes

$$\|\widetilde{L}_m^{p_m,q_m}(f)-f\|_{[c_0,u_0]}<\frac{\epsilon}{3}, \text{as } m\to\infty.\tag{5.5}$$

Hence we get the required proof by combining (5.3)-(5.5)

$$\sup_{u\in[0,\infty)}\frac{|\widetilde{L}_m^{p_m,q_m}(f;u)-f(u)|}{(1+u^2)^\alpha}<\epsilon.$$

□

Acknowledgment. The third author (M. Qasim) is grateful to Council of Scientific and Industrial Research (CSIR), India, for providing the Senior Research Fellowship under the grant number 09/1172(0001)/2017-EMR-I.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

ADDITIVE-QUADRATIC FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES AND STABILITY

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Abstract. In this paper, we investigate the following functional inequality

$$N(f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)]$$
$$-f(-x)-f(-y)-f(-z),t)\geq N(f(x+y+z),t)$$

and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

1. Introduction and preliminaries

The concept of a fuzzy norm on a linear space was introduced by Katsaras [11] in 1984. Later, Cheng and Mordeson [3] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13].

Definition 1.1. Let X be a real vector space. A function $N : X\times\mathbb{R}\longrightarrow[0,1]$ is called *a fuzzy norm on* X if for all $x, y\in X$ and all $c, s, t\in\mathbb{R}$,

(N1) $N(x,\ t) = 0$ for all $t\leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x,\frac{t}{|c|})$ if $c\neq 0$;
(N4) $N(x+y,s+t)\geq\min\{N(x,s),N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty}N(x,t) = 1$;
(N6) for any $x\neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.

In this case, the pair (X,N) is called *a fuzzy normed space.*

Let (X,N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* if there exists an $x\in X$ such that $\lim_{n\to\infty}N(x_n-x,t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* (X,N) *and one denotes it by* $N-\lim_{n\to\infty}x_n = x$. *A sequence* $\{x_n\}$ *in* (X,N) *is said to be Cauchy if for any* $\epsilon > 0$, *there is an* $m\in N$ *such that for any* $n\geq m$ *and any positive integer* p, $N(x_{n+p}-x_n,t) > 1-\epsilon$ *for all* $t > 0$. *A fuzzy normed space is said to be complete if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called a fuzzy Banach space.*

In 1940, Ulam proposed the following stability problem (cf.[21]*):*

"Let G_1 *be a group and* G_2 *a metric group with the metric* d. *Given a constant* $\delta > 0$, *does there exist a constant* $c > 0$ *such that if a mapping* $f : G_1\longrightarrow G_2$ *satisfies* $d(f(xy),f(x)f(y)) < c$ *for all* $x,y\in G_1$, *then there exists an unique homomorphism* $h : G_1\longrightarrow G_2$ *with* $d(f(x),h(x)) < \delta$ *for all* $x\in G_1$*?"*

2010 *Mathematics Subject Classification.* 39B62, 39B72, 54A40, 47H10.

Key words and phrases. Hyers-Ulam stability, additive-quadratic functional inequality, fuzzy normed space, fixed point theorem.

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In the next year, Hyers [10] *gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki* [1] *for additive mappings, and by Rassias* [17] *for linear mappings, to consider the stability problem with unbounded Cauchy differences. A generalization of the Rassias' theorem was obtained by Găvruta* [7] *by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach. In 2008, for the first time, Mirmostafaee and Moslehian* [14], [15] *used the definition of a fuzzy norm in* [2] *to obtain a fuzzy version of stability for the Cauchy functional equation*

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

Glányi [8] *and Rätz* [18] *showed that if a mapping* $f : X \longrightarrow Y$ *satisfies the following functional inequality*

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \le \|f(xy)\|, \tag{1.3}$$

then f *satisfies the following Jordan-Von Neumann functional equation*

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

for an abelian group X *divisible by* 2 *into an inner product space* Y. *Glányi* [9] *and Fechner* [6] *proved the Hyers-Ulam stability of (1.3). The stability problems of several functional equations and inequalities have been extensively investigated by a number of authors and there are many interesting results concerning the stability of various functional equations and inequalities.*

Now, we consider the following fixed point theorem on generalized metric spaces.

Definition 1.2. Let X be a non-empty set. Then a mapping $d : X^2 \longrightarrow [0, \infty]$ is called *a generalized metric on* X if d satisfies the following conditions:
(D1) $d(x, y) = 0$ if and only if $x = y$,
(D2) $d(x, y) = d(y, x)$, and
(D3) $d(x, y) \le d(x, z) + d(z, y)$.
In case, (X, d) is called *a generalized metric space.*

Theorem 1.3. [4] *Let* (X, d) *be a complete generalized metric space and let* $J : X \longrightarrow X$ *a strictly contractive mapping with some Lipschitz constant* L *with* $0 < L < 1$. *Then for each given element* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integers* n *or there exists a positive integer* n_0 *such that*
(1) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n \ge n_0$;
(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J ;
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ and
(4) $d(y, y^*) \le \dfrac{1}{1-L}\, d(y, Jy)$ *for all* $y \in Y$.

The following function equation $f : X \longrightarrow Y$ *is called the Drygas functional equation :*

$$f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y)$$

for all $x, y \in X$. The Drygas functional equation has been studied by Szabo [20] *and Ebanks, Făiziev and Sahoo* [5]. *The solutions of the Drygas functional equation in abelian group are obtained by H. Stetkær in* [19].

In this paper, we investigate the following functional inequality which is related with the Drygas type functional equation

$$\begin{aligned} &N(f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)] \\ &-f(-x)-f(-y)-f(-z),t) \geq N(f(x+y+z),t) \end{aligned} \tag{1.4}$$

and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

Throughout this paper, we assume that X is a linear space, (Y, N) is a fuzzy Banach space, and $(\mathbb{R}, N')$ is a fuzzy normed space.

2. Solutions and the stability for (1.4)

In this section, we investigate the functional equation (1.4) and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces. For any mapping $f : X \longrightarrow Y$, let

$$f_o(x) = \frac{f(x)-f(-x)}{2}, \quad f_e(x) = \frac{f(x)+f(-x)}{2}.$$

In [12], *the authors proved the following theorem:*

Lemma 2.1. [12] *Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$. Then f is quadratic if and only if f satisfies the following functional equation*

$$f(ax+by)+f(ax-by)-2a^2f(x)-2b^2f(y) = k[f(x+y)+f(x-y)-2f(x)-2f(y)]$$

for all $x, y \in X$, a fixed nonzero rational number a and fixed real numbers b, k with $a^2 \neq b^2$.

Using this, we have the following theorem:

Theorem 2.2. *If a mapping $f : X \longrightarrow Y$ saisfies (1.4), then f is an additive-quadratic mapping.*

Proof. Suppose that f satisfies (1.4). Setting $x = y = z = 0$ in (1.4), by (N3), we have

$$N(f(0),t) \leq N(6f(0),t) = N\Big(f(0), \frac{t}{6}\Big)$$

for all $t > 0$ and by (N5), $N(f(0), \frac{t}{6}) \leq N(f(0), t)$ for all $t > 0$. Hence we have

$$N(f(0),t) = N(f(0),6t)$$

for all $t > 0$. By induction, we get

$$N(f(0),t) = N(f(0),6^nt)$$

for all $t > 0$ and all $n \in \mathbb{N}$. By (N5), we get

$$N(f(0),t) = \lim_{n\to\infty} N(f(0),6^nt) = 1$$

for all $t > 0$ and hence by (N2), $f(0) = 0$. Letting $z = -x-y$ in (1.4), we have

$$\begin{aligned} &N(f(x-y)+f(x+2y)+f(-2x-y)-2f(x)-2f(y)-2f(-x-y) \\ &-f(-x)-f(-y)-f(x+y),t) \geq N(0,t) = 1 \end{aligned}$$

for all $x, y \in X$ and all $t > 0$ and so by (N2), we get

$$\begin{aligned} &f(x-y)+f(x+2y)+f(-2x-y)\\ &=2f(x)+2f(y)+2f(-x-y)+f(x+y)+f(-x)+f(-y)\end{aligned} \tag{2.1}$$

for all $x, y \in X$. By (2.1), we have

$$f_o(x-y)+f_o(x+2y)-f_o(2x+y)=-f_o(x+y)+f_o(x)+f_o(y) \tag{2.2}$$

for all $x, y \in X$ and interchanging x and y in (2.2), we have

$$-f_o(x-y)+f_o(2x+y)-f_o(x+2y)=-f_o(x+y)+f_o(x)+f_o(y) \tag{2.3}$$

for all $x, y \in X$. By (2.2) and (2.3), we have

$$f_o(x+y)=f_o(x)+f_o(y)$$

for all $x, y \in X$ and hence f_o is an additive mapping. By (2.1), we have

$$f_e(x-y)+f_e(2x+y)+f_e(x+2y)=3f_e(x+y)+3f_e(x)+3f_e(y) \tag{2.4}$$

for all $x, y \in X$ and letting $y=-y$ in (2.4), we have

$$f_e(x+y)+f_e(2x-y)+f_e(x-2y)=3f_e(x-y)+3f_e(x)+3f_e(y) \tag{2.5}$$

for all $x, y \in X$. By (2.4) and (2.5), we have

$$\begin{aligned} &f_e(2x+y)+f_e(2x-y)+f_e(x+2y)+f_e(x-2y)\\ &=2f_e(x+y)+2f_e(x-y)+6f_e(x)+6f_e(y)\end{aligned} \tag{2.6}$$

for all $x, y \in X$. Letting $y=0$ in (2.6), we get

$$f_e(2x)=4f_e(x) \tag{2.7}$$

for all $x \in X$ and letting $y=2y$ in (2.6), by (2.7), we have

$$\begin{aligned} &4f_e(x+y)+4f_e(x-y)+f_e(x+4y)+f_e(x-4y)\\ &=2f_e(x+2y)+2f_e(x-2y)+6f_e(x)+24f_e(y)\end{aligned} \tag{2.8}$$

for all $x, y \in X$. By (2.6) and (2.8), we have

$$2f_e(2x+y)+2f_e(2x-y)+f_e(x+4y)+f_e(x-4y)=18f_e(x)+36f_e(y) \tag{2.9}$$

for all $x, y \in X$. Letting $y=2y$ in (2.9), by (2.7), we have

$$f_e(x+8y)+f_e(x-8y)+8f_e(x+y)+8f_e(x-y)=18f_e(x)+144f_e(y)$$

for all $x, y \in X$. By Lemma 2.1, f_e is a quadratic mapping. Thus f is an additive-quadratic mapping. $\square$

Now, we will prove the generalized Hyers-Ulam stability of (1.4) in fuzzy normed spaces. For any mapping $f : X \longrightarrow Y$, *let*

$$Df(x,y,z)=f(x-y)+f(y-z)+f(z-x)-2[f(x)+f(y)+f(z)]-f(-x)-f(-y)-f(-z).$$

Theorem 2.3. *Assume that* $\phi : X^3 \longrightarrow [0,\infty)$ *is a function such that*

$$N'\Big(\phi\Big(\frac{x}{2},\frac{y}{2},\frac{z}{2}\Big),t\Big) \ge N'\Big(\frac{L}{4}\phi(x,y,z),t\Big) \tag{2.10}$$

for all $x, y, z \in X$, $t > 0$ *and some* L *with* $0 < L < 1$. *Let* $f : X \longrightarrow Y$ *be a mapping such that* $f(0) = 0$ *and*

$$N(Df(x,y,z),t) \ge \min\{N(f(x+y+z),t), N'(\phi(x,y,z),t)\} \tag{2.11}$$

for all $x, y, z \in X$ and all $t > 0$. Then there exists an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N\Big(f(x) - F(x), \frac{L}{4(1-L)}t\Big) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \tag{2.12}$$

for all $x \in X$ and all $t > 0$. Further, we have

$$F(x) = N - \lim_{n\to\infty} \Big[\frac{2^n(2^n+1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n-1)}{2} f\Big(-\frac{x}{2^n}\Big)\Big] \tag{2.13}$$

for all $x \in X$.

Proof. Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned} d(g, h) = \inf\{c \in [0, \infty) |\ & N(g(x) - h(x), ct) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\}, \\ & \forall x \in X, \forall t > 0\}. \end{aligned}$$

Then (S, d) is a complete metric space([16]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 3g(\frac{x}{2}) + g(-\frac{x}{2})$ for all $g \in S$ and all $x \in X$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (2.10), we have

$$\begin{aligned} & N(Jg(x) - Jh(x), cLt) \\ &= N\Big(3g\Big(\frac{x}{2}\Big) + g\Big(-\frac{x}{2}\Big) - 3h\Big(\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), cLt\Big) \\ &\geq \min\Big\{N\Big(g\Big(\frac{x}{2}\Big) - h\Big(\frac{x}{2}\Big), \frac{1}{4}cLt\Big), N\Big(g\Big(-\frac{x}{2}\Big) - h\Big(-\frac{x}{2}\Big), \frac{1}{4}cLt\Big)\Big\} \\ &\geq \min\Big\{N'\Big(\phi\Big(\frac{x}{2}, -\frac{x}{2}, 0\Big), \frac{1}{4}Lt\Big), N'\Big(\phi\Big(-\frac{x}{2}, \frac{x}{2}, 0\Big), \frac{1}{4}Lt\Big)\Big\} \\ &\geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence we have $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$ and so J is a strictly contractive mapping.

Putting $y = -x$ and $z = 0$ in (2.11), we get

$$N(f(2x) - 3f(x) - f(-x), t) \geq N'(\phi(x, -x, 0), t) \tag{2.14}$$

for all $x \in X$, $t > 0$ and hence

$$\begin{aligned} & N\Big(f(x) - Jf(x), \frac{L}{4}t\Big) = N\Big(f(x) - 3f\Big(\frac{x}{2}\Big) - f\Big(-\frac{x}{2}\Big), \frac{L}{4}t\Big) \\ &\geq N'\Big(\phi\Big(\frac{x}{2}, -\frac{x}{2}, 0\Big), \frac{L}{4}t\Big) \geq \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\} \end{aligned}$$

for all $x \in X$, $t > 0$ and so we have $d(f, Jf) \leq \frac{L}{4} < \infty$. By Theorem 1.3, there exists a mapping $F : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, F) \to 0$ as $n \to \infty$. By induction, we have

$$J^n f(x) = \frac{2^n(2^n+1)}{2} f\Big(\frac{x}{2^n}\Big) + \frac{2^n(2^n-1)}{2} f\Big(-\frac{x}{2^n}\Big)$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence we have (2.13).

Replacing x, y, z, and t by $\frac{x}{2^n}$, $\frac{y}{2^n}$, $\frac{z}{2^n}$, and $\frac{t}{2^{2n}}$ in (2.11), respectively, by (2.11), we have

$$\begin{aligned}
&N\Big(Df_e\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big)\\
&\geq \min\Big\{N\Big(Df\Big(\frac{x}{2^n},\frac{y}{2^n},-\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big),N\Big(Df\Big(-\frac{x}{2^n},-\frac{y}{2^n},\frac{z}{2^n}\Big),\frac{1}{2^{2n}}t\Big)\Big\}\\
&\geq \min\{N'(L^n\phi(x,y,z),t),N'(L^n\phi(-x,-y,z),t)\}
\end{aligned}\tag{2.15}$$

for all $x,y,z\in X$ and all $n\in\mathbb{N}$. Letting $n\to\infty$ in (2.15), we have

$$DF_e(x,y,z)=0$$

for all $x,y,z\in X$ and by Lemma 2.1, F_e is an quadratic mapping. Similarly, F_o is an additive mapping and thus F is an additive-quadratic mapping. Since $d(f,Jf)\leq\frac{L}{4}$, by Theorem 1.3, we have (2.12).

Now, we show the uniqueness of F. Let G be another additive-quadratic mapping with (2.12). Then clearly, G is a fixed point of J and

$$d(Jf,G)=d(Jf,JG)\leq Ld(f,G)\leq\frac{L^2}{4(1-L)}<\infty\tag{2.16}$$

and hence by (3) in Theorem 1.3, $F=G$. □

Similar to Theorem 2.3, we have the following threoem:

Theorem 2.4. *Assume that $\phi:X^3\longrightarrow[0,\infty)$ is a function such that*

$$N'(\phi(2x,2y,2z),t)\geq N'(2L\phi(x,y,z),t)\tag{2.17}$$

for all $x,y,z\in X$, $t>0$ and some L with $0<L<1$. Let $f:X\longrightarrow Y$ be a mapping with $f(0)=0$ and (2.11). Then there exists an unique additive-quadratic mapping $F:X\longrightarrow Y$ such that

$$N\Big(f(x)-F(x),\frac{1}{2(1-L)}t\Big)\geq\min\{N'(\phi(x,-x,0),t),N'(\phi(-x,x,0),t)\}\tag{2.18}$$

for all $x\in X$ and all $t>0$. Further, we have

$$F(x)=\lim_{n\to\infty}\Big[\frac{2^n+1}{2^{2n+1}}f(2^nx)-\frac{2^n-1}{2^{2n+1}}f(-2^nx)\Big]$$

for all $x\in X$.

Proof. Consider the set $S=\{g\mid g:X\longrightarrow Y\}$ and the generalized metric d on S defined by

$$\begin{aligned}
d(g,h)=\inf\{c\in[0,\infty)|\ &N(g(x)-h(x),ct)\geq\min\{N'(\phi(x,-x,0),t),N'(\phi(-x,x,0),t)\},\\
&\forall x\in X,\forall t>0\}.
\end{aligned}$$

Then (S,d) is a complete metric space([16]). Define a mapping $J:S\longrightarrow S$ by $Jg(x)=\frac{3}{8}g(2x)-\frac{1}{8}g(-2x)$ for all $g\in S$ and all $x\in X$. Let $g,h\in S$ and $d(g,h)\leq c$ for some $c\in[0,\infty)$. Then by (2.10), we have

$$\begin{aligned}
&N(Jg(x)-Jh(x),cLt)\\
&=N\Big(\frac{3}{8}g(2x)-\frac{1}{8}g(-2x)-\frac{3}{8}h(2x)+\frac{1}{8}h(-2x),cLt\Big)\\
&\geq\min\{N(g(2x)-h(2x),2cLt),N(g(-2x)-h(-2x),2cLt)\}\\
&\geq\min\{N'(\phi(2x,-2x,0),2Lt),N'(\phi(-2x,2x,0),2Lt)\}\\
&\geq\min\{N'\phi(x,-x,0),t),N'(\phi(-x,x,0),t)\}
\end{aligned}$$

for all $x \in X$. Hence we have $d(Jg, Jh) \le Ld(g, h)$ for any $g, h \in S$ and since $0 < L < 1$, J is a strictly contractive mapping. By (2.14), we get

$$\begin{aligned}&N\Big(f(x) - Jf(x), \frac{t}{2}\Big)\\ &= N\Big(\frac{3}{8}[f(2x) - 3f(x) - f(-x)] - \frac{1}{8}[f(-2x) - 3f(-x) - f(x)], \frac{t}{2}\Big)\\ &\ge \min\{N'(\phi(x, -x, 0), t), N'(\phi(-x, x, 0), t)\}\end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus $d(f, Jf) \le \frac{1}{2} < \infty$. The rest of proof the proof is similar to Theorem 2.3. ☐

As examples of $\phi(x, y, z)$ and $N'(x, t)$ in Theorem 2.3 and Theorem 2.4, we can take $\phi(x, y, z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ and

$$N'(x,t) = \begin{cases} \frac{t}{t+k|x|}, & if\, t > 0 \\ 0, & if\, t \le 0 \end{cases}$$

for all $x \in \mathbb{R}$, $t > 0$, and for some $\epsilon > 0$, where $k = 1, 2$. Then we can formulate the following corollary:

Corollary 2.5. *Let X be a normed space and (Y, N) a fuzzy Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(Df(x, y, z), t) \ge \min\Big\{N(f(x + y + z), t), \frac{t}{t + k\epsilon(\|x\|^p + \|y\|^p + \|z\|^p)}\Big\}$$

for all $x, y, z \in X$, $t > 0$, a fixed real number p with $0 < p < 1$ or $2 < p$. Then there is an unique additive-quadratic mapping $F : X \longrightarrow Y$ such that

$$N(f(x) - F(x), t) \ge \begin{cases} \dfrac{(2^p - 4)t}{(2^p - 4)t + 2k\epsilon\|x\|^p}, & if\ 2 < p \\[2ex] \dfrac{(2 - 2^p)t}{(2 - 2^p)t + 2k\epsilon\|x\|^p}, & if\ 0 < p < 1 \end{cases} \tag{2.19}$$

for all $x \in X$ and all $t > 0$.

For any $f : X \longrightarrow Y$, let

$$\begin{aligned} D_1f(x, y) =& f(x - y) + f(x + 2y) + f(-2x - y)\\ &- 2f(x) - 2f(y) - 2f(-x - y) - f(-x) - f(-y) - f(x + y)\end{aligned} \tag{2.20}$$

Using Corollary 2.5, we have the following corollary:

Corollary 2.6. *Let X be a normed space and (Y, N) a fuzzy Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

$$N(D_1f(x, y), t) \ge \frac{t}{t + k\epsilon(\|x\|^p + \|y\|^p + \|x + y\|^p)} \tag{2.21}$$

for all $x, y, z \in X$, $t > 0$, a fixed real number p with $0 < p < 1$ or $2 < p$. Then there is an unique additive-quadratic mapping $F : X \longrightarrow Y$ with (2.19).

We remark that the functional inequality (1.4) is not stable for $p = 1$ in Corollary 2.6. The following example shows that the inequality (2.21) is not stable for $p = 1$.

Example 2.7. Define mappings $t, s : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$t(x) = \begin{cases} x, & \text{if } |x| < 1 \\ -1, & \text{if } x \leq -1 \\ 1, & \text{if } 1 \leq x, \end{cases}$$

$$s(x) = \begin{cases} x^2, & \text{if } |x| < 1 \\ 1, & \text{ortherwise} \end{cases}$$

and a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \Big[\frac{t(2^n x)}{2^n} + \frac{s(2^n x)}{4^n} \Big]$$

We will show that f satisfies the following inequality

$$|D_1 f(x, y)| \leq 112(|x| + |y| + |x + y|) \tag{2.22}$$

for all $x, y \in \mathbb{R}$ and so f satisfies (2.21). But there do not exist an additive-quadratic mapping $F : \mathbb{R} \longrightarrow \mathbb{R}$ and a non-negative constant K such that

$$|F(x) - f(x)| \leq K|x| \tag{2.23}$$

for all $x \in \mathbb{R}$.

Proof. Note that $t_o(x) = t(x)$, $s_o(x) = 0$, and $|f_o(x)| \leq 2$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \leq |x| + |y| + |x + y|$. Then $|D_1 f_o(x, y)| \leq 48(|x| + |y| + |x + y|)$. Now suppose that $\frac{1}{4} > |x| + |y| + |x + y|$. Then there is a non-negative integer m such that

$$\frac{1}{2^{m+3}} \leq |x| + |y| + |x + y| < \frac{1}{2^{m+2}}$$

and so

$$2^m|x| < \frac{1}{4}, \quad 2^m|y| < \frac{1}{4}, \quad 2^m|x + y| < \frac{1}{4}.$$

Hence we have

$$\{2^m x, 2^m y, 2^m(x - y),\ 2^m(x + y),\ 2^m(x + 2y),\ 2^m(2x + y)\} \subseteq (-1,\ 1)$$

and so for any $n = 0, 1, 2, \cdots, m$,

$$|D_1 t_0(2^n x, 2^n y)| = 0,$$

because $t(x) = t_o(x) = x$ on $(-1, 1)$. Thus

$$\begin{aligned} |D_1 f_o(x, y)| &= \Big| \sum_{n=0}^{\infty} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y) \Big| \\ &\leq \Big| \sum_{n=0}^{m} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y) \Big| + \Big| \sum_{n=m+1}^{\infty} \frac{1}{2^n} D_1 t_o(2^n x, 2^n y) \Big| \\ &\leq \frac{12}{2^{m+1}} \leq 48(|x| + |y| + |x + y|), \end{aligned}$$

because $|D_1 t_0(2^n x, 2^n y)| \leq 6$.

Note that $t_e(x) = 0$, $s_e(x) = s(x)$, and $|f_e(x)| \leq \frac{4}{3}$ for all $x \in \mathbb{R}$. First, suppose that $\frac{1}{4} \leq |x|+|y|+|x+y|$. Then $|D_1 f_e(x,y)| \leq 64(|x|+|y|+|x+y|)$. Now suppose that $\frac{1}{4} > |x|+|y|+|x+y|$. Then there is a non-negative integer k such that

$$\frac{1}{2^{2k+4}} \leq |x|+|y|+|x+y| < \frac{1}{2^{2k+2}}$$

and so

$$2^{2k}|x| < \frac{1}{4}, \quad 2^{2k}|y| < \frac{1}{4}, \quad 2^{2k}|x+y| < \frac{1}{4}.$$

Hence we have

$$\{2^k x, 2^k y, 2^k(x-y), 2^k(x+y),\ 2^k(x+2y),\ 2^k(2x+y)\} \subseteq (-1,\ 1)$$

and so for any $n = 0, 1, 2, \cdots, k$,

$$|D_1 s_e(2^n x, 2^n y)| = 0.$$

Thus

$$\begin{aligned}
|D_1 f_e(x,y)| &= \Big|\sum_{n=0}^{\infty} \frac{1}{2^n} D_1 s_e(2^n x, 2^n y)\Big| \\
&\leq \Big|\sum_{n=0}^{k} \frac{1}{4^n} D_1 s_e(2^n x, 2^n y)\Big| + \Big|\sum_{n=k+1}^{\infty} \frac{1}{4^n} D_1 s_e(2^n x, 2^n y)\Big| \\
&\leq \frac{16}{4^{k+1}} \leq 64(|x|+|y|+|x+y|),
\end{aligned}$$

because $|D_1 s_e(2^n x, 2^n y)| \leq 12$. Hence we have

$$|D_1 f_o(x,y)| \leq 48(|x|+|y|+|x+y|),\ |D_1 f_e(x,y)| \leq 64(|x|+|y|+|x+y|)$$

for all $x, y \in X$ and so we have (2.22).

Suppose that there exist an additive mapping $A : \mathbb{R} \longrightarrow \mathbb{R}$, a quadratic mapping $Q : \mathbb{R} \longrightarrow \mathbb{R}$, and a non-negative constant K such that $A+Q$ satisfies (2.23). Since $|f(x)| \leq \frac{10}{3}$, by (2.23), we have

$$-\frac{10}{3n^2} - K\frac{|x|}{n} \leq \frac{A(x)}{n} + Q(x) \leq \frac{10}{3n^2} + K\frac{|x|}{n}$$

for all $x \in X$ and all positive integers n and so $Q(x) = 0$ for all $x \in X$. Since A is additive,

$$-\frac{10}{3n} - K|x| \leq A(x) \leq \frac{10}{3n} + K|x|$$

for all $x \in X$ and all $n \in \mathbb{N}$ and hence $|A(x)| \leq K|x|$. By (2.23), we have

$$|f(x)| \leq 2K|x| \tag{2.24}$$

for all $x \in X$. Take a positive integer l such that $l > 2K$ and $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Since $x > 0$,

$$f(x) \geq \sum_{n=0}^{\infty} \frac{t(2^n x)}{2^n} \geq \sum_{n=0}^{l-1} \frac{t(2^n x)}{2^n} = lx > 2Kx$$

which contradicts to (2.24). □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

GENERALIZED ADDITIVE-CUBIC FUCNTIONAL EQUATION AND ITS STABILITY

CHANG IL KIM

Abstract. In this paper, we establish some stability results for the following additive-cubic functional equation with an extra term G_f

$$f(2x+y)+f(2x-y)+G_f(x,y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x).$$

in Banach spaces, where G_f is a functional operator of f. Using these, we give new additive-cubic functional equations and prove their stability.

1. Introduction

In 1940, Ulam [12] raised the following question concerning the stability of group homomorphisms: "Under what conditions does there is an additive mapping near an approximately additive mapping between a group and a metric group ? "
In the next year, Hyers [5] gave a partial solution of Ulam's problem for the case of additive mappings. Hyers 's result, using unbounded Cauchy different, was generalized for additive mappings in [1] and for a linera mapping in [11]. Some stability results for additive, quardartic and mixed additve-cubic functional equations were investigated ([2], [3], [4], [6], [7], [8], [9], [10]).

The generalized Hyers–Ulam stability for the mixed additive-cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x) \tag{1.1}$$

in quasi-Banach spaces has been investigated by Najati and Eskandani [8]. Functional equation (1.1) is called *an additive-cubic functional equation*, since the function $f(x)=ax^3+bx$ is its solution. Every solution of this mixed additive-cubic functional equation is said to be *an additive-cubic mapping*.

In this paper, we are interested in what kind of a term $G_f(x,y)$ can be added to (1.1) while the solution of the new functional equation is also an additive-cubic funtional equation and the generalized Hyers-Ulam stability for it still holds, where $G_f(x,y)$ is a functional operator depending on the variables x,y, and function f. The new functional equation can be written as

$$f(2x+y)+f(2x-y)+G_f(x,y)=2f(x+y)+2f(x-y)+2f(2x)-4f(x). \tag{1.2}$$

We give some new functional equations in section 3 as examples of our results and prove the generalized Hyers-Ulam stability for these.

2010 *Mathematics Subject Classification.* 39B62, 39B72.

Key words and phrases. Hyers-Ulam stability, additive-cubic functional inequality.

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2. The generalized Hyers-Ulam stability for (1.2)

Let X be a real normed linear space and Y a real Banach space. For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \cdots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \cdots, f(\sigma_l(x, y))).$$

Throughout this section we always assume that G_f satisfies the following two conditions unless a specific expression for G_f is given.

Condition $\mathbf{P_1}$: Suppose that $f : X \longrightarrow Y$ is a mapping satisfying $f(2x) = 2f(x)$ and

$$f(2x + y) + f(2x - y) + G_f(x, y) = 2f(x + y) + 2f(x - y) \tag{2.1}$$

for all $x, y \in X$. Then f is an additive mapping.

Condition $\mathbf{P_2}$: Suppose that $f : X \longrightarrow Y$ is a mapping satisfying $f(2x) = 8f(x)$ and

$$f(2x + y) + f(2x - y) + G_f(x, y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{2.2}$$

for all $x, y \in X$. Then f is a cubic mapping.

For any $f : X \longrightarrow Y$, let

$$f_a(x) = \frac{4}{3}f(x) - \frac{1}{6}f(2x), \;\; f_c(x) = -\frac{1}{3}f(x) + \frac{1}{6}f(2x)$$

Now, we prove the following main theorem.

Theorem 2.1. *Let G_t be a functional operator satisfying* **Condition $\mathbf{P_1}$** *and* **Condition $\mathbf{P_2}$**. *Further, suppose that there is a real number $\lambda (\lambda \neq -1)$ such that*

$$G_t(x, 2x) + 2G_t(x, x) - 2G_t(0, x) = \lambda[t(4x) - 10t(2x) + 16t(x)] \tag{2.3}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that

$$\sum_{n=0}^{\infty} 2^{-n}\phi(2^n x, 2^n y) < \infty \tag{2.4}$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and

$$\begin{aligned} &\|f(2x + y) + f(2x - y) + G_f(x, y) \\ &- 2f(x + y) - 2f(x - y) - 2f(2x) + 4f(x)\| \leq \phi(x, y) \end{aligned} \tag{2.5}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned} &\|F_a(x) - f_a(x)\| \\ &\leq \frac{1}{12|\lambda + 1|}\sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\|F_c(x) - f_c(x)\| \\ &\leq \frac{1}{48|\lambda+1|}\sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]\end{aligned} \tag{2.7}$$

for all $x \in X$.

Proof. By (2.5), we have

$$\|f(x) + f(-x) - G_f(0, x)\| \leq \phi(0, x), \tag{2.8}$$

$$\|f(3x) - 4f(2x) + 5f(x) + G_f(x, x)\| \leq \phi(x, x), \tag{2.9}$$

and

$$\|f(4x) - 2f(3x) - 2f(2x) - 2f(-x) + 4f(x) + G_f(x, 2x)\| \leq \phi(x, 2x) \tag{2.10}$$

for all $x \in X$. By (2.3), (2.8), (2.9), and (2.10), we have

$$\|2^{-1}f_a(2x) - f_a(x)\| \leq \frac{1}{12|\lambda+1|}[\phi(x, 2x) + 2\phi(x, x) + 2\phi(0, x)] \tag{2.11}$$

for all $x \in X$. By (2.11), for $m, n \in \mathbb{N} \cup \{0\}$ with $0 \leq m < n$, we have

$$\begin{aligned}&\|2^{-n}f_a(2^n x) - 2^{-m}f_a(2^m x)\| \\ &= 2^{-m}\|2^{-(n-m)}f_a(2^{n-m} \cdot 2^m x) - f_a(2^m x)\| \\ &\leq \frac{1}{12|\lambda+1|}\sum_{k=m}^{n-1} 2^{-k}[\phi(2^k x, 2^{k+1}x) + 2\phi(2^k x, 2^k x) + 2\phi(0, 2^k x)]\end{aligned} \tag{2.12}$$

for all $x \in X$. By (2.12), $\{2^{-n}f_a(2^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $A : X \longrightarrow Y$ such that

$$A(x) = \lim_{n\to\infty} 2^{-n}f_a(2^n x)$$

for all $x \in X$. Moreover, by (2.12), we have

$$\begin{aligned}&\|A(x) - f_a(x)\| \\ &\leq \frac{1}{12|\lambda+1|}\sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)]\end{aligned} \tag{2.13}$$

for all $x \in X$. By (2.5), we have

$$\begin{aligned}&\|f_a(2x + y) + f_a(2x - y) + G_{f_a}(x, y) - 2f_a(x + y) - 2f_a(x - y) \\ &- 2f_a(2x) + 4f_a(x)\| \leq \frac{4}{3}\phi(x, y) + \frac{1}{6}\phi(2x, 2y)\end{aligned} \tag{2.14}$$

for all $x, y \in X$. Replacing x and y by $2^n x$ and $2^n y$ in (2.14), respectively and deviding (2.5) by 2^n, we have

$$\begin{aligned}&\|2^{-n}f_a(2^n(2x + y)) + 2^{-n}f_a(2^n(2x - y)) + 2^{-n}G_{f_a}(2^n x, 2^n y) \\ &- 2 \cdot 2^{-n}f_a(2^n(x + y)) - 2 \cdot 2^{-n}f_a(2^n(x - y)) - 2 \cdot 2^{-n}f_a(2^{n+1}x) \\ &+ 4 \cdot 2^{-n}f_a(2^n x)\| \leq \frac{4}{3} \cdot 2^{-n}\phi(2^n x, 2^n y) + \frac{1}{6} \cdot 2^{-n}\phi(2^{n+1}x, 2^{n+1}y)\end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we have

$$\begin{aligned} &A(2x+y) + A(2x-y) + \lim_{n\to\infty} 2^{-n} G_{f_a}(2^n x, 2^n y) \\ &- 2A(x+y) - 2A(x-y) - 2A(2x) + 4A(x) = 0 \end{aligned} \tag{2.15}$$

for all $x, y \in X$ and since F is continuous,

$$\begin{aligned} &\lim_{n\to\infty} 2^{-n} G_{f_a}(2^n x, 2^n y) \\ &= \lim_{n\to\infty} F(2^{-n} f_a(2^n \sigma_1(x,y)), 2^{-n} f_a(2^n \sigma_2(x,y)), \cdots, 2^{-n} f_a(2^n \sigma_l(x,y))) \\ &= G_A(x,y) \end{aligned}$$

for all $x, y \in X$. Hence by (2.15), we have

$$A(2x+y) + A(2x-y) + G_A(x,y) = 2A(x+y) + 2A(x-y) + 2A(2x) - 4A(x) \tag{2.16}$$

for all $x, y \in X$. Relpacing x by $2^n x$ in (2.11) and deviding (2.11) by 2^n, we have

$$\begin{aligned} &\|2^{-n-1} f_a(2^n \cdot 2x) - 2^{-n} f_a(2^n x)\| \\ &\le \frac{2^{-n}}{12|\lambda+1|}[\phi(2^n x, 2^{n+1} x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned}$$

for all $x \in X$ and letting $n \to \infty$ in the above inequality, we have

$$A(2x) = 2A(x) \tag{2.17}$$

for all $x, y \in X$. By (2.16) and (2.17), A satisfies (2.1). By **Condition $\mathbf{P_1}$**, A is an additive mapping .

By (2.3), (2.8), (2.9), and (2.10), we have

$$\|8^{-1} f_c(2x) - f_c(x)\| \le \frac{1}{48|\lambda+1|}[\phi(x, 2x) + 2\phi(x,x) + 2\phi(0,x)] \tag{2.18}$$

for all $x \in X$. By (2.18), for $m, n \in \mathbb{N} \cup \{0\}$ with $0 \le m < n$, we have

$$\begin{aligned} &\|2^{-3n} f_c(2^n x) - 2^{-3m} f_c(2^m x)\| \\ &= 2^{-3m} \|2^{-3(n-m)} f_c(2^{n-m} \cdot 2^m x) - f_c(2^m x)\| \\ &\le \frac{1}{48|\lambda+1|} \sum_{k=m}^{n-1} 2^{-3k}[\phi(2^k x, 2^{k+1} x) + 2\phi(2^k x, 2^k x) + 2\phi(0, 2^k x)] \end{aligned} \tag{2.19}$$

for all $x \in X$. By (2.19), $\{2^{-3n} f_c(2^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $C : X \longrightarrow Y$ such that

$$C(x) = \lim_{n\to\infty} 2^{-3n} h(2^n x)$$

for all $x \in X$. Moreover, by (2.19), we have

$$\begin{aligned} &\|C(x) - f_c(x)\| \\ &\le \frac{1}{48|\lambda+1|} \sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1} x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \end{aligned} \tag{2.20}$$

for all $x \in X$. By (2.5), we have

$$\begin{aligned} &\|f_c(2x+y) + f_c(2x-y) + G_{f_c}(x,y) - 2f_c(x+y) - 2f_c(x-y) \\ &- 2f_c(2x) + 4f_c(x)\| \le \frac{1}{3}\phi(x,y) + \frac{1}{6}\phi(2x, 2y) \end{aligned} \tag{2.21}$$

for all $x, y \in X$. Replacing x and y by $2^n x$ and $2^n y$ in (2.21), respectively and deviding (2.21) by 2^{3n}, we have

$$\begin{aligned}&\|2^{-3n}f_c(2^n(2x+y)) + 2^{-3n}f_c(2^n(2x-y)) + 2^{-3n}G_{f_c}(2^nx, 2^ny)\\ &- 2\cdot 2^{-3n}f_c(2^n(x+y)) - 2\cdot 2^{-3n}f_c(2^n(x-y)) - 2\cdot 2^{-n}f_c(2^{n+1}x)\\ &+ 4\cdot 2^{-3n}f_c(2^nx)\| \le \frac{1}{3}\cdot 2^{-3n}\phi(2^nx, 2^ny) + \frac{1}{6}\cdot 2^{-3n}\phi(2^{n+1}x, 2^{n+1}y)\end{aligned}$$

for all $x, y \in X$. Letting $n \to \infty$ in the last inequality, we have

$$\begin{aligned}&C(2x+y) + C(2x-y) + \lim_{n\to\infty} 2^{-3n}G_{f_c}(2^nx, 2^ny)\\ &- 2C(x+y) - 2C(x-y) - 2C(2x) + 4C(x) = 0\end{aligned} \tag{2.22}$$

for all $x, y \in X$ and since F is continuous,

$$\begin{aligned}&\lim_{n\to\infty} 2^{-3n}G_{f_c}(2^nx, 2^ny)\\ &= \lim_{n\to\infty} F(2^{-3n}h(2^n\sigma_1(x,y)), 2^{-3n}h(2^n\sigma_2(x,y)), \cdots, 2^{-3n}h(2^n\sigma_l(x,y)))\\ &= G_C(x,y)\end{aligned}$$

for all $x, y \in X$. Hence by (2.22), we have

$$C(2x+y) + C(2x-y) + G_C(x,y) = 2C(x+y) + 2C(x-y) + 2C(2x) - 4C(x) \tag{2.23}$$

for all $x, y \in X$. Relpacing x by 2^nx in (2.18) and deviding (2.18) by 2^{3n}, we have

$$\begin{aligned}&\|2^{-3}\cdot 2^{-3n}f_c(2^n\cdot 2x) - 2^{-3n}f_c(2^nx)\|\\ &\le \frac{2^{-3n}}{48|\lambda+1|}[\phi(2^nx, 2^{n+1}x) + 2\phi(2^nx, 2^nx) + 2\phi(0, 2^nx)]\end{aligned}$$

for all $x \in X$ and letting $n \to \infty$ in the above inequality, we have

$$C(2x) = 8C(x) \tag{2.24}$$

for all $x, y \in X$. By (2.23) and (2.24), C satisifes (2.2). By **Condition $\mathbf{P_2}$**, C is a cubic mapping.

Let $F = A + C$. Then F is an additive-cubic mapping, $F_a = A$, and $F_c = C$. By (2.13) and (2.20), we have (2.6) and (2.7).

For the uniqueness of F, let H be another additive-cubic mapping with (2.6) and (2.7). Then F_a and H_a are additive mappings and hence

$$\begin{aligned}&\|F_a(x) - H_a(x)\| = 2^{-k}\|F_a(2^kx) - H_a(2^kx)\|\\ &\le \frac{1}{6|\lambda+1|}\sum_{n=k}^{\infty} 2^{-n}[\phi(2^nx, 2^{n+1}x) + 2\phi(2^nx, 2^nx) + 2\phi(0, 2^nx)]\end{aligned}$$

for all $x \in X$. Hence, letting $k \to \infty$ in the above inequality, we have $F_a = H_a$ and similarly, we have $F_c = H_c$. Thus $F = H$. □

Similarly, we have the following theorem:

Theorem 2.2. *Let G_t be a functional operator satisfying* **Condition $\mathbf{P_1}$**, **Condition $\mathbf{P_2}$**, *and*

$$G_t(x, 0) = -G_t(0, x). \tag{2.25}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Further, suppose that there are real numbers λ, δ ($\lambda \neq -1$) such that

$$\begin{aligned} &G_t(x,2x) + 2G_t(x,x) - 2G_t(0,x) \\ &= \lambda[t(4x) - 10t(2x) + 16t(x)] + \delta[f(x) + f(-x)] \end{aligned} \tag{2.26}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (2.5). Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned} \|F_a(x) - f_a(x)\| \le \frac{1}{12|\lambda+1|} &\sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) \\ &+ 2\phi(2^n x, 2^n x) + |\delta|\phi(2^n x, 0) + (2+|\delta|)\phi(0, 2^n x)] \end{aligned}$$

and

$$\begin{aligned} \|F_c(x) - f_c(x)\| \le \frac{1}{48|\lambda+1|} &\sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) \\ &+ 2\phi(2^n x, 2^n x) + |\delta|\phi(2^n x, 0) + (2+|\delta|)\phi(0, 2^n x)] \end{aligned}$$

for all $x \in X$.

Proof. By (2.8) and (2.25), we have

$$\|f(x) + f(-x)\| \le \phi(x,0) + \phi(0,x)$$

for all $x \in X$, because $\|G_f(x,0)\| \le \phi(x,0)$ and $G_f(x,0) = -G_f(0,x)$. Similar to the proof of Theorem 2.1, we have

$$\begin{aligned} &\|(1+\lambda)[f(4x) - 10f(2x) + 16f(x)]\| \\ &\le \phi(x,2x) + 2\phi(x,x) + 2\phi(0,x) + |\delta|\|f(x) + f(-x)\| \\ &\le \phi(x,2x) + 2\phi(x,x) + |\delta|\phi(x,0) + (2+|\delta|)\phi(0,x) \end{aligned}$$

for all $x \in X$ and so we get

$$\|2^{-1}f_a(2x) - f_a(x)\| \le \frac{1}{12|\lambda+1|}[\phi(x,2x) + 2\phi(x,x) + |\delta|\phi(x,0) + (2+|\delta|)\phi(0,x)]$$

for all $x \in X$. The rest of this proof is similar to the proof of Theorem 2.1. □

3. Applications

In this section, using Theorem 2.1 and Theorem 2.2, we will prove the generalized Hyers-Ulam stability for some additive-cubic functional equations.

First, we consider the following functional equation :

$$f(2x+y) + f(2x-y) - f(4x) = 2f(x+y) + 2f(x-y) - 8f(2x) + 12f(x). \tag{3.1}$$

Theorem 3.1. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - f(4x) - 2f(x+y) - 2f(x-y) \\ &+ 8f(2x) - 12f(x)\| \le \phi(x,y) \end{aligned} \tag{3.2}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\|F_a(x) - f_a(x)\| \le \frac{1}{24}\sum_{n=0}^{\infty} 2^{-n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \tag{3.3}$$

and

$$\|F_c(x) - f_c(x)\| \le \frac{1}{96}\sum_{n=0}^{\infty} 2^{-3n}[\phi(2^n x, 2^{n+1}x) + 2\phi(2^n x, 2^n x) + 2\phi(0, 2^n x)] \tag{3.4}$$

for all $x \in X$*.*

Proof. Let $G_f(x,y) = -f(4x) + 10f(2x) - 16f(x)$. Then f satisfies (2.5) and

$$G_t(x, 2x) + 2G_t(x,x) - 2G_t(0,x) = -3[t(4x) - 10t(2x) + 16t(x)]$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. If $t : X \longrightarrow Y$ is a mapping with $t(2x) = 2t(x)$ for all $x \in X$ and (2.1), then $G_t(x,y) = 0$ for all $x, y \in X$ and so t is an additive mapping. Hence G_t satifies **Condition $\mathbf{P_1}$** and similarly G_t satifies **Condition $\mathbf{P_2}$**. By Theroem 2.1, there is an unique additive-cubic mapping $F : X \longrightarrow Y$ with (3.3) and (3.4). □

Using the above theorem, we have the following corollaries:

Corollary 3.2. *Let* $f : X \longrightarrow Y$ *be a mapping. Then* f *satisfies (3.1) if and only if* f *is an additive-cubic mapping.*

Ostadbashi and Kazemzadeh [9] investigated the following additive-cubic functinal equation :

$$\begin{aligned} &f(2x+y) + f(2x-y) - f(4x) \\ &= 2f(x+y) + 2f(x-y) - 8f(2x) + 10f(x) - 2f(-x). \end{aligned} \tag{3.5}$$

Corollary 3.3. *Let* $\phi : X^2 \longrightarrow [0,\infty)$ *be a function with (2.4). Let* $f : X \longrightarrow Y$ *be a mapping such that* $f(0) = 0$ *and*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - f(4x) - 2f(x+y) - 2f(x-y) \\ &+ 8f(2x) - 10f(x) + 2f(-x)\| \le \phi(x,y) \end{aligned} \tag{3.6}$$

for all $x, y \in X$*. Then there exists an unique additive-cubic mapping* $F : X \longrightarrow Y$ *such that*

$$\|F_a(x) - f_a(x)\| \le \frac{1}{24}\sum_{n=0}^{\infty} 2^{-n}[\phi_1(2^n x, 2^{n+1}x) + 2\phi_1(2^n x, 2^n x) + 2\phi_1(0, 2^n x)] \tag{3.7}$$

and

$$\|F_c(x) - f_c(x)\| \le \frac{1}{96}\sum_{n=0}^{\infty} 2^{-3n}[\phi_1(2^n x, 2^{n+1}x) + 2\phi_1(2^n x, 2^n x) + 2\phi_1(0, 2^n x)] \tag{3.8}$$

for all $x \in X$*, where* $\phi_1(x,y) = \phi(x,y) + \phi(0,x)$*.*

Proof. By (3.6), we have

$$\|f(x) + f(-x)\| \le \phi(0,x)$$

for all $x \in X$ and hence we have

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - f(4x) - 2f(x+y) - 2f(x-y) \\ &+ 8f(2x) - 12f(x)\| \le \phi(x,y) + \phi(0,x) = \phi_1(x,y) \end{aligned}$$

for all $x, y \in X$. By Theorem 3.3, we have the results. □

Finally, we consider the following new functional equation :

$$\begin{aligned} &f(2x+y)+f(2x-y)-2f(x+y)+3f(x-y)-5f(y-x)\\ &-10(x)+14f(y)-2f(2y)=0. \end{aligned} \tag{3.9}$$

Lemma 3.4. *Let G_f be a functional operator such that*

$$G_f(x,y)=-G_f(y,x) \tag{3.10}$$

for all mapping $f : X \longrightarrow Y$ and all $x, y \in X$. Then **Condition $\mathbf{P_1}$** *and* **Condition $\mathbf{P_2}$** *hold.*

Proof. Suppose that $f : X \longrightarrow Y$ is a mapping with $f(2x) = 2f(x)$ and (2.1). Letting $y = 0$ in (2.1), we have

$$G_f(x,0)=0 \tag{3.11}$$

for all $x \in X$ and by (3.10) and (3.11), we get

$$G_f(x,0)=-G_f(0,x)=-[f(x)+f(-x)]=0$$

for all $x \in X$. Hence

$$f(-x)=-f(x) \tag{3.12}$$

for all $x \in X$. Interchaging x and y in (2.1), by (3.12), we have

$$f(x+2y)-f(x-2y)+G_f(y,x)=2f(x+y)-2f(x-y) \tag{3.13}$$

for all $x, y \in X$ and by (2.1), (3.10), and (3.13), we have

$$f(2x+y)+f(2x-y)+f(x+2y)-f(x-2y)=4f(x+y) \tag{3.14}$$

for all $x, y \in X$. Letting $y = -y$ in (3.14), we have

$$f(2x+y)+f(2x-y)+f(x-2y)-f(x+2y)=4f(x-y) \tag{3.15}$$

for all $x, y \in X$. By (3.14) and (3.15), we have

$$f(x+y)+f(x-y)=f(x+2y)+f(x-2y) \tag{3.16}$$

for all $x, y \in X$. Letting $x = x + y$ in (3.16), we get

$$f(x+2y)+f(x)=f(x+3y)+f(x-y) \tag{3.17}$$

for all $x, y \in X$ and letting $x = 2x$ in (3.16), we get

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y) \tag{3.18}$$

for all $x, y \in X$. Letting $y = x + y$ in (3.18), we get

$$f(3x+y)+f(x-y)=2f(2x+y)-2f(y) \tag{3.19}$$

for all $x, y \in X$ and interchaging x and y in (3.19), we have

$$f(x+3y)-f(x-y)=2f(x+2y)-2f(x) \tag{3.20}$$

for all $x, y \in X$. By (3.17) and (3.20), we have

$$f(x+2y)-3f(x)+2f(x-y)=0 \tag{3.21}$$

for all $x, y \in X$. Letting $x = x - y$ in (3.21), we get

$$f(x+y)-3f(x-y)+2f(x-2y)=0 \tag{3.22}$$

for all $x, y \in X$ and letting $y = -y$ in (3.22), we get

$$f(x-y)-3f(x+y)+2f(x+2y)=0 \tag{3.23}$$

for all $x, y \in X$. By (3.21) and (3.23), we have

$$f(x+y)+f(x-y)-2f(x)=0$$

for all $x, y \in X$ and hence f is an additive mapping. Thus **Condition $\mathbf{P_1}$** holds.

(2) Suppose that $f : X \longrightarrow Y$ is a mapping with $f(2x) = 8f(x)$ and (2.2). Similar to (1), we have

$$G_f(x,0) = -G_f(0,x) = 0,\ f(-x) = -f(x)$$

for all $x, y \in X$. Interchaging x and y in (2.2), we have

$$f(x+2y)-f(x-2y)+G_f(y,x) = 2f(x+y)-2f(x-y)+12f(y) \tag{3.24}$$

for all $x, y \in X$ and by (2.2), (3.10), and (3.24), we have

$$f(2x+y)+f(2x-y)+f(x+2y)-f(x-2y) = 4f(x+y)+12f(x)+12f(y) \tag{3.25}$$

for all $x, y \in X$. Letting $y = -y$ in (3.25), we have

$$f(2x+y)+f(2x-y)+f(x-2y)-f(x+2y) = 4f(x-y)+12f(x)-12f(y) \tag{3.26}$$

for all $x, y \in X$. By (3.25) and (3.26), we have

$$f(2x+y)+f(2x-y) = 2f(x+y)+2f(x-y)+12f(x)$$

for all $x, y \in X$ and hence f is a cubic mapping. Thus **Condition $\mathbf{P_2}$** holds. □

Using Lemma 3.4, we investigate solutions and the generalized Hyers-Ulam stability for (3.9).

Theorem 3.5. *Let $\phi : X^2 \longrightarrow [0,\infty)$ be a function with (2.4). Let $f : X \longrightarrow Y$ be a mapping such that $f(0) = 0$ and*

$$\begin{aligned}&\|f(2x+y)+f(2x-y)-2f(x+y)+3f(x-y)-5f(y-x)\\&-10(x)+14f(y)-2f(2y)\| \le \phi(x,y)\end{aligned} \tag{3.27}$$

for all $x, y \in X$. Then there exists an unique additive-cubic mapping $F : X \longrightarrow Y$ such that

$$\begin{aligned}\|F_a(x)-f_a(x)\| \le \frac{1}{12}\sum_{n=0}^{\infty} 2^{-n}[&\phi(2^n x, 2^{n+1}x)+2\phi(2^n x, 2^n x)\\&+5\phi(2^n x, 0)+7\phi(0, 2^n x)]\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\|F_c(x)-f_c(x)\| \le \frac{1}{48}\sum_{n=0}^{\infty} 2^{-3n}[&\phi(2^n x, 2^{n+1}x)+2\phi(2^n x, 2^n x)\\&+5\phi(2^n x, 0)+7\phi(0, 2^n x)]\end{aligned} \tag{3.29}$$

for all $x \in X$.

Proof. Let $G_f(x,y) = 5[f(x-y)-f(y-x)]-14[f(x)-f(y)]+2[f(2x)-f(2y)]$. Then f satisfies (2.5) and

$$\begin{aligned}&G_t(x,2x)+2G_t(x,x)-2G_t(0,x)\\&= -2[t(4x)-10t(2x)+16t(x)]-5[f(x)+f(-x)]\end{aligned}$$

for all $x \in X$ and all mapping $t : X \longrightarrow Y$. Since G_f satifies (3.10), by Lemma 3.4, **Condition $\mathbf{P_1}$** and **Condition $\mathbf{P_2}$** satisfy. By Theroem 2.2, there is an unique additive-cubic mapping $F : X \longrightarrow Y$ with (3.28) and (3.29). □

Corollary 3.6. *Let $f : X \longrightarrow Y$ be a mapping. Then f satisfies (3.9) if and only if f is an additive-cubic mapping.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

TOEPLITZ DUALS OF FIBONACCI SEQUENCE SPACES

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Abstract. In this paper we introduce and study some classes of almost strongly convergent difference sequences of Fibonacci numbers defined by a sequence of modulus functions. We also make an effort to study some topological properties and inclusion relations between these classes of sequences. Further, we compute toeplitz duals of theses classes and study matrix transformations on these classes of sequences.

1. Introduction and Preliminaries

Let w be the vector space of all real sequences. We shall write c, c_0 and l_∞ for the sequence spaces of all convergent, null and bounded sequences. Moreover, we write bs and cs for the spaces of all bounded and convergent series, respectively. Also, we use the conventions that $e = (1, 1, 1, ...)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$.

Let X and Y be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix transformation from X into Y and we denote it by writing $A : X \to Y$ if for every sequence $x = (x_k) \in X$, the sequence $Ax = \{A_n(x)\}$ and the A-transform of x is in Y, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

By (X, Y) we denote the class of all matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax \in Y$ for all $x \in X$. The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\} \tag{1.2}$$

which is a sequence space. By using the matrix domain of a triangle infinite matrix, so many sequence spaces have recently been defined by several authors, (see [1], [2], [15], [25]). In the literature, the matrix domain X_Δ is called the difference sequence space whenever X is a normed or paranormed sequence space, where Δ denotes the backward difference matrix $\Delta = (\Delta_{nk})$ and $\Delta' = (\Delta'_{nk})$ denotes the forward difference matrix (the transpose of the matrix Δ), which are defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n, \\ 0 \quad , & 0 \le k < n-1 \text{ or } k > n \end{cases}$$

$$\Delta'_{nk} = \begin{cases} (-1)^{n-k}, & n \le k \le n+1, \\ 0 \quad , & 0 \le k < n \text{ or } k > n+1 \end{cases}$$

2010 *Mathematics Subject Classification.* 11B39, 46A45, 46B45.

Key words and phrases. Fibonacci numbers, difference matrix, modulus function, paranorm space, $\alpha-, \beta-, \gamma-$ duals, matrix transformations.

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for all $k, n \in \mathbb{N}$ respectively. The notion of difference sequence spaces was introduced by Kızmaz [16], who defined the sequence spaces

$$X(\Delta) = \{x = (x_k) \in w : (x_k - x_{k+1}) \in X\}$$

for $X = l_\infty, c$ and c_0. The difference space $b\nu_p$, consisting of all sequences (x_k) such that $(x_k - x_{k-1})$ is in the sequence space l_p, was studied in the case $0 < p < 1$ by Altay and Başar [3] and in the case $1 \leq p \leq \infty$ by Başar and Altay [7] and Çolak et al. [9]. Kirişçi and Başar [15] have been introduced and studied the generalized difference sequence spaces

$$\hat{X} = \{x = (x_k) \in w : B(r, s)x \in X\}$$

where X denotes any of the spaces l_∞, l_p, c and c_0, $(1 \leq p < \infty)$ and $B(r, s)x = (sx_{k-1} + rx_k)$ with $r, s \in \mathbb{R}\backslash\{0\}$. Following Kirişçi and Başar [15], Sönmez [31] have been examined the sequence space $X(B)$ as the set of all sequences whose $B(r, s, t)$-transforms are in the space $X \in \{l_\infty, l_p, c, c_0\}$, where $B(r, s, t)$ denotes the triple band matrix $B(r, s, t) = \{b_{nk}(r, s, t)\}$ defined by

$$b_{nk}(r, s, t) = \begin{cases} r, & n = k \\ s, & n = k + 1 \\ t, & n = k + 2 \\ 0, & \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s, t \in \mathbb{R}\backslash\{0\}$. Also in ([10-13], [26]) authors studied certain difference sequence spaces.

A B-space is a complete normed space. A topological sequence space in which all coordinate functionals $\pi_k, \pi_k(x) = x_k$, are continuous is called a K-space. A BK-space is defined as a K-space which is also a B-space, that is, a BK-space is a Banach space with continuous coordinates. For example, the space $l_p (1 \leq p < \infty)$ is a BK-space with $\|x\|_p = \Big(\sum_{k=0}^{\infty} |x_k|^p\Big)^{\frac{1}{p}}$ and c_0, c and l_∞ are BK-spaces with $\|x\|_\infty = \sup_k |x_k|$. The sequence space X is said to be solid (see [17, p. 48]) if and only if

$$\widetilde{X} = \{(v_k) \in w : \exists (x_k) \in X \text{ such that } |v_k| \leq |x_k| \text{ for all } k \in \mathbb{N}\} \subset X.$$

A sequence (b_n) in a normed space X is called a Schauder basis for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_m \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$.

The following lemma (known as the Toeplitz Theorem) contains necessary and sufficient condition for regularity of a matrix.

Lemma 1.1. *(Wilansky, 1984): Matrix* $A = (a_{nk})_{n,k=1}^{\infty}$ *is regular if and only if the following three conditions hold:*

(1) There exists $M > 0$ *such that for every* $n = 1, 2, ...$ *the following inequality holds:*

$$\sum_{k=1}^{\infty} |a_{nk}| \leq M;$$

(2) $\lim_{n\to\infty} a_{nk} = 0$ *for every* $k = 1, 2, ...$

(3) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1.$

The sequence $\{f_n\}_{n=0}^{\infty}$ of Fibonacci numbers is given by the linear recurrence relations $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2},\ n \geq 2$. Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also, in [18] some basic properties of Fibonacci numbers are given as follows:

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \phi \text{ (golden ratio)},$$
$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \ (n \in \mathbb{N}),$$
$$\sum_{k} \frac{1}{f_k} \text{ converges},$$
$$f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad (n \geq 1) \text{ (Cassini formula)}.$$

Substituting for f_{n+1} in Cassini's formula yields $f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}$.

Now, let $A = (a_{nk})$ be an infinite matrix and list the following conditions:

$$\sup_{n\in\mathbb{N}} \sum_k \left|a_{nk}\right| < \infty \tag{1.3}$$

$$\lim_{n\to\infty} a_{nk} = 0 \ \text{ for each } k \in \mathbb{N} \tag{1.4}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} a_{nk} = \alpha_k \ \text{ for each } k \in \mathbb{N} \tag{1.5}$$

$$\lim_{n\to\infty} \sum_k a_{nk} = 0 \tag{1.6}$$

$$\exists \alpha \in \mathbb{C} \ni \lim_{n\to\infty} \sum_k a_{nk} = \alpha \tag{1.7}$$

$$\sup_{k\in\mathcal{H}} \sum_n \Big| \sum_{k\in K} a_{nk} \Big| < \infty \tag{1.8}$$

where $\mathbb{C}$ and $\mathcal{H}$ denote the set of all complex numbers and the collection of all finite subsets of $\mathbb{N}$, respectively.
Now, we may give the following lemma on the characterization of the matrix transformations between some classical sequence spaces.

Lemma 1.2. *The following statements hold:*
(a) $A = (a_{nk}) \in (c_0, c_0)$ if and only if (1.3) and (1.4) hold.
(b) $A = (a_{nk}) \in (c_0, c)$ if and only if (1.3) and (1.5) hold.
(c) $A = (a_{nk}) \in (c, c_0)$ if and only if (1.3), (1.4) and (1.6) hold.
(d) $A = (a_{nk}) \in (c, c)$ if and only if (1.3), (1.5) and (1.7) hold.
(e) $A = (a_{nk}) \in (c_0, l_\infty) = (c, l_\infty)$ if and only if condition (1.3) holds.
(f) $A = (a_{nk}) \in (c_0, l_1) = (c, l_1)$ if and only if condition (1.8) holds.

Recently, Kara [19] has defined the sequence spaces $l_p(\hat{F})$ as follows:

$$l_p(\hat{F}) = \{x \in w : \hat{F}x \in l_p\},\ (1 \leq p \leq \infty)$$

where $\hat{F} = (\hat{f}_{nk})$ is the double band matrix defined by the sequence (f_n) of Fibonacci numbers as follows

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & k = n-1, \\ \frac{f_n}{f_{n+1}}\ , & k = n, \\ 0\ \ , & 0 \leq k < n-1 \text{ or } k > n \end{cases} \qquad (k, n \in \mathbb{N}).$$

Also, in [20] Kara et al. have characterized some classes of compact operators on the spaces $l_p(\hat{F})$ and $l_\infty(\hat{F})$, where $1 \leq p < \infty$.
The inverse $\hat{F}^{-1} = (g_{nk})$ of the Fibonacci matrix $\hat{F}$ is given by

$$g_{nk} = \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}}, & 0 \leq k \leq n, \\ 0\ \ , & k > n \end{cases} \quad (k, n \in \mathbb{N}).$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ... \\ \frac{4}{1} & \frac{4}{2} & 0 & 0 & 0 & 0 & ... \\ \frac{9}{1} & \frac{9}{2} & \frac{9}{6} & 0 & 0 & 0 & ... \\ \frac{25}{1} & \frac{25}{2} & \frac{25}{6} & \frac{25}{15} & 0 & 0 & ... \\ \frac{64}{1} & \frac{64}{2} & \frac{64}{6} & \frac{64}{15} & \frac{64}{40} & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is obvious that the matrix $\hat{F}$ is a triangular matrix, that is, $f_{nn} \neq 0$ and $f_{nk} = 0$ for $k > n$ $(n = 1, 2, 3...)$. Also, it follows by Lemma 1.1 that the method $\hat{F}$ is regular.
In [8] Başarir et al. introduce the Fibonacci difference sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$ as the set of all sequences whose $\hat{F}$-transforms are in the spaces c_0 and c, respectively, i.e.,

$$c_0(\hat{F}) = \left\{ x = (x_n) \in w : \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = 0 \right\},$$

and

$$c(\hat{F}) = \left\{ x = (x_n) \in w : \exists l \in \mathbb{C} \ni \lim_{n\to\infty} \left(\frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = l \right\}.$$

Define the sequence $y = (y_n)$ by the $\hat{F}$-transform of a sequence $x = (x_n)$, i.e.,

$$y_n = \hat{F}_n(x) = \begin{cases} x_0 & , \quad n = 0 \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & n \geq 1 \end{cases} \quad (n \in \mathbb{N}). \tag{1.9}$$

A linear functional L on l_∞ is said to be a Banach limit if it has the following properties:
(1) $L(x) \geq 0$ if $n \geq 0$ (i.e. $x_n \geq 0$ for all n),
(2) $L(e) = 1$, where $e = (1, 1, ...)$,
(3) $L(Dx) = L(x)$,
where the shift operator D is defined by $D(x_n) = \{x_{n+1}\}$ (see [6]).
Let B be the set of all Banach limits on l_∞. A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent if all Banach limits of $x = (x_k)$ coincide. In [22], it was shown that

$$\hat{c} = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_{k+s} \text{ exits, uniformly in s} \right\}$$

In ([23], [24]) Maddox defined strongly almost convergent sequences. Recall that a sequence $x = (x_k)$ is strongly almost convergent if there is a number l such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_{k+s} - l| = 0, \text{ uniformly in s.}$$

Let X be a linear metric space. A function $p: X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$ for all $x \in X$,
(2) $p(-x) = p(x)$ for all $x \in X$,
(3) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$,
(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [33], Theorem 10.4.2, pp. 183).
A modulus function is a function $f: [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,
(2) $f(x+y) \leq f(x) + f(y)$, for all $x, y \geq 0$,
(3) f is increasing,
(4) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$ then the modulus function $f(x)$ is unbounded. Subsequently, modulus function has been discussed in ([4], [27], [28], [29], [30]) and references therein.
Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions, $p = (p_k)$ be any bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. In this paper we define the following sequence spaces:

$$c_0(\hat{F}, \mathcal{F}, u, p) = \left\{ x = (x_k) \in w : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} = 0 \right\},$$

and

$$c(\hat{F}, \mathcal{F}, u, p) = \left\{ x = (x_k) \in w : \exists l \in \mathbb{C} \ni \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} = l \right\}.$$

If $F_k(x) = x$, for all $k \in \mathbb{N}$. Then above sequence spaces reduces to $c_0(\hat{F}, u, p)$ and $c(\hat{F}, u, p)$.
By taking $p_k = 1$ and $u_k = 1$, for all $k \in \mathbb{N}$, then we get the sequence spaces $c_0(\hat{F}, \mathcal{F})$ and $c(\hat{F}, \mathcal{F})$.
With the notation of (1.2), the sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ can be redefined as follows:

$$c_0(\hat{F}, \mathcal{F}, u, p) = \{c_0(\mathcal{F}, u, p)\}_{\hat{F}} \text{ and } c(\hat{F}, \mathcal{F}, u, p) = \{c(\mathcal{F}, u, p)\}_{\hat{F}}. \tag{1.10}$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.11}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper, we introduce the sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$. We investigate some topological properties of these new sequence spaces and establish some inclusion relations between these spaces. Also we determine the $\alpha-, \beta-$ and $\gamma-$ duals of these spaces and construct the matrix transformation of the spaces $(c_0(\hat{F}, \mathcal{F}, u, p), X)$ and $(c(\hat{F}, \mathcal{F}, u, p), X)$, where X denote the spaces $l_\infty, f, c, f_0, c_0, bs, fs$ and l_1.

2. Some topological properties of the spaces $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$

Theorem 2.1. *Let $\mathcal{F}=(F_k)$ be a sequence of modulus functions, $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ are linear spaces over the field $\mathbb{R}$ of real numbers.*

Proof. Let $x=(x_k),\ y=(y_k)\in c_0(\hat{F},\mathcal{F},u,p)$ and $\lambda,\ \mu\in\mathbb{C}$. Then there exist integers M_λ and N_μ such that $|\lambda|\le M_\lambda$ and $|\mu|\le N_\mu$. Using inequality (1.11) and definition of modulus function, we have

$$\tfrac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k\Big|\lambda\Big(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big)+\mu\Big(\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big)\Big|\Big]^{p_k}$$

$$\le\frac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k|\lambda|\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\Big]^{p_k}+\frac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k|\mu|\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\Big]^{p_k}$$

$$\le K\frac{1}{n}\sum_{k=1}^{n}\Big[u_kF_kM_\lambda\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\Big]^{p_k}+K\frac{1}{n}\sum_{k=1}^{n}\Big[u_kF_kN_\mu\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\Big]^{p_k}$$

$$\le KM_\lambda^H\frac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\Big]^{p_k}+KN_\mu^H\frac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\Big]^{p_k}$$

$\to 0$ as $n\to\infty$.
Thus $\lambda x+\mu y\in c_0(\hat{F},\mathcal{F},u,p)$. This proves that $c_0(\hat{F},\mathcal{F},u,p)$ is a linear space. Similarly we can prove that $c(\hat{F},\mathcal{F},u,p)$ is a linear space over the real field $\mathbb{R}$. ☐

Theorem 2.2. *Let $\mathcal{F}=(F_k)$ be a sequence of modulus functions and $p=(p_k)$ be a bounded sequence of positive real numbers and $u=(u_k)$ be a sequence of strictly positive real numbers. Then $c_0(\hat{F},\mathcal{F},u,p)$ and $c(\hat{F},\mathcal{F},u,p)$ are paranormed space with the paranorm defined by*

$$g(x)=\sup\left(\frac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\Big]^{p_k}\right)^{\frac{1}{M}}$$

where $0\le p_k\le\sup p_k=H,\ \ M=\max(1,H)$.

Proof. Since the proof is similar for the space $c(\hat{F},\mathcal{F},u,p)$, we consider only the space $c_0(\hat{F},\mathcal{F},u,p)$. Clearly $g(-x)=g(x)$, for all $x\in c_0(\hat{F},\mathcal{F},u,p)$. It is trivial that $\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}=0$, for $x=0$. Hence we get $g(0)=0$. Since $\frac{p_k}{M}\le 1$, using Minkowski's inequality, we have

$$\left(\tfrac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k\Big|\Big(\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big)+\Big(\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big)\Big|\Big]^{p_k}\right)^{\frac{1}{M}}$$

$$\le\left(\tfrac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|+u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\Big]^{p_k}\right)^{\frac{1}{M}}$$

$$\le\left(\tfrac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k\Big|\frac{f_k}{f_{k+1}}x_k-\frac{f_{k+1}}{f_k}x_{k-1}\Big|\Big]^{p_k}\right)^{\frac{1}{M}}+\left(\frac{1}{n}\sum_{k=1}^{n}\Big[u_kF_k\Big|\frac{f_k}{f_{k+1}}y_k-\frac{f_{k+1}}{f_k}y_{k-1}\Big|\Big]^{p_k}\right)^{\frac{1}{M}}.$$

Now it follows that $g(x)$ is subadditive. Finally to check the continuity of scalar multiplication let us take any real number ρ. By definition of modulus function F_k, we have

$$\begin{aligned} g(\rho x) &= \sup_k \left(\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \rho \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \right)^{\frac{1}{M}} \\ &\leq C_\rho^{\frac{H}{M}} g(x). \end{aligned}$$

where C_ρ is a positive integer such that $|\rho| \leq C_\rho$. Now, Let $\rho \to 0$ for any fixed x with $g(x) = 0$. By definition for $|\rho| < 1$, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} < \epsilon \text{ for } n > N(\epsilon). \tag{2.1}$$

Also for $1 \leq n < N$, taking ρ small enough. Since F_k is continuous, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} < \epsilon. \tag{2.2}$$

Now from equation (2.1) and (2.2), we have

$$g(\rho x) \to 0 \text{ as } \rho \to 0.$$

This completes the proof. ☐

Theorem 2.3. *Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions, $u = (u_k)$ be a sequence of strictly positive real numbers. If $p = (p_k)$ and $q = (q_k)$ are bounded sequences of positive real numbers with $0 \leq p_k \leq q_k < \infty$ for each k, then $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c(\hat{F}, \mathcal{F}, u, q)$.*

Proof. Let $x \in c_0(\hat{F}, \mathcal{F}, u, p)$. Then

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

This implies that

$$\left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \leq 1,$$

for sufficiently large values of k. Since F_k is increasing and $p_k \leq q_k$ we have

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{q_k} &\leq \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \\ &\longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $x \in c(\hat{F}, \mathcal{F}, u, q)$. This completes the proof. ☐

Theorem 2.4. *Let $\mathcal{F} = (F_k)$ be a sequence of modulus functions and $\varrho = \lim_{t \to \infty} \frac{F_k(t)}{t} > 0$. Then $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c_0(\hat{F}, u, p)$.*

Proof. In order to prove that $c_0(\hat{F}, \mathcal{F}, u, p) \subseteq c_0(\hat{F}, u, p)$. Let $\varrho > 0$. By definition of ϱ, we have $F_k(t) \geq \varrho(t)$, for all $t > 0$. Since $\varrho > 0$, we have $t \leq \frac{1}{\varrho} F_k(t)$ for all $t > 0$.
Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}, u, p)$. Thus, we have

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \leq \frac{1}{\varrho n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k}$$

which implies that $x = (x_k) \in c_0(\hat{F}, u, p)$. This completes the proof. ☐

Theorem 2.5. *Let $\mathcal{F}' = (F_k')$ and $\mathcal{F}'' = (F_k'')$ are sequences of modulus functions, then*

$$c_0(\hat{F}, \mathcal{F}', u, p) \cap c_0(\hat{F}, \mathcal{F}'', u, p) \subseteq c_0(\hat{F}, \mathcal{F}' + \mathcal{F}'', u, p).$$

Proof. Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}', u, p) \cap c_0(\hat{F}, \mathcal{F}'', u, p)$. Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k'\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k''\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$$

Then we have

$$\begin{aligned}
&\tfrac{1}{n}\sum_{k=1}^{n}\left[u_k (F_k' + F_k'')\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \\
&\qquad \leq \quad K\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k'\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right\} \\
&\qquad + \quad K\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k''\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k}\right\} \\
&\qquad \longrightarrow \quad 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $\frac{1}{n}\sum_{k=1}^{n}\left[u_k (F_k' + F_k'')\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} \longrightarrow 0 \text{ as } n \to \infty.$

Therefore $x = (x_k) \in c_0(\hat{F}, \mathcal{F}' + \mathcal{F}'', u, p)$ and this completes the proof. □

Theorem 2.6. *Let $\mathcal{F} = (F_k)$ and $\mathcal{F}' = (F_k')$ be two sequences of modulus functions, then*

$$c_0(\hat{F}, \mathcal{F}', u, p) \subseteq c_0(\hat{F}, \mathcal{F}o\mathcal{F}', u, p).$$

Proof. Let $x = (x_k) \in c_0(\hat{F}, \mathcal{F}', u, p)$. Then we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k'\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]^{p_k} = 0.$$

Let $\epsilon > 0$ and choose $\delta > 0$ with $0 < \delta < 1$ such that $F_k(t) < \epsilon$ for $0 \leq t \leq \delta$.

Write $y_k = \left[u_k F_k'\Big|\frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}x_{k-1}\Big|\right]$ and consider

$$\frac{1}{n}\sum_{k=1}^{n}[F_k(y_k)]^{p_k} = \frac{1}{n}\sum_{1}[F_k(y_k)]^{p_k} + \frac{1}{n}\sum_{2}[F_k(y_k)]^{p_k}$$

where the first summation is over $y_k \leq \delta$ and second summation is over $y_k \geq \delta$. Since F_k is continuous, we have

$$\frac{1}{n}\sum_{1}[F_k(y_k)]^{p_k} < \epsilon^H \tag{2.3}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}.$$

By the definition, we have for $y_k > \delta$

$$F_k(y_k) < 2F_k(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{n}\sum_{2}[F_k(y_k)]^{p_k} \leq \max\Big(1, (2F_k(1)\delta^{-1})^H\Big)\frac{1}{n}\sum_{k}[y_k]^{p_k}. \tag{2.4}$$

From equation (2.3) and (2.4), we have

$$c_0(\hat{F}, \mathcal{F}', u, p) \subseteq c_0(\hat{F}, \mathcal{F}o\mathcal{F}', u, p).$$

This completes the proof. ☐

Theorem 2.7. *The sets* $c_0(\hat{F}, \mathcal{F}, u, p)$ *and* $c(\hat{F}, \mathcal{F}, u, p)$ *are BK-spaces with the norm* $\|x\|_{c_0(\hat{F},\mathcal{F},u,p)} = \|x\|_{c(\hat{F},\mathcal{F},u,p)} = \|\hat{F}x\|_\infty$.

Proof. Since (1.10) holds, c_0 and c are the BK-spaces with respect to their natural norms and the matrix $\hat{F}$ is a triangle; Theorem 4.3.12 of Wilansky [33, p.63] gives the fact that the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are BK-spaces with the given norms. This completes the proof. ☐

Remark 2.8. One can easily check that the absolute property does not hold on the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$, that is, $\|x\|_{c_0(\hat{F},\mathcal{F},u,p)} \neq \||x|\|_{c_0(\hat{F},\mathcal{F},u,p)}$ and $\|x\|_{c(\hat{F},\mathcal{F},u,p)} \neq \||x|\|_{c(\hat{F},\mathcal{F},u,p)}$ for at least one sequence in the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$, and this shows that $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are the sequence spaces of non-absolute type, where $|x| = (|x_k|)$.

Theorem 2.9. *The Fibonacci difference sequence spaces* $c_0(\hat{F}, \mathcal{F}, u, p)$ *and* $c(\hat{F}, \mathcal{F}, u, p)$ *of non-absolute type are linearly isomorphic to the spaces* c_0 *and* c *respectively, i.e.,* $c_0(\hat{F}, \mathcal{F}, p, u) \cong c_0$ *and* $c(\hat{F}, \mathcal{F}, p, u) \cong c$.

Proof. To prove this, we should show the existence of a linear bijection between the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0. Consider the transformation T defined with the notation of (1.9), from $c_0(\hat{F}, \mathcal{F}, u, p)$ to c_0 by $x \to y = Tx$. The linearity of T is clear. Further it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective.
We assume that $y = (y_k) \in c_0$, for $1 \leq p \leq \infty$ and defined the sequence $x = (x_k)$ by

$$x_k = \sum_{j=0}^{k}\frac{f_{k+1}^2}{f_j f_{j+1}}y_j, \quad \text{for all } \ k \in \mathbb{N}.$$

Then we have

$$\lim_{k\to\infty}\left\{\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k\Big|\frac{f_k}{f_{k+1}}\sum_{j=0}^{k}\frac{f_{k+1}^2}{f_j f_{j+1}}y_j - \frac{f_{k+1}}{f_k}\sum_{j=0}^{k-1}\frac{f_k^2}{f_j f_{j+1}}y_j\Big|\right]^{p_k}\right\} = \lim_{k\to\infty} y_k = 0$$

which shows that $x \in c_0(\hat{F}, \mathcal{F}, p, u)$. Additionally, we have for every $x \in c_0(\hat{F}, \mathcal{F}, p, u)$ that

$$\begin{aligned}
\|x\|_{c_0(\hat{F},\mathcal{F},p,u)} &= \sup_{k\in\mathbb{N}} \left| \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \Big| \right]^{p_k} \right| \\
&= \sup_{k\in\mathbb{N}} \left| \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j - \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{f_k^2}{f_j f_{j+1}} y_j \Big| \right]^{p_k} \right| \\
&= \sup_{k\in\mathbb{N}} \Big(|y_k|^{p_k} \Big) \\
&= \|y\|_\infty < \infty.
\end{aligned}$$

Consequently, we see from here that T is surjective and norm preserving. Hence, T is a linear bijection which shows that the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0 are linearly isomorphic. It is clear here that if the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and c_0 are respectively replaced by the spaces $c(\hat{F}, \mathcal{F}, u, p)$ and c, then we obtain the fact that $c(\hat{F}, \mathcal{F}, p, u) \cong c$. This concludes the proof. ☐

Now, we give some inclusion relations concerning with the space $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.

Theorem 2.10. *The inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ strictly holds.*

Proof. It is clear that the inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ holds. Further, to show that this inclusion is strict, consider the sequence $x = (x_k) = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j^2}$. Then, we obtain (1.9) for all $k \in \mathbb{N}$ that

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j^2} - \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{f_{k+1}^2}{f_j^2} \Big| \right]^{p_k} = \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big(\frac{f_{k+1}}{f_k} \Big) \right]^{p_k}$$

which shows that $\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big(\frac{f_{k+1}}{f_k} \Big) \right]^{p_k} \to \varphi,$ as $k \to \infty$. This is to say that $\hat{F}(x) \in c \backslash c_0$. Thus, the sequence x is in the $c(\hat{F}, \mathcal{F}, u, p)$ but not in $c_0(\hat{F}, \mathcal{F}, u, p)$. Hence, the inclusion $c_0(\hat{F}, \mathcal{F}, u, p) \subset c(\hat{F}, \mathcal{F}, u, p)$ is strict. ☐

Theorem 2.11. *The space l_∞ does not include the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.*

Proof. Let us consider the sequence $x = (x_k) = (f_{k+1}^2)$. Since $f_{k+1}^2 \to \infty$ as $k \to \infty$ and $\hat{F}(x) = e^{(0)} = (1, 0, 0, ...)$, the sequence x is in the space $c_0(\hat{F}, \mathcal{F}, u, p)$ but is not in the space l_∞. This shows that the space l_∞ does not include the space $c_0(\hat{F}, \mathcal{F}, u, p)$ and the space $c(\hat{F}, \mathcal{F}, u, p)$, as desired. ☐

Theorem 2.12. *The inclusions $c_0 \subset c_0(\hat{F}, \mathcal{F}, u, p)$ and $c \subset c(\hat{F}, \mathcal{F}, u, p)$ strictly holds.*

Proof. Let $X = c_0$ or c. Since the matrix $\hat{F} = (f_{nk})$ satisfies the conditions

$$\begin{aligned}
&\sup_{n\in\mathbb{N}} \sum_{k} |f_{nk}| = \sup_{n\in\mathbb{N}} \Big(\frac{f_n}{f_{n+1}} + \frac{f_{n+1}}{f_n} \Big) = 2 + \frac{1}{2} = \frac{5}{2}, \\
&\lim_{n\to\infty} f_{nk} = 0, \\
&\lim_{n\to\infty} \sum_{k} f_{nk} = \lim_{n\to\infty} \Big(\frac{f_n}{f_{n+1}} - \frac{f_{n+1}}{f_n} \Big) = \frac{1}{\varphi} - \varphi
\end{aligned}$$

we conclude by parts (a) and (c) of Lemma 1.2 that $(\hat{F}, \mathcal{F}, u, p) \in (X, X)$. This leads that $(\hat{F}, \mathcal{F}, u, p)x \in X$ for any $x \in X$. Thus, $x \in X_{(\hat{F}, \mathcal{F}, u, p)}$. This shows that $X \subset X_{(\hat{F}, \mathcal{F}, u, p)}$. Now, let $x = (x_k) = (f_{k+1}^2)$. Then, it is clear that $x \in X_{(\hat{F}, \mathcal{F}, u, p)} \backslash X$. This says that the inclusion $X \subset X_{(\hat{F}, \mathcal{F}, u, p)}$ is strict. ☐

Theorem 2.13. *The spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ are not solid.*

Proof. Consider the sequences $r = (r_k)$ and $s = (s_k)$ defined by $r_k = f_{k+1}^2$ and $s_k = (-1)^{k+1}$ for all $k \in \mathbb{N}$. Then, it is clear that $r \in c_0(\hat{F}, \mathcal{F}, u, p)$ and $s \in l_\infty$. Nevertheless $rs = \{(-1)^{k+1} f_{k+1}^2\}$ is not in the space $c_0(\hat{F}, \mathcal{F}, u, p)$, since

$$\frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \Big| \frac{f_k}{f_{k+1}} (-1)^{k+1} f_{k+1}^2 - \frac{f_{k+1}}{f_k} (-1)^k f_k^2 \Big| \right]^{p_k}$$

$$= \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \big(2(-1)^{k+1} f_k f_{k+1} \big) \right]^{p_k} \quad \text{for all } k \in \mathbb{N}.$$

This shows that the multiplication $l_\infty c_0(\hat{F}, \mathcal{F}, u, p)$ of the spaces l_∞ and $c_0(\hat{F}, \mathcal{F}, u, p)$ is not a subset of $c_0(\hat{F}, \mathcal{F}, u, p)$. Hence, the space $c_0(\hat{F}, \mathcal{F}, u, p)$ is not solid.

It is clear here that if the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ is replaced by the space $c(\hat{F}, \mathcal{F}, u, p)$, then we obtain the fact $c(\hat{F}, \mathcal{F}, u, p)$ is not solid. This completes the proof. ☐

It is known from Theorem 2.3 of Jarrah and Malkowsky [14] that the domain X_T of an infinite matrix $T = (t_{nk})$ in a normed sequence space X has a basis if and only if X has a basis, if T is a triangle. As a direct consequence of this fact, we have

Corollary 2.14. *Define the sequences $c^{(-1)} = \{c_k^{(-1)}\}_{k \in \mathbb{N}}$ and $c^{(n)} = \{c_k^{(n)}\}_{k \in \mathbb{N}}$ for every fixed $n \in \mathbb{N}$ by*

$$c_k^{(-1)} = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} \quad \text{and} \quad c_k^{(n)} = \begin{cases} 0 & , \ 0 \leq k \leq n-1 \\ \frac{f_{k+1}^2}{f_n f_{n+1}}, & k \geq n \end{cases}$$

Then, the following statements hold:

(a) The sequence $\{c^{(n)}\}_{n=0}^{\infty}$ is a basis for the space $c_0(\hat{F}, \mathcal{F}, u, p)$ and every sequence $x \in c_0(\hat{F}, \mathcal{F}, u, p)$ has a unique representation $x = \sum_n \hat{F}_n(x) c^{(n)}$.

(b) The sequence $\{c^{(n)}\}_{n=-1}^{\infty}$ is a basis for the space $c(\hat{F}, \mathcal{F}, u, p)$ and every sequence $z = (z_n) \in c(\hat{F}, \mathcal{F}, u, p)$ has a unique representation $z = lc^{(-1)} + \sum_n [\hat{F}_n(z) - l] c^{(n)}$, where $l = \lim_{n \to \infty} \hat{F}_n(z)$.

3. The $\alpha-, \beta-$ and $\gamma-$ duals of the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ and some matrix transformations

The $\alpha-, \beta-$ and $\gamma-$ duals of the sequence space X are respectively defined by

$$X^\alpha = \{a = (a_k) \in w : ax = (a_k x_k) \in l_1 \ \text{ for all } x = (x_k) \in X\},$$

$$X^\beta = \{a = (a_k) \in w : ax = (a_k x_k) \in cs \ \text{ for all } x = (x_k) \in X\}$$

and

$$X^\gamma = \{a = (a_k) \in w : ax = (a_k x_k) \in bs \ \text{ for all } x = (x_k) \in X\}$$

In this section, we determine $\alpha-, \beta-$ and $\gamma-$ duals of the sequence spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$, and characterize the classes of infinite matrices from the spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$ to the spaces $c_0, c, l_\infty, f, f_0, bs, fs, cs$ and l_1, and from the space f to the

spaces $c_0(\hat{F}, \mathcal{F}, u, p)$ and $c(\hat{F}, \mathcal{F}, u, p)$.
The following two lemmas are essential for our results.

Lemma 3.1. [8] *Let X be any of the spaces c_0 or c and $a = (a_n) \in w$, and the matrix $B = (b_{nk})$ be defined by $B_n = a_n \hat{F}_n^{-1}$, that is ,*

$$b_{nk} = \begin{cases} a_n g_{nk}, & 0 \leq k \leq n, \\ 0 \quad , & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then $a \in X_{\hat{F}}^{\beta}$ if and only if $B \in (X, l_1)$.

Lemma 3.2 (5, Theorem 3.1)**.** *Let $C = (c_{nk})$ be defined via a sequence $a = (a_k) \in w$ and the inverse matrix $V = (v_{nk})$ of the triangle matrix $Z = (z_{nk})$ by*

$$c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & 0 \leq k \leq n, \\ \qquad 0 \qquad , & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Then for any sequence space X,

$$X_Z^{\gamma} = \{a = (a_k) \in w : C \in (X, l_\infty)\},$$

$$X_Z^{\beta} = \{a = (a_k) \in w : C \in (X, c)\}.$$

Combining Lemmas (1.2), (3.1), and (3.2), we have

Corollary 3.3. *Consider the sets d_1, d_2, d_3 and d_4 defined as follows:*

$$d_1 = \left\{ a = (a_k) \in w : \sup_{k \in \mathcal{H}} \sum_n \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{k \in K} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n \right| \right]^{p_k} < \infty \right\},$$

$$d_2 = \left\{ a = (a_k) \in w : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} < \infty \right\},$$

$$d_3 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} \textit{ exists for each } k \in \mathbb{N} \right\},$$

$$d_4 = \left\{ a = (a_k) \in w : \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{n} \sum_{k=1}^{n} \left[u_k F_k \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \right| \right]^{p_k} \textit{ exists} \right\}.$$

Then the following statements hold:
(a) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\alpha} = \{c(\hat{F}, \mathcal{F}, u, p)\}^{\alpha} = d_1$.
(b) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\beta} = d_2 \cap d_3$ and $\{c(\hat{F}, \mathcal{F}, u, p)\}^{\beta} = d_2 \cap d_3 \cap d_4$.
(c) $\{c_0(\hat{F}, \mathcal{F}, u, p)\}^{\gamma} = \{c(\hat{F}, \mathcal{F}, u, p)\}^{\gamma} = d_2$.

Theorem 3.4. *Let $X = c_0$ or c and Y be an arbitrary subset of w. Then, we have $A = (a_{nk}) \in (X_{\hat{F}}, Y)$ if and only if*

$$D^{(m)} = (d_{nk}^{(m)}) \in (X, c) \textit{ for all } n \in \mathbb{N}, \tag{3.1}$$

$$D = (d_{nk}) \in (X, Y), \tag{3.2}$$

where

$$d_{nk}^{(m)} = \begin{cases} \frac{1}{n} \sum_{k=1}^{n} \left(u_k F_k \left| \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right| \right)^{p_k}, & 0 \leq k \leq m \\ 0 & , \quad k > m \end{cases}$$

and

$$d_{nk} = \frac{1}{n}\sum_{k=1}^{n}\left(u_k F_k\left|\sum_{j=k}^{\infty}\frac{f_{j+1}^2}{f_k f_{k+1}}a_{nj}\right|\right)^{p_k} \text{ for all } k, m, n \in \mathbb{N}.$$

By changing the roles of the spaces $X_{\hat{F}}$ and X with Y in Theorem 3.4, we have

Theorem 3.5. *Suppose that the elements of the infinite matrices $A = (a_{nk})$ and $B = (b_{nk})$ are connected with the relation*

$$b_{nk} = \frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k\left| -\frac{f_{n+1}}{f_n}a_{n-1,k} + \frac{f_n}{f_{n+1}}a_{nk}\right|\right]^{p_k} \tag{3.3}$$

for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (Y, X_{\hat{F}})$ if and only if $B \in (Y, X)$.

Proof. Let $z = (z_k) \in Y$. Then, by taking into account the relation (3.3) one can easily derive the following equality

$$\sum_{k=0}^{m} b_{nk}z_k = \sum_{k=0}^{m}\left(\frac{1}{n}\sum_{k=1}^{n}\left[u_k F_k\left| -\frac{f_{n+1}}{f_n}a_{n-1,k} + \frac{f_n}{f_{n+1}}a_{nk}\right|\right]^{p_k}\right)z_k \text{ for all } m, n \in \mathbb{N}$$

which yields as $m \to \infty$ that $(Bz)_n = [\hat{F}(Az)]_n$. Therefore, we conclude that $Az \in X_{\hat{F}}$ whenever $z \in Y$ if and only if $Bz \in X$ whenever $z \in Y$. This completes the proof. □

By f_0, f and fs we denote the spaces of almost null and almost convergent sequences and series respectively. Now, the following two lemmas characterizing the strongly and almost conservative matrices:

Lemma 3.6. *(see [32]) $A = (a_{nk}) \in (f, c)$ if and only if (1.3), (1.5), and (1.7) hold, and*

$$\lim_{n\to\infty}\sum_k \Delta(a_{nk} - \alpha_k) = 0 \tag{3.4}$$

also holds, where $\Delta(a_{nk} - \alpha_k) = a_{nk} - \alpha_k - (a_{n,k+1} - \alpha_{k+1})$ for all $k, n \in \mathbb{N}$.

Lemma 3.7. *(see [21]) $A = (a_{nk}) \in (c, f)$ if and only if (1.3) holds, and*

$$\exists \alpha_k \in \mathbb{C} \ni f - \lim a_{nk} = \alpha_k \text{ for each fixed } k \in \mathbb{N}, \tag{3.5}$$

$$\exists \alpha \in \mathbb{C} \ni f - \lim \sum_k a_{nk} = \alpha. \tag{3.6}$$

Now, we list the following conditions:

$$\sup_{m\in\mathbb{N}}\sum_{k=0}^{m}\left|d_{mk}^{(n)}\right| < \infty \tag{3.7}$$

$$\exists d_{nk} \in \mathbb{C} \ni \lim_{m\to\infty} d_{mk}^{(n)} = d_{nk} \text{ for each } k, n \in \mathbb{N} \tag{3.8}$$

$$\sup_{n\in\mathbb{N}}\sum_k |d_{nk}| < \infty \tag{3.9}$$

$$\exists \alpha_k \in \mathbb{C} \ni \lim_{n\to\infty} d_{nk} = \alpha_k \text{ for each } k \in \mathbb{N} \tag{3.10}$$

$$\sup_{N,K\in\mathcal{H}}\Big|\sum_{n\in\mathbb{N}}\sum_{k\in K} d_{nk}\Big| < \infty \tag{3.11}$$

$$\exists\beta_n \in \mathbb{C} \ni \lim_{m\to\infty}\sum_{k=0}^{m} d_{mk}^{(n)} = \beta_n \text{ for each } n \in \mathbb{N} \tag{3.12}$$

$$\exists\alpha \in \mathbb{C} \ni \lim_{n\to\infty}\sum_{k} d_{nk} = \alpha \tag{3.13}$$

It is trivial that Theorem 3.4 and Theorem 3.5 have several consequences. Indeed, combining Theorem 3.4, 3.5 and Lemmas 1.1, 3.6 and 3.7 we derive the following results:

Corollary 3.8. *Let $A = (a_{nk})$ be an infinite matrix and $a(n,k) = \sum_{j=0}^{n} a_{jn}$ for all $k, n \in \mathbb{N}$. Then, the following statements hold:*
(a) $A = (a_{nk}) \in (c_0(\hat{F}, \mathcal{F}, u, p), c_0)$ if and only if (3.7), (3.8), (3.9) hold and (3.10) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.
(b) $A = (a_{nk}) \in (c_0(\hat{F}, \mathcal{F}, u, p), cs_0)$ if and only if (3.7), (3.8), (3.9) hold and (3.10) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ with $a(n,k)$ instead of a_{nk}.
(c) $A = (a_{nk}) \in (c_0(\hat{F}, \mathcal{F}, u, p), c)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold.
(d) $A = (a_{nk}) \in (c_0(\hat{F}, \mathcal{F}, u, p), cs)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $a(n,k)$ instead of a_{nk}.
(e) $A = (a_{nk}) \in (c_0(\hat{F}, \mathcal{F}, u, p), l_\infty)$ if and only if (3.7), (3.8) and (3.9) hold.
(f) $A = (a_{nk}) \in (c_0(\hat{F}, \mathcal{F}, u, p), bs)$ if and only if (3.7), (3.8) and (3.9) hold with $a(n,k)$ instead of a_{nk}.
(g) $A = (a_{nk}) \in (c_0(\hat{F}, \mathcal{F}, u, p), l_1)$ if and only if (3.7), (3.8) and (3.11) hold.
(h) $A = (a_{nk}) \in (c_0(\hat{F}, \mathcal{F}, u, p), bv_1)$ if and only if (3.7), (3.8) and (3.11) hold with $a_{nk} - a_{n-1,k}$ instead of a_{nk}.

Corollary 3.9. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*
(a) $A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), l_\infty)$ if and only if (3.7), (3.8), (3.9) and (3.12) hold.
(b) $A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), bs)$ if and only if (3.7), (3.8), (3.9) and (3.12) hold with $a(n,k)$ instead of a_{nk}.
(c) $A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), c)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold.
(d) $A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), cs)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold with $a(n,k)$ instead of a_{nk}.
(e) $A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), c_0)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$, (3.12) and (3.13) also hold with $\alpha = 0$.
(f) $A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), cs_0)$ if and only if (3.7), (3.8), (3.9) and (3.10) hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$, (3.12) and (3.13) also hold with $\alpha = 0$ with $a(n,k)$ instead of a_{nk}.
(g) $A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), l_1)$ if and only if (3.7), (3.8), (3.11) and (3.12)hold.
(h) $A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), bv_1)$ if and only if (3.7), (3.8), (3.11) and (3.12) hold with $a_{nk} - a_{n-1,k}$ instead of a_{nk}..

Corollary 3.10. *$A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), f)$ if and only if (3.7), (3.8), (3.12) and (3.13) hold, and (3.9), (3.10) also hold with d_{nk} instead of a_{nk}.*

Corollary 3.11. *$A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), f_0)$ if and only if (3.7), (3.8), (3.12) and (3.13) hold, and (3.9), (3.10) also hold with d_{nk} instead of a_{nk} and $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Corollary 3.12. *$A = (a_{nk}) \in (c(\hat{F}, \mathcal{F}, u, p), f_s)$ if and only if (3.7), (3.8), (3.9), (3.10), (3.12) and (3.13) hold with $a(n, k)$ instead of a_{nk} and (3.9), (3.10) also hold with $d(n, k)$ instead of d_{nk}.*

Corollary 3.13. *$A = (a_{nk}) \in (f, c(\hat{F}, \mathcal{F}, u, p))$ if and only if (1.3), (1.5), (1.7) and (3.8) hold with b_{nk} instead of a_{nk}, where $b(n, k)$ is defined by (3.3) .*

Corollary 3.14. *$A = (a_{nk}) \in (f, c_0(\hat{F}, \mathcal{F}, u, p))$ if and only if (1.3) and (1.7) hold, (1.5) and (3.8) also hold with b_{nk} instead of a_{nk} and $\alpha_k = 0$ for all $k \in \mathbb{N}$, where $b(n, k)$ is defined by (3.3).*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.5, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Completely monotonic functions involving Bateman's $G-$function

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Abstract

In this paper, we prove the complete monotonicity of some functions involving Bateman's $G-$function and show that

$$\frac{1}{2x^2+\alpha} < G(x) - \frac{1}{x} < \frac{1}{2x^2+\beta}, \qquad x > 0$$

where $\alpha = 1$ and $\beta = 0$ are the best possible constants, which is a refinement of a recent result. Then, we give a new proof of Slavić inequality about Wallis ratio W_m and provide a new inequality for W_m. Our new inequality improves some recent related works. We also present two inequalities for the hyperbolic tangent function.

2010 Mathematics Subject Classification: 26A48, 26D15, 33B15.

Key Words: Bateman's $G-$function, completely monotonic, best possible constant, Bernoulli numbers, Wallis ratio, hyperbolic tangent function.

1 Introduction

A function $H : J \to \mathbb{R}$ is said to be completely monotonic (see [45] and [11]), if $H^{(m)}(x)$ exists on J for all $m \geq 0$ and

$$(-1)^m H^{(m)}(x) \geq 0 \qquad x \in J;\ m \geq 0. \tag{1}$$

For $x > 0$, the necessary and sufficient condition for the function $H(x)$ to be completely monotonic is the convergence of the following integral

$$H(x) = \int_0^\infty e^{-xt} dv(t), \tag{2}$$

where $v(t)$ is a nonnegative measure on $t \geq 0$. The function $H(x)$ is said to be strictly completely monotonic if the inequality (1) is strict for all $x \in J$ and $m \geq 0$. The concept of completely monotonic function is the continuous analogue of the totally monotone sequence presented by Hausdorff in 1921 [15] (see also [45]). These functions find applications in several diverse fields such as in the theory of special functions, asymptotic analysis, probability, physics, and the list continues, see [2], [5], [6], [12] , [13], [32], [34], [35], [38], [44] and the references therein.

The Bateman's $G-$function is defined by (see Erdélyi [10])

$$G(t) = \psi\left(\frac{t}{2} + \frac{1}{2}\right) - \psi\left(\frac{t}{2}\right), \qquad t \neq 0, -1, -2, ... \tag{3}$$

where $\psi(t)$ is the digamma (Psi) function which is defined by

$$\psi(t) = \frac{d}{dt} \ln \Gamma(t)$$

and $\Gamma(z)$ is the classical Euler gamma function which is defined for $Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-w} w^{z-1} dw.$$

For more details on bounds, identities, properties and applications of Bateman's $G-$function, refer to [10], [21]-[25], [31], [39] and the references therein. The following relations hold for the function $G(x)$ [10]:

$$G(x+1) = -G(x) + 2x^{-1}, \tag{4}$$

$$G(x) = \int_0^\infty \frac{2e^{-xv}}{1+e^{-v}} dv, \qquad x > 0 \tag{5}$$

$$G(x) = x^{-1} \, {}_2F_1\left(1, 1; 1+x; \frac{1}{2}\right), \tag{6}$$

where

$${}_lF_m(v_1, ..., v_l; w_1, ..., w_m; z) = \sum_{k=0}^\infty \frac{(v_1)_k ... (v_l)_k}{(w_1)_k ... (w_m)_k} \frac{z^k}{k!}$$

is the generalized hypergeometric function [3] defined for $l, m \in \mathbb{N}$, $v_j, w_j \in \mathbb{C}$, $w_j \neq 0, -1, -2, ...$ and

$$(v)_0 = 1 \quad \text{and} \quad (v)_n = \frac{\Gamma(v+m)}{\Gamma(v)}, \qquad n \in \mathbb{N}.$$

Qiu and Vuorinen [39] established the inequality

$$\frac{(6 - 4\ln 4)}{x^2} < G(x) - \frac{1}{x} < \frac{1}{2x^2}, \qquad x > 1/2 \tag{7}$$

and Mortici [25] improved the inequality (7) to the double inequality

$$0<\psi(x+h)-\psi(x)\leq\psi(h)+\gamma-h+h^{-1},\qquad x\geq 1;\ h\in(0,1) \tag{8}$$

where γ is the Euler constant. Mahmoud and Agarwal [21] deduced the following asymptotic formula for $x\to\infty$

$$G(x)-\frac{1}{x}\sim\sum_{k=1}^{\infty}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k}, \tag{9}$$

where $B_m{}'s$ are the Bernoulli numbers [17] and they also presented the following inequality

$$\frac{1}{2x^2+\frac{3}{2}}<G(x)-x^{-1}<\frac{1}{2x^2},\qquad x>0 \tag{10}$$

which improves the lower bound of the inequality (7) for $x>\left(\frac{9-12\ln 2}{16\ln 2-11}\right)^{1/2}$. In [22] Mahmoud and Almuashi proved the following inequality

$$\sum_{n=1}^{2m}\frac{(2^{2n}-1)}{n}B_{2n}x^{-2n}<G(x)-x^{-1}<\sum_{n=1}^{2m-1}\frac{(2^{2n}-1)}{n}B_{2n}x^{-2n},\qquad m\in\mathbb{N} \tag{11}$$

where $\frac{(2^{2n}-1)}{n}B_{2n}$ are the best possible constants. Also, Mahmoud, Talat and Moustafa [23] studied the following family of approximations of Bateman's $G-$function

$$\chi(\rho,x)=\ln\left(1+\frac{1}{x+\rho}\right)+\frac{2}{x(x+1)},\qquad 1\leq\rho\leq 2;\ x>0$$

which is asymptotically equivalent to the function $G(x)$ for $x\to\infty$.

Recently, Mahmoud and Almuashi [24] presented some identities, functional equations and an asymptotic expansion of the generalized Bateman's $G-$function $G_\sigma(x)$ defined by

$$G_\sigma(x)=\psi\left(\frac{x+\sigma}{2}\right)-\psi\left(\frac{x}{2}\right),\qquad x\neq -2r,-2r-\sigma;\ \sigma\in(0,2);\ \text{for}\ \ r=0,1,2,\ldots\ .$$

Also, they presented the double inequality

$$\ln\left(1+\frac{\sigma}{x+\phi}\right)<G_\sigma(x)-\frac{2\sigma}{x(x+\sigma)}<\ln\left(1+\frac{\sigma}{x+\theta}\right),\qquad x>0;\ \sigma\in(0,2)$$

where $\phi=\frac{\sigma}{e^{\gamma+\frac{2}{\sigma}+\psi(\frac{\sigma}{2})}-1}$ and $\theta=1$ are the best possible constants.

In this paper, we will study the complete monotonicity of some functions involving the function $G(x)$ and as a consequence, we will deduce a double inequality of it. Also, we will prove that the function

$$q(x)=\frac{1}{G(x)-\frac{1}{x}}-2x^2,\qquad x>0$$

is strictly increasing and present a refinement of the lower bound of the inequality (10). We will apply our results to present a new proof of Slavić inequality about Wallis ratio $W_m=\frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}$ for $m\in N$. We will also present a new inequality of W_m, which improves some recent results. Further, we will present two inequalities involving the hyperbolic tangent function.

2 Main Results

We begin by proving some auxiliary results involving Bernoulli numbers.

Lemma 2.1. *For any positive integer $s \geq 1$, we have*

$$B_{2s} = \frac{1}{2(2^{2s}-1)} \left[1 - \frac{1}{2s+1} \sum_{k=1}^{s-1} 2(2^{2k}-1) \binom{2s+1}{2k} B_{2k} \right] \tag{12}$$

and

$$B_{2s} = \frac{1}{2(2^{2s}-1)} \left[s - \sum_{k=1}^{s-1} (2^{2k}-1) \binom{2s}{2k} B_{2k} \right]. \tag{13}$$

Proof. The identity [30]

$$B_m = \frac{1}{2(1-2^m)} \sum_{j=0}^{m-1} 2^j \binom{m}{j} B_j, \qquad m \in \mathbb{N} \tag{14}$$

can be rewritten as

$$B_m = \frac{1}{2(1-2^m)} [1 - m + \sum_{j=1}^{[\frac{m-1}{2}]} 2^{2j} \binom{m}{2j} B_{2j}], \qquad m \geq 1$$

where $B_{2r+1} = 0$ for $r \in \mathbb{N}$ and hence

$$\sum_{k=1}^{s} 2^{2k} \binom{2s+1}{2k} B_{2k} = 2s, \quad s \geq 1 \tag{15}$$

$$\sum_{k=1}^{s-1} 2^{2k} \binom{2s}{2k} B_{2k} = (2s-1) + 2(1-2^{2s})B_{2s}, \quad s \geq 2. \tag{16}$$

Also, Bernoulli numbers satisfy [4]

$$s - \frac{1}{2} = \sum_{k=1}^{s} \binom{2s+1}{2k} B_{2k}, \quad s \geq 1 \tag{17}$$

$$s - 1 = \sum_{k=1}^{s-1} \binom{2s}{2k} B_{2k}, \quad s \geq 2. \tag{18}$$

From the two identities (15) and (17), we get

$$\sum_{k=1}^{s} 2(2^{2k}-1) \binom{2s+1}{2k} B_{2k} = 2s+1 \qquad s \geq 1 \tag{19}$$

and the two identities (16) and (18) give us

$$2(2^{2s}-1)B_{2s} + \sum_{k=1}^{s-1} (2^{2k}-1) \binom{2s}{2k} B_{2k} = s \qquad s \geq 1. \tag{20}$$

□

Lemma 2.2. *For $v = 2, 3, 4, \cdots$, Bernoulli numbers satisfy*

$$\frac{(2^{2v+2}-1)}{(2^{2v}-1)}\pi^2 < \frac{|B_{2v}|}{|B_{2v+2}|}(2v+1)(2v+2) < \frac{(2^{2v+2}-1)}{(2^{2v}-1)}(\pi^2+1). \tag{21}$$

Proof. The function

$$f(x) = x(8x-(9+3\pi^2))+1, \qquad x \geq \frac{9+3\pi^2+\sqrt{49+54\pi^2+9\pi^4}}{16} \approx 4.80006...$$

is increasing and positive, and hence

$$2^{2v-1}(2^{2v+2}-(9+3\pi^2))+1 > 0, \qquad v \geq 2.$$

Then

$$(\pi^2+1)(2^{2v+2}-1)(2^{2v-1}-1)-\pi^2(2^{2v+1}-1)(2^{2v}-1) > 0, \qquad v \geq 2$$

or

$$\frac{(2^{2v+1}-1)}{(2^{2v-1}-1)}\pi^2 < \frac{(2^{2v+2}-1)}{(2^{2v}-1)}(\pi^2+1), \qquad v \geq 2. \tag{22}$$

From the Qi's result [36]

$$\frac{(2^{2v+2}-1)}{(2^{2v}-1)}\frac{\pi^2}{(2v+1)(2v+2)} < \frac{|B_{2v}|}{|B_{2v+2}|} < \frac{(2^{2v+1}-1)}{(2^{2v-1}-1)}\frac{\pi^2}{(2v+1)(2v+2)}, \qquad v \geq 1 \tag{23}$$

and the inequality (22), we complete the proof. □

Now we will prove the complete monotonicity of some functions involving the function $G(x)$.

Lemma 2.3. *For a positive integer m, the function*

$$F(x) = G(x) - \frac{1}{x} - \sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}}{kx^{2k}}, \qquad x > 0 \tag{24}$$

is strictly completely monotonic.

Proof. Using the formula [1]

$$\frac{1}{x^k} = \frac{1}{(k-1)!}\int_0^\infty t^{k-1}e^{-xt}dt, \qquad k \in \mathbb{N} \tag{25}$$

and the integral representation of $G(x)$, we get

$$\begin{aligned} F(x) &= \int_0^\infty \left[e^t - 1 - (1+e^t)\sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}t^{2k-1}}{k(2k-1)!}\right]\frac{e^{-xt}}{1+e^t}dt \\ &= \int_0^\infty \varphi(t)\frac{e^{-xt}}{1+e^t}dt, \end{aligned}$$

where

$$\varphi(t) = e^t - 1 - (1+e^t)\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}. \tag{26}$$

Now

$$
\begin{aligned}
\varphi(t) &= \sum_{r=1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{r=0}^{\infty}\frac{t^{r+2k-1}}{r!}\\
&= \sum_{r=1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{\infty}\frac{t^s}{(s-2k+1)!}\\
&= \sum_{r=1}^{4m}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{4m}\frac{t^s}{(s-2k+1)!}\\
&\quad+\sum_{r=4m+1}^{\infty}\frac{t^r}{r!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=4m+1}^{\infty}\frac{t^s}{(s-2k+1)!}.
\end{aligned}
$$

Rewrite infinite summations from 0 and split finite summations by even and odd power of t we obtain

$$
\begin{aligned}
\varphi(t) &= \sum_{s=1}^{2m}\frac{t^{2s-1}}{(2s-1)!}-\sum_{s=1}^{2m}\frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m}\frac{t^{2s-1}}{(2s-2k)!}\\
&\quad+\sum_{s=1}^{2m}\frac{t^{2s}}{(2s)!}-\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m}\frac{t^{2s}}{(2s-2k+1)!}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{2m}\frac{2(2^{2k}-1)B_{2k}}{(2k)!(s+4m-2k+2)!}t^{s+4m+1},
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
\varphi(t) &= \sum_{s=1}^{2m}\frac{t^{2s-1}}{(2s-1)!}-\sum_{s=1}^{2m}\frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1}-\sum_{s=1}^{2m}\frac{1}{(2s)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)(2s!)B_{2k}}{(2k)!(2s-2k)!}t^{2s-1}\\
&\quad+\sum_{s=1}^{2m}\frac{t^{2s}}{(2s)!}-\sum_{s=1}^{2m}\frac{1}{(2s+1)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)((2s+1)!)B_{2k}}{(2k)!(2s-2k+1)!}t^{2s}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{m}\left(\frac{2(2^{4k}-1)B_{4k}}{(4k)!(s+4(m-k)+2)!}+\frac{2(2^{4k-2}-1)B_{4k-2}}{(4k-2)!(s+4(m-k)+4)!}\right)t^{s+4m+1}\\
&= \sum_{s=1}^{2m}\left[2s-4(2^{2s}-1)B_{2s}-\sum_{k=1}^{s-1}2(2^{2k}-1)\binom{2s}{2k}B_{2k}\right]\frac{t^{2s-1}}{(2s)!}\\
&\quad+\sum_{s=1}^{2m}\left[2s+1-\sum_{k=1}^{s}2(2^{2k}-1)\binom{2s+1}{2k}B_{2k}\right]\frac{t^{2s}}{(2s+1)!}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\
&\quad-\sum_{s=0}^{\infty}\sum_{k=1}^{m}\left[\left(1+\frac{(2^{4k-2}-1)(4k)(4k-1)B_{4k-2}}{(2^{4k}-1)(s+4(m-k)+3)(s+4(m-k)+4)B_{4k}}\right)\right.\\
&\qquad\qquad\left.\frac{2(2^{4k}-1)t^{s+4m+1}B_{4k}}{(4k)!(s+4(m-k)+2)!}\right].
\end{aligned}
$$

Using the identities (12) and (13) with the relation

$$(-1)^{r+1}B_{2r} > 0, \qquad r \in \mathbb{N} \tag{27}$$

we obtain

$$\begin{aligned}\varphi(t) &= \sum_{s=0}^{\infty}\sum_{k=1}^{m}\left[\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{(2^{4k}-1)(s+4(m-k)+3)(s+4(m-k)+4)|B_{4k}|}\right)\right.\\ &\quad \left.\frac{2(2^{4k}-1)|B_{4k}|t^{s+4m+1}}{(4k)!(s+4(m-k)+2)!}\right]+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}.\end{aligned}$$

For $s \geq 0$ and $m \geq k \geq 1$, we have

$$(s+4(m-k)+3)(s+4(m-k)+4) \geq (s+3)(s+4) \geq 12$$

and then

$$\begin{aligned}\varphi(t) &\geq \sum_{s=0}^{\infty}\sum_{k=1}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{12(2^{4k}-1)|B_{4k}|}\right)t^{s+4m+1}\\ &\quad +\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}\\ &\geq \sum_{s=0}^{\infty}\sum_{k=2}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{(2^{4k-2}-1)(4k)(4k-1)|B_{4k-2}|}{12(2^{4k}-1)|B_{4k}|}\right)t^{s+4m+1}\\ &\quad +\sum_{s=0}^{\infty}\frac{30|B_4|}{(4!)(s+4m-2)!)}\left(1-\frac{|B_2|}{5|B_4|}\right)t^{s+4m+1}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!}.\end{aligned}$$

Using inequality (21) with $v = 2k-1$ for $k \in \mathbb{N}$, we get

$$\varphi(t) > \sum_{s=0}^{\infty}\sum_{k=2}^{m}\frac{2(2^{4k}-1)|B_{4k}|}{(4k)!(s+4(m-k)+2)!}\left(1-\frac{\pi^2+1}{12}\right)t^{s+4m+1}+\sum_{s=0}^{\infty}\frac{t^{s+4m+1}}{(s+4m+1)!} > 0,$$

which complete the proof. □

Lemma 2.4. *For a positive integer m, the function*

$$M(x) = \frac{1}{x} - G(x) + \sum_{k=1}^{2m-1}\frac{(2^{2k}-1)B_{2k}}{kx^{2k}}, \qquad x > 0 \tag{28}$$

is strictly completely monotonic.

Proof. Using the formula (25) and the integral representation of $G(x)$, we have

$$\begin{aligned}M(x) &= \int_0^{\infty}\left[(1+e^t)\sum_{k=1}^{2m-1}\frac{(2^{2k}-1)B_{2k}t^{2k-1}}{k(2k-1)!}-(e^t-1)\right]\frac{e^{-xt}}{1+e^t}dt\\ &= \int_0^{\infty}\mu(t)\frac{e^{-xt}}{1+e^t}dt,\end{aligned}$$

where

$$\mu(t) = (1+e^t)\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} - (e^t-1).$$

Now

$$\begin{aligned}
\mu(t) &= \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} + \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{r=0}^{\infty}\frac{t^{r+2k-1}}{r!} - \sum_{r=1}^{\infty}\frac{t^r}{r!}\mu(t) \\
&= \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} + \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{\infty}\frac{t^s}{(s-2k+1)!} - \sum_{r=1}^{\infty}\frac{t^r}{r!} \\
&= \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} + \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=2k-1}^{4m-2}\frac{t^s}{(s-2k+1)!} - \sum_{r=1}^{4m-2}\frac{t^r}{r!} \\
&\quad + \sum_{s=4m-1}^{\infty}\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^s}{(s-2k+1)!} - \sum_{r=4m-1}^{\infty}\frac{t^r}{r!}.
\end{aligned}$$

Rewrite infinite summations from 0 and split finite summations by even and odd power of t, we obtain

$$\begin{aligned}
\mu(t) &= \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}t^{2k-1} + \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m-1}\frac{t^{2s-1}}{(2s-2k)!} - \sum_{r=1}^{2m-1}\frac{t^{2r-1}}{(2r-1)!} \\
&\quad + \sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\sum_{s=k}^{2m-1}\frac{t^{2s}}{(2s-2k+1)!} - \sum_{r=1}^{2m-1}\frac{t^{2r}}{(2r)!} \\
&\quad + \sum_{s=0}^{\infty}\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^{s+4m-1}}{(s+4m-2k)!} - \sum_{r=0}^{\infty}\frac{t^{r+4m-1}}{(r+4m-1)!},
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
\mu(t) &= \sum_{s=1}^{2m-1}\frac{2(2^{2s}-1)B_{2s}}{(2s)!}t^{2s-1} + \sum_{s=1}^{2m-1}\frac{1}{(2s)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)(2s!)B_{2k}}{(2k)!(2s-2k)!}t^{2s-1} - \sum_{s=1}^{2m-1}\frac{t^{2s-1}}{(2s-1)!} \\
&\quad + \sum_{s=1}^{2m-1}\frac{1}{(2s+1)!}\sum_{k=1}^{s}\frac{2(2^{2k}-1)(2s+1)!B_{2k}}{(2k)!(2s-2k+1)!}t^{2s} - \sum_{s=1}^{2m-1}\frac{t^{2s}}{(2s)!} \\
&\quad + \sum_{s=0}^{\infty}\sum_{k=1}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!}\frac{t^{s+4m-1}}{(s+4m-2k)!} - \sum_{s=0}^{\infty}\frac{t^{s+4m-1}}{(s+4m-1)!} \\
&= \sum_{s=1}^{2m-1}\left[2(2^{2s}-1)B_{2s} - s + \sum_{k=1}^{s-1}(2^{2k}-1)\tbinom{2s}{2k}B_{2k}\right]\frac{2t^{2s-1}}{(2s)!} \\
&\quad + \sum_{s=1}^{2m-1}\left[-(2s+1) + \sum_{k=1}^{s}2(2^{2k}-1)\tbinom{2s+1}{2k}B_{2k}\right]\frac{t^{2s}}{(2s+1)!} \\
&\quad + \sum_{s=0}^{\infty}\left[\frac{1}{2}\frac{s+4m-3}{(s+4m-1)!} + \sum_{k=2}^{2m-1}\frac{2(2^{2k}-1)B_{2k}}{(2k)!(s+4m-2k)!}\right]t^{s+4m-1}.
\end{aligned}$$

Using the identities (12) and (13) with the relation (27), $\mu(t)$ satisfies

$$\begin{aligned}\mu(t) &> \sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(\frac{2(2^{4k}-1)B_{4k}}{(4k)!(s+4(m-k))!}+\frac{2(2^{4k+2}-1)B_{4k+2}}{(4k+2)!(s+4(m-k)-2)!}\right)t^{s+4m-1}\\ &> \left[\sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(1-\frac{(2^{4k}-1)(4k+1)(4k+2)|B_{4k}|}{(2^{4k+2}-1)(s+4(m-k)-1)(s+4(m-k))|B_{4k+2}|}\right)\right.\\ &\qquad\left.\frac{2(2^{4k+2}-1)|B_{4k+2}|t^{s+4m-1}}{(4k+2)!(s+4(m-k)-2)!}\right].\end{aligned}$$

For $s \geq 0$ and $m-k \geq 1$, we have

$$(s+4(m-k)-1)(s+4(m-k)) \geq (s+3)(s+4) \geq 12$$

and then μ satisfies

$$\mu(t) > \sum_{s=0}^{\infty}\sum_{k=1}^{m-1}\left(1-\frac{\pi^2+1}{12}\right)\frac{2(2^{4k+2}-1)|B_{4k+2}|t^{s+4m-1}}{(4k+2)!(s+4(m-k)-2)!} > 0,$$

which complete the proof. ☐

From the complete monotonicity of the two functions $F(x)$ and $M(x)$ with the asymptotic expansion (9), we get the following double inequality which posed as a conjecture in [21].

Lemma 2.5. *The following double inequality holds*

$$\sum_{k=1}^{2m}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k} < G(x)-x^{-1} < \sum_{k=1}^{2l-1}\frac{(2^{2k}-1)B_{2k}}{k}x^{-2k}, \qquad l,m \in N;\ x>0. \tag{29}$$

From the positivity of the two functions $\varphi(t)$ and $\mu(t)$ in the proofs of Lemmas 2.3 and 2.4, we obtain the following result:

Lemma 2.6. *The following double inequality holds*

$$\sum_{k=1}^{2m}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1} \leq \tanh(x) \leq \sum_{k=1}^{2l-1}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1}, \qquad l,m \in N;\ x \geq 0 \tag{30}$$

and the inequality is reversed if $x \leq 0$. Equality holds if $x=0$.

Remark 1. In the case $|x| < \frac{\pi}{2}$ and l or $m =$ tends to ∞, in the inequality (30) in fact equality holds, since

$$\tanh(x) = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!}x^{2k-1}, \qquad |x| < \frac{\pi}{2}.$$

Elbert and Laforgia established the following lemma to study the monotonicity of some functions involving gamma function [9] (see also [48]).

Lemma 2.7. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x\to\infty} K(x) = 0$. Then $K(x) > 0$, if $K(x) > K(x+1)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+1)$ for all $x > a$.*

To present our next result, we can easily prove the following simple modification on Lemma 2.7:

Corollary 2.8. *Let K be a real-valued function defined on $x > a$, $a \in \mathbb{R}$ with $\lim_{x\to\infty} K(x) = 0$. Then for $m \in \mathbb{N}$, $K(x) > 0$, if $K(x) > K(x+m)$ for all $x > a$ and $K(x) < 0$, if $K(x) < K(x+m)$ for all $x > a$.*

Proof. For $m \in \mathbb{N}$, if we have $K(x) > K(x+m)$ and $\lim_{x\to\infty} K(x) = 0$, then

$$K(x) > K(x+m) > \ldots > K(x+rm) > \ldots > \lim_{r\to\infty} K(x+rm) = \lim_{y\to\infty} K(y) = 0.$$

The other case is similarly treated. □

Lemma 2.9. *The function*

$$q(x) = \frac{1}{G(x) - \frac{1}{x}} - 2x^2, \qquad x > 0 \tag{31}$$

is strictly increasing.

Proof. For $x > 0$, we have

$$q'(x) = \frac{L(x)}{[G(x) - \frac{1}{x}]^2},$$

where

$$L(x) = -G'(x) - 4xG^2(x) + 8G(x) - \frac{(4x+1)}{x^2}.$$

Now,

$$\begin{aligned} L(x+1) - L(x) &= G'(x) - G'(x+1) + 4x\left[G^2(x) - G^2(x+1)\right] - 4G^2(x+1) \\ &- 8\left[G(x) - G(x+1)\right] + \frac{4x^2+6x+1}{x^2(x+1)^2} \end{aligned}$$

and using equation (4) and its derivative, we get

$$L(x+1) - L(x) = 2G'(x) - 4G^2(x+1) + \frac{6x^2+10x+3}{x^2(x+1)^2} \triangleq L_1(x).$$

Consider the difference

$$\begin{aligned} L_1(x+2) - L_1(x) &= 2\left[G'(x+2) - G'(x)\right] - 4\left[G^2(x+3) - G^2(x+1)\right] \\ &- \frac{4\left(27 + 135x + 220x^2 + 158x^3 + 51x^4 + 6x^5\right)}{x^2(x+1)^2(x+2)^2(x+3)^2} \end{aligned}$$

and using equation (4) and its derivative, we obtain

$$\begin{aligned} L_1(x+2)-L_1(x) &= \frac{16}{(x+1)(x+2)}\left\{G(x+1)-\frac{4x^5+34x^4+98x^3+99x^2+3x-9}{4x^2(x+1)(x+2)(x+3)^2}\right\} \\ &\triangleq \frac{16}{(x+1)(x+2)}L_2(x). \end{aligned}$$

Using equation (4), the function $L_2(x)$ satisfies

$$L_2(x+2)-L_2(x)=-\frac{3(7x+15)(7x+20)}{2x^2(x+1)(x+2)^2(x+3)^2(x+4)(x+5)^2}<0.$$

From the asymptotic formula (9) and its derivative

$$G'(x)\sim -\frac{1}{x^2}-\sum_{k=1}^{\infty}\frac{2(2^{2k}-1)B_{2k}}{x^{2k+1}}, \qquad x\to\infty \tag{32}$$

we have

$$\lim_{x\to\infty}L(x)=\lim_{x\to\infty}L_1(x)=\lim_{x\to\infty}L_2(x)=0.$$

Hence, using Corollary 2.8, we get that $L(x)>0$ for all $x>0$ which completes the proof. □

As a consequence of the monotonicity of the function $q(x)$ with the asymptotic expansion (9), we obtain the following inequality:

Lemma 2.10. *The following double inequality holds*

$$\frac{1}{2x^2+\alpha}<G(x)-\frac{1}{x}<\frac{1}{2x^2+\beta}, \qquad x>0 \tag{33}$$

where $\alpha=1$ and $\beta=0$ are the best possible constants.

Remark 2. The double inequality (33) is a refinement of the double inequality (10).

Lemma 2.11. *The function*

$$U(x)=G(x)-\frac{1}{x}-\frac{1}{2x^2+1}, \qquad x>0 \tag{34}$$

is strictly completely monotonic.

Proof. Using the formula (25), the integral representation of $G(x)$ and the Laplace transform of sine function, we have

$$U(x)=\int_0^{\infty}\lambda(t)e^{-xt}dt,$$

where

$$\lambda(t)=\frac{e^t-1}{e^t+1}-\frac{1}{\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right).$$

Since $\sin z < 1$, we get

$$\lambda(t) > \frac{e^t-1}{e^t+1} - \frac{1}{\sqrt{2}} > 0, \quad t > \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) \approx 1.76275\ .$$

Also, from the generalization of Redheffer-Williams's inequality [40], [41], [42], [46]

$$\frac{\pi^2-x^2}{\pi^2+x^2} \le \frac{\sin x}{x} \le \frac{12-x^2}{12+x^2}, \qquad 0 < x \le \pi$$

and the inequality (30) for $m = 4$, we obtain $\lambda(t) > \frac{t^5(2352-240t^2-17t^4)}{40320(24+t^2)} > 0$ for $0 < t < \sqrt{\frac{4\sqrt{3399}-120}{17}} \approx 2.58051$. □

As a consequence of the Lemma 2.11, we get

Lemma 2.12.

1. *For odd positive integer* r , *we have*

$$G^{(r)}(x) < -\frac{r!}{x^{r+1}} + \frac{r!(\sqrt{2})^r}{(2x^2+1)^{r+1}} \sum_{l=1}^{\frac{r+1}{2}} (-1)^l \binom{r+1}{2l-1} (\sqrt{2}x)^{r-2l+2} \quad x > 0 \tag{35}$$

2. *For even positive integer* r , *we have*

$$G^{(r)}(x) > \frac{r!}{x^{r+1}} + \frac{r!(\sqrt{2})^r}{(2x^2+1)^{r+1}} \sum_{l=1}^{\frac{r}{2}+1} (-1)^{l+1} \binom{r+1}{2l-1} (\sqrt{2}x)^{r-2l+2} \quad x > 0 \tag{36}$$

Also, as a consequence of the proof of Lemma 2.11, we obtain the following inequality:

Lemma 2.13. *The following double inequality holds*

$$\tanh(x) \ge \frac{1}{\sqrt{2}} \sin(\sqrt{2}x), \qquad x \ge 0. \tag{37}$$

Equality holds iff $x = 0$.

3 Applications: Some inequalities of Wallis ratio

The Wallis ratio

$$W_m = \frac{1.3.5...(2m-1)}{2.4.6...(2m)} = \frac{\Gamma(m+1/2)}{\sqrt{\pi}\,\Gamma(m+1)}, \qquad m \in N \tag{38}$$

plays an important role in mathematics especially in special functions, combinatorics, graph theory and many other branches. For further details about its history and applications, we refer to [7], [16], [18], [20], [26]-[29].

Guo, Xu and Qi [14] deduced the inequality

$$\frac{C_1}{m}\left(1-\frac{1}{2m}\right)^m\sqrt{m-1}<W_m\leq\frac{C_2}{m}\left(1-\frac{1}{2m}\right)^m\sqrt{m-1},\quad m\geq 2 \tag{39}$$

with the best possible constants $C_1=\sqrt{\frac{e}{\pi}}$ and $C_2=\frac{4}{3}$.

Recently, Qi and Mortici [37] presented the following improvement of the double inequality (39)

$$\sqrt{\frac{e}{\pi\, m}}\left[1-\frac{1}{2(m+1/3)}\right]^{m+1/3}<W_m<\sqrt{\frac{e}{\pi\, m}}\left[1-\frac{1}{2(m+1/3)}\right]^{m+1/3}e^{\frac{1}{144m^3}},\qquad m\in N. \tag{40}$$

Also, Zhang, Xu and Situ [47] presented the inequality

$$\frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m}}<W_m\leq\frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m+16}},\qquad m\in N. \tag{41}$$

Recently, Cristea [8] improved the upper bound of the inequality (41) by

$$W_m\leq\frac{1}{\sqrt{e\pi m}}\left(1+\frac{1}{2m}\right)^{m-\frac{1}{12m}+\frac{1}{48m^2}-\frac{1}{2880m^3}},\qquad m\in N \tag{42}$$

which is better than the upper bound of the inequality (40).

3.1 New proof of Slavić inequality

Slavić [43] presented the following double inequality

$$\frac{1}{\sqrt{x}}\exp\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{k(1-2k)x^{2k-1}}\right)<\frac{\Gamma(x+1/2)}{\Gamma(x+1)}<\frac{1}{\sqrt{x}}\exp\left(\sum_{k=1}^{2m}\frac{(1-2^{-2k})B_{2k}}{k(1-2k)x^{2k-1}}\right), \tag{43}$$

where $x>0$ and $l,m\in N$. In the following sequel, we will present a new proof of Slavić inequality (43). Consider the two functions

$$S_L(x)=\frac{\Gamma(x+1/2)}{\Gamma(x+1)}\sqrt{x}\exp\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right),\qquad l\in N$$

and

$$S_U(x)=\frac{\Gamma(x+1/2)}{\Gamma(x+1)}\sqrt{x}\exp\left(\sum_{k=1}^{2m}\frac{(1-2^{-2k})B_{2k}}{k(2k-1)x^{2k-1}}\right),\qquad m\in N.$$

Using Lemma 2.5, we obtain

$$\frac{S_L'(x)}{S_L(x)}=G(2x)-\frac{1}{2x}-\left(\sum_{k=1}^{2l-1}\frac{(1-2^{-2k})B_{2k}}{kx^{2k}}\right)<0,\qquad l\in N$$

and

$$\frac{S_U'(x)}{S_U(x)} = G(2x) - \frac{1}{2x} - \left(\sum_{k=1}^{2m} \frac{(1-2^{-2k})B_{2k}}{kx^{2k}} \right) > 0, \qquad m \in N.$$

Then the function $S_L(x)$ is decreasing and the function $S_U(x)$ is increasing and using the asymptotic expansion of the ratio of two gamma functions [19]

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b} \left[1 + \frac{(a-b)(a+b-1)}{2x} + O(x^{-2}) \right], \qquad a, b \geq 0 \tag{44}$$

as $x \to \infty$, we have

$$\lim_{x \to \infty} S_L(x) = \lim_{x \to \infty} S_U(x) = 1.$$

Hence we get

$$S_L(x) > 1 \quad \text{and} \quad S_U(x) < 1,$$

which complete the proof of Slavić inequality (43).

Remark 3. In the case of $l = 1$, $m = 1$ and $x = m$, the inequality (43) will gives

$$\frac{e^{\frac{-1}{8m}}}{\sqrt{\pi m}} < W_m < \frac{e^{\frac{-1}{8m} + \frac{1}{192m^3}}}{\sqrt{\pi m}}, \qquad m \in N \tag{45}$$

which is better than inequality (40) of Qi and Mortici [37].

3.2 New upper bound of W_n

Consider the function

$$M_L(x) = \frac{\Gamma(x+1/2)}{\Gamma(x+1)} \sqrt{x} e^{\frac{-1}{2\sqrt{2}} \left[\tan^{-1}(2\sqrt{2}x) - \frac{\pi}{2} \right]}, \qquad x > 0.$$

Using the inequality (33), we get

$$\frac{M_L'(x)}{M_L(x)} = G(2x) - \frac{1}{2x} - \frac{1}{8x^2+1} > 0$$

and using the expansion (44), we have $\lim_{x \to \infty} M_L(x) = 1$. Then

$$M_L(x) < 1$$

and we obtain the following result:

Lemma 3.1. *The following double inequality holds*

$$\frac{\Gamma(x+1/2)}{\Gamma(x+1)} < \frac{e^{\frac{1}{2\sqrt{2}} \left[\tan^{-1}(2\sqrt{2}x) - \frac{\pi}{2} \right]}}{\sqrt{x}}, \qquad x > 0. \tag{46}$$

Remark 4. In the case of $x = m$ in the inequality (46), we have

$$W_m < \frac{e^{\frac{1}{2\sqrt{2}} \left[\tan^{-1}(2\sqrt{2}m) - \frac{\pi}{2} \right]}}{\sqrt{\pi m}}, \qquad m \in N \tag{47}$$

which is better than inequality (42) of Cristea [8].

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On Ramanujan's asymptotic formula for $n!$

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Abstract

In this paper, we present the following new asymptotic formula of factorial n

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-U(n)}, \qquad n\to\infty$$

where $U(n)=\left(\frac{240}{11}n+\frac{9480}{847}+\frac{919466}{65219\,n}+\frac{1455925}{5021863\,n^2}-\frac{6391301400029}{92804028240\,n^3}+...\right)^{-1}$ depending on Ramanujan's approximation formula for $n!$ and we deduce the following upper bound for gamma function $\Gamma(x+1)<\sqrt{\pi}\,(x/e)^x\left[8x^3+4x^2+x+\frac{1}{30}+\frac{1}{\frac{240x}{11}+\frac{9480}{847}}\right]^{1/6}$, $x>0$.

2010 Mathematics Subject Classification: 41A60, 41A25, 33B15.

Key Words: Factorial, Ramanujan's formula, asymptotic formula, best possible constant, rate of convergence,bounds.

1 Introduction.

In many science branches, we need estimations of big factorials. Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n\to\infty$$

is the most well known and used approximation formula for factorial n, which is satisfactory in many branches such as statistical physics and statistics but we need more precise estimates in many pure mathematics studies. For more details about Stirling's formula refinements and its related inequalities, we refer to [2], [12], [22].

Other known formula for estimating $n!$ for large values of n is Ramanujan formula:

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}}, \tag{1}$$

which is a refinement of Stirling's formula and was recorded in the book "The lost notebook and other unpublished papers" as a conjecture of Srinivasa Ramanujan based on some numerical evidence. For more details please refer to [1], [4], [13], [24], [29].

Starting from Ramanujan formula (1), Karatsuba presented the following asymptotic formula [13]

$$\Gamma(x+1) \sim \sqrt{\pi}\,(x/e)^x \left[8x^3+4x^2+x+\frac{1}{30}-\frac{11}{240x}+\frac{79}{3360x^2}+\frac{3539}{201600x^3}+...\right]^{1/6}, \qquad (2)$$

where $\Gamma(x)=\int_0^\infty e^{-r}r^{x-1}dr$, $x>0$ is the ordinary gamma function and $n! = \Gamma(n+1)$ for $n \in N$. Mortici [23] improve the Ramanujan formula by establishing the following asymptotic formula:

$$\Gamma(x+1) \sim \sqrt{\pi}\,(x/e)^x \left[8x^3+4x^2+x+\frac{1}{30}\right]^{1/6} \exp\left[-\frac{11}{11520x^4}+\frac{13}{3440x^5}+\frac{1}{691200x^6}+...\right], \qquad (3)$$

which is faster than formula (2).

Dumitrescu and Mortici [9] introduced the following class of approximations:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\,(x/e)^x \sqrt[6]{1+\frac{1}{2(x-\delta)}+\frac{\alpha}{2(x-\delta)^2}+\frac{\beta}{2(x-\delta)^3}}, \qquad \alpha,\beta,\delta \in R \qquad (4)$$

which is a generalization of the Ramanujan's formula (1) at $\delta = 0$, $\alpha = 1/8$ and $\beta = 1/240$.

More various results involving approximations for the gamma function and the factorial can be found in [7], [8], [15], [16], [25], [26], [30] and the references therein.

In sequel, we need the following important Lemma, which is due to Mortici in 2010 and is a very useful tool for constructing asymptotic expansions and measuring the convergence rate of a family of null sequences [19]:

Lemma 1.1. *If* $\{\sigma_m\}_{m\in N}$ *is a null sequence and there is* $s \in R$ *and* $n > 1$ *such that*

$$\lim_{m\to\infty} m^n(\sigma_m - \sigma_{m+1}) = s, \qquad (5)$$

then we have

$$\lim_{m\to\infty} m^{n-1}\sigma_m = \frac{s}{n-1}.$$

From Lemma (1.1), we can conclude that the convergence rate of the sequence $\{\sigma_m\}_{m\in N}$ will increase with the increasing of the value of n in relation (5). Several approximations, formulas and inequalities have been produced using the technique developed by this Lemma. For more details please refer to [5], [6], [11], [14], [17], [20], [21], [28] and the references therein.

In the rest of this paper, we will present a new asymptotic formula of $n!$ depending on Ramanujan's asymptotic formula (1) and we deduce a new upper bound for the ordinary gamma function related to our new asymptotic formula.

2 Main results.

In our first step, we will try to find the best possible constants k_1 and k_2 in the approximation formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{k_1 n+k_2}}, \qquad n\to\infty \tag{6}$$

by defining a sequence A_n satisfies

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{k_1 n+k_2}}\, e^{A_n}, \qquad n\geq 1.$$

Then

$$\begin{aligned} A_n - A_{n+1} &= \left(\frac{1}{12k_1}-\frac{11}{2880}\right)\frac{1}{n^5}+\left(-\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}\right)\frac{1}{n^6} \\ &+\left(\frac{-9031k_1^3+158200k_1^2+100800k_1k_2+33600k_2^2}{268800k_1^3}\right)\frac{1}{n^7}+O(n^{-8}). \end{aligned}$$

If $\left(\frac{1}{12k_1}-\frac{11}{2880}\right)\neq 0$ and $\left(-\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}\right)\neq 0$, then the sequence $A_n - A_{n+1}$ has a rate of worse than n^{-6}. So, we will consider

$$\begin{cases} \frac{1}{12k_1}-\frac{11}{2880}=0 \\ -\frac{5k_2}{48k_1^2}-\frac{25}{96k_1}+\frac{29}{2016}=0 \end{cases}$$

that is, $k_1=\frac{240}{11}$ and $k_2=\frac{9480}{847}$. Now by Lemma (1.1), we obtain the following result:

Lemma 2.1. *The sequence*

$$A_n = \ln n! - \ln\sqrt{\pi} - n\ln n - n - \frac{1}{6}\ln n\left(8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}}\right) \tag{7}$$

has a rate of convergence equal to n^{-6}, *where*

$$\lim_{n\to\infty} n^7(A_n - A_{n+1}) = \frac{459733}{124185600}.$$

In our second step, we will try to find the best possible constants T_1, T_2 and T_3 in the approximation formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}+\frac{T_1}{n}+\frac{T_2}{n^2}+\frac{T_3}{n^3}}}, \qquad n\to\infty \tag{8}$$

by defining a sequence B_n satisfies

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3+4n^2+n+\frac{1}{30}-\frac{1}{\frac{240}{11}n+\frac{9480}{847}+\frac{T_1}{n}+\frac{T_2}{n^2}+\frac{T_3}{n^3}}}\, e^{B_n}, \qquad n\geq 1.$$

Hence

$$\begin{aligned} B_n - B_{n+1} &= \frac{(919466 - 65219T_1)}{248371200\, n^7} + \frac{(454576437T_1 - 10043726T_2 - 637955952)}{32784998400\, n^8} \\ &+ \frac{1}{265066712064000\, n^9}(42535179 61T_1^2 - 1277759560770T_1 + 4664306354 40T_2 \\ &- 92804028240T_3 + 16394247383595) \\ &+ \frac{1}{54427031543808000\, n^{10}}(-59336575555595T_1^2 + 1965125297982T_1T_2 \\ &+ 7507357980624 81T_1 - 361540539736530T_2 + 118464342048360T_3 \\ &- 8420494064916176) \\ &+ \frac{1}{30174346287887155 2000\, n^{11}}(-277410187898459T_1^3 + 1431360261443828 10T_1^2 \\ &- 79155247002714960T_1T_2 + 12105171835569120T_1T_3 \\ &- 10550047712231492850T_1 + 60525859177845 60T_2^2 + 6180552136457196960T_2 \\ &- 2679997511635567200T_3 + 1013933646178352555 40) \\ &+ O(n^{-12}). \end{aligned}$$

To obtain the best possible values of the constants T_1, T_2 and T_3, we put

$$\begin{cases} 65219T_1 = 919466 \\ 45457643T_1 - 10043726T_2 = 637955952 \\ 4253517961T_1^2 - 1277759560770T_1 + 4664306354 40T_2 - 92804028240T_3 = -16394247383595 \end{cases},$$

that is, $T_1 = \frac{919466}{65219}$, $T_2 = \frac{1455925}{5021863}$ and $T_3 = -\frac{639130140029}{92804028240}$. Hence by Lemma (1.1), we get the following result:

Lemma 2.2. *The sequence*

$$B_n = \ln n! - \ln\sqrt{\pi} - n\ln n - n - \frac{1}{6}\ln n\left(8n^3 + 4n^2 + n + \frac{1}{30}\right.$$
$$\left. - \frac{1}{\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\, n} + \frac{1455925}{5021863\, n^2} - \frac{639130140029}{92804028240\, n^3}}\right) \tag{9}$$

has a rate of convergence equal to n^{-9}, *where*

$$\lim_{n\to\infty} n^{10}(B_n - B_{n+1}) = \frac{142970656174139}{108854063087616000}.$$

In our third step, we can follow the same technique to get the following result:

Lemma 2.3. *The sequence* C_n *defined by*

$$n! = \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3 + 4n^2 + n + \frac{1}{30} - V(n)}\; e^{C_n},$$

where

$$V(n) = \frac{1}{\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\, n} + \frac{1455925}{5021863\, n^2} - \frac{639130140029}{92804028240\, n^3} + \frac{T_4}{n^4} + \frac{T_5}{n^5} + \frac{T_6}{n^6}},$$

converges to zero as n^{-12} *with the best possible constants* $T_4 = \frac{142970656174139}{42875461046880}$, $T_5 = \frac{288878734012247231}{22009403337398400}$ *and* $T_6 = -\frac{54220526084844409095873}{3965654293332443712 00}$ *since*

$$\lim_{n\to\infty} n^{13}(C_n - C_{n+1}) = -\frac{384377015548794481311979}{191419595788599033856000000}.$$

Hence, we get the asymptotic formula

$$n! \sim \sqrt{\pi}\left(\frac{n}{e}\right)^n \sqrt[6]{8n^3 + 4n^2 + n + \frac{1}{30} - U(n)}, \qquad n \to \infty \tag{10}$$

where

$$U(n) = \left(\frac{240}{11}n + \frac{9480}{847} + \frac{919466}{65219\, n} + \frac{1455925}{5021863\, n^2} - \frac{639130140029}{92804028240\, n^3} + \frac{142970656174139}{42875461046880\, n^4}\right.$$
$$\left. + \frac{288878734012247231}{22009403337398400\, n^5} - \frac{54220526084844409095873}{396565429333244371200\, n^6} + \ldots\right)^{-1}.$$

3 An inequality of Gamma function.

In this section, we will follow a method presented by Elbert and Laforgia in their paper [10] (see also, [3], [27], [32] and its simple modification in [18]):

Corollary 3.1. *Let* $T(t)$ *be a real-valued function defined on* $t > t_0 \in \mathbb{R}$ *with* $\lim_{t\to\infty} T(t) = 0$. *Then* $T(t) > 0$, *if* $T(t) > T(t+1)$ *for all* $t > t_0$ *and* $T(t) < 0$, *if* $T(t) < T(t+1)$ *for all* $t > t_0$.

Now, Consider the following function

$$F(x) = -\frac{1}{6}\ln\left(8x^3 + 4x^2 + x + \frac{1}{\frac{240x}{11} + \frac{9480}{847}} + \frac{1}{30}\right) + x - x\ln(x) + \ln\Gamma(x+1) - \ln(\sqrt{\pi}), \qquad x > 0$$

which satisfies

$$\lim_{x\to\infty} F(x) = 0.$$

$$\begin{aligned} F(x) - F(x+1) &= \frac{-1}{6}\ln\left(8x^3 + 4x^2 + x + \frac{847}{18480x + 9480} + \frac{1}{30}\right) - x\ln(x) + x\ln(x+1) \\ &+ \frac{1}{6}\ln\left(8(x+1)^3 + 4(x+1)^2 + x + \frac{847}{18480x + 27960} + \frac{31}{30}\right) - 1 \\ &\doteqdot H(x) \end{aligned}$$

The function $H(x)$ satisfies

$$H''(x) = \frac{H_1(x)}{H_2(x)} < 0, \qquad x > 0$$

where

$$\begin{aligned} H_1(x) = & - 1.84724 \times 10^{29} x^{16} - 2.37023 \times 10^{30} x^{15} - 1.39723 \times 10^{31} x^{14} - 5.01631 \times 10^{31} x^{13} \\ & - 1.22596 \times 10^{32} x^{12} - 2.15964 \times 10^{32} x^{11} - 2.83269 \times 10^{32} x^{10} - 2.81806 \times 10^{32} x^{9} \\ & - 2.14586 \times 10^{32} x^{8} - 1.25283 \times 10^{32} x^{7} - 5.57791 \times 10^{31} x^{6} - 1.86841 \times 10^{31} x^{5} \\ & - 4.59618 \times 10^{30} x^{4} - 7.9786 \times 10^{29} x^{3} - 9.149 \times 10^{28} x^{2} - 6.15185 \times 10^{27} x \\ & - 1.83421 \times 10^{26} < 0 \end{aligned}$$

and

$$H_2(x) = 3x(x+1)^2(154x+79)^2(154x+233)^2 \left(147840x^4 + 149760x^3 + 56400x^2 + 10096x + 1163\right)^2$$

$$\left(147840x^4 + 741120x^3 + 1392720x^2 + 1163536x + 365259\right)^2 .$$

Then $H(x)$ is strictly concave function satisfies

$$\lim_{x \to 0} H(x) = \frac{1}{6} \left(\log \left(\frac{28855461}{270979} \right) - 6 \right) < 0$$

and

$$\lim_{x \to \infty} H(x) = 0.$$

So, $F(x) < 0$ for $x > 0$ and hence we get the following inequality

Lemma 3.2.

$$\Gamma(x+1) < \sqrt{\pi}\,(x/e)^x \left[8x^3 + 4x^2 + x + \frac{1}{30} + \frac{1}{\frac{240x}{11} + \frac{9480}{847}} \right]^{1/6}, \qquad x > 0. \tag{11}$$

Remark 1. In 2018, Yang and Tian [31] presented the inequality

$$\Gamma(x+1) < \left(\frac{x^2 + \frac{6\gamma}{\pi^2 - 12\gamma}}{x + \frac{6\gamma}{\pi^2 - 12\gamma}} \right)^{\frac{6\gamma^2}{\pi^2 - 12\gamma}}, \qquad 0 < x < 1 \tag{12}$$

which is not included in inequality (11).

Remark 2. From the spirit of the previous inequality (11), we can suggest the following inequality:

$$\Gamma(x+1) > \sqrt[6]{\frac{9480}{1163}}\,(x/e)^x \left[8x^3 + 4x^2 + x + \frac{1}{30} + \frac{1}{\frac{240x}{11} + \frac{9480}{847}} \right]^{1/6}, \qquad x > 0$$

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An inertial extragradient subgradient method for solving bilevel equilibrium problems

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Abstract

In this paper, we propose an algorithm by combining an inertial term with the extragradient subgradient method for finding some solutions of bilevel equilibrium problems in a real Hilbert space. Then, we establish a strongly convergent theorem of the proposed algorithm under some sufficient assumptions on the bifunctions involving pseudomonotone and Lipschitz-type conditions. Some numerical experiments are tested to illustrate the advantage performance of our algorithm.

Keywords: Bilevel equilibrium problem; Extragradient subgradient method; Inertial method; Strong convergence

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H, and let f and g be bifunctions from $H \times H$ to $\mathbb{R}$ such that $f(x,x) = 0$ and $g(x,x) = 0$ for all $x \in H$. The equilibrium problem associated with g and C is denoted by $EP(C,g)$: Find $x^* \in C$ such that

$$g(x^*, y) \geq 0 \quad \text{for every} \quad y \in C, \tag{1.1}$$

which was considered by Blum and Oettli [4]. The solution set of problem (1.1) is denoted by Ω.

It can be seen that the equilibrium problem is related to science in various fields and is very important because many problems arise in applied areas such as the fixed point problem, the (generalized) Nash equilibrium problem in game theory, the saddle point problem, the variational inequality problem, the optimization problem and others.

The simple basic method for solving some monotone equilibrium problems is the proximal point method (see [20, 22, 27]). In 2008, Tran et al. [37] proposed the extragradient algorithm for solving the equilibrium problem by using the strongly convex minimization problem to solve at each iteration. Furthermore, Hieu [16] introduced subgradient extragradient methods for pseudomonotone equilibrium problem and the other methods (see the details in [1, 12, 21, 23, 31, 39]).

In this paper, we consider the bilevel equilibrium problems, that is, the equilibrium problem whose constraints are the solution sets of equilibrium problems: Find $x^* \in \Omega$ such that

$$f(x^*, y) \geq 0 \quad \text{for every} \quad y \in \Omega. \tag{1.2}$$

The solution set of problem (1.2) is denoted by Ω^*.

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[1] Supported by The Royal Golden Jubilee Project Grant no. PHD/0219/2556, Thailand.

The bilevel equilibrium problems were introduced by Chadli et al. [7] in 2000. This kind of problems is very important and interesting because it is a generalization class of problems such as optimization problems over equilibrium constraints, variational inequality over equilibrium constraints, hierarchical minimization problems, and complementarity problems. Furthermore, the particular case of the bilevel equilibrium can be applied to a real word model such as the variational inequality over the fixed point set of a firmly nonexpansive mapping applied to the power control problem of CDMA networks which were introduced by Iiduka [18]. For more on the relation of bilevel equilibrium with particular cases, see [10, 19, 30].

Methods for solving bilevel equilibrium problems have been studied extensively by many authors. In 2010, Moudafi [28] introduced a simple proximal method and proved the weak convergence to a solution of problem (1.2). In 2014, Quy [33] introduced the algorithm by combining the proximal method with the Halpern method for solving bilevel monotone equilibrium and fixed point problem. For more details and most recent works on the methods for solving bilevel equilibrium problems, we refer the reader to [2, 8, 36]. The authors considered the method for monotone and pseudoparamonotone equilibrium problem. If a bifunction is more generally monotone, we cannot use the above methods for solving bilevel equilibrium problem, for example, the pseudomonotone property.

In 2018, Yuying et. al [40] proposed a method for finding the solution for bilevel equilibrium problems where f is strongly monotone and g is pseudomonotone and Lipschitz-type continuous. They obtained the convergent sequence by combining an extragradient subgradient method with the Halpern method.

On the other hand, an inertial-type algorithm was first proposed by Polyak [32] as an acceleration process in solving a smooth convex minimisation problem. An inertial-type algorithm is a two-step iterative method in which the next iterate is defined by making use of the previous two iterates. It is well known that incorporating an inertial term in an algorithm speeds up or accelerates the rate of convergence of the sequence generated by the algorithm. Consequently, a lot of research interest is now devoted to the inertial-type algorithm(see e.g. [5, 13, 24] and the references contained in them).

Motivated and inspired by the research work in this direction, in this work, we provide an algorithm which is generated by an inertial term and the extragradient subgradient method for solving bilevel equilibrium problems in a real Hilbert space. Then, the strong convergence theorem of the proposed algorithm are established under some sufficient assumptions on the bifunctions involving pseudomonotone and Lipschitz-type conditions. The numerical experiments are investigated to illustrate the advantage performance together with some improvement of our algorithm.

2. Preliminaries

Throughout this paper, H is a real Hilbert space, C is a nonempty closed convex subset of H. Denote that $x_n \rightharpoonup x$ and $x_n \to x$ are the weak convergence and the strong convergence of a sequence $\{x_n\}$ to x, respectively. For every $x \in H$, there exists a unique element $P_C x$ defined by

$$P_C x = \operatorname{argmin}\{\|x-y\| : y \in C\},$$

which can be found, e.g., in [[6], Sect. 1.2.2, Theorem 1.7], [[11], Theorem 3.4(2)], [[14], Theorem 7.43], [[17], Chap. III, Sect. 3.1] or [[29], Theorem 8.25].

Lemma 2.1 ([15]). *The metric projection P_C has the following basic properties:*

(i) $\|x-y\|^2 \geq \|x-P_C x\|^2 + \|y-P_C x\|^2$ *for all* $x \in H$ *and* $y \in C$;
(ii) $\langle x-P_C x, P_C x - y\rangle \geq 0$ *for all* $x \in H$ *and* $y \in C$;
(iii) $\|P_C(x)-P_C(y)\| \leq \|x-y\|$ *for all* $x, y \in H$.

We now recall the concept of proximity operator introduced by Moreau [26]. For a proper, convex and lower semicontinuous function $g : H \to (-\infty, \infty]$ and $\gamma > 0$, the Moreau envelope of g of parameter γ is the convex function

$$^{\gamma}g(x) = \inf_{y\in H} \left\{ g(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\} \quad \forall x \in H.$$

For all $x \in H$, the function $y \mapsto g(y)+\frac{1}{2\gamma}\|y-x\|^2$ is proper, strongly convex and lowe semicontinuous, thus the infimum is attained, i.e. $^{\gamma}g : H \to \mathbb{R}$.

The unique minimum of $y \mapsto g(y) + \frac{1}{2\gamma}\|y - x\|^2$ is called proximal point of g at x and it is denoted by $\text{prox}_g(x)$. The operator

$$\text{prox}_g(x) : H \to H$$

$$x \mapsto \operatorname*{arg\,min}_{y\in H} \left\{ g(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\}$$

is well-defined and is said to be the proximity operator of g. When $g = i_C$ (the indicator function of the convex set C), one has

$$\text{prox}_{i_C}(x) = P_C(x)$$

for all $x \in H$.

We also recall that the subdifferential of $g : H \to (-\infty, \infty]$ at $x \in \text{dom}g$ is defined as the set of all subgradient of g at x

$$\partial g(x) := \{w \in H : g(y) - g(x) \geq \langle w, y - x\rangle \, \forall y \in H\}.$$

The function g is called subdifferentiable at x if $\partial g(x) \neq \emptyset$, g is said to be subdifferentiable on a subset $C \subset H$ if it is subdifferentiable at each point $x \in C$, and it is said to be subdifferentiable, if it is subdifferentiable at each point $x \in H$, i.e., if $\text{dom}(\partial g) = H$.

The normal cone of C at $x \in C$ is defined by

$$N_C(x) := \{q \in H : \langle q, y - x\rangle \leq 0, \forall y \in C\}.$$

Definition 2.2 ([34, 35])**.** A bifunction $\psi : H \times H \to \mathbb{R}$ is called:

(i) β-strongly monotone on C if there exists $\beta > 0$ such that

$$\psi(x, y) + \psi(y, x) \leq -\beta\|x - y\|^2 \quad \forall x, y \in C;$$

(ii) monotone on C if

$$\psi(x, y) + \psi(y, x) \leq 0 \quad \forall x, y \in C;$$

(iii) pseudomonotone on C if

$$\psi(x, y) \geq 0 \Rightarrow \psi(y, x) \leq 0 \quad \forall x, y \in C.$$

(iv) β-strongly pseudomonotone on C if there exists $\beta > 0$ such that

$$\psi(x, y) \geq 0 \Rightarrow \psi(y, x) \leq -\beta\|x - y\|^2 \quad \forall x, y \in C.$$

It is easy to see from the aforementioned definitions that the following implications hold,

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \quad \text{and} \quad \text{(i)} \Rightarrow \text{(iv)} \Rightarrow \text{(iii)}$$

The converses in general are not true.

In this paper, we consider the bifunctions f and g under the following conditions.

Condition A

(A1) $f(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.
(A2) $f(\cdot,y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.
(A3) f is β-strongly monotone on $H \times H$.
(A4) For each $x, y \in H$, there exists $L > 0$ such that

$$\|w - v\| \leq L\|x - y\|, \quad \forall w \in \partial f(x,\cdot)(x), v \in \partial f(y,\cdot)(y).$$

(A5) The function $x \mapsto \partial f(x,\cdot)(x)$ is bounded on the bounded subsets of H.

Condition B

(B1) $g(x,\cdot)$ is convex, weakly lower semicontinuous and subdifferentiable on H for every fixed $x \in H$.
(B2) $g(\cdot,y)$ is weakly upper semicontinuous on H for every fixed $y \in H$.
(B3) g is pseudomonotone on C with respect to Ω, i.e.,

$$g(x, x^*) \leq 0, \quad \forall x \in C,\, x^* \in \Omega.$$

(B4) g is Lipschitz-type continuous, i.e., there exist two positive constants L_1, L_2 such that

$$g(x,y) + g(y,z) \geq g(x,z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \forall x, y, z \in H.$$

(B5) g is jointly weakly continuous on $H \times H$ in the sense that, if $x, y \in H$ and $\{x_n\}, \{y_n\} \in H$ converge weakly to x and y, respectively, then $g(x_n, y_n) \to g(x,y)$ as $n \to +\infty$.

Example 2.3 ([40]). Let $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $f(x,y) = 5y^2 - 7x^2 + 2xy$ and $g(x,y) = 2y^2 - 7x^2 + 5xy$. It follows that f and g satisfy Condition A and Condition B, respectively.

Lemma 2.4 ([3], Propositions 3.1, 3.2). *If the bifunction g satisfies Assumptions (B1), (B2), and (B3), then the solution set Ω is closed and convex.*

Remark 2.5. Let the bifunction f satisfy Condition A and the bifunction g satisfy Condition B. If $\Omega \neq \emptyset$, then the bilevel equilibrium problem (1.2) has a unique solution, see the details in [33].

Lemma 2.6 ([9]). *Let $\phi : C \to \mathbb{R}$ be a convex, lower semicontinuous, and subdifferentiable function on C. Then x^* is a solution to the convex optimization problem*

$$\min\{f(x) : x \in C\}$$

if and only if

$$0 \in \partial\phi(x^*) + N_C(x^*).$$

The following lemmas will be used in the proof of the convergence result.

Lemma 2.7 ([38]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$, and $\{\xi_n\}$ be a sequence in $\mathbb{R}$ satisfying the condition*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\xi_n, \quad \forall n \geq 0,$$

where $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \xi_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8 ([25]). *Let $\{a_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that*

$$a_{n_j} < a_{n_j+1} \quad \textit{for all} \quad j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ *defined, for all* $n \geq n_0$*, by*

$$\tau(n) = \max\{k \leq n \mid a_k < a_{k+1}\}.$$

Then $\{\tau(n)\}_{n\geq n_0}$ *is a nondecreasing sequence verifying*

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and, for all $n \geq n_0$*, the following two estimates hold:*

$$a_{\tau(n)} \leq a_{\tau(n)+1} \quad \textit{and} \quad a_n \leq a_{\tau(n)+1}.$$

Lemma 2.9 ([40])**.** *Suppose that* f *is* β*-strongly monotone on* H *and satisfies (A4). Let* $0 < \alpha < 1$*,* $0 \leq \eta \leq 1-\alpha$*, and* $0 < \mu < \frac{2\beta}{L^2}$*. For each* $x, y \in H$*,* $w \in \partial f(x,\cdot)(x)$*, and* $v \in \partial f(y,\cdot)(y)$*, we have*

$$\|(1-\eta)x - \alpha\mu w - [(1-\eta)y - \alpha\mu v]\| \leq (1-\eta-\alpha\sigma)\|x-y\|,$$

where $\sigma = 1 - \sqrt{1-\mu(2\beta-\mu L^2)} \in (0,1]$.

3. Main Result

In this section, we propose the algorithm for finding the solution of a bilevel equilibrium problem under the strong monotonicity of f and the pseudomonotonicity and Lipschitztype continuous conditions on g.

Algorithm 3.1. Initialization: Choose $x_0, x_1 \in H$, $0 < \mu < \frac{2\beta}{L^2}$, $\theta \in [0,1)$, the sequences $\{\alpha_n\} \subset (0,1)$, $\{\epsilon_n\} \subset [0,+\infty)$ and $\{\eta_n\}$ are such that

$$\begin{cases} \lim_{n\to\infty}\alpha_n = 0, \ \sum_{n=0}^{\infty} \alpha_n = \infty, \\ 0 \leq \eta_n \leq 1-\alpha_n \ \forall n \geq 0, \ \lim_{n\to\infty}\eta_n = \eta < 1, \\ \sum_{n=0}^{\infty} \epsilon_n < \infty. \end{cases}$$

Select initial $x_0, x_1 \in C$ and set $n \geq 1$.
Step 1.: Given x_{n-1} and x_n $(n \geq 1)$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta_n}$, where

$$\theta_n = \begin{cases} \min\left\{\theta, \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } x_n \neq x_{n-1}, \\ \theta & \text{if otherwise.} \end{cases} \tag{3.1}$$

Choose $\{\lambda_n\}$ such that

$$0 < \underline{\lambda} \leq \lambda_n \leq \bar{\lambda} < \min\left(\frac{1+\theta_n}{2L_1}, \frac{1+\theta_n}{2L_2}\right).$$

Compute

$$s_n = x_n + \theta_n(x_n - x_{n-1}),$$

$$y_n = \arg\min_{y\in C}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - s_n\|^2\right\},$$

$$z_n = \arg\min_{y\in C}\left\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2\right\}.$$

Step 2. Compute $w_n \in \partial f(z_n,\cdot)(z_n)$ and

$$x_{n+1} = \eta_n x_n + (1-\eta_n)z_n - \alpha_n\mu w_n.$$

Set $n := n+1$ and return to **Step 1**.

Remark 3.2. Some remarks on the algorithm are in order now.

(1) Evidently, we have from (3.1) that

$$\sum\nolimits_{n=0}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty, \tag{3.2}$$

due to $\theta_n \|x_n - x_{n-1}\| \le \bar{\theta_n} \|x_n - x_{n-1}\| \le \epsilon_n$.

(2) When $\theta_n = 0$, Algorithm 3.1 reduces to Algorithm 1 of [40].

Theorem 3.3. *Let bifunctions f and g satisfy Condition A and Condition B, respectively. Assume that $\Omega \neq \emptyset$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution of the bilevel equilibrium problem* (1.2).

Proof. Under assumptions of two bifunctions f and g, we get the unique solution of the bilevel equilibrium problem (1.2), denoted by x^*.

Step 1: Show that

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - (1+\theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 - (1+\theta_n - 2\lambda_n L_2)\|y_n - z_n\|^2 - \theta_n \|x_n - x_{n-1}\|^2. \tag{3.3}$$

The definition of y_n and Lemma 2.6 imply that

$$0 \in \partial \left\{ \lambda_n g(x_n, y) + \frac{1}{2}\|y - s_n\|^2 \right\} (y_n) + N_C(y_n).$$

There are $w \in \partial g(x_n, \cdot)(y_n)$ and $\bar{w} \in N_C(y_n)$ such that

$$\lambda_n w + y_n - s_n + \bar{w} = 0. \tag{3.4}$$

Since $\bar{w} \in N_C(y_n)$, we have

$$\langle \bar{w}, y - y_n \rangle \le 0 \quad \text{for all}\, y \in C. \tag{3.5}$$

By using (3.4) and (3.5), we obtain $\lambda_n \langle w, y - y_n \rangle \ge \langle s_n - y_n, y - y_n \rangle$ for all $y \in C$. Since $z_n \in C$, we have

$$\lambda_n \langle w, z_n - y_n \rangle \ge \langle s_n - y_n, z_n - y_n \rangle. \tag{3.6}$$

It follows from $w \in \partial g(x_n, \cdot)(y_n)$ that

$$g(x_n, y) - g(x_n, y_n) \ge \langle w, y - y_n \rangle \quad \text{for all}\, y \in H. \tag{3.7}$$

By using (3.6) and (3.7), we get

$$\lambda_n \{ g(x_n, z_n) - g(x_n, y_n) \} \ge \langle s_n - y_n, z_n - y_n \rangle. \tag{3.8}$$

Similarly, the definition of z_n implies that

$$0 \in \partial \left\{ \lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\} (z_n) + N_C(z_n).$$

There are $u \in \partial g(y_n, \cdot)(z_n)$ and $\bar{u} \in N_C(x)$ such that

$$\lambda_n u + z_n - x_n + \bar{u} = 0. \tag{3.9}$$

Since $\bar{u} \in N_C(z_n)$, we have

$$\langle \bar{u}, y - z_n \rangle \le 0 \quad \text{for all}\, y \in C. \tag{3.10}$$

By using (3.9) and (3.10), we obtain $\lambda_n\langle u, y - z_n\rangle \geq \langle x_n - z_n, y - z_n\rangle$ for all $y \in C$. Since $x^* \in C$, we have

$$\lambda_n\langle u, x^* - z_n\rangle \geq \langle x_n - z_n, x^* - z_n\rangle \tag{3.11}$$

It follows from $u \in \partial g(y_n, \cdot)(z_n)$ that

$$g(y_n, y) - g(y_n, z_n) \geq \langle u, y - z_n\rangle \quad \text{for all}\, y \in H. \tag{3.12}$$

By using (3.11) and (3.12), we get

$$\lambda_n\{g(y_n, x^*) - g(y_n, z_n)\} \geq \langle x_n - z_n, x^* - z_n\rangle.$$

Since $x^* \in \Omega$, we have $g(x^*, y_n) \geq 0$. If follows from the pseudomonotonicity of g on C with respect to Ω that $g(y_n, x^*) \leq 0$. This implies that

$$\langle x_n - z_n, z_n - x^*\rangle \geq \lambda_n g(y_n, z_n). \tag{3.13}$$

Since g is Lipschitz-type continuous, there exist two positive constants L_1, L_2 such that

$$g(y_n, z_n) \geq g(x_n, z_n) - g(x_n, y_n) - L_1\|x_n - y_n\|^2 - L_2\|y_n - z_n\|^2. \tag{3.14}$$

By using (3.13) and (3.14), we get

$$\langle x_n - z_n, z_n - x^*\rangle \geq \lambda_n\{g(x_n, z_n) - g(x_n, y_n)\} - \lambda_n L_1\|x_n - y_n\|^2 - \lambda_n L_2\|y_n - z_n\|^2.$$

From (3.8) and the above inequality, we obtain

$$2\langle x_n - z_n, z_n - x^*\rangle \geq 2\langle s_n - y_n, z_n - y_n\rangle - 2\lambda_n L_1\|x_n - y_n\|^2 - 2\lambda_n L_2\|y_n - z_n\|^2. \tag{3.15}$$

By the definition of s_n, we have that

$$\begin{aligned} 2\langle s_n - y_n, z_n - y_n\rangle &= 2\langle x_n + \theta_n(x_n - x_{n-1}) - y_n, z_n - y_n\rangle \\ &= -2\langle x_n - y_n, y_n - z_n\rangle + 2\theta_n\langle x_n - x_{n-1}, z_n - y_n\rangle. \end{aligned}$$

We know that

$$\begin{aligned} 2\langle x_n - z_n, z_n - x^*\rangle &= \|x_n - x^*\|^2 - \|z_n - x_n\|^2 - \|z_n - x^*\|^2 \\ -2\langle x_n - y_n, y_n - z_n\rangle &= -\|x_n - z_n\|^2 + \|x_n - y_n\|^2 + \|y_n - z_n\|^2 \\ 2\theta_n\langle x_n - x_{n-1}, z_n - y_n\rangle &= \theta_n(\|x_n - y_n\|^2 - \|x_n - x_{n+1}\|^2 - \|y_n - z_n\|^2). \end{aligned}$$

From (3.15), we can conclude that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1+\theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 - (1+\theta_n - 2\lambda_n L_2)\|y_n - z_n\|^2 - \theta_n\|x_n - x_{n-1}\|^2. \tag{3.16}$$

Step 2: The sequences $\{x_n\}$, $\{w_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.
Since $0 < \lambda_n < a$, where $a = \min\left\{\frac{1+\theta_n}{2L_1}, \frac{1+\theta_n}{2L_2}\right\}$, we have

$$(1 + \theta_n - 2\lambda_n L_1) > 0 \quad \text{and} \quad (1 + \theta_n - 2\lambda_n L_2) > 0.$$

It follows from (3.3) and the above inequalities that

$$\|z_n - x^*\| \leq \|x_n - x^*\| \quad \text{for all}\, n \in \mathbb{N}. \tag{3.17}$$

By Lemma 2.9 and (3.17), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\| &= \|\eta_n x_n + (1-\eta_n)z_n - \alpha\mu w_n - x^* + \eta_n x^* - \eta_n x^* + \alpha_n\mu v - \alpha_n\mu v\| \\ &= \|(1-\eta_n)z_n - \alpha_n\mu w_n - (1-\eta_n)x^* + \alpha_n\mu v + \eta_n(x_n - x^*) - \alpha_n\mu v\| \\ &\le \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v]\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\ &\le (1-\eta_n-\alpha_n\sigma)\|z_n - x^*\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\ &\le (1-\eta_n-\alpha_n\sigma)\|x_n - x^*\| + \eta_n\|x_n - x^*\| + \alpha_n\mu\|v\| \\ &= (1-\alpha_n\sigma)\|x_n - x^*\| + \alpha_n\tau\left(\frac{\mu\|v\|}{\sigma}\right), \end{aligned} \tag{3.18}$$

where $w_n \in \partial f(z_n,\cdot)(z_n)$ and $v \in \partial f(x^*,\cdot)(x^*)$. This implies that

$$\|x_{n+1}-x^*\| \le \max\left\{\|x_n - x^*\|, \frac{\mu\|v\|}{\sigma}\right\}.$$

By induction, we obtain

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\mu\|v\|}{\sigma}\right\}.$$

Thus the sequence $\{x_n\}$ is bounded. By using (3.17), we have $\{z_n\}$, and using Condition (A5), we can conclude that $\{w_n\}$ is also bounded.

Step 3: Show that the sequence $\{x_n\}$ converges strongly to x^*.

Since $x^* \in \Omega^*$, we have $f(x^*, y) \ge 0$ for all $y \in \Omega$. Note that $f(x^*, x^*) = 0$. Thus x^* is a minimum of the convex function $f(x^*,\cdot)$ over Ω. By Lemma 2.6, we obtain $0 \in \partial f(x^*,\cdot)(x^*) + N_\Omega(x^*)$. Then there exists $v \in \partial f(x^*,\cdot)(x^*)$ such that

$$\langle v, z - x^*\rangle \ge 0 \quad \text{for all}\, z \in \Omega. \tag{3.19}$$

Note that

$$\|x-y\|^2 \le \|x\|^2 - 2\langle y, x-y\rangle \quad \text{for all}\, x, y \in H. \tag{3.20}$$

From Lemma 2.9 and (3.20), we obtain

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 \\ &= \|\eta_n x_n + (1-\eta_n)z_n - \alpha\mu w_n - x^*\|^2 \\ &= \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*) - \alpha_n\mu v\|^2 \\ &\le \|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*)\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &\le \left\{\|(1-\eta_n)z_n - \alpha_n\mu w_n - [(1-\eta_n)x^* + \alpha_n\mu v] + \eta_n(x_n - x^*)\|^2\right\} - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &\le [(1-\eta_n-\alpha_n\sigma)\|z_n - x^*\| + \eta_n\|x_n - x^*\|]^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &\le (1-\eta_n-\alpha_n\sigma)\|z_n - x^*\|^2 + \eta_n\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &\le (1-\eta_n-\alpha_n\sigma)\|x_n - x^*\|^2 + \eta_n\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \\ &= (1-\alpha_n\sigma)\|x_n - x^*\|^2 - 2\alpha_n\mu\langle v, x_{n+1} - x^*\rangle \end{aligned} \tag{3.21}$$

It follows that

$$\|x_{n+1}-x^*\|^2 \le (1-\alpha_n\sigma)\|x_n - x^*\|^2 + 2\alpha_n\mu\langle v, x^* - x_{n+1}\rangle. \tag{3.22}$$

Let us consider two cases.

Case 1: There exists n_0 such that $\|x_n - x^*\|$ is decreasing for $n \geq n_0$. Therefore the limit of sequence $\|x_n - x^*\|$ exists. By using (3.17) and (3.22), we obtain

$$\begin{aligned}0 &\leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2 \\ &\leq -\frac{\alpha_n \sigma}{1-\eta_n}\|z_n - x^*\|^2 - \frac{2\alpha_n \mu}{1-\eta_n}\langle v, x_{n+1} - x^*\rangle \\ &\quad + \frac{1}{1-\eta_n}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2).\end{aligned}$$

Since $\lim_{n\to\infty} = \eta_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$ and the limit of $\|x_n - x^*\|$ exists, we have

$$\lim_{n\to\infty}(\|x_n - x^*\|^2 - \|z_n - x^*\|^2) = 0. \tag{3.23}$$

From $0 < \lambda_n < a$ and inequality (3.3), we get

$$(1 + \theta_n - a)\|x_n - y_n\|^2 \leq (1 + \theta_n - 2\lambda_n L_1)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x^*\|^2.$$

By using (3.23), we obtain $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Next, we show that

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle \leq 0. \tag{3.24}$$

Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle = \limsup_{k\to\infty}\langle v, x^* - x_{n_k}\rangle.$$

Since $\{x_{n_k}\}$ is bounded, we may assume that $\{x_{n_k}\}$ converges weakly to some $\bar{x} \in H$. Therefore

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle = \limsup_{k\to\infty}\langle v, x^* - x_{n_k}\rangle = \langle v, x^* - \bar{x}\rangle. \tag{3.25}$$

Since $\lim_{n\to\infty}\|x_n - y_n\| = 0$ and $x_{n_k} \rightharpoonup \bar{x}$, we have $y_{n_k} \rightharpoonup \bar{x}$. Let us consider that

$$\lim_{n\to\infty}\|s_n - y_n\| \leq \lim_{n\to\infty}\|s_n - x_n\| + \lim_{n\to\infty}\|x_n - y_n\|.$$

By the definition of s_n, we have that

$$\begin{aligned}\lim_{n\to\infty}\|s_n - x_n\| &= \lim_{n\to\infty}\|x_n - \theta_n(x_n - x_{n-1}) - x_n\| \\ &= \lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\|.\end{aligned}$$

Using the assumption $\sum_{n=1}^{\infty}\theta_n\|x_n - x_{n-1}\| < \infty$, it implies that $\lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\| = 0$. Thus $\lim_{n\to\infty}\|s_n - x_n\| = 0$. Since $\lim_{n\to\infty}\|s_n - x_n\| = 0$ and $x_{n_k} \rightharpoonup \bar{x}$, we have $s_{n_k} \rightharpoonup \bar{x}$. Since C is closed and convex, it is also weakly closed and thus $\bar{x} \in C$. Next, we show that $\bar{x} \in \Omega$. From the definition of $\{y_n\}$ and Lemma 2.6, we obtain

$$0 \in \partial\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|s_n - y\|^2\right\}(y_n) + N_C(y_n).$$

There exist $\bar{w} \in N_C(y_n)$ and $w \in \partial g(x_n, \cdot)(y_n)$ such that

$$\lambda_n w + y_n - s_n + \bar{w} = 0. \tag{3.26}$$

Since $\bar{w} \in N_C(y_n)$, we have $\langle \bar{w}, y - y_n\rangle \leq 0$ for all $y \in C$. From (3.26), we obtain

$$\lambda_n\langle w, y - y_n\rangle \geq \langle s_n - y_n, y - y_n\rangle \quad \text{for all}\, y \in C. \tag{3.27}$$

Since $w \in \partial g(x_n, \cdot)(y_n)$, we have

$$g(x_n, y) - g(x_n, y_n) \geq \langle w, y - y_n\rangle \quad \text{for all}\, y \in H. \tag{3.28}$$

Combining (3.27) and (3.28), we get

$$\lambda_n\{g(x_n, y) - g(x_n, y_n)\} \geq \langle s_n - y_n, y - y_n\rangle \quad \text{for all}\, y \in C. \tag{3.29}$$

Taking $n = n_k$ and $k \to \infty$ in (3.29), the assumption of λ_n and (B5), we obtain $g(\bar{x}, y) = 0$ for all $y \in C$. This implies that $\bar{x} \in \Omega$. By inequality (3.19), we obtain $\langle v, \bar{x} - x^*\rangle \geq 0$. It follows from (3.25) that

$$\limsup_{n\to\infty}\langle v, x^* - x_{n+1}\rangle \leq 0. \tag{3.30}$$

We can write inequality (3.22) in the following form:

$$\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_n\sigma)\|x_n - x^*\|^2 + \alpha_n\sigma\xi_n,$$

where $\xi_n = \frac{2\mu}{\sigma}\langle v, x^* - x_{n+1}\rangle$. It follows from (3.30) that $\limsup_{n\to\infty} \xi_n \leq 0$. By Lemma 2.7, we can conclude that $\lim_{n\to\infty} \|x_n - x^*\|^2 = 0$. Hence $x_n \to x^*$ as $n \to \infty$.

Case 2: There exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\|x_{n_j} - x^*\| \leq \|x_{n_{j+1}} - x^*\|$ for all $j \in \mathbb{N}$. By Lemma 2.8, there exists a nondecreasing sequence $\{\tau(n)\}$ of N such that $\lim_{n\to\infty} \tau(n) = \infty$, and for each sufficiently large $n \in \mathbb{N}$, we have

$$\|x_{\tau(n)} - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \quad \text{and} \quad \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\|. \tag{3.31}$$

Combining (3.18) and (3.31), we have

$$\begin{aligned} \|x_{\tau(n)} - x^*\| &\leq \|x_{\tau(n)+1} - x^*\| \\ &\leq (1 - \eta_{\tau(n)} - \alpha_{\tau(n)}\sigma)\|z_{\tau(n)} - x^*\| + \eta_{\tau(n)}\|x_{\tau(n)} - x^*\| + \alpha_{\tau(n)}\mu\|v\|. \end{aligned} \tag{3.32}$$

From (3.17) and (3.32), we get

$$0 \leq \|x_{\tau(n)} - x^*\| - \|z_{\tau(n)} - x^*\| \leq -\frac{\alpha_{\tau(n)}\sigma}{1 - \eta_{\tau(n)}}\|z_{\tau(n)-x^*}\| + \frac{\alpha_{\tau(n)}\sigma}{1 - \eta_{\tau(n)}}\|v\|. \tag{3.33}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \eta_n = \eta < 1$, $\{z_n\}$ is bounded, and (3.33), we have $\lim_{n\to\infty}(\|x_{\tau(n)} - x^*\| - \|z_{\tau(n)} - x^*\|) = 0$. It follows from the boundedness of $\{x_n\}$ and $\{z_n\}$ that

$$\lim_{n\to\infty} (\|x_{\tau(n)} - x^*\|^2 - \|z_{\tau(n)} - x^*\|^2) = 0. \tag{3.34}$$

By using the assumption of $\{\lambda_n\}$, we get the following two inequalities:

$$1 + \theta_n - 2\lambda_{\tau(n)}L_1 > 1 + \theta_n - 2aL_1 > 0 \quad \text{and} \quad 1 + \theta_n - 2\lambda_{\tau(n)}L_2 > 1 + \theta_n - 2aL_2 > 0.$$

From (3.3), we obtain

$$\begin{aligned} \|z_{\tau(n)} - x^*\|^2 &\leq \|x_{\tau(n)} - x^*\|^2 - (1 + \theta_n - 2\lambda_{\tau(n)}L_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\quad - (1 + \theta_n - 2\lambda_{\tau(n)}L_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\leq \|x_{\tau(n)} - x^*\|^2 - (1 + \theta_n - 2aL_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\quad - (1 + \theta_n - 2aL_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} 0 &< (1 + \theta_n - 2aL_1)\|x_{\tau(n)} - y_{\tau(n)}\|^2 + (1 + \theta_n - 2aL_2)\|y_{\tau(n)} - z_{\tau(n)}\|^2 \\ &\leq \|x_{\tau(n)} - x^*\|^2 - \|z_{\tau(n)} - x^*\|^2. \end{aligned}$$

It follows from (3.34) and the above inequality that

$$\lim_{n\to\infty}\|x_{\tau(n)}-y_{\tau(n)}\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|y_{\tau(n)}-z_{\tau(n)}\|=0. \tag{3.35}$$

Note that $\|x_{\tau(n)}-z_{\tau(n)}\|\leq\|x_{\tau(n)}-y_{\tau(n)}\|+\|y_{\tau(n)}-z_{\tau(n)}\|$. From (3.35), we have

$$\lim_{n\to\infty}\|x_{\tau(n)}-z_{\tau(n)}\|=0. \tag{3.36}$$

By using the definition of x_{n+1} and Lemma 2.9, we obtain

$$\begin{aligned}
\|x_{\tau(n)+1}-x_{\tau(n)}\|^2 &= \|\eta_{\tau(n)}x_{\tau(n)}+(1-\eta_{\tau(n)})z_{\tau(n)}-\alpha_{\tau(n)}\mu t_{\tau(n)}-x_{\tau(n)}\| \\
&= \|(1-\eta_{\tau(n)})z_{\tau(n)}-\alpha_{\tau(n)}\mu t_{\tau(n)} \\
&\quad -[(1-\eta_{\tau(n)})x_{\tau(n)}-\alpha_{\tau(n)}w_{\tau(n)}]-\alpha_{\tau(n)}w_{\tau(n)}\| \\
&\leq \|(1-\eta_{\tau(n)})z_{\tau(n)}-\alpha_{\tau(n)}t_{\tau(n)} \\
&\quad -[(1-\eta_{\tau(n)})x_{\tau(n)}-\alpha_{\tau(n)}w_{\tau(n)}]\|+\alpha_{\tau(n)}\|w_{\tau(n)}\| \\
&\leq (1-\eta_{\tau(n)}-\alpha_{\tau(n)}\sigma)\|z_{\tau(n)}-x_{\tau(n)}\|+\alpha_{\tau(n)}\|w_{\tau(n)}\| \\
&\leq \|z_{\tau(n)}-x_{\tau(n)}\|+\alpha_{\tau(n)}\|w_{\tau(n)}\|,
\end{aligned}$$

where $t_{\tau(n)}\in\partial f(z_{\tau(n)},\cdot)(z_{\tau(n)})$ and $w_{\tau(n)}\in\partial f(x_{\tau(n)},\cdot)(x_{\tau(n)})$. Since $\lim_{n\to\infty}\alpha_n=0$, the boundedness of $\{w_{\tau(n)}\}$ and (3.36), we have $\lim_{n\to\infty}\|x_{\tau(n)+1}-x_{\tau(n)}\|=0$. As proved in the first case, we can conclude that

$$\limsup_{n\to\infty}\langle v,x^*-x_{\tau(n)+1}\rangle=\limsup_{n\to\infty}\langle v,x^*-x_{\tau(n)}\rangle\leq 0. \tag{3.37}$$

Combining (3.22) and (3.31), we obtain

$$\begin{aligned}
\|x_{\tau(n)+1}-x^*\|^2 &\leq (1-\alpha_{n(\tau)}\sigma)\|x_{\tau(n)}-x^*\|^2+2\alpha_{n(\tau)}\mu\langle v,x^*-x_{\tau(n)+1}\rangle \\
&\leq (1-\alpha_{n(\tau)}\sigma)\|x_{\tau(n)+1}-x^*\|^2+2\alpha_{n(\tau)}\mu\langle v,x^*-x_{\tau(n)+1}\rangle.
\end{aligned}$$

By using (3.31) again, we have

$$\|x_n-x^*\|^2\leq\|x_{\tau(n)+1}-x^*\|^2\leq\frac{2\mu}{\sigma}\langle v,x^*-x_{\tau(n)+1}\rangle.$$

From (3.37), we can conclude that $\limsup_{n\to\infty}\|x_n-x^*\|^2\leq 0$. Hence $x_n\to x^*$ as $n\to\infty$. This completes the proof. ■

4. Numerical example

In this section, we provide a numerical example to test our algorithm. All Matlab colds were performed on a computer with CPU Intel Core i7-7500U, up to 3.5GHz, 4GB of RAM under version MATLAB R2015b. In the following example, we use the standard Euclidean norm and inner product.

Example 4.1. We compare our algorithm with Algorithm 1 proposed in Yuying et al. [40]. Let us consider a problem when $H=\mathbb{R}^n$ and $C=\{x\in\mathbb{R}^n : -5\leq x_i\leq 5, \forall i\in\{1,2,...,n\}\}$. Let the bifunction $g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be defined by

$$g(x,y)=\langle Px+Qy,y-x\rangle \quad\text{for all}\quad x,y\in\mathbb{R}^n,$$

where P and Q are randomly symmetric positive semidefinite matrices such that $P-Q$ is positive semidefinite. Then g is pseudomonotone on $\mathbb{R}^n$. Next, we obtain that g is Lipschitz-type continuous with $L_1=L_2=\frac{1}{2}\|P-Q\|$. Furthermore, we define the bifunction $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ as

$$f(x,y)=\langle Ax+By,y-x\rangle \quad\text{for all}\quad x,y\in\mathbb{R}^n,$$

Table 1: Comparison: proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall i = 1, 2, ..., n\}$.

n	Algorithm 3.1				Yuying et al. Algorithm	
	$\theta = 0.6$		$\theta = 0.9$			
	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)
5	29	1.0015	28	1.0178	34	1.3618
10	43	1.7310	38	1.3645	54	1.9099
50	90	4.3222	88	4.6822	98	5.5028

Table 2: Comparison: proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall i = 1, 2, ..., n\}$.

n	Algorithm 3.1				Yuying et al. Algorithm	
	$\theta = 0.6$		$\theta = 0.9$			
	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)	No. of Iter.	CPU (Time)
5	32	1.1074	30	1.0388	37	1.3528
10	50	1.8239	45	1.8472	61	2.3260
50	108	6.6858	105	6.5254	116	6.7247

with A and B being positive definite matices defined by

$$B = N^T N + nI_n \quad \text{and} \quad A = B + M^T M + nI_n, \tag{4.1}$$

where M, N are randomly $n \times n$ matrices and I_n is the identity matrix.
Moreover, $\partial f(x, \cdot)(x) = \{(A+B)x\}$ and $\|(A+B)x-(A+B)y\| \leq \|A+B\|\|x-y\|$ for all $x, y \in \mathbb{R}^n$. Thus the mapping $x \to \partial f(x, \cdot)(x)$ is bounded and $\|A+B\|$-Lipschitz continuous on every bounded subset of H.

It is easy to see that all the conditions of Theorem 3.3 and of Theorem 3.1 in [40] are satisfied. New, we compare the performance of our algorithm and algorithm of Yuying et al. [40], we take $\lambda_k = \frac{1}{k+5}$, $\alpha_k = \frac{1}{k+4}$, $\eta_k = \frac{k+1}{3(k+4)}$, $\mu = \frac{2}{\|A+B\|^2}$, the same starting point $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall\, i = 1, 2, ..., n\}$ and $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall\, i = 1, 2, ..., n\}$ for all the algorithms. For Algorithm 3.1, we choose $\epsilon_k = \frac{1}{k^{1.1}}$, $\theta \in [0, 1)$ and θ_k such that $0 \leq \theta_k \leq \bar{\theta_k}$, where

$$\theta_k = \begin{cases} \min\left\{\theta, \dfrac{1}{k^{1.1}\|x_k - x_{k-1}\|}\right\} & \text{if } x_k \neq x_{k-1}, \\ \theta & \text{if otherwise.} \end{cases}$$

To terminate the algorithm, we used the stopping criteria $\|x_{k+1} - x_k\| < \varepsilon$ with $\varepsilon = 10^{-6}$ is a tolerance. The results are reported in the Table 1 and Table 2, we can see that the number of iterations (No. of Iter.) by Algorithm 3.1 with different inertial parameters ($\theta = 0.6$ and $\theta = 0.9$) is less than that of Yuying et al. Algorithm [40], for two different starting points, we can see that in this example the starting points $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = 1, \forall\, i = 1, 2, ..., n\}$ give better performance than $x_0 = x_1 \in \{x \in \mathbb{R}^n : x_i = i, \forall\, i = 1, 2, ..., n\}$. Moreover, Figure 1 and Figure 2 illustrate the numerical behavior of both algorithms. In these figures, the value of errors $\|x_{k+1}-x_k\|$ is represented by the y-axis, number of iterations is represented by the x-axis.

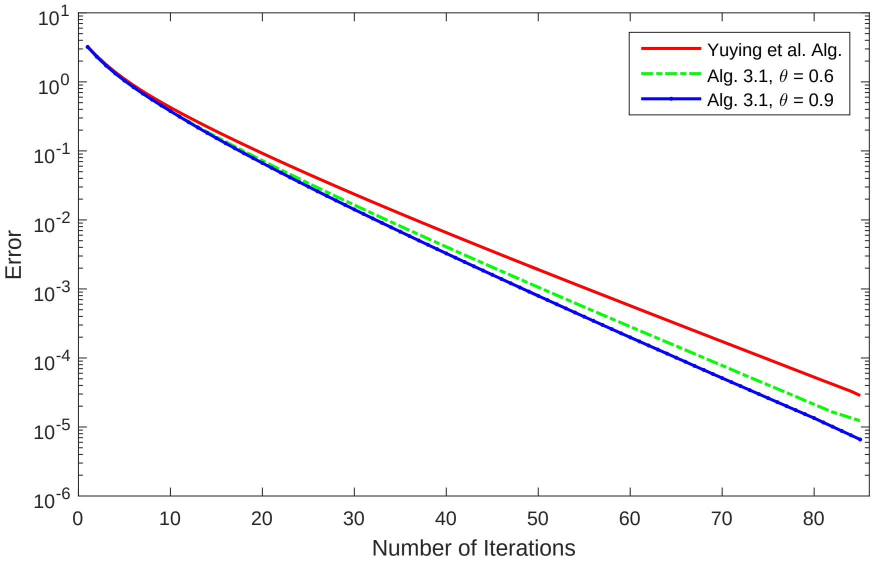

Figure 1: Comparison of proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = (1, 1, ..., 1)^T$ and n=50.

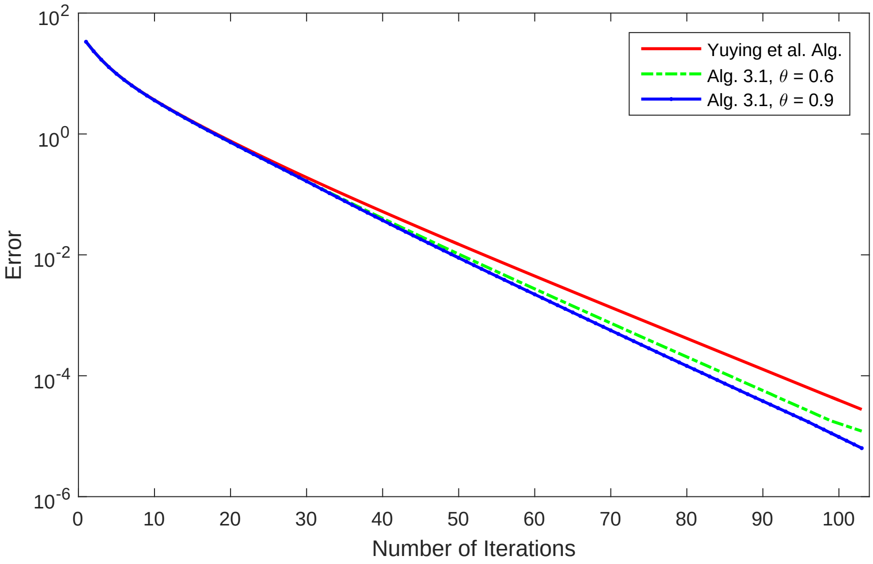

Figure 2: Comparison of proposed Algorithm 3.1 and Yuying et al. [40] with $x_0 = (1, 2, ..., 50)^T$ and n=50.

5. Conclusions

In this article, we introduced an iterative algorithm for finding the solution of a bilevel equilibrium problem in real Hilbert space. Under some suitable conditions imposed on parameters, we proved the strong convergence of the algorithm. We showed the efficiency of the proposed algorithm is verified by a numerical experiment and preliminary comparison. These numerical results have also confirmed that the algorithm with inertial effects seems to work better than without inertial effects.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgements

The authors would like to thank Naresuan University and The Thailand Research Fund for financial support. Moreover, J. Munkong is also supported by Naresuan University and The Royal Golden Jubilee Program under Grant Ph.D/0219/2556, Thailand.

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Volume 29, Number 6 December, 2021
ISSN:1521-1398 PRINT,1572-9206 ONLINE

Journal of Computational Analysis and Applications

EUDOXUS PRESS, LLC

Journal of Computational Analysis and Applications
ISSNno.'s:1521-1398 PRINT,1572-9206 ONLINE

SCOPE OF THE JOURNAL

An international publication of Eudoxus Press, LLC
(six times annually)

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Journal of Computational Analysis and Applications

An international publication of Eudoxus Press, LLC, of TN.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Daubechies Wavelet Method for Second Kind Fredholm Integral Equations with Weakly Singular Kernel *

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Abstract

In this paper, the weakly singular Fredholm integral equations of the second kind are solved by the periodized Daubechies wavelets method. In order to obtain a good degree of accuracy of the numerical solutions, the Sidi-Israeli quadrature formulae are used to construct the approximation of the singular kernel functions. By applying the asymptotically compact theory, we prove the convergence of approximate solutions. In addition, the sidi transformation can be used to degrade the singularities when the kernel function is non-periodic. At last, numerical examples show the method is efficient and errors of the numerical solutions possess high accuracy order $O\left(h^{3+\alpha}\right)$, where h is the mesh size.

Keyword: Daubechies wavelets; weakly singular kernel; Fredholm integral equation of the second; linear and nonlinear integral equations; convergence rate.

1 Introduction

Many problems in science and engineering such as Lapalace's equation, problems in elasticity, conformal mapping, free surface flows and so on, result in Fredholm integral equations with singular or weakly singular (in general logarithmic) and periodic kernels [11]. Therefore, singular or weakly Singular Fredholm linear equations and its nonlinear counterparts are most frequently studied for decades.

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*This work is supported by Project (NO. KYTZ201505) Supported by the Scientific Research Foundation of CUIT

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Generally, the weakly singular Fredholm integral equation of the second can be converted into the following form

$$u(x)-\int_0^1 k(x,y)g(u(y))\mathrm{d}y=f(x),\quad x\in[0,1],\tag{1.1}$$

where

$$k(x,y)=H_1(x,y)|x-y|^{\alpha}(\ln|x-y|)^{\beta}+H_2(x,y),\ \alpha>-1,\ \beta\geq 0,\tag{1.2}$$

$u(x)$ is an unknown function and $f\in L^2[0,1]$, and $H_j(x,y)$ $(j=1,2)$ are continuous on $[0,1]$. The integral equation (1.1) is linear when $g(u(y))=u(y)$, and when $g(u(y))\neq u(y)$ the equation is nonlinear.

As is known, several different orthonormal basis functions, for example, Chebyshev polynomial [8], Fourier functions [2], and wavelets [3, 4, 5, 6, 7, 9, 10, 13, 14, 16, 17], can be used to approximate the solutions of integral equations. However, for large scale problems, the most attractive one among them may be the wavelet bases, in which the kernel can be transformed to a sparse matrix after discretization. This is mainly due to functions with fast oscillations, or even discontinuities, in localized regions may be approximated well by a linear combination of relatively few wavelets [3].

This paper is organized as follows: in Section 2, the periodized Daubechies wavelets is introduced for solving weakly singular Fredholm integral equations of the second in detail. In Section 3, the convergence and error analysis are investigated. In Section 4, numerical examples are provided to verify the theoretical results. Some useful conclusions are made in Section 5.

2 Periodized Daubechies wavelets method

2.1 Multiresolution analysis and function expansions

Wavelets are attractive for the numerical solution of integrations, because their vanishing moments property leads to operator compression. Especially, Daubechies wavelets [12, 15] have many good properties and can deal with some types of kernels arising from boundary integral formulation of elliptic PDEs, and the coefficient are often numerically sparse. In fact, there are only $O\ (n\log n)$ significant elements. Supposed that ψ and ϕ be the the wavelet of genus N and Daubechies scaling function respectively. Thus their support are $\mathrm{supp}(\phi)=\mathrm{supp}(\psi)=[0,N-1]$. For any $j,k\in Z$, we introduce the notations $\phi_{j,k}(x)=2^{j/2}\phi(2^jx-k)$ and $\psi_{j,k}(x)=2^{j/2}\psi(2^jx-k)$, then their periodic kin with period-1 can be described by

$$\tilde{\phi}_{j,k}(x)=\sum_{n\in Z}\phi_{j,k}(x+n),\quad \tilde{\psi}_{j,k}(x)=\sum_{n\in Z}\psi_{j,k}(x+n),\quad x\in R,\quad 0\leq k<2^j.\tag{2.1}$$

Here $\{\tilde{\phi}_{j,k}(x)\}_{k\in Z}$ and $\{\tilde{\psi}_{j,k}(x)\}_{k\in Z}$ are orthogonal [17]. Defining the periodic spaces $\tilde{V}_j=\mathrm{span}\{\tilde{\phi}_{j,k}\}_{k=0}^{2^j-1}$ and $\tilde{W}_j=\mathrm{span}\{\tilde{\psi}_{j,k}\}_{k=0}^{2^j-1}$. A chain of spaces $\tilde{V}_0\subset\tilde{V}_1\cdots\subset$

$L^2[0,1]$ can be constructed, which subject to the following conditions: (a) $\cup_{j\geq 0}\tilde{V}_j = L^2[0,1]$, $\cap_{j\in Z}\tilde{V}_j=\{0\}$; (b) $h(x)\in\tilde{V}_j \Leftrightarrow h(2x)\in\tilde{V}_{j+1}$; (c) $\tilde{V}_j\oplus\tilde{W}_j=\tilde{V}_{j+1}$, $\tilde{W}_j\perp\tilde{V}_j$. The Daubechies wavelets and scaling functions described above result in the wavelet theory (i.e., multiresolution analysis (MRA)) of $L^2[0,1]$.

Supposed that function $p(x)\in L^2[0,1]$ be approximated by scaling series at resolution J as

$$p(x)=\sum_{k=0}^{2^J-1} c_{J,k}\tilde{\phi}_{J,k}(x)=\Phi^t(x)c, \quad x\in[0,1], \tag{2.2}$$

where

$$\Phi(x)=[\tilde{\phi}_{J,0}(x),\tilde{\phi}_{J,1}(x),\cdots,\tilde{\phi}_{J,2^{J-1}}(x)], \tag{2.3}$$

and

$$c=(c_{J,0},c_{J,1},\cdots,c_{J,2^{J-1}})^t, \quad c_{J,k}=\int_0^1 p(x)\tilde{\phi}_{J,k}(x)\mathrm{d}x. \tag{2.4}$$

First, we calculate the wavelet coefficient $c_{J,k}$ for nonsingular function $p(x)\in L^2[0,1]$. Let $x_i=i/2^J$, $i=0,1,\cdots,2^{J-1}$. Substituting $x=x_i$ into Eq. (2.2), we have

$$p(x)=\sum_{k=0}^{2^J-1} c_{J,k}\tilde{\phi}_{J,k}(i/2^J)=2^{J/2}\sum_{k=0}^{2^J-1} c_{J,k}\sum_{n\in Z}\phi_{j,k}(2^Jn+i-k). \tag{2.5}$$

By using the relationship between $\mathrm{supp}(\phi)$ and $[0,1]$, we know when $J\geq J_0$ only finite terms of the inner summation in (2.5) contribute the following result

$$n=\begin{cases} 0 \text{ or } 1, & \text{if } 2^J-N+2\leq N-1,\\ 0, & \text{if } 0\leq k\leq 2^Js-N+1.\end{cases}$$

Now we write (2.5) as the matrix form

$$p=Tc, \tag{2.6}$$

where $p=[p(0),p(1/2^J),\cdots,p((2^J-1)/2^J)]^t$, T is the nonsingular matrix which entries are the function values of $\phi(x)$ at integers (i.e., $\phi(0)$,$\phi(1)$,$\cdots$,$\phi(N-2)$) appear in it, and hence it satisfies

$$\sum_{i=1}^{2^J} T_{ij}=\sum_{j=1}^{2^J} T_{ij}=2^{J/2}, \quad i,j=1,2,\cdots,2^J. \tag{2.7}$$

Consequently, the function $k(x,y)\in L^2([0,1]\times[0,1])$ in Eq.(1.1) can be approximated at resolution J as

$$k(x,y)=\Phi^t(x)Q\Phi(y), \tag{2.8}$$

where Q is the $2^J\times 2^J$ coefficient matrix. Eq. (2.8) can be written as the following form

$$Q=T^{-1}KT^{-t}, \tag{2.9}$$

where K is the $2^J\times 2^J$ kernel matrix with $K_{i,j}=k(i/2^J,j/2^J)$.

Secondly, if the function $p(x)$ is singular on $[0, 1]$, some values of $p(x)$ at the dyadic points $x_i = i/2^J$, $i = 0, 1, \cdots, 2^J - 1$ may be unbounded and then c can not be immediately solved from the Eq.(2.6). In order to avoid the Eq.(2.6) being invalid, we can use the method in the literature [17] to compute the values of $p(x)$. Without loss of generality, we assumed that function $p(x) \in L^2[0, 1]$ has only one singular point $x_i = i/2^J$, $i \in \{0, 1, \cdots, 2^J - 1\}$. Then the function value $p(x_i)$ in Eq.(2.6) can be computed via on the following (see [17])

$$p(i/2^J) = 2^J \int_0^1 p(x)\mathrm{d}x - \sum_{j=0, j\neq i}^{2^J-1} p(j/2^J), \quad i \in \{0, 1, \cdots, 2^J - 1\}, \tag{2.10}$$

where integration $\int_0^1 p(x)\mathrm{d}x$ can be calculated by Sidi-Israeli quadrature formulae [11].

2.2 Kernel function approximation and discretization of singular integral equation

Motivated by the Eq.(2.10) and by thinking $k(x, y)$ as a one-dimensional function of variable x and y respectively, we also have

$$k(x, m/2^J) = 2^J \int_0^1 k(x, y)\mathrm{d}y - \sum_{j=0, j\neq m}^{2^J-1} k(x, j/2^J), \quad m \in \{0, 1, \cdots, 2^J - 1\}. \tag{2.11}$$

The following Theorem 2.1 can be used to construct the kernel approximation of Eq.(1.1).

Theorem 2.1 [11] Assume that the functions $H_1(x, y)$ and $H_2(x, y)$ are 2ℓ times differentiable on $[a, b]$. Assume also that the functions $k(x, y)$ are periodic with period $T = b - a$, and that they are 2ℓ times differentiable on $\widetilde{R} = (-\infty, \infty)\backslash\{x + jT\}_{j=-\infty}^{\infty}$. If $k(x, y) = H_1(x, y)|x - y|^\alpha(\ln|x - y|)^\beta + H_2(x, y)$, $s > -1$, $\beta = 0, 1$, then the quadrature rules of the following integral

$$I[k(x, y)] = \int_a^b k(x, y)\mathrm{d}y, \tag{2.12}$$

are

$$I_n[k(x, y)] = h \sum_{j=1, y_j\neq x}^{n} k(x, y_j) + 2[\beta\zeta^{'}(-\alpha) - \zeta(-\alpha)(\ln h)^\beta]H_1(x, x)h^{\alpha+1} + H_2(x, x)h, \tag{2.13}$$

and the quadrature errors are

$$E_n[k(x, y)] = 2\sum_{\mu=1}^{\ell-1}[\beta\zeta^{'}(-\alpha - 2\mu) - \zeta(-\alpha - 2\mu)(\ln h)^\beta]\frac{H_1^{(2\mu)}(x, y_j)}{(2\mu)!}h^{2\mu+\alpha+1} + o\,(h^{2\ell}), \tag{2.14}$$

where $E_n[k(x,y] = I[k(x,y)] - I_n[k(x,y)]$, and the mesh size is $h = (b-a)/n$.

Let $n = 2^J$ and by (2.13), we can get the Nyström approximation for the kernel function $k(x,y)$

$$k_D(x_i, y_j) = \begin{cases} 2[\beta\zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^\beta]H_1(x_i,x_i)h^\alpha + H_2(x_i,x_i), & \text{if } i = j, \\ k(x_i,y_j), \quad \text{if } i \neq j. \end{cases} \tag{2.15}$$

Supposed that the kernel function $k(x,y)$, $u(x)$ and $f(x)$ be approximated at resolution J as

$$k(x,y) = \Phi^t(x)Q\Phi(y), \quad f(x) = \Phi^t(x)b \text{ and } u(x) = \Phi^t(x)c, \tag{2.16}$$

where $c = [c(0), c(1/2^J), \cdots, c((2^J-1)/2^J)]^t$ is the expansion coefficient vector of $u(x)$. By the orthonormality of periodized wavelets, the integration of the product of the same two scaling function vectors is achieved as

$$\int_0^1 \Phi(x)\Phi^t(x)\mathrm{d}x = I, \tag{2.17}$$

where I is the 2^J by 2^J identity matrix. For the linear integral equation, we have

$$\Phi^t(x)c - \int_0^1 \Phi^t(x)Q\Phi(y)\Phi^t(y)c\mathrm{d}y = \Phi^t(x)b. \tag{2.18}$$

Substituting (2.15), (2.16) and (2.17) into (2.18), and by invoking (2.9), we get a linear system

$$(I - K_D(T^tT)^{-1})u_D = f, \tag{2.19}$$

where $f = [f(0), f(1/2^J), \cdots, f((2^J-1)/2^J)]^t$ and $K_D = T^tQT$. Similarly, the nonlinear case for Eq. (1.1) can be transformed into the following by the wavelet method

$$u_D - K_D(TT^t)^{-1}g(u_D) = f, \tag{2.20}$$

Eq. (2.20) is a system of nonlinear equations about u and can be computed by Newton iteration method.

3 Convergence and error analysis

In this section, we mainly study the convergence and error for the linear case of (1.1) by wavelet method.

We write Eq. (1.1) as the operate form

$$(I - \tilde{K})u = f, \tag{3.1}$$

where

$$(\tilde{K}u)(x) = \int_0^1 k(x,y)u(y)\mathrm{d}y, \tag{3.2}$$

with the kernel

$$k(x,y)=H_1(x,y)|x-y|^{\alpha}(\ln|x-y|)^{\beta}+H_2(x,y),\ \alpha>-1,\ \beta\geq 0, \tag{3.3}$$

and the approximation of $\tilde{K}$ is defined by

$$(\tilde{K}_n u)(x)=h\sum_{j=1,y_j\neq x}^{n}k(x,y_j)u(y_j)+\omega_n(x)u(x), \tag{3.4}$$

where the weight function

$$\omega(x)=2[\beta\zeta^{'}(-\alpha)-\zeta(-\alpha)(\ln h)^{\beta}]H_1(x,x)h^{\alpha+1}+H_2(x,x)h. \tag{3.5}$$

Supposed that the approximation of (3.1) is

$$(I-\tilde{K}_n)u_n(x)=g. \tag{3.6}$$

Lemma 3.1 Supposed the the operator $\tilde{K}_n$ is defined by (3.4), then the operator sequence $\{\tilde{K}_n\}$ is asymptotically compactly convergent to $\tilde{K}$, *i.e.*,

$$\tilde{K}_n\overset{a.c}{\rightarrow}\tilde{K}, \tag{3.7}$$

where $\overset{a.c}{\rightarrow}$ denotes the asymptotically compact convergence.

Proof. Let the continuous kernel approximation of $\tilde{K}$ be defined by

$$k_n^c(x,y)=\begin{cases} k(x,y), & \text{if } |x-y|\geq h,\\ H_1(x,x)h^{\alpha}(\ln h)^{\beta}+H_2(x,x), & \text{if } |x-y|<h,\end{cases} \tag{3.8}$$

and the corresponding operator approximation be

$$(K_n^c u)(x)=h\sum_{j=1}^{n}k_n^c(x,y_j)u(y_j). \tag{3.9}$$

For any $v\in C[0,1]$, we have

$$\begin{aligned}\|(\tilde{K}-K_n^c)v\| &= \sup_{\|v\|_\infty\leq 1}\int_0^1|(k(x,y)-k_n^c(x,y))v(y)|\mathrm{d}y\\ &\leq \int_0^1|k(x,y)-k_n^c(x,y)|\mathrm{d}y\|v\|_\infty\\ &\leq \int_{|x-y|\leq h}|H_1(x,y)|||x-y|^{\alpha}(\ln|x-y|)^{\beta}-h^{\alpha}(\ln h)^{\beta}|\mathrm{d}y\|v\|_\infty\\ &\leq \max_{x,y\in C[0,1]}|H_1(x,y)|\int_{|x-y|\leq h}||x-y|^{\alpha}(\ln|x-y|)^{\beta}-h^{\alpha}(\ln h)^{\beta}|\mathrm{d}y\|v\|_\infty\\ &= O((\ln h)^{\beta}h^{\alpha})\|v\|_\infty,\end{aligned} \tag{3.10}$$

hence, we can obtain

$$\|\tilde{K}-K_n^c\| = O((\ln h)^{\beta}h^{\alpha}) \to 0, \quad \text{as } h \to 0. \tag{3.11}$$

On the other hand, we know $\omega(x) \to 0$ as $h \to 0$ by (3.5), then

$$\|K_n^c-\tilde{K}_n\| \to 0, \quad \text{as } h \to 0. \tag{3.12}$$

First, there exists a subsequence in $\{K_n^c y_n\}$ for any $y_n \subset C[0,1]$ by (3.11),. Without loss of generality, assumed that $K_n^c y_n \to z$ and by (3.12), then

$$\begin{aligned}\|\tilde{K}_n y_n - z\| &\le \|\tilde{K}_n y_n - K_n^c y_n\| + \|K_n^c y_n - z\| \\ &\le \|\tilde{K}_n - K_n^c\|\|y_n\| + \|K_n^c y_n - z\| \to 0,\end{aligned} \tag{3.13}$$

that is to say, the sequence $\{\tilde{K}_n\}$ is asymptotically compactly convergent. Secondly, for any $y \in C[0,1]$, we have

$$\|\tilde{K}_n y - \tilde{K}y\| \le \|\tilde{K}_n - K_n^c\|\|y\| + \|K_n^c y - \tilde{K}y\| \to 0. \tag{3.14}$$

The proof of Lemma 3.1 is completed. □

Corollary 3.2 The operator sequence $\{\tilde{K}_n(I-o(h)E)\}$ is asymptotically compactly convergent to $\tilde{K}$, *i.e.*,

$$\tilde{K}_n(I-o(h)E) \overset{a.c}{\to} \tilde{K}. \tag{3.15}$$

where E is a matrix and every element in it is one.

Proof. By Lemma 3.1, we know

$$\tilde{K}_n \overset{a.c}{\to} \tilde{K}, \tag{3.16}$$

that is,

$$\|\tilde{K}_n - \tilde{K}\| \to 0. \tag{3.17}$$

Hence, we immediately have

$$\|\tilde{K}_n(I-o(h)E) - \tilde{K}\| \le \|\tilde{K}_n - \tilde{K}\| + \|\tilde{K}_n\|\|o(h)E\| \to 0. \tag{3.18}$$

The proof of Corollary 3.2 is completed. □

Let $x = (i-1)h,\ i = 1, 2, \cdots, 2^J$, where $h = 1/2^J$. Using the trapezoidal rule to approximate Eq.(2.17), then we have $hTT^t = I + o(h)E$, where E is a matrix and every element in it is one. By $(hTT^t)^{-1} = I + o(h)E$, (2.15) and (2.19), we get

$$(I - \tilde{K}_n(hT^{ht}T)^{-1})u_D = f, \tag{3.19}$$

which is equivalent to

$$(I - \tilde{K}_n(I-o(h)E)u_D = f. \tag{3.20}$$

Hence, by the Corollary 3.2 we get the following remark.

Remark 1 According to the Corollary 3.2, the solutions u_D of Eq.(3.20) by

Daubechies wavelet method are convergent to the solutions u_n of Eq.(3.6) when $h \rightarrow 0$.

Theorem 3.3 The solutions of Eq.(3.6) have asymptotic expansions hold at nodes

$$u_n(x) = u(x) + \sigma_1(x)h^{\alpha+3} + \sigma_2(x)h^{\alpha+3}\ln h + o(h^{\alpha+5}\ln h), \tag{3.21}$$

where $\sigma_j(x) \in C[0,1]$, $j = 1, 2$ are independent of h, and $\sigma_2 = 0$ when $\beta = 0$ and $\alpha > -1$, or $\beta = 1$ and $\alpha = 0$.

Proof. We construct the auxiliary equation

$$(I - \tilde{K})\sigma = P(x), \tag{3.22}$$

where

$$P(x) = [\beta\zeta'(-\alpha-2) - \zeta(-\alpha-2)(\ln h)^{\beta}](H_1 u)^{(2)}h^{3+\alpha}. \tag{3.23}$$

By invoking Eq.(2.14), we have

$$(\tilde{K}_n - \tilde{K})u(x) = -P(x) + o(h^{5+\alpha}\ln h). \tag{3.24}$$

Using (3.22), we get

$$\begin{aligned}&(I - \tilde{K}_n)(u_n - u - h^{3+\alpha}\sigma) = f - u + \tilde{K}_n u + h^{3+\alpha}(I - \tilde{K}_n)\sigma \\ &= (\tilde{K}_n - \tilde{K})u + h^{3+\alpha}(I - \tilde{K})\sigma + h^{3+\alpha}(\tilde{K}_n - \tilde{K})\sigma \\ &= o(h^{\alpha+5}\ln h),\end{aligned} \tag{3.25}$$

that is,

$$u_n - u - h^{3+\alpha}\sigma = o(h^{\alpha+5}\ln h). \tag{3.26}$$

From (3.22), we obtain

$$\sigma = -\sigma_1 - \sigma_2(\ln h)^{\beta}, \tag{3.27}$$

where

$$\sigma_1 = -\beta\zeta'(-\alpha-2)(I-\tilde{K})^{-1}(H_1 u)^{(2)}, \text{ and } \sigma_2 = \zeta(-\alpha-2)(I-\tilde{K})^{-1}(H_1 u)^{(2)}. \tag{3.28}$$

Substituting (3.28) into (3.26), and by $\zeta(-2) = 0$ (see [1]), we know that (3.21) holds. The proof of Theorem 3.3 is completed. ☐

Remark 2 According to the Theorem 3.3 and Remark 1, the numerical solutions u_D of Eq.(3.19) possess high accuracy order $O\ (h^{3+\alpha})$ as $h \rightarrow 0$.

4 Numerical experiments

In this section, two numerical examples about the Fredholm equations are computed by Daubechies wavelet method. Let $err_n^u(x) = |u(x) - u_n(x)|$ be the errors by Daubechies wavelet method using n $(= 2^J\ J = 3, \cdots, 8)$ nodes, and let $EOC = \log(err_n/err_{2n})/\log 2$ be the estimated order of convergence.

If the kernel function $k(x,y)$ of Eq.(1.1) is not periodic, we can apply the Sidi transformation for Eq.(1.1) and make the kernel be periodic. The Sidi transformation is defined by (see [18])

$$\psi_\gamma(t) = \int_0^t (\sin\pi\tau)^\gamma \mathrm{d}\tau \Big(\int_0^1 (\sin\pi\tau)^\gamma \mathrm{d}\tau \Big)^{-1} : [0,1] \to [0,1], \quad \gamma \geq 1.$$

In the following three examples, the errors and error ratio of numerical solutions at the selected points $x_1 = 0$, $x_2 = 0.25$ and $x_3 = 0.5$ by Daubechies wavelet method using transformation $\psi_6(t)$ are listed in tables.

Example 1. Consider the linear Fredholm equation of the first kind

$$u(x) + \int_0^1 \ln|x-y| u(y)\mathrm{d}y = g(x)$$

where $g(x) = x^2 \ln x/2 + (1-x^2)\ln(1-x)/2 + x/2 - 1/4$ and the exact solution is $u(x) = x$. We use periodic Daubechies wavelet of genus $D = 12$ as basis functions to compute the errors for Example 1 using different resolutions. The plots of computed errors are shown in Figure 1 and the errors and error ratio of numerical solutions are listed in Table 1. From the results in Table 1, we can see $EOC \approx 3$.

Table 1: The Errors of u.

J	3	4	5	6	7	8
$err_n^u(x_1)$	2.058-02	2.111-03	3.214-04	3.935-05	4.896-06	6.116-07
$EOC(x_1)$	–	3.2857	2.715	3.030	3.007	3.001
$err_n^u(x_2)$	2.328-02	2.485-03	3.607-04	4.401-05	5.481-06	6.847-07
$EOC(x_2)$	–	3.228	2.784	3.035	3.006	3.001
$err_n^u(x_3)$	2.278-02	9.764-05	1.752-05	2.222-06	2.778-07	3.474-08
$EOC(x_3)$	–	7.866	2.478	2.979	3.000	3.000

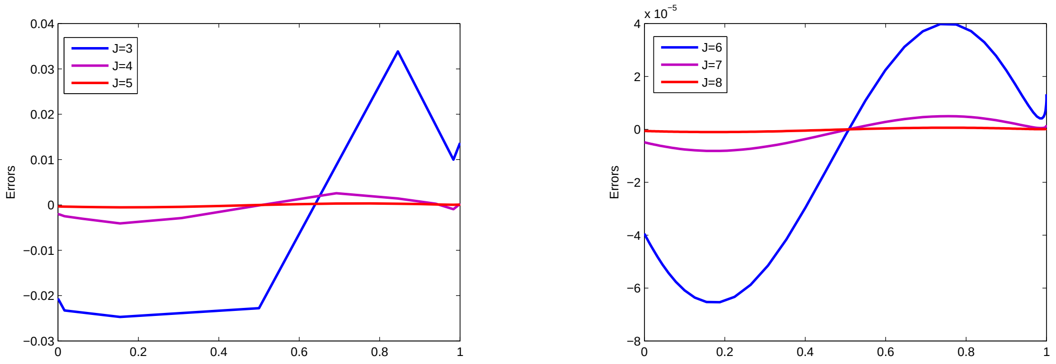

Figure 1: *The error distributions of Example 1 at different resolutions.*

Example 2. Solving the following non-periodic second kind Fredholm integral equation with algebraic singular kernel

$$u(x) + \int_0^1 |x-y|^{-1/2} u(y)\mathrm{d}y = g(x),$$

where $g(x) = x + 2(x\sqrt{(}x) - x^{2/3}/3 + x\sqrt{(}1-x) + (1-x)^{2/3}/3)$ and the exact solution is $u(x) = x$. We also use periodic Daubechies wavelet of genus $D = 12$ as basis functions to compute the errors using different resolutions. The plots of computed errors are shown in Figure 2 and the errors and error ratio of numerical solutions are listed in Table 2. From the results in Table 2, we can see $EOC \approx 2.5$.

Table 2: The Errors of u.

J	3	4	5	6	7	8
$err_n^u(x_1)$	1.713-03	3.964-05	1.638-05	2.627-06	4.505-07	7.898-08
$EOC(x_1)$	–	5.433	1.275	2.640	2.544	2.512
$err_n^u(x_2)$	8.526-03	3.899-04	3.847-05	4.843-06	8.529-07	1.508-07
$EOC(x_2)$	–	4.451	3.341	2.990	2.505	2.500
$err_n^u(x_3)$	4.698-03	2.975-04	6.878-05	1.216-05	2.150-06	3.800-07
$EOC(x_3)$	–	3.981	2.113	2.499	2.500	2.500

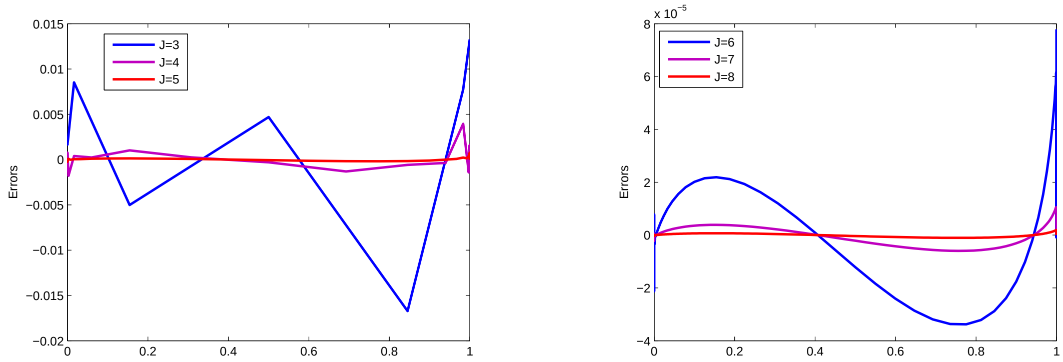

Figure 2: *The error distributions of Example 2 at different resolutions.*

Example 3. Solving the following nonlinear second kind Fredholm integral equation with weakly singular kernel

$$u(x) + \int_0^1 \ln|x-y| g(u(y))\mathrm{d}y = f(x),$$

where

$$f(x) = (x-0.5)^{2/3} + \frac{1}{3}(x^2 - x + 1/3) - x(x^2/3 - x/2 + 0.25)\ln(\frac{x}{1-x}) - \frac{1}{12}\ln(1-x).$$

The exact solution is $u(x) = (x-0.5)^{2/3}$. The periodic Daubechies wavelets of genus $D = 12$ as basis functions are used to compute the errors. The Newton iteration method is used for solve Example 3 and the initial vector of u_0 is given by $u_0 = (1, 1, \cdots, 1)^t_{2^J \times 1}$. After 4 iterations the errors are shown in Fig.3. The errors and error ratio of numerical solutions are listed in Table 3. From Table 3, we can see $EOC \approx 3$.

Table 3: The Errors of u.

J	3	4	5	6	7	8
$err_n^u(x_1)$	4.900-04	4.755-05	4.371-06	5.481-07	6.865-08	8.582-09
$EOC(x_1)$	–	3.365	3.443	2.996	2.997	3.000
$err_n^u(x_2)$	1.659-03	1.565-04	1.714-05	2.141-06	2.673-07	3.340-08
$EOC(x_2)$	–	3.407	3.190	3.001	3.002	3.001
$err_n^u(x_3)$	6.442-04	2.102-04	6.570-05	8.145-06	1.004-06	1.252-07
$EOC(x_3)$	–	1.616	1.678	3.012	3.019	3.003

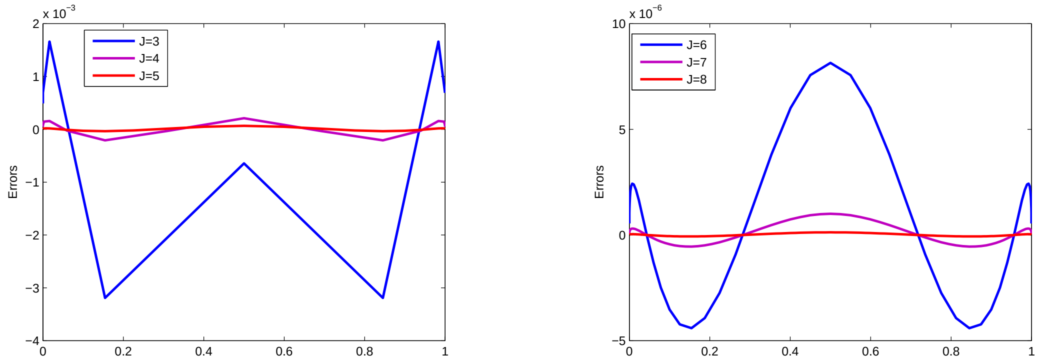

Figure 3: *The error distributions of Example 3 at different resolutions.*

5 Conclusions

In this paper, the Sidi-Israeli quadrature formula is used to construct the approximation of kernel functions and then the Daubechies wavelet method is used to solve Eq. (1.1). when the kernel functions are not periodic, we can apply the Sidi transformation for Eq.(1.1) and make the kernels be periodic. Because the wavelet integrations are completely avoided and the expansion coefficients obtained here are exact, which makes the wavelets method has a good degree of accuracy. In addition, the Daubechies wavelets method is used for linear Fredholm integration equation, the discrete matrix of the associated linear system can be transformed into a very sparse and symmetrical one. Accordingly, many preconditioners can be used to reduce the computational cost.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Some fixed point results in ordered complete dislocated quasi G_d metric space

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Abstract: In this paper, we discuss the fixed points of mappings satisfying contractive type condition on a closed ball in an ordered complete dislocated quasi G metric space. The notion of dominated mappings is applied to approximate the unique solution of non linear functional equations. An example is given to show the validity of our work. Our results improve/generalize several well known recent and classical results.

2010 Mathematics Subjects Classification: 46S70; 47H10; 54H25.

Keywords and phrases: fixed point; contractive dominated mappings; closed ball; ordered complete dislocated quasi metric spaces.

1 Introduction and Preliminaries

Let $T : X \to X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$. Let x_0 be an arbitrary chosen point in X. Define a sequence $\{x_n\}$ in X by a simple iterative method given by $x_{n+1} = Tx_n$, where $n \in \{0, 1, 2, 3, ...\}$. Such a sequence is called a picard iterative sequence and its convergence plays a very important role in proving existence of fixed point of a mapping T. A self mapping T on a metric space X is said to be a Banach contraction mapping if,

$$d(Tx, Ty) \leq kd(x, y)$$

holds for all $x, y \in X$ where $0 \leq k < 1$. Recently, many results appeared in literature related to fixed point results in complete metric spaces endowed with a partial ordering . Ran and Reurings [17] proved an analogue of Banach's fixed point theorem in metric space endowed with partial order and gave applications to matrix equations. Subsequently, Nieto et. al. [12] extended the results of [17] for non decreasing mappings and applied this results obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions. On the other hand in 2005, Mustafa and Sims in [14] introduce the notion of a generalized metric space as generalization the usual metric space. Mustafa and others studied fixed point theorems for mappings satisfying different contractive conditions for further useful results can be seen in [3, 8, 9, 10, 15, 16, 21]. Recently, Arshad et. al. [4] proved a result concerning the existence of fixed points of a mapping satisfying a contractive condition on closed ball in a complete dislocated metric space. For further results on closed ball we refer the reader to [5, 6, 7, 13, 20] and references their in. The dominated mapping [2] which satisfies the condition $fx \preceq x$ occurs very naturally in several practical problems . For example x denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$.

In this paper we have obtained fixed point results for dominated self- mappings in an ordered complete dislocated quasi G_d metric space on a closed ball under contractive condition to generalize, extend and improve some classical fixed point results. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point. Our results do not exists even yet in metric spaces. An example shows how this result can be used when the corresponding results cannot.

Definition 1 *Let X be a nonempty set and let $G_d : X \times X \times X \rightarrow R^+$ be a function satisfying the*

following axioms:

(i) If $G_d(x,y,z) = G_d(y,z,x) = G_d(z,x,y) = 0$,then $x = y = z$,

(ii) $G_d(x,y,z) \leq G_d(x,a,a) + G_d(a,y,z)$ for all $x,y,z,a \in X$ (rectangle inequality).

Then the pair (X,G_d) is called the dislocated quasi G_d-metric space. It is clear that if

$G_d(x,y,z) = G_d(y,z,x) = G_d(z,x,y) = 0$ then from (i) $x = y = z$. But if $x = y = z$ then $G_d(x,y,z)$ may not be 0. It is observed that if $G_d(x,y,z) = G_d(y,z,x) = G_d(z,x,y)$ for all $x,y,z \in X$, then (X,G_d) becomes a dislocated G_d-metric space.

Example 2 *If $X = R^+ \cup \{0\}$ then $G_d(x,y,z) = x + \max\{x,y,z\}$ defines a dislocated quasi metric G on X.*

Definition 3 *Let (X,G_d) be a G_d-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\rightarrow\infty} G_d(x,x_n,x_m) = 0$, and one says that sequence $\{x_n\}$ is G_d-convergent to x.Thus, if $x_n \rightarrow x$ in a G_d-metric space (X,G_d), then for any $\in > 0$, there exists $n,m \in \mathbb{N}$ such that $G_d(x,x_n,x_m) < \in$, for all $n,m \geq N$.*

Definition 4 *Let (X,G_d) be a G_d-metric space. A sequence $\{x_n\}$ is called G_d-Cauchy sequence if, for each $\in > 0$ there exists a positive integer $n^\star \in \mathbb{N}$ such that $G_d(x_n,x_m,x_l) < \in$ for all $n,l,m \geq n^\star$; i.e. if $G_d(x_n,x_m,x_l) \rightarrow 0$ as $n,m,l \rightarrow \infty$.*

Definition 5 *G_d-metric space (X,G_d) is said to be G_d-complete if every G_d-Cauchy sequence in (X,G_d) is G_d-convergent in X.*

Proposition 6 *Let (X,G_d) be a G_d-metric space, then the following are equivalent:*

(1) $\{x_n\}$ is G_d convergent to x.

(2) $G_d(x_n,x_n,x) \rightarrow 0$ as $n \rightarrow \infty$.

(3) $G_d(x_n,x,x) \rightarrow 0$ as $n \rightarrow \infty$.

(4) $G_d(x_n,x_m,x) \rightarrow 0$ as $m\ n \rightarrow \infty$.

Definition 7 *Let (X,G_d) be a G_d-metric space then for $x_0 \in X$, $r > 0$, the closed ball with centre x_0 and radius r is,*

$$\overline{B(x_0,r)} = \{y \in X : G_d(x_0,y,y) \leq r\}.$$

Definition 8 *[2] Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.*

Definition 9 *[2] Let $(X, \preceq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \preceq x$ for each x in X.*

Example 10 *[2] Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, therefore f is a dominated map.*

2 Fixed Points of Contractive Mapping

Theorem 11 *Let $(X, \preceq, G_d)$ be an ordered complete dislocated quasi G_d metric space, and $T : X \to X$ be a dominated mapping. Suppose there exists a, b such that $a + 3b < 1$ and for all comparable elements x, y and z in $\overline{B(x_0, r)}$, with $x_0 \in \overline{B(x_0, r)}$, $r > 0$,.*

$$\begin{aligned} G_d(Tx, Ty, Tz) &\leq a\ G_d(x, y, z) + b\ [G_d(x, Tx, Tx) \\ &\quad + G_d(y, Ty, Ty) + G_d(z, Tz, Tz)] \end{aligned} \tag{2.1}$$

$$\text{where } \lambda = \frac{a + b}{1 - 2b}$$

$$\text{and } G_d(x_0, Tx_0, Tx_0) \leq (1 - \lambda)r. \tag{2.2}$$

If for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$ and

$$\begin{aligned} & G(x_0, Tx_0, Tx_0) + G(v, Tv, Tv) + G(v, Tv, Tv) \\ \leq\ & G(x_0, v, v) + G(Tx_0, Tv, Tv) + G(Tx_0, Tv, Tv) \end{aligned} \tag{2.3}$$

then there exists a point $x^\star$ in $\overline{B(x_0, r)}$ such that $G_d(x^\star, x^\star, x^\star) = 0$ and $x^\star = Tx^\star$.

Proof. *Consider a picard sequence $x_{n+1} = Tx_n$ with initial guess x_0. As $x_{n+1} = Tx_n \preceq x_n$ for all $n \in \{0\} \cup \mathbb{N}$. By inequality (2.2), $G_d(x_0, x_1, x_1) \leq r$. It implies that $x_1 \in \overline{B(x_0, r)}$. Similarly $x_2 \ldots x_j \in \overline{B(x_0, r)}$ for some $j \in \mathbb{N}$.*

$$\begin{aligned} G_d(x_j, x_{j+1}, x_{j+1}) &= G_d(Tx_{j-1}, Tx_j, Tx_j) \leq a\ G_d(x_{j-1}, x_j, x_j) \\ &\quad + b[G_d(x_{j-1}, Tx_{j-1}, Tx_{j-1}) + G_d(x_j, x_{j+1}, x_{j+1}) \\ &\quad + G_d(x_j, x_{j+1}, x_{j+1})] \\ (1 - 2b)G_d(x_j, x_{j+1}, x_{j+1}) &\leq (a + b)G_d(x_{j-1}, x_j, x_j) \\ G_d(x_j, x_{j+1}, x_{j+1}) &\leq \frac{(a + b)}{(1 - 2b)} G_d(x_{j-1}, x_j, x_j) \\ &\ \vdots \\ G_d(x_j, x_{j+1}, x_{j+1}) &\leq \lambda^j G_d(x_0, x_1, x_1). \end{aligned} \tag{2.4}$$

Now by using the inequality (2.2) *and* (2.4) *we have*

$$\begin{aligned}
G_d(x_j, x_{j+1}, x_{j+1}) &\leq G_d(x_0, x_1, x_1) + G_d(x_1, x_2, x_2) + \cdots + G_d(x_j, x_{j+1}, x_{j+1})\\
G_d(x_j, x_{j+1}, x_{j+1}) &\leq (1-\lambda)r + \lambda(1-\lambda)r + \cdots + \lambda^j(1-\lambda)r\\
G_d(x_j, x_{j+1}, x_{j+1}) &\leq r(1-\lambda)[1 + \lambda + \lambda^2 + \cdots + \lambda^j]\\
G_d(x_j, x_{j+1}, x_{j+1}) &\leq r(1-\lambda)\frac{(1-\lambda^{j+1})}{(1-\lambda)} \leq r\\
&\Rightarrow G_d(x_j, x_{j+1}, x_{j+1}) \leq r.
\end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0, r)}$. *Hence* $x_n \in \overline{B(x_0, r)}$ *for all* $n \in \mathbb{N}$. *Now inequality* (2.4) *can be written as in the form of*

$$G_d(x_n, x_{n+1}, x_{n+1}) \leq \lambda^n G_d(x_0, x_1, x_1) \text{ for all } n \in \mathbb{N}. \tag{2.5}$$

By using inequality (2.5) *we get*

$$\begin{aligned}
G_d(x_n, x_{n+i}, x_{n+i}) &\leq G_d(x_n, x_{n+1}, x_{n+1}) + \cdots + G_d(x_{n+i-1}, x_{n+i}, x_{n+i})\\
G_d(x_n, x_{n+i}, x_{n+i}) &\leq \frac{\lambda^n(1-\lambda^i)}{(1-\lambda)}\, G_d(x_0, x_1, x_1) \to 0 \text{ as } n \to \infty \qquad (2.6)
\end{aligned}$$

Notice that the sequence $\{x_n\}$ *is Cauchy sequence in* $(\overline{B(x_0, r)}\ , G_d)$. *Therefore there exist a point* $x^\star \in \overline{B(x_0, r)}$.

$$\begin{aligned}
\lim_{n\to\infty} G_d(x_n, x^\star, x^\star) &= \lim_{n\to\infty} G_d(x^\star, x^\star, x_n) = 0\\
G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + G_d(x_n, Tx^\star, Tx^\star)
\end{aligned}$$

By assumption $x^\star \preceq x_n \preceq x_{n-1}$, *therefore,*

$$\begin{aligned}
G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + G_d(Tx_{n-1}, Tx^\star, Tx^\star)\\
G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + a\ G_d(x_{n-1}, x^\star, x^\star)\\
&\quad + b[G_d(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G_d(x^\star, Tx^\star, Tx^\star)\\
&\quad G_d(x^\star, Tx^\star, Tx^\star)]\\
G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + a\ G_d(x_{n-1}, x^\star, x^\star)\\
&\quad + b[G_d(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + 2G_d(x^\star, Tx^\star, Tx^\star)\\
(1-2b)G_d(x^\star, Tx^\star, Tx^\star) &\leq G_d(x^\star, x_n, x_n) + a\ G_d(x_{n-1}, x^\star, x^\star)\\
&\quad + b\ G_d(x_{n-1}, x_n, x_n)
\end{aligned}$$

Taking $\lim_{n\to\infty}$ *both sides and using* (2.6) *we have*

$$\begin{aligned}
(1-2b)G_d(x^\star, Tx^\star, Tx^\star) &\leq 0 + a(0) + b(0)\\
&\Rightarrow G_d(x^\star, Tx^\star, Tx^\star) \leq 0\\
&\Rightarrow x^\star = Tx^\star. \qquad (2.7)
\end{aligned}$$

Similarly $G_d(Tx^\star, Tx^\star, x^\star) = 0$ *and* $G_d(Tx^\star, x^\star, Tx^\star) = 0$ *and hence* $x^\star = Tx^\star$*.Now*

$$\begin{aligned} G_d(x^\star, x^\star, x^\star) &= G_d(Tx^\star, Tx^\star, Tx^\star) \leq a\ G_d(x^\star, x^\star, x^\star) \\ &\quad +3bG_d(x^\star, Tx^\star, Tx^\star) \\ (1-a-3b)G_d(x^\star, x^\star, x^\star) &\leq 0 \\ &\Rightarrow G_d(x^\star, x^\star, x^\star) \leq 0. \end{aligned}$$

This implies that $G_d(x^\star, x^\star, x^\star) = 0$.

Uniqueness:

Let $y^\star$ *be another point in* $\overline{B(x_0, r)}$ *such that*

$$\begin{aligned} y^\star &= Ty^\star. & (2.8) \\ G_d(y^\star, y^\star, y^\star) &= G_d(Ty^\star, Ty^\star, Ty^\star) \leq a\ G_d(y^\star, y^\star, y^\star) \\ &\quad +3b[G_d(y^\star, Ty^\star, Ty^\star)] \\ (1-a-3b)G_d(y^\star, y^\star, y^\star) &\leq 0 \\ &\Rightarrow G_d(y^\star, y^\star, y^\star) \leq 0. \\ &\Rightarrow G_d(y^\star, y^\star, y^\star) = 0. \end{aligned}$$

If $x^\star$ *and* $y^\star$ *are comparable then*

$$\begin{aligned} G_d(x^\star, y^\star, y^\star) &= G_d(Tx^\star, Ty^\star, Ty^\star) \leq a\ G_d(x^\star, y^\star, y^\star) \\ &\quad +b[G_d(x^\star, Tx^\star, Tx^\star) + 2G_d(y^\star, Ty^\star, Ty^\star)] \\ (1-a)G_d(x^\star, y^\star, y^\star) &\leq 0 \\ &\Rightarrow G_d(x^\star, y^\star, y^\star) = 0. \end{aligned}$$

Similarly, $G_d(y^\star, y^\star, x^\star) = 0$*. This shows that* $x^\star = y^\star$.
If $x^\star$ *and* $y^\star$ *are not comparable then there exist a point* $v \in \overline{B(x_0, r)}$ *which is a lower bound of both* $x^\star$ *and* $y^\star$*. Now we will to prove that* $T^n v \in \overline{B(x_0, r)}$*. Moreover by assumptions* $v \preceq x^\star \preceq x_n \preceq \cdots \preceq x_0$*. Now by using* (2.1)*, we have,*

$$G_d(Tx_0, Tv, Tv) \leq a\ G_d(x_0, v, v) + b\ [G_d(x_0, x_1, x_1) + 2G_d(v, Tv, Tv)].$$

By using (2.3)*, we have*

$$\begin{aligned} G_d(Tx_0, Tv, Tv) &\leq a\ G_d(x_0, v, v) + b\ [G_d(x_0, v, v) + 2G_d(x_1, Tv, Tv)] \\ (1-2b)G_d(Tx_0, Tv, Tv) &\leq (a+b)\ G_d(x_0, v, v) \\ G_d(Tx_0, Tv, Tv) &\leq \frac{(a+b)}{(1-2b)}\ G_d(x_0, v, v) \\ G_d(Tx_0, Tv, Tv) &\leq \lambda\ G_d(x_0, v, v). & (2.9) \end{aligned}$$

Now,

$$\begin{aligned} G_d(x_0, Tv, Tv) &\leq G_d(x_0, x_1, x_1) + G_d(x_1, Tv, Tv) \\ G_d(x_0, Tv, Tv) &\leq G_d(x_0, x_1, x_1) + \lambda\ G_d(x_0, v, v)\ \text{by } (2.9) \\ G_d(x_0, Tv, Tv) &\leq (1-\lambda)r + \lambda r \\ G_d(x_0, Tv, Tv) &\leq r. \end{aligned}$$

It follows that $Tv \in \overline{B(x_0,r)}$. *Now we will prove that* $T^n v \in \overline{B(x_0,r)}$. *By using mathematical induction to apply inequality* (2.1). *Let* $T^2v,\ T^3v, \cdots T^jv \in \overline{B(x_0,r)}$ *for some* $j \in N$. *As*

$$T^jv \preceq T^{j-1}v \preceq \cdots \preceq v \preceq x^\star \preceq x_n \preceq \cdots \preceq x_0.$$

Then,

$$\begin{aligned}
G_d(T^jv,T^{j+1}v,T^{j+1}v) &= G_d(T(T^{j-1}v),T(T^jv),T(T^jv)) \\
G_d(T^jv,T^{j+1}v,T^{j+1}v) &\leq a\ G_d(T^{j-1}v,T^jv,T^jv) + b\ [G_d(T^{j-1}v,T^jv,T^jv) \\
&\quad +2G_d(T^jv,T^{j+1}v,T^{j+1}v)] \\
(1-2b)G_d(T^jv,T^{j+1}v,T^{j+1}v) &\leq (a+b)G_d(T^{j-1}v,T^jv,T^jv) \\
G_d(T^jv,T^{j+1}v,T^{j+1}v) &\leq \lambda G_d(T^{j-1}v,T^jv,T^jv) \\
G_d(T^jv,T^{j+1}v,T^{j+1}v) &\leq \lambda^2 G_d(T^{j-2}v,T^{j-1}v,T^{j-1}v) \\
G_d(T^jv,T^{j+1}v,T^{j+1}v) &\leq \lambda^3 G_d(T^{j-3}v,T^{j-2}v,T^{j-2}v) \\
&\ \vdots \\
G_d(T^jv,T^{j+1}v,T^{j+1}v) &\leq \lambda^j G_d(T^{j-j}v,T^{j-(j-1)}v,T^{j-(j-1)}v) \\
G_d(T^jv,T^{j+1}v,T^{j+1}v) &\leq \lambda^j G_d(v,Tv,Tv) \qquad (2.10)
\end{aligned}$$

Now,

$$\begin{aligned}
G_d(x_{j+1},T^{j+1}v,T^{j+1}v) &\leq G_d(Tx_j,T(T^jv),T(T^jv)) \\
G_d(x_{j+1},T^{j+1}v,T^{j+1}v) &\leq a\ G_d(x_j,T^jv,T^jv) \\
&\quad +b\ [G_d(x_j,Tx_j,Tx_j) + 2G_d(T^jv,T^{j+1}v,T^{j+1}v)].
\end{aligned}$$

By using (2.4) *and* (2.10)

$$\begin{aligned}
G_d(x_{j+1},T^{j+1}v,T^{j+1}v) &\leq a\lambda^j G_d(x_0,v,v) \\
&\quad +b[\lambda^j G_d(x_0,x_1,x_1) + 2\lambda^j G_d(v,Tv,Tv)] \\
G_d(x_{j+1},T^{j+1}v,T^{j+1}v) &\leq a\lambda^j G_d(x_0,v,v) \\
&\quad +b\lambda^j[G_d(x_0,x_1,x_1) + 2G_d(v,Tv,Tv)]
\end{aligned}$$

By using the condition (2.3)

$$\begin{aligned}
G_d(x_{j+1},T^{j+1}v,T^{j+1}v) &\leq a\lambda^j G_d(x_0,v,v) \\
&\quad +b\lambda^j[G_d(x_0,v,v) + 2\lambda G_d(x_0,v,v)] \\
G_d(x_{j+1},T^{j+1}v,T^{j+1}v) &\leq \lambda^j(a+b+2b\lambda)G_d(x_0,v,v) \\
G_d(x_{j+1},T^{j+1}v,T^{j+1}v) &\leq \lambda^{j+1} G_d(x_0,v,v) \qquad (2.11)
\end{aligned}$$

Now ,

$$\begin{aligned}
G_d(x_0,T^{j+1}v,T^{j+1}v) &\leq G_d(x_0,x_{j+1},x_{j+1})+G_d(x_{j+1},T^{j+1}v,T^{j+1}v)\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\leq G_d(x_0,x_1,x_1)+\cdots+G_d(x_j,x_{j+1},x_{j+1})\\
&\quad +G_d(x_{j+1},T^{j+1}v,T^{j+1}v)\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\leq G_d(x_0,x_1,x_1)+\lambda G_d(x_0,x_1,x_1)\\
&\quad +\cdots+\lambda^{j+1}G_d(x_0,v,v) \text{ by } (2.5) \text{ and } (2.11)\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\leq G_d(x_0,x_1,x_1)[1+\lambda+\lambda^2+\cdots+\lambda^j]+\lambda^{j+1}r \text{ as } v\in\overline{B(x_0,r)}\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\leq (1-\lambda)r\frac{(1-\lambda^{j+1})}{(1-\lambda)}+\lambda^{j+1}r=r\\
G_d(x_0,T^{j+1}v,T^{j+1}v) &\leq r.
\end{aligned}$$

It follows that $T^{j+1}v\in\overline{B(x_0,r)}$ *and hence* $T^jv\in\overline{B(x_0,r)}$*. Now the inequality* (2.10) *can be written as*

$$G_d(T^nv,T^{n+1}v,T^{n+1}v)\leq\lambda^nG_d(v,Tv,Tv)\to 0 \text{ as } n\to\infty \tag{2.12}$$

Now,

$$\begin{aligned}
G_d(x^\star,y^\star,y^\star) &= G_d(Tx^\star,Ty^\star,Ty^\star)\\
G_d(x^\star,y^\star,y^\star) &\leq G_d(Tx^\star,T^{n+1}v,T^{n+1}v)+G_d(T^{n+1}v,Ty^\star,Ty^\star)\\
G_d(x^\star,y^\star,y^\star) &\leq a\ G_d(x^\star,T^nv,T^nv)+b\ [G_d(x^\star,Tx^\star,Tx^\star)\\
&\quad +2G_d(T^nv,T^{n+1}v,T^{n+1}v)]+a\ G_d(T^nv,Ty^\star,Ty^\star)\\
&\quad +b\ [G_d(T^nv,T^{n+1}v,T^{n+1}v)+2G_d(y^\star,Ty^\star,Ty^\star)]
\end{aligned}$$

By using $(2.7),(2.8)$ *and* (2.12) *we have*

$$\begin{aligned}
G_d(x^\star,y^\star,y^\star) &\leq a\ G_d(x^\star,T^nv,T^nv)+a\ G_d(T^nv,y^\star,y^\star)\\
G_d(x^\star,y^\star,y^\star) &\leq a\ [G_d(Tx^\star,T^nv,T^nv)+G_d(T^nv,Ty^\star,Ty^\star)]\\
G_d(x^\star,y^\star,y^\star) &\leq a\ [a\ G_d(x^\star,T^{n-1}v,T^{n-1}v)+b\ G_d(x^\star,Tx^\star,Tx^\star)\\
&\quad +2b\ G_d(T^{n-1}v,T^nv,T^nv)+a\ G_d(T^{n-1}v,y^\star,y^\star)\\
&\quad +b\ G_d(T^{n-1}v,T^nv,T^nv)+2b\ G_d(y^\star,Ty^\star,Ty^\star)].
\end{aligned}$$

By using $(2.7),(2.8)$ *and* (2.12) *we have*

$$\begin{aligned}
G_d(x^\star,y^\star,y^\star) &\leq a^2\ [G_d(x^\star,T^{n-1}v,T^{n-1}v)+G_d(T^{n-1}v,y^\star,y^\star)]\\
G_d(x^\star,y^\star,y^\star) &\leq a^3\ [G_d(x^\star,T^{n-2}v,T^{n-2}v)+G_d(T^{n-2}v,y^\star,y^\star)]\\
&\ \ \vdots\\
G_d(x^\star,y^\star,y^\star) &\leq a^n\ [G_d(x^\star,Tv,Tv)+G_d(Tv,y^\star,y^\star)]\\
G_d(x^\star,y^\star,y^\star) &\to 0 \text{ as } n\to\infty\\
G_d(x^\star,y^\star,y^\star) &= 0\\
x^\star &= y^\star.
\end{aligned}$$

This proves the uniqueness of the fixed point. ■

Now we give an example of an ordered complete dislocated quasi G_d-metric space in which the contraction does not hold on the whole space rather it holds on a closed ball only.

Example 12 *Let* $X = \mathbb{R}^+ \cup \{0\}$ *be endowed with usual order and* $G_d : X \times X \times X \to X$ *be a complete dislocated quasi* G_d *metric space defined by,*

$$G_d(x, y, z) = \left\{ \begin{array}{c} 0 \text{ if } x = y = z \\ \max\ \{2x, y, z\} \text{ otherwise.} \end{array} \right\}$$

Then (X, G_d) is a G_d complete G dislocated quasi metric space.

Let $T : X \to X$ be defined by,

$$Tx = \left\{ \begin{array}{c} \frac{x}{5} \text{ if } x \in [0, \frac{3}{2}] \\ x - \frac{1}{3} \text{ if } x \in [\frac{3}{2}, \infty) \end{array} \right\}.$$

Clearly, T is a dominated mappings. Take $x_0 = \frac{1}{3}$, $r = \frac{3}{2}$, $\overline{B(x_0, r)} = [0, \frac{3}{2}]$ and $\lambda = \frac{1}{4}$, $a + 3b < 1$, where $a = \frac{1}{10}$, and $b = \frac{1}{10}$.

$$\begin{array}{rcl} G_d(x_0, Tx_0, Tx_0) & \leq & (1 - \lambda)r \\ G_d(\dfrac{1}{3}, T\dfrac{1}{3}, T\dfrac{1}{3}) & = & \max\{\dfrac{2}{3}, \dfrac{1}{15}, \dfrac{1}{15}\} = \dfrac{2}{3} \\ \text{Since } (1 - \lambda)r & = & (1 - \dfrac{1}{4})\dfrac{3}{2} = \dfrac{9}{8} \\ & \Rightarrow & \dfrac{2}{3} \leq \dfrac{9}{8} \\ & \Rightarrow & 16 \leq 27 \end{array}$$

Also if x, y and $z \in [\frac{3}{2}, \infty)$. We assume that $x > y$, and $y > z$, then

$$\begin{array}{rcl} \max\{2x - \dfrac{2}{3}, y - \dfrac{1}{3}, z - \dfrac{1}{3}\} & \geq & \dfrac{1}{10} \max\{2x, y, z\} \\ & & \dfrac{1}{10}[\max\{2x, x - \dfrac{1}{3}, x - \dfrac{1}{3}\} \\ & & + \max\{2y, y - \dfrac{1}{2}, y - \dfrac{1}{2}\} \\ & & + \max\{2z, z - \dfrac{1}{2}, z - \dfrac{1}{2}\}] \\ G_d(Tx, Ty, Tz) & \geq & a\ G_d(x, y, z) + b\ [G_d(x, Tx, Tx) \\ & & + G_d(y, Ty, Ty) + G_d(z, Tz, Tz)] \end{array}$$

So the contractive conditions does not holds in X. Now if x, y and $z \in \overline{B(x_0, r)}$

then,

$$\begin{aligned} G_d(Tx,Ty,Tz) &= \max\{\frac{2x}{5},\frac{y}{5},\frac{z}{5}\} \leq \frac{1}{10}\{2x,y,z\} \\ &+\frac{1}{10}[\max\{2x,\frac{x}{5},\frac{x}{5}\}+\max\{2y,\frac{y}{5},\frac{y}{5}\} \\ &+\max\{z,\frac{z}{5},\frac{z}{5}\}] \\ &\Rightarrow G_d(Tx,Ty,Tz) \leq a\ G_d(x,y,z)+b\ [G_d(x,Tx,Tx) \\ &+G_d(y,Ty,Ty)+G_d(z,Tz,Tz)]. \end{aligned}$$

Hence it satisfies all the requirements of Theorem11. If we take $b=0$ in inequality (2.1) then we obtain the following corollary.

Corollary 13 *Let $(X,\preceq,G)$ be an ordered complete dislocated quasi $G-$metric space, $T:X\rightarrow X$ be a dominated mapping and x_0 be any arbitrary point in X. Suppose there exists $a\in[0,1)$ with,*

$$G(Tx,Ty,Tz)\leq a\ G(x,y,z),\ \textit{for all } x,\ y \textit{ and } z\in Y=\overline{B(x_0,r),}$$

and

$$G(x_0,Tx_0,Tx_0)\leq(1-a)r.$$

If for a nonincreasing sequence $\{x_n\}\rightarrow u$ implies that $u\preceq x_n$. Then there exists a point $x^\star$ in $\overline{B(x_0,r)}$ such that $x^\star=Sx^\star$ and $G(x^\star,x^\star,x^\star)=0$. Moreover if for any three points x,y and z in $\overline{B(x_0,r)}$ such that there exists a point $v\in\overline{B(x_0,r)}$ such that $v\preceq x$, $v\preceq y$ and $v\preceq z$, that is, every three of elements in $\overline{B(x_0,r)}$ has a lower bound, then the point $x^\star$ is unique.
Similarly if we take $a=0$ in inequality (2.1) then we obtain the following corollary.

Corollary 14 *Let $(X,\preceq,G)$ be an ordered complete dislocated quasi G-metric space $T:X\rightarrow R$ be a mapping and x_0 be an arbitrary point in X. Suppose there exists $b\in\left[0,\frac{1}{3}\right)$ with*

$$G(Tx,Ty,Tz)\leq b\ (G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz))$$

for all comparable elements $x,y,z\in\overline{B(x_0,r)}$ and

$$G(x_0,Tx_0,Tx_0)\leq(1-\lambda)r,$$

where $\lambda=\frac{b}{1-2b}$. If for non increasing sequence $\{x_n\}\rightarrow u$ implies that $u\preceq x_n$. Then there exists a point $x^\star$ in $\overline{B(x_0,r)}$ such that $x^\star=Sx^\star$ and $G(x^\star,x^\star,x^\star)=0$. Moreover, if for any three points $x,y,z\in\overline{B(x_o,r)}$, there exists a point v in $\overline{B(x_0,r)}$ such that $v\preceq x$ and $v\preceq y$, $v\preceq z$.

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

MIZOGUCHI- TAKAHASHI'S FIXED POINT THEOREM IN ν-GENERALIZED METRIC SPACES

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Abstract. Our main work is to prove Mizogchi- Takahashi theorem in ν-generalized metric space in the sense of Brancairi. In the same setting we prove two more theorems which are generalizations of the main one.

1. Introduction

A metric is defined as a mapping $d : X \times X \to [0, \infty)$, for any non-empty set X which satisfying the following axioms, for any $x, y, z \in X$

(i) $d(x, y) = 0$ iff $x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

We said that the pair (X, d) is a metric space. The theory of metric spaces form a basic environment for a lot of concepts in mathematics such as the fixed point theorems which have an important rules in various branches of mathematical analysis. One of the famous result of fixed point theorems is Banach Contraction Principle which state that,

Theorem 1.1. [11]*(Banach Contraction Principle)*
Let (X, d) be a complete metric space. Let $T : X \to X$ be a self map on X such that

$$d(Tx, Ty) \leq rd(x, y),$$

hold for any $x, y \in X$, where $r \in [0, 1)$. Then T has a unique fixed point.

Many authors explored the importance of this theorem and extended it in different directions. For examples, we refer the reader to the following papers [2, 9, 8, 6], and the references therein. In (1969) Nadler extended theorem 1.1 for multi-valued mapping. Recall that the set of all non- empty, closed and bounded subsets of X is denoted by $CB(X)$ and let A, B be any sets in $CB(X)$. A Hausdorff metric is defined as

$$\mathcal{H}(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

Theorem 1.2. [12]*(Nadler's theorem) Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multi-valued map. Assume that*

$$\mathcal{H}(Tx, Ty) \leq rd(x, y),$$

holds for each $x, y \in X$ and $r \in [0, 1)$. Then T has a fixed point.

Key words and phrases. Mizoguchi-Takahashi's theorem, ν- generalized metric space,Generalized of MT- theorem, Fixed point theory.

Many attempts have been done to generalize Nadler's theorem. One of these generalizations is Mizoguchi- Takahashi's theorem which stats that:

Theorem 1.3. [10] *Let (X, d) be a complete metric space. Let $T : X \to CB(X)$ be a multi-valued mapping. Assume that*

$$\mathcal{H}(Tx, Ty) \leq \beta(d(x, y))d(x, y),$$

hold for each $x, y \in X$, where $\beta : [0, \infty) \to [0, 1)$ is a function such that $\limsup_{s\to t^+} \beta(s) < 1$. Then T has a fixed point.

Remark 1.4. *The function β in theorem 1.3, which satisfies $\limsup_{s\to t^+} \beta(s) < 1$ is called Mizoguchi- Takahashi function (MT- function for short).*

Starting with Mizoguchi and Takahashi's paper, many generalizations of their theorem have been established see [3, 13]. Recently, Eldred et al [4], claimed that Nadler's and Mizoguchi- Takahashi's theorems are equivalent. However, in [14], Suzuki proved that their claim is not true and he shown that Mizoguchi- Takahashi's theorem (1.3) is a real extension of Nadler's theorem. This is why we are interesting in such theorem.

In another direction, in (2000) Branciari created a new concept of generalized metric spaces by modifying the triangle inequality to involve more points.

Definition 1.5. [1] *Let X be a non- empty set and $d : X \times X \to [0, \infty)$. For $\nu \in \mathbb{N}$, a pair (X, d) is called a ν- generalized metric space if the following hold:*
(M1) $d(x, y) = 0$ *iff* $x = y$
(M2) $d(x, y) = d(y, x)$
(M3) $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + ... + d(u_\nu, y)$,
for any $x, u_1, u_2, ...u_\nu, y \in X$, such that $x, u_1, u_2, ...u_\nu, y$ are all different.

It is not difficult to show that the new space is not the same as the original one. Moreover, the new space is hard to deal with because it does not satisfy all topological properties that metric space has, see [15] for more details. Recently, in [16], Suzuki proved Nadler's theorem in ν- generalized metric spaces. The main work of this paper is to prove Mizoguchi -Takahashi's theorem in ν- generalized metric spaces. Firstly, we will list all the necessary definitions and some results that we will need. Then, we will be able to prove our main results.

2. PRELIMINARY

Definition 2.1. *A point $x \in X$ is said to be a fixed point of multi-valued map T if $x \in Tx$.*

Definition 2.2. [1] *Let (X, d) be a ν- generalized metric space. A sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ is said to be Cauchy sequence if*

$$\lim_{n} \sup_{n>m} d(x_n, x_m) = 0$$

Definition 2.3. [16] *A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $(\sum, \neq)$- Cauchy sequence if all x_n's are different and*

$$\sum_{j=1}^{\infty} d(x_j, x_{j+1}) < \infty$$

Definition 2.4. [16] *Let (X, d) be a ν- generalized metric space. We said that, X is a $(\sum, \neq)$- complete if every $(\sum, \neq)$- Cauchy sequence converges.*

Lemma 2.5. [16, 5] *Let (X, d) be a ν- generalized metric space.*

- *Every converge $(\sum, \neq)$- Cauchy sequence is Cauchy.*
- *Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence converges to some $y \in X$ and $\{y_n\} \in X$ be a sequence such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. Then, $\{y_n\}$ also converges to y.*

Lemma 2.6. [14] *Let $\beta : [0, \infty) \to [0, 1)$ is a MT-function. Then, for all $s \in [0, \infty)$, there exist $r_s \in [0, 1)$ and $\varepsilon_s > 0$ such that $\beta(t) \leq r_s$ for all $t \in [s, s + \varepsilon_s)$*

Lemma 2.7. [12] *Let (X, d) be a metric space. For any $A, B \in CB(X)$ and $\varepsilon > 0$, there exist $a \in A$ and $b \in B$ such that $d(a, b) \leq \mathcal{H}(A, B) + \varepsilon$*

3. Main Result

In this section we prove Mizoguchi -Takahashi's theorem in ν -generalized metric spaces and some of its generalizations in the space.

Theorem 3.1. *Let (X, d) be a $(\sum, \neq)$ complete, ν- generalized metric space. and let T be a multi-valued map defined from X into $CB(X)$ satisfies the following:*

(i) If $\{y_n\} \in Tx$ and $\{y_n\}$ converges to y then $y \in Tx$.
(ii) For any $x, y \in X$, $\mathcal{H}(Tx, Ty) \leq \alpha(d(x, y))d(x, y)$,

where α is MT-function. Then T has a fixed point.

Proof. Let define a function $\gamma : [0, \infty) \to [0, 1)$ as $\gamma(t) = \dfrac{1 + \alpha(t)}{2}$. It is not difficult to show that $\alpha(t) < \gamma(t)$, for any $t \in [0, \infty)$ and $\lim_{s\to t^+} \sup \gamma(s) < 1$. Moreover, for each $x, y \in X$ and $v \in Tx$, there exist $u \in Ty$ such that

$$d(v, u) \leq \gamma(d(x, y))d(x, y).$$

Putting $v = y$, we will get that

$$d(y, u) \leq \gamma(d(x, y))d(x, y)$$

Define $f(x) = \inf\{d(x, b) : b \in Tx\}$ and suppose that T does not have a fixed point (i.e., for all $x \in X, f(x) > 0$). Let $x_1 \in X$ be arbitrary and choose $x_2 \in Tx_1$ satisfying

$$d(x_1, x_2) < \frac{1}{\gamma(d(x_1, x_2))} f(x_1). \tag{1}$$

Since $Tx_2 \neq \emptyset$, we can choose an arbitrary element $x_3 \in Tx_2$ such that

$$f(x_2) \leq d(x_2, x_3) \leq \gamma(d(x_1, x_2))d(x_1, x_2). \tag{2}$$

Also, as in equation (1), we have

$$d(x_2, x_3) < \frac{1}{\gamma(d(x_2, x_3))} f(x_2). \tag{3}$$

From (2) and (3), we have

$$d(x_2, x_3) \leq \min\{\gamma(d(x_1, x_2))d(x_1, x_2), \frac{1}{\gamma(d(x_2, x_3))} f(x_2)\}.$$

Thus

$$\gamma(d(x_2, x_3))d(x_2, x_3) < f(x_2) \leq \gamma(d(x_1, , x_2))d(x_1, x_2) < f(x_1).$$

Continuously, $\{x_n\}_{n\in\mathbb{N}} \in X$ is a sequence constructed such that $x_{n+1} \in Tx_n$ and satisfying

(4)

$$\gamma(d(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) < f(x_{n+1}) \leq \gamma(d(x_n, x_{n+1}))d(x_n, x_{n+1}) < f(x_n),$$

and

$$d(x_{n+1}, x_{n+2}) \leq \gamma(d(x_{n+1}, x_n))d(x_{n+1}, x_n). \tag{5}$$

Since $\gamma(t) < 1$, we have $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$. Hence, from (4) and (5), the sequences $\{f(x_n)\}$ and $\{d(x_n, x_{n+1})\}$ are strictly decreasing. Next, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a $(\sum, \neq)$- Cauchy sequence in two steps:

Step 1 we show that all terms different. Suppose not i.e suppose $x_n = x_m$ for some $n > m$, where $m, n \in \mathbb{N}$. Hence

$$\begin{aligned} f(x_m) &= \inf\{d(x_m, b) : b \in Tx_m\} \\ &= \inf\{d(x_n, b) : b \in Tx_n\} \\ &= f(x_n), \end{aligned}$$

which contradicts $\{f(x_n)\}$ being strictly decreasing.

Step 2 We show that $\sum d(x_n, x_{n+1}) < \infty$. Since $\{d(x_n, x_{n+1})\}$ is an decreasing sequence in R and bounded below, it converges to some positive real number (say δ). Also, we have $\lim_{s\to t^+} \sup \gamma(s) < 1$, thus, there exist $r \in [0, 1)$ and $\varepsilon > 0$ such that $\gamma(s) \leq r$ for all $s \in [\delta, \delta + \varepsilon)$. For any $n \in \mathbb{N}$, we can choose $\mu \in \mathbb{N}$ satisfying $\delta \leq d(x_n, x_{n+1}) \leq \delta + \varepsilon$ with $n \geq \mu$. So,

$$\begin{aligned} \sum_{n=1}^{\infty} d(x_n, x_{n+1}) &\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=\mu+1}^{\infty} d(x_n, x_{n+1}) \\ &\leq \sum_{n=1}^{\mu} d(x_n, x_{n+1}) + \sum_{n=1}^{\infty} r^n d(x_\mu, x_{\mu+1}) \\ &< \infty. \end{aligned}$$

Thus $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence in $(\sum, \neq)$ complete ν- generalize metric space. Then, it is converge to some $z \in X$ and by lemma (2.5), $\{x_n\}$ is a Cauchy sequence. From our assumption we choose $\{u_n\} \in Tz$ satisfy

$$d(x_{n+1}, u_n) \leq \mathcal{H}(Tx_n, Tz) \leq \gamma(d(x_n, z))d(x_n, z),$$

for any $n \in \mathbb{N}$. But $\{x_n\}$ converges to z, so $d(x_{n+1}, u_n) \to 0$ as $n \to \infty$. Thus we have $x_{n+1} \to z$ and $x_{n+1} \to u_n$. Therefore, by lemma(2.5) $d(u_n, z) = 0$ as $n \to \infty$. So $d(Tz, z) = 0$ implies $f(z) = 0$ which is a contradiction. Therefore, there exist $z \in X$ such that $f(z) = 0$ and hence $z \in Tz$ is a fixed point. □

Definition 3.2. [7] *A multi- valued map T from X into $CB(X)$ is called α- admissible if for any $x \in X$ and $y \in Tx$, $\alpha(x, y) \geq 1$ implies $\alpha(y, z) \geq 1$ for any $z \in Ty$, where $\alpha : X \times X \to [0, \infty)$.*

The up coming lemma proved in [18], for single-valued map here, we prove it for multi- valued map.

Lemma 3.3. *Let (X,d) be a ν- generalized metric space. Let T be a multi-valued mapping from X into 2^X and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X defined by $x_{n+1}\in Tx_n$ such that $x_n\neq x_{n+1}$. Assume that*

$$d(x_n,x_{n+1})\leq \delta d(x_{n-1},x_n) \tag{6}$$

hold for any $\delta\in[0,1)$. Then $x_n\neq x_m$ $\forall n\neq m\in\mathbb{N}$.

Proof. We prove that $x_{n+\ell}\neq x_n$ for all $n\in\mathbb{N}$ and $\ell\geq 1$. Suppose the contrary that is $x_{n+\ell}=x_n$ for some $n\in\mathbb{N}$ and $\ell\geq 1$. By assumption, we have that $x_{n+\ell+1}=x_{n+1}$. Then from (6) we get

$$d(x_n,x_{n+1})=d(x_{n+\ell},x_{n+\ell+1})\leq \delta d(x_{n+\ell-1},x_{n+\ell})\leq ...\leq \delta^{\ell} d(x_n,x_{n+1})<d(x_n,x_{n+1}) \tag{7}$$

which is contradiction. Thus, we get $x_m\neq x_n$ for all $m\neq n$ in $\mathbb{N}$. □

Let Φ be the family of all functions $\varphi:[0,\infty)\to[0,\infty)$ which satisfying the following conditions:

(a) $\varphi(s)=0$ iff $s=0$.
(b) φ is non-decreasing and lower semi-continuous
(c) $\lim_{s\to 0^+}\sup\dfrac{s}{\varphi(s)}<\infty$.

Theorem 3.4. *Let (X,d) be a $(\sum,\neq)$ complete ν- generalized metric space. Let $T:X\to CB(X)$ be an α- admissible multi-valued mapping satisfying:*

(i) There exist $x_0\in X$ and $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$
(ii) If $(y_n)\in Tx$ and (y_n) converge to y then $y\in Tx$
(iii) $\alpha(x,y)\mathcal{H}(Tx,Ty)\leq\phi(d(x,y))d(x,y)$ for any $x,y\in X$, and ϕ is MT-function.

Then T has a fixed point.

Proof. Let $\beta:[0,\infty)\to[0,1)$ as $\beta(t)=\dfrac{1+\phi(t)}{2}$ such that $\lim_{s\to t^+}\sup\beta(s)<1$. Clearly $\phi(t)<\beta(t)$ for each $t\in[0,\infty)$. Let $x_0\in X$ and choose $x_1\in Tx_0$ such that $\alpha(x_0,x_1)\geq 1$. Assume $x_0\neq x_1$ so, $\dfrac{1-\phi(d(x_0,x_1))}{2}d(x_0,x_1)>0$. Since $Tx_1\neq\emptyset$, choose $x_2\in Tx_1$ such that

$$\begin{aligned} d(x_1,x_2)&\leq \mathcal{H}(Tx_0,Tx_1)+\frac{1-\phi(d(x_0,x_1))}{2}d(x_0,x_1)\\ &\leq \alpha(x_0,x_1)\mathcal{H}(Tx_0,Tx_1)+\frac{1-\phi(d(x_0,x_1))}{2}d(x_0,x_1)\\ &\leq \phi(d(x_0,x_1))d(x_0,x_1)+\frac{1-\phi(d(x_0,x_1))}{2}d(x_0,x_1)\\ &\leq \beta(d(x_0,x_1))d(x_0,x_1). \end{aligned}$$

Since T is α- admissible, $x_1\in Tx_0$ and $\alpha(x_0,x_1)\geq 1$ then, $\alpha(Tx_0,Tx_1)\geq 1$ which implies $\alpha(x_1,x_2)\geq 1$. Similarly assume $x_1\neq x_2$ we have $\dfrac{1-\phi(d(x_1,x_2))}{2}d(x_1,x_2)>$

0 and choose $x_3 \in Tx_2$ such that

$$\begin{aligned} d(x_2,x_3) &\leq \mathcal{H}(Tx_1,Tx_2) + \frac{1-\phi(d(x_1,x_2))}{2} d(x_1,x_2) \\ &\leq \alpha(x_1,x_2)\mathcal{H}(Tx_1,Tx_2) + \frac{1-\phi(d(x_1,x_2))}{2} d(x_1,x_2) \\ &\leq \phi(d(x_1,x_2))d(x_1,x_2) + \frac{1-\phi(d(x_1,x_2))}{2} d(x_1,x_2) \\ &\leq \beta(d(x_1,x_2))d(x_1,x_2). \end{aligned}$$

Similarly, using the same method of proving theorem (3.1), we have our result. □

Theorem 3.5. *Let (X,d) be a $(\sum,\neq)$ complete ν- generalized metric space. Let $T: X \to CB(X)$ be a multi-valued map satisfying:*

$$\varphi(\mathcal{H}(Tx,Ty)) \leq \alpha(\varphi(d(x,y)))\varphi(d(x,y)),$$

for each $x,y \in X$, where α is a MT- function and $\varphi \in \Phi$. Then T has a fixed point.

Proof. Let $\gamma : [0,\infty) \to [0,1)$ defined by $\gamma(t) = \dfrac{1+\alpha(t)}{2}$. Since φ is non- decreasing function, then

$$\begin{aligned} &\max\left\{\sup_{v\in Tx}\varphi(d(v,Ty)), \sup_{u\in Ty}\varphi(d(u,Tx))\right\} \\ &\quad= \max\left\{\varphi(\sup_{v\in Tx} d(v,Ty)), \varphi(\sup_{u\in Ty} d(u,Tx))\right\} \\ &\quad= \varphi\left(\mathcal{H}(Tx,Ty)\right) \leq \gamma(\varphi(d(x,y)))\varphi(d(x,y)). \end{aligned} \tag{8}$$

There exist an element $z \in Ty$ such that

$$\varphi(d(y,z)) \leq \gamma(\varphi(d(x,y)))\varphi(d(x,y)),$$

for each $x \in X$ and $y \in Tx$. Thus, in the same way a sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ defined as $x_{n+1} \in Tx_n$ is constructed such that

$$\varphi(d(x_n,x_{n+1}) \leq \gamma(\varphi(d(x_{n-1},x_n))\varphi(d(x_{n-1},x_n)) \tag{9}$$

for all $n \in \mathbb{N}$. Since $\gamma(t) < 1$ for any $t \in [0,\infty)$, hence from (9) we get

$$\varphi(d(x_n,x_{n+1}) < \varphi(d(x_{n-1},x_n)). \tag{10}$$

Clearly $\{\varphi(d(x_{n-1},x_n))\}$ is decreasing sequence of positive real numbers. Hence it is converge to some non- negative real number, say ϵ. By contradiction, it is easy to show that $\epsilon = 0$. Note that, φ is a non- decreasing function which implies to $d(x_n,x_{n+1}) < d(x_{n-1},x_n)$. Thus the sequence $\{d(x_n,x_{n+1})\}$ is also decreasing. Hence by lemma (3.3), the terms of the sequence all are different. Now, show that $\sum_{n=0}^{\infty} d(x_n,x_{n+1}) < \infty$. Note that the sequence $\{d(x_n,x_{n+1})\}$ is decreasing and bounded. Thus, it is converges to a positive real number (say δ) which implies that $\varphi(\delta) \leq \varphi(d(x_n,x_{n+1}))$. Thus,

$$\varphi(\delta) \leq \lim_{n\to\infty} \varphi(d(x_n,x_{n+1})) = \epsilon = 0.$$

Since $\varphi(s) = 0$ if and only if $s = 0$ then, $\delta = 0$. By lemma (2.6), there exist $r \in [0, 1)$ such that, $\varphi(d(x_n, x_{n+1})) \leq r\varphi(d(x_{n-1}, x_n))$. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} \varphi(d(x_n, x_{n+1})) &\leq \sum_{n=1}^{\mu} \varphi(d(x_n, x_{n+1})) + \sum_{n=\mu+1}^{\infty} \varphi(d(x_n, x_{n+1})) \\ &\leq \sum_{n=1}^{\mu} \varphi(d(x_n, x_{n+1})) + \sum_{n=1}^{\infty} r^n \varphi(d(x_\mu, x_{\mu+1})) \\ &< \infty.\end{aligned}$$

By defintion of φ, we have

$$\lim_{n\to\infty} \sup \frac{d(x_n, x_{n+1})}{\varphi(d(x_n, x_{n+1}))} \leq \lim_{s\to 0^+} \frac{s}{\varphi(s)} < \infty.$$

Thus, the sequence $\{x_n\}$ is a $(\sum, \neq)$- Cauchy sequence. Since X is a $(\sum, \neq)$ complete ν- generalized metric space and by lemma (2.5), it is Cauchy and then it is converge to some $z \in X$. From the definition of φ and its increasing we conclude that,

$$\begin{aligned}\varphi d(z, Tz)) &\leq \lim_{n\to\infty} \inf \varphi(d(x_{n+1}, Tz) \leq \lim_{n\to\infty} \inf \varphi(\mathcal{H}(Tx_n, Tz)) \\ &\leq \lim_{n\to\infty} \inf \gamma(\varphi(d(x_n, z)))\varphi(d(x_n, z)) \leq \lim_{n\to\infty} \inf \varphi(d(x_n, z)) \\ &= \lim_{s\to 0^+} \varphi(s) = \lim_{n\to\infty} \varphi(d(x_n, x_{n+1})) = 0.\end{aligned}$$

Therefore, $\varphi(d(z, Tz)) = 0$. Thus by the definition of φ and since Tz closed we have $z \in Tz$ is a fixed point. □

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

On Multiresolution Analyses Of Multiplicity n

Richard A. Zalik [*]

Abstract

This paper studies multiresolution analyses in $L^2(\mathbb{R}^d)$ that have more than one scaling function and are generated by an arbitrary dilation matrix. It provides a further analysis of a representation theorem obtained by the author for such MRA's.

1 Introduction

The concept of multiresolution analysis of multiplicity n is due to Alpert [1, 2, 3] who introduced his now well known dyadic multiresolution analysis with an arbitrary number of filters in $L^2(\mathbb{R})$. Alpert's results motivated a number of papers, focused on the univariate case, such as Hervé [12, 13], Donovan, Geronimo and Hardin [6, 7], Geronimo and Marcellán [10], Goodman, Lee and Tang [9], Goodman and Lee [8], and Hardin, Kessler and Massopoust [11]. Multiresolution analyses of multiplicity 1 (i.e., with a single scaling function) with arbitrary expansive matrices in $L^2(\mathbb{R})$ were studied by Lemarié [15, 16] and Madych [17], among others, and we should also mention Wojtaszczyk's excellent textbook [25]. Properties of low pass filters and scaling functions in this context were studied by San Antolín [19, 20, 21, 22] and Cifuentes, Kazarian and San Antolín [5]. Saliani [18] extended these results to multiresolution analyses of multiplicity n generated by an expansive matrix. These results were further extended by Soto–Bajo [24] to multiresolution analyses having an arbitrary (not necessarily finite) set of generator functions. In [4], Behera studied multiwavelet packets and frame packets of $L^2(\mathbb{R}^d)$ associated with multiresolution analyses of multiplicity n generated by an expansive matrix. In [26] the author presented a representation theorem for such multiresolution analyses, in [27] he gave some simple examples for the case $n = 1$, and in [23] San Antolín and the author obtained representation theorems for vector valued wavelets. Some of the authors cited above have showed that by using more than one scaling function it is possible to

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construct wavelets that have a set of properties not available for wavelets associated with a multiresolution analysis having a single scaling function; for instance in [6] they constructed wavelets associated with more than two scaling functions having compact support, arbitrary regularity, orthogonality, and symmetry. These results would indicate that the further study of multiresolution analyses of multiplicity n may lead to other interesting results.

In what follows, $\mathbb{Z}$ will denote the set of integers, $\mathbb{Z}_+$ the set of strictly positive integers and $\mathbb{R}$ the set of real numbers; $\mathbb{C}$ will denote the set of complex numbers, and $\mathbf{I}$ will stand for the identity matrix. Boldface lowcase letters will denote elements of $\mathbb{R}^d$; $\mathbf{x}\cdot\mathbf{y}$ will stand for the standard dot product of the vectors $\mathbf{x}$ and $\mathbf{y}$; the vector norm $||\cdot||$ is defined by $||\mathbf{x}||^2 := \mathbf{x}\cdot\mathbf{x}$. If $\mathbf{A}$ is a matrix $||\mathbf{A}||$ will denote the matrix norm induced by the vector norm $||\cdot||$. The inner product of two functions $f, g \in L^2(\mathbb{R}^d)$ will be denoted by $\langle f, g\rangle$, their bracket product by $[f, g]$, and the norm of f by $||f||$; thus,

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(\mathbf{t})\overline{g(\mathbf{t})}\, d\mathbf{t},$$

$$[f, g](\mathbf{t}) := \sum_{\mathbf{k}\in\mathbb{Z}^d} f(\mathbf{t}+\mathbf{k})\overline{g(\mathbf{t}+\mathbf{k})},$$

and

$$||f|| := \sqrt{\langle f, f\rangle}.$$

The Fourier transform of a function f will be denoted by $\widehat{f}$. If $f \in L(\mathbb{R}^d)$,

$$\widehat{f}(\mathbf{x}) := \int_{\mathbb{R}^d} e^{-i2\pi\mathbf{x}\cdot\mathbf{t}} f(\mathbf{t})\, d\mathbf{t}.$$

Let $\mathbf{A} \in \mathbb{C}^{d\times d}$ and $|a| := \det(\mathbf{A})$. For every $j \in \mathbb{Z}$ and $\mathbf{k} \in \mathbb{Z}^d$ the dilation operator $D_j^{\mathbf{A}}$ and the translation operator $T_{\mathbf{k}}$ are defined on $L^2(\mathbb{R}^d)$ by

$$D_j^{\mathbf{A}} f(\mathbf{t}) := |a|^{j/2} f(\mathbf{A}^j\mathbf{t})$$

and

$$T_{\mathbf{k}} f(\mathbf{t}) := f(\mathbf{t}+\mathbf{k})$$

respectively. Let $\mathbb{T} := [0, 1]$, and let $\mathbb{T}^d$ denote the d–fold cartesian product of $\mathbb{T}$. A function f will be called $\mathbb{Z}^d$*–periodic* if it is defined on $\mathbb{R}^d$ and $T_{\mathbf{k}} f = f$ for every $\mathbf{k} \in \mathbb{Z}^d$.

Let $\mathbf{u} = \{u_1, \cdots, u_m\} \subset L^2(\mathbb{R}^d)$; then

$$T(\mathbf{u}) = T(u_1, \cdots, u_m) := \{T_{\mathbf{k}} u; u \in \mathbf{u}, \mathbf{k} \in \mathbb{Z}^d\}$$

and

$$S(\mathbf{u}) = S(u_1, \cdots, u_m) := \overline{\operatorname{span}}\, T(\mathbf{u}),$$

where the closure is in $L^2(\mathbb{R}^d)$. $S(\mathbf{u})$ is called a *finitely generated shift–invariant space* or FSI and the functions u_ℓ are called the *generators* of $S(\mathbf{u})$. In this case we will also use the symbols $T(u_1, \cdots, u_n)$ and $S(u_1, \cdots, u_n)$ to denote $S(\mathbf{u})$ and $T(\mathbf{u})$ respectively.

We also define

$$T(\mathbf{A}^j;\mathbf{u}) = T(\mathbf{A}^j;u_1,\cdots,u_m) := \{D_j^{\mathbf{A}}T_{\mathbf{k}}u_\ell; \ell=1,\cdots m, \mathbf{k}\in\mathbb{Z}^d\},$$

and

$$S(\mathbf{A}^j;\mathbf{u}) = S_j(\mathbf{A}^j;u_1,\cdots,u_m) := \overline{\operatorname{span}}\, T(\mathbf{A}^j;\mathbf{u}).$$

Given a sequence of functions $\mathbf{u} := \{u_1,\cdots,u_m\}$ in $L^2(\mathbb{R}^d)$, by $G[u_1,\cdots,u_m](\mathbf{x})$, $G_{\mathbf{u}}(\mathbf{x})$ or $G(\mathbf{x})$ we will denote its *Gramian matrix*, viz.

$$G(\mathbf{x}) := \Big([\widehat{u_\ell},\widehat{u_j}](\mathbf{x})\Big)_{\ell,j=1}^m.$$

Let $\Lambda\subset\mathbb{Z}$ and $\mathbf{u}=\{u_k; k\in\Lambda\}\subset S\subset L^2(\mathbb{R}^d)$. If S is a shift–invariant space then $\mathbf{u}$ is called a *basis generator* of S, if for every $f\in S$ there are $\mathbb{Z}^d$–periodic functions p_k, uniquely determined by f (up to a set of measure 0), such that

$$\widehat{f} = \sum_{k\in\Lambda} p_k\widehat{u_k}.$$

In what follows we will assume that $\mathbf{A}$ is a fixed matrix preserving the lattice $\mathbb{Z}^d$, i.e. $\mathbf{A}\mathbb{Z}^d\subset\mathbb{Z}^d$. We will also assume that $\mathbf{A}$ is expansive, that is, there exist constants $C>0$ and $\delta>1$ such that for every $j\in\mathbb{Z}_+$ and $\mathbf{x}\in\mathbb{R}^d$

$$||\mathbf{A}^j\mathbf{x}|| \geq C\delta^j||\mathbf{x}||.$$

Lemma 1. $\mathbf{A}$ *is expansive if and only if all its eigenvalues have modulus larger that* 1.

Proof. Suppose first that all the eigenvalues of $\mathbf{A}$ have modulus larger that 1. If $\mathbf{A}$ is a Jordan block, then $\mathbf{A}=\lambda\mathbf{I}+\mathbf{N}$, where $\mathbf{N}$ has 1's on the superdiagonal and 0's elsewhere, and from e.g. [14, Lemma 3.1.4] we deduce that

$$\mathbf{A}^j\mathbf{x} = \sum_{k=0}^d \binom{j}{k}\lambda^k N^{j-k}\mathbf{x}$$

whence the assertion readily follows, and therefore it also follows when $\mathbf{A}$ is in Jordan canonical form. In general, if $\mathbf{Q}$ is the Jordan form of $\mathbf{A}$, then $\mathbf{Q}=\mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ and we have

$$C\delta^j||\mathbf{y}|| \leq ||\mathbf{Q}^j\mathbf{y}|| = ||\mathbf{B}^{-1}\mathbf{A}^j\mathbf{B}\mathbf{y}|| \leq ||\mathbf{B}^{-1}||\,||\mathbf{A}^j\mathbf{B}\mathbf{y}||.$$

Setting $\mathbf{x}=\mathbf{B}\mathbf{y}$ the assertion readily follows.

Conversely, if $\mathbf{A}$ has an eigenvalue λ with modulus less or equal to 1 and $\mathbf{v}$ is an eigenvector for λ with $||\mathbf{v}||=1$, then $\mathbf{A}\mathbf{v}=\lambda\mathbf{I}$; hence $||\mathbf{A}^j\mathbf{v}||=|\lambda|^j\leq|\lambda|$, and $||\mathbf{A}^j\mathbf{v}||$ remains bounded. So $\mathbf{A}$ is not expansive. ☐

The previous proof was suggested by Wayne Lawton. An elementary proof may be found in San Antolín's thesis [19, Lema A.12].

A *multiresolution analysis* (MRA) of multiplicity n in $L^2(\mathbb{R}^d)$ (generated by $\mathbf{A}$) is a sequence $\{V_j; j \in \mathbb{Z}\}$ of closed linear subspaces of $L^2(\mathbb{R}^d)$ such that:

(i) $V_j \subset V_{j+1}$ for every $j \in \mathbb{Z}$.

(ii) For every $j \in \mathbb{Z}, f(\mathbf{t}) \in V_j$ if and only if $f(\mathbf{At}) \in V_{j+1}$.

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$.

(iv) There are functions $\mathbf{u} := \{u_1, \cdots, u_n\}$ such that $T(\mathbf{u})$ is an orthonormal basis of V_0.

From [18, Lemma 17] we know that if $\{V_j; j \in \mathbb{Z}\}$ is a multiresolution analysis, then

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}. \tag{1}$$

This generalizes a result of Cifuentes, Kazarian and San Antolín, which was established for multiresolution analyses of multiplicity 1 (cf. [5, Lemma 4]).

It follows from the definition of multiresolution analysis that there are $\mathbb{Z}^d$–periodic functions $p_{\ell,j} \in L^2(\mathbb{T}^d)$ such that the functions u_ℓ satisfy the *scaling identity*

$$\widehat{u}_\ell(\mathbf{A}^*\mathbf{x}) = \sum_{j=1}^{n} p_{\ell,j}(\mathbf{x})\widehat{u}_j(\mathbf{x}), \quad j, \ell = 1, \cdots, n \quad a.e.,$$

where $\mathbf{A}^*$ is the transpose of $\mathbf{A}$. The functions u_ℓ are called *scaling functions* for the multiresolution analysis, and the functions $p_{\ell,j}$ are called the *low pass filters* associated with $\mathbf{u}$.

A finite set of functions $\boldsymbol{\psi} = \{\psi_1, \cdots, \psi_m\} \in L^2(\mathbb{R}^d)$ will be called an orthonormal wavelet system if the affine sequence

$$\{D_j^{\mathbf{A}} T_{\mathbf{k}} \psi_\ell; j \in \mathbb{Z}, \mathbf{k} \in \mathbb{Z}^d, \ell = 1, \cdots, m\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Let $\boldsymbol{\psi} := \{\psi_1, \cdots, \psi_m\}$ be an orthogonal wavelet system in $L^2(\mathbb{R}^d)$ generated by a matrix $\mathbf{A}$, let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis and let W_j denote the orthogonal complement of V_j in V_{j+1}. We say that $\boldsymbol{\psi}$ is *associated* with an MRA, if $T(\boldsymbol{\psi})$ is an orthonormal basis of W_0.

2 Representation of orthonormal wavelets.

For $k > 1$ let $\mathrm{diag}\,\{-\mathrm{e}^{\mathrm{i}\omega}, 1, \cdots, 1\}_{\mathrm{k}}$ denote the $k \times k$ diagonal matrix with $-e^{i\omega}, 1, \cdots, 1$ as its diagonal entries. With the convention that $\mathrm{Arg}\ 0 = 0$ we have

Theorem 1. *Let $M := \{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis of multiplicity n with scaling functions $\mathbf{u} := \{u_1, \cdots, u_n\}$, generated by a matrix $\mathbf{A}$ that preserves the lattice*

$\mathbb{Z}^d$. For $1 \le \ell \le n$, let $\{v_{\ell,1}, \cdots, v_{\ell,|a|}\}$ be an orthonormal basis generator of $S(\mathbf{A}, u_\ell)$, let $\mathbf{e} := (1, 0, \cdots, 0) \in \mathbb{R}^k$, and

$$\widehat{u_\ell}(\mathbf{x}) = \sum_{j=1}^{|a|} b_{\ell,j}(\mathbf{x}) \widehat{v}_{\ell,j}(\mathbf{x}), \tag{2}$$

$$\mathbf{b}_\ell(\mathbf{x}) := (b_{\ell,1}(\mathbf{x}), \cdots, b_{\ell,|a|}(\mathbf{x}))^T, \quad \delta_\ell(\mathbf{x}) := e^{iArg\, b_{\ell,1}(\mathbf{x})}, \quad \mathbf{q}_\ell(\mathbf{x}) := \mathbf{b}_\ell(\mathbf{x}) + \delta_\ell(\mathbf{x})\mathbf{e},$$

$$\widehat{\mathbf{v}}(\mathbf{x}) := (\widehat{v}_{1,1}(\mathbf{x}), \cdots, \widehat{v}_{1,|a|}(\mathbf{x}), \cdots, \widehat{v}_{n,1}(\mathbf{x}), \cdots, \widehat{v}_{n,|a|}(\mathbf{x}))^T,$$

and

$$\mathbf{Q}_\ell(\mathbf{x}) := \operatorname{diag}\{-\delta_\ell(\mathbf{x}), 1, \cdots, 1\}_{|a|} \left[\overline{\mathbf{I} - 2\mathbf{q}_\ell(\mathbf{x})\mathbf{q}_\ell(\mathbf{x})^* / \mathbf{q}_\ell(\mathbf{x})^* \mathbf{q}_\ell(\mathbf{x})}\right].$$

Let $a := \det \mathbf{A}$, $m := |a|n$, and let

$$\mathbf{Q}(\mathbf{x}) = \Big(q_{\ell,k}(\mathbf{x})\Big)_{\ell,k=1}^{m}$$

be the $m \times m$ block diagonal matrix

$$\mathbf{Q}_1(\mathbf{x}) \oplus \mathbf{Q}_2(\mathbf{x}) \oplus \cdots \oplus \mathbf{Q}_m(\mathbf{x}) = \begin{pmatrix} \mathbf{Q}_1(\mathbf{x}) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{Q}_n(\mathbf{x}) \end{pmatrix}.$$

If

$$(\widehat{y}_1(\mathbf{x}), \cdots, \widehat{y}_m(\mathbf{x}))^T := \mathbf{Q}(\mathbf{x})\widehat{\mathbf{v}}(\mathbf{x}),$$

then

$$y_{(\ell-1)|a|+1} = u_\ell; \quad 1 \le \ell \le n, \tag{3}$$

and

$$\{y_{(\ell-1)|a|+k}; 1 \le \ell \le n,\ 2 \le k \le |a|\}$$

is an orthonormal wavelet system associated with M.

The preceding theorem was proved in [26, Theorem 9] but there was arguably a gap in the proof, which is bridged by the following

Lemma 2. *Let $m = n(|a| - 1)$, let $\{V_j; j \in \mathbb{Z}\}$ be a multiresolution analysis and assume that $\{u_\ell; \ell = 1, \cdots, n\}$ is an orthonormal basis generator of V_0. Then*

$$V_1 = S(\mathbf{A}, u_1) \oplus S(\mathbf{A}, u_2) \oplus \cdots \oplus S(\mathbf{A}, u_n).$$

Proof. From [26, Theorem 3] we know that there exist functions $v_{\ell,k}$, $\ell = 1, \cdots n$, such that $\{v_{\ell,1}, \cdots v_{\ell,|a|}\}$ is an orthonormal basis generator of $S(\mathbf{A}, u_\ell)$. Therefore $\{v_{\ell,k}; 1 \le \ell \le n, 1 \le k \le |a|\}$ is an orthogonal basis generator of $S(\mathbf{A}, u_1) \oplus S(\mathbf{A}, u_2) \oplus \cdots \oplus S(\mathbf{A}, u_n)$, which is a subspace of V_1. But [26, Theorem 3] also tells us that every Riesz generator of V_1 has $|a|n$ functions, and the assertion readily follows from [26, Theorem 1]. □

We can actually say more: the following theorem elucidates the structure of a multiresolution analysis of multiplicity n:

Theorem 2. *Let* $\{V_j; j \in \mathbb{Z}\}$ *be a multiresolution analysis and assume that* $\{u_\ell; \ell = 1, \cdots, n\}$ *is an orthonormal basis generator of* V_0*. Then*
(a) If $j > 0$*,*

$$V_j = S(\mathbf{A}^j, u_1) \oplus S(\mathbf{A}^j, u_2) \oplus \cdots \oplus S(\mathbf{A}^j, u_n).$$

(b)

$$L^2(\mathbb{R}^d) = \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_1)} + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_2)} + \cdots\cdots + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_n)}$$

Proof. From [26, Theorem 4] we know that every orthonormal basis generator of $V_j, j > 0$, must have $n|a|^j$ functions, and an argument similar to the one employed in the proof of Lemma 2 yields (a).

To prove (b), let $f \in L^2(\mathbb{R}^d)$ and let $\varepsilon > 0$ be given; then there is a $j \in \mathbb{Z}_+$ and a $g \in V_j$ such that $||f - g|| < \varepsilon$. Since *a fortiori* g belongs to the closed set

$$\overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_1)} + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_2)} + \cdots\cdots + \overline{\bigcup_{j=0}^{\infty} S(\mathbf{A}^j; u_n)}$$

and ε is arbitrary, the assertion follows. □

Theorem 3. *Let* $m = n(|a| - 1)$*, let the functions* y_k*,* $k = 1 \cdots |a|n$ *be constructed as in Theorem 1, and let* $q_j = y_{j+\ell}$ *for* $\ell = 1, \cdots n$*,* $j = \ell|a|, \cdots, (\ell + 1)(|a| - 1)$*. Then* $\{w_1, \cdots w_r\}$ *is an orthonormal wavelet system if and only if* $r = m$ *and there exists an orthogonal matrix* $Q(x)$ *such that*

$$(w_1, \cdots w_m)^T = Q(x)(q_1, \cdots q_m)^T.$$

Proof. From Theorem 1 we know that $\{q_1, \cdots q_m\}$ is an orthonormal basis system or, equivalently, that it is an orthonormal basis generator of $S(\mathbf{u})$. The assertion now readily follows from [26, Corollary 3 and Theorem 5]. □

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Generalized g–Fractional vector Representation Formula and integral Inequalities for Banach space valued functions

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Abstract

Here we give a very general fractional Bochner integral representation formula for Banach space valued functions. We derive generalized left and righ fractional Opial type inequalities, fractional Ostrowski type inequalities and fractional Grüss type inequalities. All these inequalities are very general having in their background Bochner type integrals.

2010 AMS Mathematics Subject Classification : 26A33, 26B40, 26D10, 26D15, 46B25, 46E40.
Keywords and Phrases: Bamach space valued functions, vector generalized fractional derivative, Caputo fractional derivative, generalized vector fractional integral inequalities, Bochner integral, fractional vector representation formula.

1 Background

We need

Definition 1 *([2]) Let* $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ *a Banach space,* $g \in C^1([a,b])$ *and increasing,* $f \in C([a,b], X)$, $\nu > 0$.

We define the left Riemann-Liouville generalized fractional Bochner integral operator

$$\left(I^{\nu}_{a+;g} f\right)(x) := \frac{1}{\Gamma(\nu)} \int_a^x (g(x) - g(z))^{\nu-1} g'(z) f(z)\, dz, \tag{1}$$

$\forall\ x \in [a,b]$, *where* Γ *is the gamma function.*

The last integral is of Bochner type. Since $f \in C([a,b], X)$, *then* $f \in L_{\infty}([a,b], X)$. *By [2] we get that* $I^{\nu}_{a+;g} f \in C([a,b], X)$. *Above we set* $I^{0}_{a+;g} f := f$ *and see that* $\left(I^{\nu}_{a+;g} f\right)(a) = 0$.

When g is the identity function id, we get that $I_{a+;id}^{\nu}=I_{a+}^{\nu}$, the ordinary left Riemann-Liouville fractional integral

$$\left(I_{a+}^{\nu}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_{a}^{x}(x-t)^{\nu-1}f(t)\,dt, \tag{2}$$

$\forall\ x\in[a,b]$, $\left(I_{a+}^{\nu}f\right)(a)=0$.

We need

Theorem 2 *([2]) Let $\mu,\nu>0$ and $f\in C([a,b],X)$. Then*

$$I_{a+;g}^{\mu}I_{a+;g}^{\nu}f=I_{a+;g}^{\mu+\nu}f=I_{a+;g}^{\nu}I_{a+;g}^{\mu}f. \tag{3}$$

We need

Definition 3 *([2]) Let $[a,b]\subset\mathbb{R}$, $(X,\|\cdot\|)$ a Banach space, $g\in C^{1}([a,b])$ and increasing, $f\in C([a,b],X)$, $\nu>0$.*

We define the right Riemann-Liouville generalized fractional Bochner integral operator

$$\left(I_{b-;g}^{\nu}f\right)(x):=\frac{1}{\Gamma(\nu)}\int_{x}^{b}(g(z)-g(x))^{\nu-1}g'(z)f(z)\,dz, \tag{4}$$

$\forall\ x\in[a,b]$, where Γ is the gamma function.

The last integral is of Bochner type. Since $f\in C([a,b],X)$, then $f\in L_{\infty}([a,b],X)$. By [2] we get that $I_{b-;g}^{\nu}f\in C([a,b],X)$. Above we set $I_{b-;g}^{0}f:=f$ and see that $\left(I_{b-;g}^{\nu}f\right)(b)=0$.

When g is the identity function id, we get that $I_{b-;id}^{\nu}=I_{b-}^{\nu}$, the ordinary right Riemann-Liouville fractional integral

$$\left(I_{b-}^{\nu}f\right)(x)=\frac{1}{\Gamma(\nu)}\int_{x}^{b}(t-x)^{\nu-1}f(t)\,dt, \tag{5}$$

$\forall\ x\in[a,b]$, with $\left(I_{b-}^{\nu}f\right)(b)=0$.

We need

Theorem 4 *([2]) Let $\mu,\nu>0$ and $f\in C([a,b],X)$. Then*

$$I_{b-;g}^{\mu}I_{b-;g}^{\nu}f=I_{b-;g}^{\mu+\nu}f=I_{b-;g}^{\nu}I_{b-;g}^{\mu}f. \tag{6}$$

We will use

Definition 5 *([2]) Let $\alpha>0$, $\lceil\alpha\rceil=n$, $\lceil\cdot\rceil$ the ceiling of the number. Let $f\in C^{n}([a,b],X)$, where $[a,b]\subset\mathbb{R}$, and $(X,\|\cdot\|)$ is a Banach space. Let $g\in C^{1}([a,b])$, strictly increasing, such that $g^{-1}\in C^{n}([g(a),g(b)])$.*

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{a+;g}^{\alpha}f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x)-g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{7}$$

$\forall$ $x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [2], we have that $\left(D_{a+;g}^{\alpha}f\right) \in C([a,b],X)$.

We see that

$$\left(I_{a+;g}^{n-\alpha}\left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)\right)(x) = \left(D_{a+;g}^{\alpha}f\right)(x), \quad \forall\ x \in [a,b]. \tag{8}$$

We set

$$D_{a+;g}^{n}f(x) := \left(\left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) \in C([a,b],X), \quad n \in \mathbb{N}, \tag{9}$$

$$D_{a+;g}^{0}f(x) = f(x), \quad \forall\ x \in [a,b].$$

When $g = id$, then

$$D_{a+;g}^{\alpha}f = D_{a+;id}^{\alpha}f = D_{*a}^{\alpha}f, \tag{10}$$

the usual left X-valued Caputo fractional derivative, see [3].

We will use

Definition 6 *([2]) Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a,b],X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a),g(b)])$.*

We define the right generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{b-;g}^{\alpha}f\right)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t)-g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{11}$$

$\forall$ $x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by [2], we have that $\left(D_{b-;g}^{\alpha}f\right) \in C([a,b],X)$.

We see that

$$I_{b-;g}^{n-\alpha}\left((-1)^n \left(f \circ g^{-1}\right)^{(n)} \circ g\right)(x) = \left(D_{b-;g}^{\alpha}f\right)(x), \quad a \le x \le b. \tag{12}$$

We set

$$D_{b-;g}^{n}f(x) := (-1)^n \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b],X), \quad n \in \mathbb{N}, \tag{13}$$

$$D_{b-;g}^{0}f(x) := f(x), \quad \forall\ x \in [a,b].$$

When $g = id$, then

$$D_{b-;g}^{\alpha}f(x) = D_{b-;id}^{\alpha}f(x) = D_{b-}^{\alpha}f, \tag{14}$$

the usual right X-valued Caputo fractional derivative, see [3].

We make

Remark 7 *All as in Definition 5. We have (by Theorem 2.5, p. 7, [5])*

$$\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\| \leq \frac{1}{\Gamma(n-\alpha)}\int_a^x (g(x)-g(t))^{n-\alpha-1} g'(t) \left\|\left(f\circ g^{-1}\right)^{(n)}(g(t))\right\| dt$$

$$\leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)}\int_{g(a)}^{g(x)} (g(x)-g(t))^{n-\alpha-1} dg(t) =$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(x)-g(a))^{n-\alpha}. \tag{15}$$

That is

$$\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\| \leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(x)-g(a))^{n-\alpha}, \tag{16}$$

$\forall\ x \in [a,b]$.

If $\alpha \notin \mathbb{N}$, *then* $\left(D_{a+;g}^{\alpha}f\right)(a) = 0$.

Similarly, by Definition 6 we derive

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\| \leq \frac{1}{\Gamma(n-\alpha)}\int_x^b (g(t)-g(x))^{n-\alpha-1} g'(t) \left\|\left(f\circ g^{-1}\right)^{(n)}(g(t))\right\| dt$$

$$\leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha)}\int_{g(x)}^{g(b)} (g(t)-g(x))^{n-\alpha-1} dg(t) =$$

$$\frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(b)-g(x))^{n-\alpha}. \tag{17}$$

That is

$$\left\|\left(D_{b-;g}^{\alpha}f\right)(x)\right\| \leq \frac{\left\|\left(f\circ g^{-1}\right)^{(n)}\circ g\right\|_{\infty,[a,b]}}{\Gamma(n-\alpha+1)} (g(b)-g(x))^{n-\alpha}, \tag{18}$$

$\forall\ x \in [a,b]$.

If $\alpha \notin \mathbb{N}$, *then* $\left(D_{b-;g}^{\alpha}f\right)(b) = 0$.

Notation 8 *We denote by*

$$D_{a+;g}^{n\alpha} := D_{a+;g}^{\alpha}D_{a+;g}^{\alpha}...D_{a+;g}^{\alpha} \quad (n \text{ times}),\ n \in \mathbb{N}, \tag{19}$$

$$I_{a+;g}^{n\alpha} := I_{a+;g}^{\alpha}I_{a+;g}^{\alpha}...I_{a+;g}^{\alpha}, \tag{20}$$

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha}D_{b-;g}^{\alpha}...D_{b-;g}^{\alpha}, \tag{21}$$

and

$$I_{b-;g}^{n\alpha} := I_{b-;g}^{\alpha}I_{b-;g}^{\alpha}...I_{b-;g}^{\alpha}, \tag{22}$$

(n times), $n \in \mathbb{N}$.

We are motivated by the following generalized fractional Ostrowski type inequality:

Theorem 9 *([2]) Let $g \in C^1([a,b])$ and strictly increasing, such that $g^{-1} \in C^1([g(a), g(b)])$, and $0 < \alpha < 1$, $n \in \mathbb{N}$, $f \in C^1([a,b], X)$, where $(X, \|\cdot\|)$ is a Banach space. Let $x_0 \in [a,b]$ be fixed. Assume that $F_k^{x_0} := D_{x_0-;g}^{k\alpha} f$, for $k = 1, ..., n$, fulfill $F_k^{x_0} \in C^1([a, x_0], X)$ and $\left(D_{x_0-;g}^{i\alpha} f\right)(x_0) = 0$, $i = 2, ..., n$.*

Similarly, we assume that $G_k^{x_0} := D_{x_0+;g}^{k\alpha} f$, for $k = 1, ..., n$, fulfill $G_k^{x_0} \in C^1([x_0, b], X)$ and $\left(D_{x_0+;g}^{i\alpha} f\right)(x_0) = 0$, $i = 2, ..., n$.

Then

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(x_0) \right\| \leq \frac{1}{(b-a)\,\Gamma((n+1)\alpha+1)} \cdot$$

$$\left\{ (g(b) - g(x_0))^{(n+1)\alpha} (b - x_0) \left\| D_{x_0+;g}^{(n+1)\alpha} f \right\|_{\infty,[x_0,b]} + \right.$$

$$\left. (g(x_0) - g(a))^{(n+1)\alpha} (x_0 - a) \left\| D_{x_0-;g}^{(n+1)\alpha} f \right\|_{\infty,[a,x_0]} \right\}. \tag{23}$$

In this work we will present several generalized fractional Bochner integral inequalities.

We mention the following g-left generalized X-valued Taylor's formula:

Theorem 10 *([2]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Then*

$$f(x) = f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) +$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \left(D_{a+;g}^{\alpha} f\right)(t)\, dt =$$

$$f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) + \tag{24}$$

$$\frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D_{a+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\, dz, \quad \forall\ x \in [a,b].$$

We mention the following g-right generalized X-valued Taylor's formula:

Theorem 11 *([2]) Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Then*

$$f(x) = f(b) + \sum_{i=1}^{n-1} \frac{(g(x) - g(b))^i}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(b)) +$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{b-;g}^{\alpha} f\right)(t)\,dt =$$
$$f(b)+\sum_{i=1}^{n-1}\frac{(g(x)-g(b))^i}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(b))+ \tag{25}$$
$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}(z-g(x))^{\alpha-1}\left(\left(D_{b-;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz,\quad \forall\ x\in[a,b].$$

For the Bochner integral excellent resources are [4], [6], [7] and [1], pp. 422-428.

2 Main Results

We give the following representation formula:

Theorem 12 *All as in Theorem 10. Then*

$$f(y)=\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx+R_1(y), \tag{26}$$

for any $y\in[a,b]$, where

$$R_1(y)=-\frac{1}{\Gamma(\alpha)(b-a)}$$
$$\left[\int_a^b \chi_{[a,y)}(x)\left(\int_x^y |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt\right)dx\right.$$
$$\left.\int_a^b \chi_{[y,b]}(x)\left(\int_y^x |g(x)-g(t)|^{\alpha-1}g'(t)\left(D_{y+;g}^{\alpha}f\right)(t)\,dt\right)dx\right]. \tag{27}$$

here χ_A stands for the characteristic function set A, where A is an arbitrary set.

One may write also that

$$R_1(y)=-\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_x^y (g(t)-g(x))^{\alpha-1}g'(t)\left(D_{y-;g}^{\alpha}f\right)(t)\,dt\right)dx\right. \tag{28}$$
$$\left.+\int_y^b\left(\int_y^x (g(x)-g(t))^{\alpha-1}g'(t)\left(D_{y+;g}^{\alpha}f\right)(t)\,dt\right)dx\right],$$

for any $y\in[a,b]$.

Putting things together, one has

$$f(y)=\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx$$

$$-\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^b \chi_{[a,y)}(x)\left(\int_x^y |g(x)-g(t)|^{\alpha-1} g'(t)\left(D_{y-;g}^{\alpha} f\right)(t)\,dt\right)dx\right.$$

$$\left.\int_a^b \chi_{[y,b]}(x)\left(\int_y^x |g(x)-g(t)|^{\alpha-1} g'(t)\left(D_{y+;g}^{\alpha} f\right)(t)\,dt\right)dx\right]. \tag{29}$$

In particular, one has

$$f(y)-\frac{1}{b-a}\int_a^b f(x)\,dx+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx = R_1(y), \tag{30}$$

for any $y\in[a,b]$.

Proof. Here $x, y\in[a,b]$. We keep y as fixed.
By Theorem 10 we get:

$$f(x)=f(y)+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^k+ \tag{31}$$

$$\frac{1}{\Gamma(\alpha)}\int_y^x (g(x)-g(t))^{\alpha-1} g'(t)\left(D_{y+;g}^{\alpha} f\right)(t)\,dt, \quad \text{for any } x\geq y.$$

By Theorem 11 we get:

$$f(x)=f(y)+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}(g(x)-g(y))^k+ \tag{32}$$

$$\frac{1}{\Gamma(\alpha)}\int_x^y (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{y-;g}^{\alpha} f\right)(t)\,dt, \quad \text{for any } x\leq y.$$

By (31), (32) we notice that

$$\int_a^b f(x)\,dx=\int_a^y f(x)\,dx+\int_y^b f(x)\,dx= \tag{33}$$

$$\int_a^y f(y)\,dx+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}\int_a^y (g(x)-g(y))^k\,dx+$$

$$\frac{1}{\Gamma(\alpha)}\int_a^y\left(\int_x^y (g(t)-g(x))^{\alpha-1} g'(t)\left(D_{y-;g}^{\alpha} f\right)(t)\,dt\right)dx+$$

$$\int_y^b f(y)\,dx+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!}\int_y^b (g(x)-g(y))^k\,dx+$$

$$\frac{1}{\Gamma(\alpha)}\int_y^b\left(\int_y^x (g(x)-g(t))^{\alpha-1} g'(t)\left(D_{y+;g}^{\alpha} f\right)(t)\,dt\right)dx.$$

Hence it holds

$$\begin{gathered}
\frac{1}{b-a}\int_a^b f(x)\,dx = f(y)+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx+ \\
\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_x^y |g(x)-g(t)|^{\alpha-1} g'(t)\left(D^{\alpha}_{y-;g}f\right)(t)\,dt\right)dx+\right. \\
\left.\int_y^b\left(\int_y^x |g(x)-g(t)|^{\alpha-1} g'(t)\left(D^{\alpha}_{y+;g}f\right)(t)\,dt\right)dx\right].
\end{gathered} \tag{34}$$

Therefore we obtain

$$\begin{gathered}
f(y)=\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx- \\
\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_x^y |g(x)-g(t)|^{\alpha-1} g'(t)\left(D^{\alpha}_{y-;g}f\right)(t)\,dt\right)dx+\right. \\
\left.\int_y^b\left(\int_y^x |g(x)-g(t)|^{\alpha-1} g'(t)\left(D^{\alpha}_{y+;g}f\right)(t)\,dt\right)dx\right].
\end{gathered} \tag{35}$$

Hence the remainder

$$\begin{gathered}
R_1(y):=-\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_x^y |g(x)-g(t)|^{\alpha-1} g'(t)\left(D^{\alpha}_{y-;g}f\right)(t)\,dt\right)dx\right. \\
\left.+\int_y^b\left(\int_y^x |g(x)-g(t)|^{\alpha-1} g'(t)\left(D^{\alpha}_{y+;g}f\right)(t)\,dt\right)dx\right]= \\
-\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^b \chi_{[a,y)}(x)\left(\int_x^y |g(x)-g(t)|^{\alpha-1} g'(t)\left(D^{\alpha}_{y-;g}f\right)(t)\,dt\right)dx\right. \\
\left.+\int_a^b \chi_{[y,b]}(x)\left(\int_y^x |g(x)-g(t)|^{\alpha-1} g'(t)\left(D^{\alpha}_{y+;g}f\right)(t)\,dt\right)dx\right].
\end{gathered} \tag{36}$$

The theorem is proved. ■

Next we present a left fractional Opial type inequality:

Theorem 13 *All as in Theorem 10. Additionally assume that* $\alpha\geq 1$, $g\in C^1([a,b])$, *and* $\left(f\circ g^{-1}\right)^{(k)}(g(a))=0$, *for* $k=0,1,...,n-1$. *Let* $p,q>1$: $\frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\int_a^x \|f(w)\|\left\|\left(D^{\alpha}_{a+;g}f\right)(w)\right\| g'(w)\,dw\leq\frac{1}{\Gamma(\alpha)\,2^{\frac{1}{q}}}\cdot \tag{37}$$

$$\left(\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x (g'(w))^q\left\|\left(D^{\alpha}_{a+;g}f\right)(w)\right\|^q dw\right)^{\frac{2}{q}},$$

$\forall\ x\in[a,b]$.

Proof. By Theorem 10, we have that

$$f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (g(x)-g(t))^{\alpha-1} g'(t)\left(D_{a+;g}^{\alpha}f\right)(t)\,dt, \quad \forall\, x\in[a,b]. \tag{38}$$

Then, by Hölder's inequality we obtain,

$$\|f(x)\|\leq\frac{1}{\Gamma(\alpha)}\left(\int_a^x (g(x)-g(t))^{p(\alpha-1)}dt\right)^{\frac{1}{p}}\left(\int_a^x (g'(t))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(t)\right\|^q dt\right)^{\frac{1}{q}}. \tag{39}$$

Call

$$z(x):=\int_a^x (g'(t))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(t)\right\|^q dt, \tag{40}$$

$z(a)=0$.

Thus

$$z'(x)=(g'(x))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\|^q\geq 0, \tag{41}$$

and

$$(z'(x))^{\frac{1}{q}}=g'(x)\left\|\left(D_{a+;g}^{\alpha}f\right)(x)\right\|\geq 0,\ \ \forall\, x\in[a,b]. \tag{42}$$

Consequently, we get

$$\|f(w)\|\,g'(w)\left\|\left(D_{a+;g}^{\alpha}f\right)(w)\right\|\leq \tag{43}$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)^{\frac{1}{p}}(z(w)\,z'(w))^{\frac{1}{q}}, \quad \forall\, w\in[a,b].$$

Then

$$\int_a^x \|f(w)\|\left\|\left(D_{a+;g}^{\alpha}f\right)(w)\right\|g'(w)\,dw\leq \tag{44}$$

$$\frac{1}{\Gamma(\alpha)}\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)^{\frac{1}{p}}(z(w)\,z'(w))^{\frac{1}{q}}dw\leq$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x z(w)\,z'(w)\,dw\right)^{\frac{1}{q}}= \tag{45}$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)dw\right)^{\frac{1}{p}}\left(\frac{z^2(x)}{2}\right)^{\frac{1}{q}}=$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_a^x\left(\int_a^w (g(w)-g(t))^{p(\alpha-1)}dt\right)dw\right)^{\frac{1}{p}}\cdot$$

$$\left(\int_a^x (g'(t))^q\left\|\left(D_{a+;g}^{\alpha}f\right)(t)\right\|^q dt\right)^{\frac{2}{q}}\cdot 2^{-\frac{1}{q}}. \tag{46}$$

The theorem is proved. ■

We also give a right fractional Opial type inequality:

Theorem 14 *All as in Theorem 11. Additionally assume that* $\alpha \geq 1$, $g \in C^1([a,b])$, *and* $\left(f \circ g^{-1}\right)^{(k)}(g(b)) = 0$, $k = 0, 1, ..., n-1$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_x^b \|f(w)\| \left\|\left(D_{b-;g}^{\alpha} f\right)(w)\right\| g'(w)\, dw \leq \frac{1}{2^{\frac{1}{q}} \Gamma(\alpha)} \cdot \tag{47}$$

$$\left(\int_x^b \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)} dt\right) dw\right)^{\frac{1}{p}} \left(\int_x^b (g'(w))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(w)\right\|^q dw\right)^{\frac{2}{q}},$$

all $a \leq x \leq b$.

Proof. By Theorem 11, we have that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt, \quad \text{all } a \leq x \leq b. \tag{48}$$

Then, by Hölder's inequality we obtain,

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (g(t) - g(x))^{p(\alpha-1)} dt\right)^{\frac{1}{p}} \left(\int_x^b (g'(t))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(t)\right\|^q dt\right)^{\frac{1}{q}}. \tag{49}$$

Call

$$z(x) := \int_x^b (g'(t))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(t)\right\|^q dt, \tag{50}$$

$z(b) = 0$.

Hence

$$z'(x) = -(g'(x))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(x)\right\|^q \leq 0, \tag{51}$$

and

$$-z'(x) = (g'(x))^q \left\|\left(D_{b-;g}^{\alpha} f\right)(x)\right\|^q \geq 0, \tag{52}$$

and

$$(-z'(x))^{\frac{1}{q}} = g'(x) \left\|\left(D_{b-;g}^{\alpha} f\right)(x)\right\| \geq 0, \quad \forall\, x \in [a,b]. \tag{53}$$

Consequently, we get

$$\|f(w)\| g'(w) \left\|\left(D_{b-;g}^{\alpha} f\right)(w)\right\| \leq$$

$$\frac{1}{\Gamma(\alpha)} \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)} dt\right)^{\frac{1}{p}} (z(w)(-z'(w)))^{\frac{1}{q}}, \quad \forall\, w \in [a,b]. \tag{54}$$

Then

$$\int_x^b \|f(w)\| \left\|\left(D_{b-;g}^{\alpha} f\right)(w)\right\| g'(w)\, dw \leq \tag{55}$$

$$\frac{1}{\Gamma(\alpha)} \int_x^b \left(\int_w^b (g(t) - g(w))^{p(\alpha-1)} dt\right)^{\frac{1}{p}} (-z(w) z'(w))^{\frac{1}{q}} dw \leq$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_x^b\left(\int_w^b (g(t)-g(w))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}}\left(-\int_x^b z(w)\,z'(w)\,dw\right)^{\frac{1}{q}} = \tag{56}$$

$$\frac{1}{\Gamma(\alpha)}\left(\int_x^b\left(\int_w^b (g(t)-g(w))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}}\left(\frac{z^2(x)}{2}\right)^{\frac{1}{q}} =$$

$$\frac{1}{2^{\frac{1}{q}}\Gamma(\alpha)}\left(\int_x^b\left(\int_w^b (g(t)-g(w))^{p(\alpha-1)}\,dt\right)dw\right)^{\frac{1}{p}}\cdot$$

$$\left(\int_x^b (g'(t))^q\left\|\left(D_{b-;g}^{\alpha}f\right)(t)\right\|^q dt\right)^{\frac{2}{q}}. \tag{57}$$

The theorem is proved. ■

Two extreme fractional Opial type inequalities follow (case $p=1$, $q=\infty$).

Theorem 15 *All as in Theorem 10. Assume that* $\left(f\circ g^{-1}\right)^{(k)}(g(a))=0$, $k=0,1,...,n-1$. *Then*

$$\int_a^x \|f(w)\|\left\|D_{a+;g}^{\alpha}f(w)\right\|dw \leq \frac{\left\|D_{a+;g}^{\alpha}f\right\|_{\infty}^2}{\Gamma(\alpha+1)}\left(\int_a^x (g(w)-g(a))^{\alpha}\,dw\right), \tag{58}$$

all $a\leq x\leq b$.

Proof. For any $w\in[a,b]$, we have that

$$f(x)=\frac{1}{\Gamma(\alpha)}\int_a^w (g(w)-g(t))^{\alpha-1}g'(t)\left(D_{a+;g}^{\alpha}f\right)(t)\,dt, \tag{59}$$

and

$$\|f(x)\| \leq \frac{1}{\Gamma(\alpha)}\left(\int_a^w (g(w)-g(t))^{\alpha-1}g'(t)\,dt\right)\left\|D_{a+;g}^{\alpha}f\right\|_{\infty}$$
$$=\frac{\left\|D_{a+;g}^{\alpha}f\right\|_{\infty}}{\Gamma(\alpha+1)}(g(w)-g(a))^{\alpha}. \tag{60}$$

Hence we obtain

$$\|f(w)\|\left\|D_{a+;g}^{\alpha}f(w)\right\| \leq \frac{\left\|D_{a+;g}^{\alpha}f\right\|_{\infty}^2}{\Gamma(\alpha+1)}(g(w)-g(a))^{\alpha}. \tag{61}$$

Integrating (61) over $[a,x]$ we derive (58). ■

Theorem 16 *All as in Theorem 11. Assume that* $\left(f\circ g^{-1}\right)^{(k)}(g(b))=0$, $k=0,1,...,n-1$. *Then*

$$\int_x^b \|f(w)\|\left\|D_{b-;g}^{\alpha}f(w)\right\|dw \leq \frac{\left\|D_{b-;g}^{\alpha}f\right\|_{\infty}^2}{\Gamma(\alpha+1)}\left(\int_x^b (g(b)-g(w))^{\alpha}\,dw\right), \tag{62}$$

all $a\leq x\leq b$.

Proof. For any $w \in [a,b]$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_w^b (g(t) - g(w))^{\alpha-1} g'(t) \left(D_{b-;g}^{\alpha} f\right)(t)\, dt, \tag{63}$$

and

$$\begin{aligned} \|f(x)\| &\leq \frac{1}{\Gamma(\alpha)} \left(\int_w^b (g(t) - g(w))^{\alpha-1} g'(t)\, dt \right) \left\| D_{b-;g}^{\alpha} f \right\|_{\infty} \\ &= \frac{\left\| D_{b-;g}^{\alpha} f \right\|_{\infty}}{\Gamma(\alpha+1)} (g(b) - g(w))^{\alpha}. \end{aligned} \tag{64}$$

Hence we obtain

$$\|f(w)\| \left\| D_{b-;g}^{\alpha} f(w) \right\| \leq \frac{\left\| D_{b-;g}^{\alpha} f \right\|_{\infty}^2}{\Gamma(\alpha+1)} (g(b) - g(w))^{\alpha}. \tag{65}$$

Integrating (65) over $[x,b]$ we derive (62). ■

Next we present three fractional Ostrowski type inequalities:

Theorem 17 *All as in Theorem 10. Then*

$$\left\| f(y) - \frac{1}{b-a} \int_a^b f(x)\, dx + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)} (g(y))}{k!\,(b-a)} \int_a^b (g(x) - g(y))^k\, dx \right\| \leq \tag{66}$$

$$\frac{1}{\Gamma(\alpha+1)(b-a)} \cdot$$

$$\left[(g(y) - g(a))^{\alpha} (y-a) \left\| D_{y-;g}^{\alpha} f \right\|_{\infty} + (g(b) - g(y))^{\alpha} (b-y) \left\| D_{y+;g}^{\alpha} f \right\|_{\infty} \right],$$

$\forall\, y \in [a,b]$.

Proof. Define

$$\begin{gathered} \left(D_{y+;g}^{\alpha} f\right)(t) = 0, \text{ for } t < y, \\ \text{and} \\ \left(D_{y-;g}^{\alpha} f\right)(t) = 0, \text{ for } t > y. \end{gathered} \tag{67}$$

Notice for $0 < \alpha \notin \mathbb{N}$ by Remark 7 we have

$$\left(D_{a+;g}^{\alpha} f\right)(a) = 0. \tag{68}$$

Similarly it holds $(0 < \alpha \notin \mathbb{N})$ by Remark 7 that

$$\left(D_{b-;g}^{\alpha} f\right)(b) = 0. \tag{69}$$

Thus

$$\left(D_{y+;g}^{\alpha} f\right)(y) = 0, \quad \left(D_{y-;g}^{\alpha} f\right)(y) = 0, \tag{70}$$

$0 < \alpha \notin \mathbb{N}$, any $y \in [a,b]$.

We observe that

$$\|R_1(y)\| \overset{(28)}{\leq} \frac{1}{\Gamma(\alpha)(b-a)}\left[\left(\int_a^y\left(\int_x^y (g(t)-g(x))^{\alpha-1} g'(t)\,dt\right)dx\right)\left\|D_{y-;g}^{\alpha}f\right\|_\infty \right. \tag{71}$$

$$\left. +\left(\int_y^b\left(\int_y^x (g(x)-g(t))^{\alpha-1} g'(t)\,dt\right)dx\right)\left\|D_{y+;g}^{\alpha}f\right\|_\infty\right] =$$

$$\frac{1}{\Gamma(\alpha)(b-a)}\left[\left(\int_a^y \frac{(g(y)-g(x))^{\alpha}}{\alpha}dx\right)\left\|D_{y-;g}^{\alpha}f\right\|_\infty\right.$$

$$\left.+\left(\int_y^b \frac{(g(x)-g(y))^{\alpha}}{\alpha}dx\right)\left\|D_{y+;g}^{\alpha}f\right\|_\infty\right] \leq$$

$$\frac{1}{\Gamma(\alpha+1)(b-a)}\left[(g(y)-g(a))^{\alpha}(y-a)\left\|D_{y-;g}^{\alpha}f\right\|_\infty +\right.$$

$$\left.(g(b)-g(y))^{\alpha}(b-y)\left\|D_{y+;g}^{\alpha}f\right\|_\infty\right]. \tag{72}$$

We have proved that

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha+1)(b-a)}\left[(g(y)-g(a))^{\alpha}(y-a)\left\|D_{y-;g}^{\alpha}f\right\|_\infty\right. \tag{73}$$

$$\left.+(g(b)-g(y))^{\alpha}(b-y)\left\|D_{y+;g}^{\alpha}f\right\|_\infty\right],$$

any $y \in [a,b]$.

We have established the theorem. ■

Theorem 18 *All as in Theorem 10. Here we take* $\alpha \geq 1$. *Then*

$$\left\|f(y)-\frac{1}{b-a}\int_a^b f(x)\,dx+\sum_{k=1}^{n-1}\frac{\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx\right\| \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)}\left[\left\|\left(D_{y-;g}^{\alpha}f\right)\circ g^{-1}\right\|_{1,[g(a),g(y)]}(y-a)(g(y)-g(a))^{\alpha-1}\right.$$

$$\left.+\left\|\left(D_{y+;g}^{\alpha}f\right)\circ g^{-1}\right\|_{1,[g(y),g(b)]}(b-y)(g(b)-g(y))^{\alpha-1}\right], \tag{74}$$

$\forall\ y \in [a,b]$.

Proof. We can rewrite

$$R_1(y) = -\frac{1}{\Gamma(\alpha)(b-a)}\left[\int_a^y\left(\int_{g(x)}^{g(y)}(z-g(x))^{\alpha-1}\left(\left(D_{y-;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz\right)dx\right. \tag{75}$$

$$\left.+\int_y^b\left(\int_{g(y)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{y+;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz\right)dx\right].$$

We assumed $\alpha \geq 1$, then

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)} \cdot$$

$$\left[\int_a^y \left(\int_{g(x)}^{g(y)} (z-g(x))^{\alpha-1} \left\|\left(\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\| dz\right) dx \right. \tag{76}$$

$$\left. + \int_y^b \left(\int_{g(y)}^{g(x)} (g(x)-z)^{\alpha-1} \left\|\left(\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\| dz\right) dx\right] \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[\left(\int_a^y \left(\int_{g(x)}^{g(y)} \left\|\left(\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\| dz\right) dx\right) (g(y)-g(a))^{\alpha-1}\right.$$

$$\left. + \left(\int_y^b \left(\int_{g(y)}^{g(x)} \left\|\left(\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right)(z)\right\| dz\right) dx\right) (g(b)-g(y))^{\alpha-1}\right] \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[\left\|\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(a),g(y)]} (y-a)(g(y)-g(a))^{\alpha-1}\right. \tag{77}$$

$$\left. + \left\|\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(y),g(b)]} (b-y)(g(b)-g(y))^{\alpha-1}\right].$$

So when $\alpha \geq 1$, we obtained

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)} \left[\left\|\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(a),g(y)]} (y-a)(g(y)-g(a))^{\alpha-1}\right. \tag{78}$$

$$\left. + \left\|\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(y),g(b)]} (b-y)(g(b)-g(y))^{\alpha-1}\right].$$

Clearly here g^{-1} is continuous, thus $\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1} \in C([g(a), g(y)], X)$, and $\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1} \in C([g(y), g(b)], X)$. Therefore

$$\left\|\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(a),g(y)]}, \quad \left\|\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right\|_{1,[g(y),g(b)]} < \infty. \tag{79}$$

The proof of the theorem now is complete. ■

Theorem 19 *All as in Theorem 10. Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$. *Then*

$$\left\| f(y) - \frac{1}{b-a} \int_a^b f(x)\,dx + \sum_{k=1}^{n-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)} \int_a^b (g(x)-g(y))^k\,dx \right\| \tag{80}$$

$$\leq \frac{1}{\Gamma(\alpha)(b-a)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot$$

$$\left[(g(y)-g(a))^{\alpha-1+\frac{1}{p}} (y-a) \left\|\left(D_{y-;g}^{\alpha} f\right) \circ g^{-1}\right\|_{q,[g(a),g(y)]}\right.$$

$$\left. + (g(b)-g(y))^{\alpha-1+\frac{1}{p}} (b-y) \left\|\left(D_{y+;g}^{\alpha} f\right) \circ g^{-1}\right\|_{q,[g(y),g(b)]}\right],$$

$\forall\, y \in [a,b]$.

Proof. Here we use (75).

We get that

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)} \left[\int_a^y \left(\int_{g(x)}^{g(y)} (z - g(x))^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \cdot \right.$$

$$\left(\int_{g(x)}^{g(y)} \left\| \left(\left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|^q dz \right)^{\frac{1}{q}} dx + \int_y^b \left(\int_{g(y)}^{g(x)} (g(x) - z)^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \cdot$$

$$\left. \left(\int_{g(y)}^{g(x)} \left\| \left(\left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right)(z) \right\|^q dz \right)^{\frac{1}{q}} dx \right] \leq$$

$$\frac{1}{\Gamma(\alpha)(b-a)} \left[\left(\int_a^y \frac{(g(y) - g(x))^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} dx \right) \left\| \left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(a),g(y)]} \right.$$

$$\left. + \left(\int_a^y \frac{(g(x) - g(y))^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} dx \right) \left\| \left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(y),g(b)]} \right]. \quad (81)$$

(here it is $\alpha - 1 + \frac{1}{p} > 0$)

Hence it holds

$$\|R_1(y)\| \leq \frac{1}{\Gamma(\alpha)(b-a)(p(\alpha-1)+1)^{\frac{1}{p}}} \cdot \quad (82)$$

$$\left[(g(y) - g(a))^{\alpha-1+\frac{1}{p}} (y-a) \left\| \left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(a),g(y)]} + \right.$$

$$\left. (g(b) - g(y))^{\alpha-1+\frac{1}{p}} (b-y) \left\| \left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(y),g(b)]} \right].$$

Clearly here

$$\left\| \left(D_{y-;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(a),g(y)]}, \quad \left\| \left(D_{y+;g}^{\alpha} f \right) \circ g^{-1} \right\|_{q,[g(y),g(b)]} < \infty.$$

We have proved the theorem. ■

Next we give some fractional Grüss type inequalities:

Theorem 20 *Let f, h as in Theorem 10. Here $R_1(y)$ will be renamed as $R_1(f, y)$, so we can consider $R_1(h, y)$. Then*

1)

$$\Delta_n(f,h) := \frac{1}{b-a} \int_a^b f(x) h(x) dx - \frac{\left(\int_a^b f(x) dx \right) \left(\int_a^b h(x) dx \right)}{(b-a)^2} +$$

$$\frac{1}{2(b-a)^2} \sum_{k=1}^{n-1} \frac{1}{k!} \left[\int_a^b \left(\int_a^b \left(h(y) \left(f \circ g^{-1} \right)^{(k)} (g(y)) + \right. \right. \right.$$

$$f(y)\left(h\circ g^{-1}\right)^{(k)}(g(y))\Big)(g(x)-g(y))^k\,dx\Big)\,dy\Big]=$$

$$\frac{1}{2(b-a)}\left[\int_a^b\left(h(y)R_1(f,y)+f(y)R_1(h,y)\right)dy\right]=:K_n(f,h), \tag{83}$$

2) it holds

$$\|\Delta_n(f,h)\|\leq\frac{(g(b)-g(a))^{\alpha}}{2\Gamma(\alpha+1)}\left[\|h\|_\infty\left(\sup_{y\in[a,b]}\left(\left\|D^{\alpha}_{y-;g}f\right\|_\infty+\left\|D^{\alpha}_{y+;g}f\right\|_\infty\right)\right)\right.$$

$$\left.+\|f\|_\infty\left(\sup_{y\in[a,b]}\left(\left\|D^{\alpha}_{y-;g}h\right\|_\infty+\left\|D^{\alpha}_{y+;g}h\right\|_\infty\right)\right)\right], \tag{84}$$

3) if $\alpha\geq 1$, we get:

$$\|\Delta_n(f,h)\|\leq\frac{1}{2\Gamma(\alpha)(b-a)}(g(b)-g(a))^{\alpha-1}\cdot \tag{85}$$

$$\left\{\|h\|_1\left(\sup_{y\in[a,b]}\left(\left\|\left(D^{\alpha}_{y-;g}f\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}+\left\|\left(D^{\alpha}_{y+;g}f\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}\right)\right)+\right.$$

$$\left.\|f\|_1\left(\sup_{y\in[a,b]}\left(\left\|\left(D^{\alpha}_{y-;g}h\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}+\left\|\left(D^{\alpha}_{y+;g}h\right)\circ g^{-1}\right\|_{1,[g(a),g(b)]}\right)\right)\right\},$$

4) if $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$, we get:

$$\|\Delta_n(f,h)\|\leq\frac{(g(b)-g(a))^{\alpha-1+\frac{1}{p}}}{2\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\cdot \tag{86}$$

$$\left\{\|h\|_\infty\left(\sup_{y\in[a,b]}\left(\left\|\left(D^{\alpha}_{y-;g}f\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}+\left\|\left(D^{\alpha}_{y+;g}f\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}\right)\right)+\right.$$

$$\left.\|f\|_\infty\left(\sup_{y\in[a,b]}\left(\left\|\left(D^{\alpha}_{y-;g}h\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}+\left\|\left(D^{\alpha}_{y+;g}h\right)\circ g^{-1}\right\|_{q,[g(a),g(b)]}\right)\right)\right\}.$$

All right hand sides of (84)-(86) are finite.

Proof. By Theorem 10 we have

$$h(y)f(y)=\frac{h(y)}{b-a}\int_a^b f(x)\,dx-$$

$$\sum_{k=1}^{n-1}\frac{h(y)\left(f\circ g^{-1}\right)^{(k)}(g(y))}{k!(b-a)}\int_a^b(g(x)-g(y))^k\,dx+h(y)R_1(f,y), \tag{87}$$

and

$$f(y)\,h(y) = \frac{f(y)}{b-a}\int_a^b h(x)\,dx -$$

$$\sum_{k=1}^{n-1} \frac{f(y)\left(h\circ g^{-1}\right)^{(k)}(g(y))}{k!\,(b-a)}\int_a^b (g(x)-g(y))^k\,dx + f(y)\,R_1(h,y), \tag{88}$$

$\forall\ y \in [a,b]$.

Then integrating (87) we find

$$\int_a^b h(y)\,f(y)\,dy = \frac{\left(\int_a^b h(y)\,dy\right)}{b-a}\left(\int_a^b f(x)\,dx\right) -$$

$$\sum_{k=1}^{n-1}\frac{1}{k!\,(b-a)}\int_a^b\int_a^b h(y)\left(f\circ g^{-1}\right)^{(k)}(g(y))\,(g(x)-g(y))^k\,dxdy$$

$$+\int_a^b h(y)\,R_1(f,y)\,dy, \tag{89}$$

and integrating (88) we obtain

$$\int_a^b f(y)\,h(y)\,dy = \frac{\left(\int_a^b f(y)\,dy\right)\left(\int_a^b h(x)\,dx\right)}{b-a} -$$

$$\sum_{k=1}^{n-1}\frac{1}{k!\,(b-a)}\int_a^b\int_a^b f(y)\left(h\circ g^{-1}\right)^{(k)}(g(y))\,(g(x)-g(y))^k\,dxdy$$

$$+\int_a^b f(y)\,R_1(h,y)\,dy. \tag{90}$$

Adding the last two equalities (89) and (90), we get:

$$2\int_a^b f(x)\,h(x)\,dx = \frac{2\left(\int_a^b f(x)\,dx\right)\left(\int_a^b h(x)\,dx\right)}{b-a} -$$

$$\sum_{k=1}^{n-1}\frac{1}{k!\,(b-a)}\left[\left[\int_a^b\int_a^b h(y)\left(f\circ g^{-1}\right)^{(k)}(g(y)) + f(y)\left(h\circ g^{-1}\right)^{(k)}(g(y))\right]\cdot\right.$$

$$\left.(g(x)-g(y))^k\,dxdy\right] + \int_a^b \left(h(y)\,R_1(f,y) + f(y)\,R_1(h,y)\right)dy. \tag{91}$$

Divide the last (91) by $2(b-a)$ to obtain (83).

Then, we upper bound $K_n(f,h)$ using Theorems 17, 18, 19, to obtain (84)-(86), respectively.

We use also that a norm is a continuous function. The theorem is proved. ■

We make

Remark 21 *(in support of the proof of Theorem 20) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $\lceil \alpha \rceil = n$. We have*

$$\left(D_{y+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{y}^{x} (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{92}$$

$\forall\ x \in [y, b]$, and

$$\left(D_{y-;g}^{\alpha} f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{x}^{y} (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{93}$$

$\forall\ x \in [a, y]$, both are Bochner type integrals.

By change of variables for Bochner integrals, see [6], Lemma B. 4.10 and [7], p. 158, we get:

$$\left(D_{y+;g}^{\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_{g(y)}^{g(x)} (g(x) - z)^{n-\alpha-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz =$$

$$\left(D_{g(y)+}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(x)), \quad \forall\ x \in [y, b], \tag{94}$$

and

$$\left(D_{y-;g}^{\alpha} f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{g(x)}^{g(y)} (z - g(x))^{n-\alpha-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz =$$

$$\left(D_{g(y)-}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(x)), \quad \forall\ x \in [a, y]. \tag{95}$$

Here $D_{g(y)+}^{\alpha}, D_{g(y)-}^{\alpha}$ are the left and right X-valued Caputo fractional differentiation operators.

Fix $w : w \geq x_0 \geq y_0$; $w, x_0, y_0 \in [a, b]$, then $g(w) \geq g(x_0) \geq g(y_0)$. Hence

$$\left\| \left(D_{y_0+;g}^{\alpha} f\right)(w) - \left(D_{x_0+;g}^{\alpha} f\right)(w) \right\| =$$

$$\left\| \left(D_{g(y_0)+}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(w)) - \left(D_{g(x_0)+}^{\alpha} \left(f \circ g^{-1}\right)\right)(g(w)) \right\| =$$

$$\frac{1}{\Gamma(n-\alpha)} \left\| \int_{g(y_0)}^{g(x_0)} (g(w) - z)^{n-\alpha-1} \left(f \circ g^{-1}\right)^{(n)} (z)\, dz \right\| \leq \tag{96}$$

$$\frac{1}{\Gamma(n-\alpha)} \int_{g(y_0)}^{g(x_0)} (g(w) - z)^{n-\alpha-1} \left\| \left(f \circ g^{-1}\right)^{(n)} (z) \right\| dz \leq$$

$$\frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \right\|_{\infty, [g(a), g(b)]}}{\Gamma(n-\alpha)} \int_{g(y_0)}^{g(x_0)} (g(w) - z)^{n-\alpha-1} dz =$$

$$\frac{\left\| \left(f \circ g^{-1}\right)^{(n)} \right\|_{\infty, [g(a), g(b)]}}{\Gamma(n-\alpha+1)} \left[(g(y_0) - z)^{n-\alpha} - (g(x_0) - z)^{n-\alpha} \right] \to 0,$$

as $y_0 \to x_0$, *then* $g(y_0) \to g(x_0)$, *proving continuity of* $\left(D^{\alpha}_{g(y)+}\left(f \circ g^{-1}\right)\right)(g(x))$ *with respect to* $g(y)$, *and of course continuity of* $\left(D^{\alpha}_{y+;g}f\right)(x)$ *in* $y \in [a,b]$.

Similarly, it is proved that $\left(D^{\alpha}_{y-;g}f\right)(x)$ *is continuous in* $y \in [a,b]$, *the proof is omitted.*

Remark 22 *Some examples for* g *follow:*

$$\begin{aligned} g(x) &= e^{x}, \;\; x \in [a,b] \subset \mathbb{R}, \\ g(x) &= \sin x, \\ g(x) &= \tan x, \\ \text{where } x &\in \left[-\frac{\pi}{2}+\varepsilon, \frac{\pi}{2}-\varepsilon\right], \text{ where } \varepsilon > 0 \text{ small.} \end{aligned}$$

Indeed, the above examples of g *are strictly increasing and contimuous functions.*

One can apply all of our results here for the above specific choices of g. *We choose to omit this job.*

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On solutions of semilinear second-order impulsive functional differential equations

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Abstract

This paper deals with the regularity for solutions of second-order semilinear impulsive differential equations contained the nonlinear convolution with cosine families, and obtain a variation of constant formula for solutions of the given equations.

*Keywords:*semilinear second-order equations, regularity for solutions, cosine family, sine family

AMS Classification Primary 35F25; Secondary 35K55

1 Introduction

In this paper we are concerned with the regularity of the following second-order semilinear impulsive differential system

$$\begin{cases} w^{''}(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad w^{'}(0) = y_0, \\ \Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m \end{cases} \tag{1.1}$$

in a Banach space X. Here k belongs to $L^2(0,T)$ and $g : [0,T] \times D(A) \to X$ is a nonlinear mapping such that $w \mapsto g(t,w)$ satisfies Lipschitz continuous. In (1.1),

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This work was supported by a Research Grant of Pukyong National University(2021Year).

the principal operator A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$. The impulsive condition

$$\Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1, 2, ..., m$$

is combination of traditional evolution systems whose duration is negligible in comparison with duration of the process, such as biology, medicine, bioengineering etc.

In recent years the theory of impulsive differential systems has been emerging as an important area of investigation in applied sciences. The reason is that it is richer than the corresponding theory of classical differential equations and it is more adequate to represent some processes arising in various disciplines. The theory of impulsive systems provides a general framework for mathematical modeling of many real world phenomena(see [1, 2] and references therein). The theory of impulsive differential equations has seen considerable development. Impulsive differential systems have been studied in [3, 4, 5, 6], second-order impulsive integrodifferential systems in [7, 8], and Stochastic differential systems with impulsive conditions in [9, 10, 11].

In this paper, we allow implicit arguments about L^2-regularity results for semilinear hyperbolic equations with impulsive condition. These consequences are obtained by showing that results of the linear cases [12, 13] and semilinear case [14] on the L^2-regularity remain valid under the above formulation of (1.1). Earlier works prove existence of solution by using Azera Ascoli theorem. But we propose a different approach from that of earlier works to study mild, strong and classical solutions of Cauchy problems by using the properties of the linear equation in the hereditary part.

This paper is organized as follows. In Section 2, we give some definition, notation and the regularity for the corresponding linear equations. In Section 3, by using properties of the strict solutions of linear equations in dealt in Section 2, we will obtain the L^2-regularity of solutions of (1.1), and a variation of constant formula of solutions of (1.1). Finally, we also give an example to illustrate the applications of the abstract results..

2 Preliminaries

In this section, we give some definitions, notations, hypotheses and Lemmas. Let X be a Banach space with norm denoted by $||\cdot||$.

Definition 2.1. *[15] A one parameter family $C(t)$, $t \in \mathbb{R}$, of bounded linear operators in X is called a strongly continuous cosine family if* *c(1)* $C(s+t)+C(s-t)=2C(s)C(t), \quad \text{for all } s, \ t\in\mathbb{R}$,

c(2) $C(0)=I$,

c(3) $C(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x\in X$.

If $C(t),\ t\in\mathbb{R}$ is a strongly continuous cosine family in X , then $S(t),\ t\in\mathbb{R}$ is the one parameter family of operators in X defined by

$$S(t)x=\int_0^t C(s)xds,\ \ x\in X,\ \ t\in\mathbb{R}. \tag{2.1}$$

The infinitesimal generator of a strongly continuous cosine family $C(t),\ t\in\mathbb{R}$ is the operator $A: X\to X$ defined by

$$Ax=\frac{d^2}{dt^2}C(0)x.$$

We endow with the domain $D(A)=\{x\in X: C(t)x$ is a twice continuously differentiable function of $t\}$ with norm

$$||x||_{D(A)}=||x||+\sup\{||\frac{d}{dt}C(t)x||: t\in\mathbb{R}\}+||Ax||.$$

We shall also make use of the set

$$E=\{x\in X: C(t)x \text{ is a once continuously differentiable function of } t\}$$

with norm

$$||x||_E=||x||+\sup\{||\frac{d}{dt}C(t)x||: t\in\mathbb{R}\}.$$

It is not difficult to show that $D(A)$ and E with given norms are Banach spaces.

The following Lemma is from Proposition 2.1 and Proposition 2.2 of [1].

Lemma 2.1. *Let* $C(t)(t\in\mathbb{R})$ *be a strongly continuous cosine family in* X*. The following are true :*

c(4) $C(t)=C(-t)$ *for all* $t\in\mathbb{R}$,

c(5) $C(s), S(s), C(t)$ *and* $S(t)$ *commute for all* $s,t\in\mathbb{R}$,

c(6) $S(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x\in X$, *c(7)* *there exist constants* $K \geq 1$ *and* $\omega \geq 0$ *such that*

$$||C(t)|| \leq Ke^{\omega|t|} \text{ for all } t \in \mathbb{R},$$
$$||S(t_1) - S(t_2)|| \leq K\Big|\int_{t_2}^{t_1} e^{\omega|s|}ds\Big| \text{ for all } t_1, t_2 \in \mathbb{R},$$

c(8) *if* $x \in E$, *then* $S(t)x \in D(A)$ *and*

$$\frac{d}{dt}C(t)x = AS(t)x = S(t)Ax = \frac{d^2}{dt^2}S(t)x,$$

c(9) *if* $x \in D(A)$, *then* $C(t)x \in D(A)$ *and*

$$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax,$$

c(10) *if* $x \in X$ *and* $r, s \in \mathbb{R}$, *then*

$$\int_r^s S(\tau)xd\tau \in D(A) \quad \text{and} \quad A(\int_r^s S(\tau)xd\tau) = C(s)x - C(r)x,$$

c(11) $C(s+t) + C(s-t) = 2C(s)C(t)$ *for all* $s, t \in \mathbb{R}$,

c(12) $S(s+t) = S(s)C(t) + S(t)C(s)$ *for all* $s, t \in \mathbb{R}$,

c(13) $C(s+t) = C(t)C(s) - S(t)S(s)$ *for all* $s, t \in \mathbb{R}$,

c(14) $C(s+t) - C(t-s) = 2AS(t)S(s)$ *for all* $s, t \in \mathbb{R}$.

The following Lemma is from Proposition 2.4 of [15].

Lemma 2.2. *Let* $C(t)(t \in \mathbb{R})$ *be a strongly continuous cosine family in* X *with infinitesimal generator* A. *If* $f : \mathbb{R} \to X$ *is continuously differentiable,* $x_0 \in D(A)$, $y_0 \in E$, *and*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R},$$

then $w(t) \in D(A)$ *for* $t \in \mathbb{R}$, w *is twice continuously differentiable, and* w *satisfies*

$$w^{''}(t) = Aw(t) + f(t), \ \ t \in R, \ \ w(0) = x_0, \ \ w^{'}(0) = y_0. \tag{2.2}$$

Conversely, if $f : \mathbb{R} \to X$ *is continuous,* $w(t) : \mathbb{R} \to X$ *is twice continuously differentiable,* $w(t) \in D(A)$ *for* $t \in \mathbb{R}$, *and* w *satisfies* (2.2), *then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R}.$$

Proposition 2.1. *Let* $f : \mathbb{R} \to X$ *is continuously differentiable,* $x_0 \in D(A)$*,* $y_0 \in E$*. Then*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)ds, \ \ t \in \mathbb{R}$$

is a solution of (2.2) belonging to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$*. Moreover, we have that there exists a positive constant* C_1 *such that for any* $T > 0$*,*

$$||w||_{L^2(0,T;D(A))} \le C_1(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}). \tag{2.3}$$

3 Nonlinear equations

This section is to investigate the regularity of solutions of a second-order nonlinear impulsive differential system

$$\begin{cases} w^{''}(t) = Aw(t) + \int_0^t k(t-s)g(s,w(s))ds + f(t), \quad 0 < t \le T, \\ w(0) = x_0, \quad w^{'}(0) = y_0, \\ \Delta w(t_k) = I_k^1(w(t_k)), \quad \Delta w'(t_k) = I_k^2(w'(t_k^+)), \quad k = 1,2,...,m \end{cases} \tag{3.1}$$

in a Banach space X.

Assumption (G) Let $g : [0,T] \times D(A) \to X$ be a nonlinear mapping such that $t \mapsto g(t,w)$ is measurable and

(g1) $||g(t,w_1) - g(t,w_2)||_{D(A)} \le L||w_1 - w_2||$,

for a positive constant L.

Assumption (I) Let $I_k^1 : D(A) \to X$, $I_k^2 : E \to X$ be continuous and there exist positive constants $L(I_k^1)$, $L(I_k^2)$ such that

(i1) $||I_k^1(w_1) - I_k^1(w_2)|| \le L(I_k^1)||w_1 - w_2||_{D(A)}$, for each $w_1, w_2 \in D(A)$
$||I_k^1(w)|| \le L(I_k^1)$, for $w \in D(A)$

(i2) $||I_k^2(w_1') - I_k^2(w_2')|| \le L(I_k^2)||w_1' - w_2'||_E$, for each $w_1', w_2' \in E$
$||I_k^2(w')|| \le L(I_k^2)||$, for $w' \in E$.

For $w \in L^2(0,T : D(A))$, we set

$$F(t,w) = \int_0^t k(t-s)g(s,w(s))ds$$

where k belongs to $L^2(0,T)$. Then we will seek a mild solution of (3.1), that is, a solution of the integral equation

$$\begin{aligned} w(t) =& C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s,w)+f(s)\}ds \\ &+ \sum_{0<t_k<t} C(t-t_k)I_k^1(w(t_k)) + \sum_{0<t_k<t} S(t-t_k)I_k^2(w'(t_k^+)), \;\; t\in\mathbb{R}. \end{aligned} \tag{3.2}$$

Remark 3.1. *If* $g : [0,T]\times X \to X$ *is a nonlinear mapping satisfying*

$$||g(t,w_1)-g(t,w_2)|| \le L||w_1-w_2||$$

for a positive constant L, then our results can be obtained immediately.

Lemma 3.1. *Let* $w \in L^2(0,T;D(A))$, $T>0$. *Then* $F(\cdot,w)\in L^2(0,T;X)$ *and*

$$||F(\cdot,w)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)}\sqrt{T}||w||_{L^2(0,T;D(A))}.$$

Moreover if $w_1,\ w_2 \in L^2(0,T;D(A))$, *then*

$$||F(\cdot,w_1)-F(\cdot,w_2)||_{L^2(0,T;X)} \le L||k||_{L^2(0,T)}\sqrt{T}||w_1-w_2||_{L^2(0,T;D(A))}.$$

Lemma 3.2. *If* $k\in W^{1,2}(0,T)$, $T>0$, *then*

$$\begin{aligned} A\int_0^t S(t-s)F(s,w)ds =& -F(t,w) \\ &+ \int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\, ds \\ &+ \int_0^t (C(t-s)-I)k(0)g(s,w(s))ds. \end{aligned} \tag{3.3}$$

Theorem 3.1. *Suppose that the Assumptions (G) and Assumption (I) are satisfied. If* $f:\mathbb{R}\to X$ *is continuously differentiable,* $x_0\in D(A)$, $y_0\in E$, *and* $k\in W^{1,2}(0,T)$, $T>0$, *then there exists a time* $T\ge T_0>0$ *such that the functional differential equation (3.1) admits a unique solution* w *in* $L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)$.

Proof. Let us fix $T_0 > 0$ so that

$$\begin{aligned} C_2 \equiv& \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)} \qquad (3.4)\\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||||k||_{W^{1,2}(0,T_0)}\\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||||k(0)||\\ &+\{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t}L(I_k^1)Ke^{wT_0}\\ &+\{2w^{-1}K(e^{wT_0}-1)+1\}\sum_{0<t_k<t}L(I_k^2)<1 \end{aligned}$$

where $K,\ L,\ L(I_k^1)$ and $L(I_k^2)$ are constants in c(7), (g1) and Assumption (I) respectively. Invoking Proposition 2.1, for any $v \in L^2(0,T_0;D(A))$ we obtain the equation

$$\begin{cases} w''(t) = Aw(t) + F(t,v) + f(t), \quad 0 < t \le T_0,\\ w(0) = x_0, \quad w'(0) = y_0 \\ \Delta w(t_k) = I_k^1(v(t_k)), \quad \Delta w'(t_k) = I_k^2(v'(t_k^+)), \quad k = 1,2,...,m \end{cases} \qquad (3.5)$$

has a unique solution $w \in L^2(0,T_0;D(A)) \cap W^{1,2}(0,T_0;E)$. Let $w_1,\ w_2$ be the solutions of (3.5) with v replaced by $v_1,\ v_2 \in L^2(0,T_0;D(A))$, respectively. Put

$$\begin{aligned} J(w)(t) = C(t)x_0 + S(t)y_0 &+ \int_0^t S(t-s)\{F(s,v)+f(s)\}ds\\ &+\sum_{0<t_k<t}C(t-t_k)I_k^1(v(t_k)) + \sum_{0<t_k<t}S(t-t_k)I_k^2(v'(t_k^+)). \end{aligned}$$

Then

$$\begin{aligned} J(w_1)(t)-J(w_2)(t) =& \int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds\\ &+\sum_{0<t_k<t}C(t-t_k)\{I_k^1(v_1(t_k))-I_k^1(v_2(t_k))\}\\ &+\sum_{0<t_k<t}S(t-t_k)\{I_k^2(v_1'(t_k^+))-I_k^1(v_2'(t_k^+))\},\\ =& I_1+I_2+I_3. \end{aligned}$$

So, from Lemmas 3.1, 3.2, it follows that for $0 \le t \le T_0$,

$$\begin{aligned} &||\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||\\ &\quad\le \omega^{-1}KLT_0(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}, \end{aligned}$$

$$\begin{aligned}
&||\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||\\
&\le ||AS(t)\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||\\
&= ||S(t)A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||,
\end{aligned}$$

and

$$\begin{aligned}
&||A\int_0^t S(t-s)\{F(s,v_1)-F(s,v_2)\}ds||\\
&\le ||\int_0^t (C(t-s)-I)\int_0^s \frac{d}{ds}k(s-\tau)(g(\tau,v_1(\tau))-g(\tau,v_2(\tau)))d\tau\, ds||\\
&\quad + ||\int_0^t (C(t-s)-I)k(0)(g(s,v_1(s))-g(s,v_2(s)))ds||\\
&\le tL||Ke^{\omega t}+1||||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&\quad + \sqrt{t}L||Ke^{\omega t}+1||||k(0)||||v_1-v_2||_{L^2(0,T_0;D(A))}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
||I_1||_{L^2(0,T_0;D(A))} &\le \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \qquad (3.6)\\
&+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||\,||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))}\\
&+ \{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||\,||k(0)||\,||v_1-v_2||_{L^2(0,T_0;D(A))}.
\end{aligned}$$

By Assumption (i1), we obtain

$$||\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1(t_k^-))-I_k^1(v_2(t_k^-))\}|| \le \sum_{0<t_k<T_0} Ke^{wT_0}L(I_k^1)||v_1-v_2||_{D(A)},$$

$$\begin{aligned}
&||\frac{d}{dt}C(t)\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||\\
&\le ||AS(t)\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||\\
&= ||S(t)A\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1)-I_k^1(v_2)\}||,
\end{aligned}$$

and

$$\begin{aligned}
||A\sum_{0<t_k<t} C(t-t_k)\{I_k^1(v_1(t_k^-)) - I_k^1(v_2(t_k^-))\}|| &= ||\sum_{0<t_k<t} C(t-t_k)A\{I_k^1(v_1) - I_k^1(v_2)\}|| \\
&\leq \sum_{0<t_k<t} Ke^{wt}||I_k^1(v_1) - I_k^1(v_2)||_{D(A)} \\
&\leq \sum_{0<t_k<t} Ke^{wt}L(I_k^1)||v_1 - v_2||_{D(A)}.
\end{aligned}$$

Therefore, we have

$$||I_2||_{L^2(0,T_0;D(A))} \leq \{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t} L(I_k^1)Ke^{wT_0}||v_1 - v_2||_{L^2(0,T_0;D(A))}. \tag{3.7}$$

We also obtain from Assumption (i2),

$$||\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1'(t_k^+)) - I_k^1(v_2'(t_k^+))\}|| \leq \sum_{0<t_k<T_0} Kw^{-1}(e^{wT_0}-1)L(I_k^2)||v_1 - v_2||_{D(A)},$$

$$\begin{aligned}
&||\frac{d}{dt}C(t)\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1') - I_k^1(v_2')\}|| \\
&\leq ||AS(t)\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1') - I_k^1(v_2')\}|| \\
&= ||S(t)A\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1') - I_k^1(v_2')\}||,
\end{aligned}$$

and

$$\begin{aligned}
||A\sum_{0<t_k<t} S(t-t_k)\{I_k^2(v_1'(t_k^+)) - I_k^1(v_2'(t_k^+))\}|| &= ||\sum_{0<t_k<t} \frac{d}{dt}C(t)\{I_k^2(v_1') - I_k^1(v_2')\}|| \\
&\leq \sum_{0<t_k<t} ||I_k^2(v_1') - I_k^1(v_2')||_E \\
&\leq \sum_{0<t_k<t} L(I_k^2)||v_1' - v_2'||_E.
\end{aligned}$$

Therefore, we have

$$||I_3||_{L^2(0,T_0;D(A))} \le \{w^{-1}K(e^{wT_0}-1)+2\} \sum_{0<t_k<t} L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))}. \tag{3.8}$$

Thus, from (3.6),(3.7), and (3.8), we conclude that

$$\begin{aligned} &||J(w_1)-J(w_2)||_{L^2(0,T_0;D(A))} \\ &\le \omega^{-1}KLT_0^{3/2}(e^{\omega T_0}-1)||k||_{L^2(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}L||k||_{L^2(0,T_0)}\sqrt{T_0}||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0^{3/2}/\sqrt{3}L||Ke^{\omega T_0}+1||\,||k||_{W^{1,2}(0,T_0)}||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{\omega^{-1}K(e^{\omega T_0}-1)+1\}T_0/\sqrt{2}L||Ke^{\omega T_0}+1||\,||k(0)||\,||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{w^{-1}K(e^{wT_0}-1)+2\}\sum_{0<t_k<t} L(I_k^1)Ke^{wT_0}||v_1-v_2||_{L^2(0,T_0;D(A))} \\ &+\{2w^{-1}K(e^{wT_0}-1)+1\}\sum_{0<t_k<t} L(I_k^2)||v_1-v_2||_{W^{1,2}(0,T_0;D(A))}. \end{aligned} \tag{3.9}$$

Moreover, it is easily seen that

$$||J(w_1)-J(w_2)||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \le C_2||v_1-v_2||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}.$$

So by virtue of the condition (3.4) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $[0,T_0]$. ☐

Theorem 3.2. *Suppose that the Assumptions (G) and (I) are satisfied. If $f:\mathbb{R}\to X$ is continuously differentiable, $x_0\in D(A), y_0\in E$, and $k\in W^{1,2}(0,T)$, $T>0$, then the solution w of (3.1) exists and is unique in $L^2(0,T;D(A))\cap W^{1,2}(0,T;E)$, and there exists a constant C_3 depending on T such that*

$$||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)} \le C_3(1+||x_0||_{D(A)}+||y_0||_E+||f||_{W^{1,2}(0,T;X)}). \tag{3.10}$$

Proof. Let $w(\cdot)$ be the solution of (3.1) in the interval $[0,T_0]$ where T_0 is a constant in (3.4) and $v(\cdot)$ be the solution of the following equation

$$\begin{aligned} v^{''}(t) &= Av(t)+f(t),\ \ 0<t, \\ v(0) &= x_0,\ \ v^{'}(0)=y_0. \end{aligned}$$

Then

$$(w-v)(t)=\int_0^t S(t-s)F(s,w)ds+\sum_{0<t_k<t}C(t-t_k)I_k^1(w(t_k))+\sum_{0<t_k<t}S(t-t_k)I_k^2(w'(t_k^+)),$$

and in view of (3.9)

$$||w-v||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}\le C_2||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}, \tag{3.11}$$

that is, combining (3.11) with Proposition 2.1 we have

$$\begin{aligned}||w||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}&\le\frac{1}{1-C_2}||v||_{L^2(0,T_0;D(A))\cap W^{1,2}(0,T_0;E)}\\&\le\frac{C_1}{1-C_2}(1+||x_0||_{D(A)}+||y_0||_E+||f||_{W^{1,2}(0,T_0;X)}).\end{aligned} \tag{3.12}$$

Now from

$$\begin{aligned}&A\int_0^{T_0}S(T_0-s)\{F(s,w)+f(s)\}ds\\&=C(T_0)f(0)-f(T_0)+\int_0^{T_0}(C(T_0-s)-I)f^{'}(s)ds\\&-F(T_0,w)+\int_0^{T_0}(C(T_0-s)-I)\int_0^s\frac{d}{ds}k(s-\tau)g(\tau,w(\tau))d\tau\,ds\\&+\int_0^{T_0}(C(T_0-s)-I)k(0)g(s,w(s))ds,\end{aligned}$$

$$||A\sum_{0<t_k<t}C(t-t_k)I_k^1(w_1)||\le Kw^{-1}(e^{wT_0-1})Ke^{wT_0}\sum_{0<t_k<t}L(I_k^1)||w(t_k)||_{D(A)},$$

$$||\sum_{0<t_k<t}S(t-t_k)I_k^2(v_1')||\le\sum_{0<t_k<t}L(I_k^2)||w'(t_k^+)||_E,$$

and since

$$\frac{d}{dt}C(t)\int_0^t S(t-s)\{F(s,w)+f(s)\}ds=S(t)A\int_0^t S(t-s)\{F(s,w)+f(s)\}ds,$$

$$\frac{d}{dt}C(t)\sum_{0<t_k<t}C(t-t_k)I_k^1(w)\le S(t)A\sum_{0<t_k<t}C(t-t_k)I_k^1(w).$$

$$\frac{d}{dt}C(t)\sum_{0<t_k<t}S(t-t_k)I_k^2(w')\le S(t)A\sum_{0<t_k<t}S(t-t_k)I_k^2(w').$$

We have

$$\begin{aligned}||w(T_0)||_{D(A)}&=||C(T_0)x_0+S(T_0)y_0+\int_0^{T_0}S(T_0-s)\{F(s,w)+f(s)\}ds\\&\qquad+\sum_{0<t_k<t}C(t-t_k)I_k^1(w)+\sum_{0<t_k<t}S(t-t_k)I_k^2(w')||_{D(A)}\end{aligned}$$

$$\begin{aligned}&\le(\omega^{-1}K(e^{\omega T_0}-1)+1)\{Ke^{\omega T_0}||x_0||_{D(A)}+||y_0||_E+T_0L||k||_{L^2(0,T_0)}||w||_{L^2(0,T_0;D(A))}\\&\quad+||Ke^{\omega T_0}f(0)||+||f(0)||+||K(e^{\omega T_0}+1)\sqrt{T_0}||f||_{W^{1,2}(0,T;X)}\\&\quad+tL||Ke^{\omega t}+1||\,||k||_{W^{1,2}(0,T_0)}||w||_{L^2(0,T_0;D(A))}\\&\quad+\sqrt{t}L||Ke^{\omega t}+1||\,||k(0)||||w||_{L^2(0,T_0;D(A))}\}\\&\quad+\{2+Kw^{-1}(e^{wT_0}-1)\}\sum_{0<t_k<t}Ke^{wT_0}L(I_k^1)\\&\quad+\{1+2Kw^{-1}(e^{wT_0}-1)\}\sum_{0<t_k<t}L(I_k^2).\end{aligned}$$

Hence, from (3.12), there exists a positive constant $C>0$ such that

$$||w(T_0)||_{D(A)}\le C(1+||x_0||_{D(A)}+||y_0||_E+||f||_{W^{1,2}(0,T_0;X)}).$$

Since the condition (3.4) is independent of initial values, the solution of (3.1) can be extended to the interval $[0,nT_0]$ for every natural number n. An analogous estimate to (3.12) holds for the solution in $[0,nT_0]$, and hence for the initial value $(w(nT_0),w^{'}(nT_0))\in D(A)\times E$ in the interval $[nT_0,(n+1)T_0]$. □

Example. We consider the following partial differential equation

$$\begin{cases}w^{''}(t,x)=Aw(t,x)+F(t,w)+f(t),\quad 0<t,\quad 0<x<\pi,\\w(t,0)=w(t,\pi)=0,\quad t\in\mathbb{R}\\w(0,x)=x_0(x),\quad w^{'}(0,x)=y_0(x),\quad 0<x<\pi\\\Delta w(t_k,x)=I_k^1(w(t_k))=(\gamma_k||w''(t_k,x)||+t_k),\quad 1\le k\le m,\\\Delta w'(t_k,x)=I_k^2(w'(t_k))=\delta_k||w'(t_k,x)||,\end{cases}\tag{E}$$

where constants γ_k and $\delta_k(k=1,\cdots,m)$ are small.

Let $X = L^2([0,\pi];\mathbb{R})$, and let $e_n(x) = \sqrt{\frac{2}{\pi}}\sin nx$. Then $\{e_n : n = 1, \cdots\}$ is an orthonormal base for X. Let $A : X \to X$ be defined by

$$Aw(x) = w''(x),$$

where $D(A) = \{w \in X : w,\ w^{'}$ are absolutely continuous, $w^{''} \in X,\ w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2(w, e_n)e_n, \quad w \in D(A),$$

and A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt(w, e_n)e_n, \quad w \in X.$$

The associated sine family is given by

$$S(t)w = \sum_{n=1}^{\infty} \frac{\sin nt}{n}(w, e_n)e_n, \quad w \in X.$$

Let $g_1(t, x, w, p), \quad p \in \mathbb{R}^m$, be assumed that there is a continuous $\rho(t, \delta) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and a real constant $1 \le \delta$ such that

(f1) $g_1(t, x, 0, 0) = 0$,

(f2) $|g_1(t, x, w, p) - g_1(t, x, w, q)| \le \rho(t, |w|)|p - q|$,

(f3) $|g_1(t, x, w_1, p) - g_1(t, x, w_2, p)| \le \rho(t, |w_1| + |w_2|)|w_1 - w_2|$.

Let

$$g(t, w)x = g_1(t, x, w, Dw, D^2w).$$

Then noting that

$$\begin{aligned}||g(t, w_1) - g(t, w_2)||_{0,2}^2 \le 2\int_\Omega |g_1(t, x, w_1, p) - g_1(t, x, w_2, q)|^2 dx \\ + 2\int_\Omega |g_1(t, x, w_1, q) - g_1(t, x, w_2, q)|^2 dx\end{aligned}$$

where $p = (Dw_1, D^2w_1)$ and $q = (Dw_2, D^2w_2)$, it follows from (f1), (f2) and (f3) that

$$||g(t, w_1) - g(t, w_2)||_{0,2}^2 \le L(||w_1||_{D(A)}, ||w_2||_{D(A)})||w_1 - w_2||_{D(A)}$$

where $L(||w_1||_{D(A)}, ||w_2||_{D(A)})$ is a constant depending on $||w_1||_{D(A)}$ and $||w_2||_{D(A)}$. We set

$$F(t,w) = \int_0^t k(t-s)g(s,w(s))ds$$

where k belongs to $L^2(0,T)$. Then, from the results in section 3, the solution w of (E) exists and is unique in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$, and there exists a constant C_3 depending on T such that

$$||w||_{L^2(0,T;D(A))} \leq C_3(1 + ||x_0||_{D(A)} + ||y_0||_E + ||f||_{W^{1,2}(0,T;X)}).$$

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Caputo ψ-fractional Ostrowski and Grüss inequalities for several functions

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Abstract

Very general univariate mixed Caputo ψ-fractional Ostrowski and Grüss type inequalities for several functions are presented. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$. We give also applications.

2010 AMS Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: Ostrowski and Grüss inequalities, right and left Caputo ψ-fractional derivatives.

1 Introduction

In 1938, A. Ostrowski [5] proved the following important inequality.

Theorem 1 *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - f(x)\right| \leq \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\,\|f'\|_\infty, \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

In this article we are greatly motived and inspired by Theorem 1, see also [2]. Here we present various ψ-fractional Ostrowski and Grüss type inequalities for several functions and we give interesting applications.

2 Background

Here we follow [1].

Let $\alpha > 0$, $[a,b] \subset \mathbb{R}$, $f : [a,b] \to \mathbb{R}$ which is integrable and $\psi \in C^1([a,b])$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in [a,b]$. Consider $n = \lceil \alpha \rceil$, the ceiling of α. The left and right fractional integrals are defined, respectively, as follows:

$$I_{a+}^{\alpha,\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t)\, dt, \tag{2}$$

and

$$I_{b-}^{\alpha,\psi} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{\alpha-1} f(t)\, dt, \tag{3}$$

for any $x \in [a,b]$, where Γ is the gamma function.

The following semigroup property is valid for fractional integrals: if $\alpha, \beta > 0$, then

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f(x) = I_{a+}^{\alpha+\beta,\psi} f(x), \text{ and } I_{b-}^{\alpha,\psi} I_{b-}^{\beta,\psi} f(x) = I_{b-}^{\alpha+\beta,\psi} f(x).$$

We mention

Definition 2 *([1]) Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f, \psi \in C^n([a,b])$ with ψ being increasing and $\psi'(x) \neq 0$, for all $x \in [a,b]$. The left ψ-Caputo fractional derivative of f of order α is given by*

$$^{C}D_{a+}^{\alpha,\psi} f(x) := I_{a+}^{n-\alpha,\psi} \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x), \tag{4}$$

and the right ψ-Caputo fractional derivative of f is given by

$$^{C}D_{b-}^{\alpha,\psi} f(x) := I_{b-}^{n-\alpha,\psi} \left(-\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x). \tag{5}$$

To simplify notation, we will use the symbol

$$f_{\psi}^{[n]}(x) := \left(\frac{1}{\psi'(x)} \frac{d}{dx} \right)^n f(x), \tag{6}$$

with $f_{\psi}^{[0]}(x) = f(x)$.

By the definition, whrn $\alpha = m \in \mathbb{N}$, we have

$$\begin{array}{l} ^{C}D_{a+}^{\alpha,\psi} f(x) = f_{\psi}^{[m]}(x) \\ \text{and} \\ ^{C}D_{b-}^{\alpha,\psi} f(x) = (-1)^m f_{\psi}^{[m]}(x). \end{array} \tag{7}$$

If $\alpha \notin \mathbb{N}$, we have

$$^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{n-\alpha-1} f_{\psi}^{[n]}(t)\, dt, \tag{8}$$

and

$$ {}^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \psi'(t) (\psi(t) - \psi(x))^{n-\alpha-1} f_{\psi}^{[n]}(t)\, dt, \tag{9} $$

$\forall\ x \in [a,b]$.

In particular, when $\alpha \in (0,1)$, we have

$$ \begin{array}{l} {}^{C}D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (\psi(x) - \psi(t))^{-\alpha} f'(t)\, dt, \\ \text{and} \\ {}^{C}D_{b-}^{\alpha,\psi} f(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (\psi(t) - \psi(x))^{-\alpha} f'(t)\, dt \end{array} \tag{10} $$

$\forall\ x \in [a,b]$.

Clearly the above is a generalization of left and right Caputo fractional derivatives.

For more see [1].

Still we need from [1] the following left and right fractional Taylor's formulae:

Theorem 3 *([1]) Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$, $[a,b] \subset \mathbb{R}$ and $f, \psi \in C^n([a,b])$ with ψ being increasing and $\psi'(x) \neq 0$, for all $x \in [a,b]$. Then, the left fractional Taylor formula follows,*

$$ f(x) = \sum_{k=0}^{n-1} \frac{f_{\psi}^{[k]}(a)}{k!} (\psi(x) - \psi(a))^k + I_{a+}^{\alpha,\psi}\ {}^{C}D_{a+}^{\alpha,\psi} f(x), \tag{11} $$

and the right fractional Taylor formula follows,

$$ f(x) = \sum_{k=0}^{n-1} (-1)^k \frac{f_{\psi}^{[k]}(b)}{k!} (\psi(b) - \psi(x))^k + I_{b-}^{\alpha,\psi}\ {}^{C}D_{b-}^{\alpha,\psi} f(x), \tag{12} $$

$\forall\ x \in [a,b]$.

In particular, given $\alpha \in (0,1)$, we have

$$ \begin{array}{l} f(x) = f(a) + I_{a+}^{\alpha,\psi}\ {}^{C}D_{a+}^{\alpha,\psi} f(x), \\ \textit{and} \\ f(x) = f(b) + I_{b-}^{\alpha,\psi}\ {}^{C}D_{b-}^{\alpha,\psi} f(x), \end{array} \tag{13} $$

$\forall\ x \in [a,b]$.

Remark 4 *For convenience we can rewrite (11)-(13) as follows:*

$$ f(x) = \sum_{k=0}^{n-1} \frac{f_{\psi}^{[k]}(a)}{k!} (\psi(x) - \psi(a))^k + \tag{14} $$

$$ \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1}\ {}^{C}D_{a+}^{\alpha,\psi} f(t)\, dt, $$

and

$$f(x)=\sum_{k=0}^{n-1}\frac{(-1)^k f_{\psi}^{[k]}(b)}{k!}(\psi(b)-\psi(x))^k+ \tag{15}$$

$$\frac{1}{\Gamma(\alpha)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{b-}^{\alpha,\psi}f(t)\,dt,$$

$\forall\ x\in[a,b]$.

When $\alpha\in(0,1)$*, we get:*

$$\begin{array}{l} f(x)=f(a)+\frac{1}{\Gamma(\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{a+}^{\alpha,\psi}f(t)\,dt, \\ \text{and} \\ f(x)=f(b)+\frac{1}{\Gamma(\alpha)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{b-}^{\alpha,\psi}f(t)\,dt, \end{array} \tag{16}$$

$\forall\ x\in[a,b]$.

Again from [1] we have the following:

Consider the norms $\|\cdot\|_{\infty}:C([a,b])\to\mathbb{R}$ and $\|\cdot\|_{C_{\psi}^{[n]}}:C^n([a,b])\to\mathbb{R}$, where $\|f\|_{C_{\psi}^{[n]}}:=\sum_{k=0}^{n}\left\|f_{\psi}^{[k]}\right\|_{\infty}$.

We have

Theorem 5 *([1]) The* ψ*-Caputo fractional derivatives are bounded operators. For all* $\alpha>0$ *(*$n=\lceil\alpha\rceil$*)*

$$\left\|{}^{C}D_{a+}^{\alpha,\psi}\right\|_{\infty}\le K\|f\|_{C_{\psi}^{[n]}} \tag{17}$$

and

$$\left\|{}^{C}D_{b-}^{\alpha,\psi}\right\|_{\infty}\le K\|f\|_{C_{\psi}^{[n]}}, \tag{18}$$

where

$$K=\frac{(\psi(b)-\psi(a))^{n-\alpha}}{\Gamma(n+1-\alpha)}>0. \tag{19}$$

3 Main Results

At first we present the following ψ-fractional Ostrowski type inequalities for several functions:

Theorem 6 *Let* $\alpha>0$*,* $n\in\mathbb{N}:n=\lceil\alpha\rceil$*,* $[a,b]\subset\mathbb{R}$ *and* $f_i,\psi\in C^n([a,b])$*,* $i=1,...,r$*; with* ψ *being increasing and* $\psi'(x)\neq0$*, for all* $x\in[a,b]$*. Let* $x_0\in[a,b]$ *and assume that* $f_{i\psi}^{[k]}(x_0)=0$*, for* $k=1,...,n-1$*;* $i=1,...,r$*. Set*

$$\Phi(f_1,...,f_r)(x_0):=r\int_a^b\left(\prod_{k=1}^{r}f_k(x)\right)d\psi(x)- \tag{20}$$

$$\sum_{i=1}^{r}\left[f_i\left(x_0\right)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)d\psi\left(x\right)\right].$$

Then

$$\left|\Phi\left(f_1,...,f_r\right)\left(x_0\right)\right|\leq\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,\left[a,x_0\right]}I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right. \tag{21}$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,\left[x_0,b\right]}I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right].$$

If $0<\alpha\leq1$, *then (21) is valid without any initial conditions.*

Proof. By Theorem 3 we have that

$$f_i\left(x\right)-f_i\left(x_0\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1}\ {}^{C}D_{x_0+}^{\alpha,\psi}f_i\left(t\right)dt, \tag{22}$$

$\forall\ x\in\left[x_0,b\right]$,

and

$$f_i\left(x\right)-f_i\left(x_0\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{x_0}\psi'\left(t\right)\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha-1}\ {}^{C}D_{x_0-}^{\alpha,\psi}f_i\left(t\right)dt, \tag{23}$$

$\forall\ x\in\left[a,x_0\right]$;

for all $i=1,...,r$.

Multiplying (22) and (23) by $\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)$ we obtain, respectively,

$$\prod_{k=1}^{r}f_k\left(x\right)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x_0\right)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)}{\Gamma\left(\alpha\right)}\int_{x_0}^{x}\psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1}\ {}^{C}D_{x_0+}^{\alpha,\psi}f_i\left(t\right)dt, \tag{24}$$

$\forall\ x\in\left[x_0,b\right]$,

$$\prod_{k=1}^{r}f_k\left(x\right)-\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x_0\right)=$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{\Gamma(\alpha)} \int_{x}^{x_0} \psi'(t)\,(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{x_0-}^{\alpha,\psi} f_i(t)\,dt, \tag{25}$$

$\forall\ x \in [a, x_0]$;

for all $i = 1, ..., r$.

Adding (24) and (25), separately, we obtain

$$r\left(\prod_{k=1}^{r} f_k(x)\right) - \sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\int_{x_0}^{x} \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{x_0+}^{\alpha,\psi} f_i(t)\,dt\right], \tag{26}$$

$\forall\ x \in [x_0, b]$,

and

$$r\left(\prod_{k=1}^{r} f_k(x)\right) - \sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\int_{x}^{x_0} \psi'(t)\,(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{x_0-}^{\alpha,\psi} f_i(t)\,dt\right], \tag{27}$$

$\forall\ x \in [a, x_0]$.

Next we integrate (26) and (27) with respect to $\psi(x)$, $x \in [a, b]$. We have

$$r\int_{x_0}^{b}\left(\prod_{k=1}^{r} f_k(x)\right) d\psi(x) - \sum_{i=1}^{r}\left[f_i(x_0)\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) d\psi(x)\right] =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)[\int_{x_0}^{x} \psi'(t)\,(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{x_0+}^{\alpha,\psi} f_i(t)\,dt] d\psi(x)\right], \tag{28}$$

and

$$r\int_{a}^{x_0}\left(\prod_{k=1}^{r} f_k(x)\right) d\psi(x) - \sum_{i=1}^{r}\left[f_i(x_0)\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) d\psi(x)\right] =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)[\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{x_0-}^{\alpha,\psi}f_i(t)\,dt]d\psi(x)\right]. \tag{29}$$

Adding (28) and (29) we derive the identity:

$$\Phi(f_1,...,f_r)(x_0):=r\int_a^b\left(\prod_{k=1}^{r}f_k(x)\right)d\psi(x)-$$

$$\sum_{i=1}^{r}\left[f_i(x_0)\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)d\psi(x)\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ {}^{C}D_{x_0-}^{\alpha,\psi}f_i(t)\,dt\right)d\psi(x)\right]\right. \tag{30}$$

$$\left.+\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{x_0}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ {}^{C}D_{x_0+}^{\alpha,\psi}f_i(t)\,dt\right)d\psi(x)\right]\right].$$

Hence it holds

$$|\Phi(f_1,...,f_r)(x_0)|\leq$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)\left(\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\ \left|{}^{C}D_{x_0-}^{\alpha,\psi}f_i(t)\right|dt\right)d\psi(x)\right]\right.$$

$$\left.+\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)\left(\int_{x_0}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\ \left|{}^{C}D_{x_0+}^{\alpha,\psi}f_i(t)\right|dt\right)d\psi(x)\right]\right]=:(*). \tag{31}$$

We observe that

$$(*)\leq\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)(\psi(x_0)-\psi(x))^{\alpha}d\psi(x)\right]\right.$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)(\psi(x)-\psi(x_0))^{\alpha}d\psi(x)\right]\right]= \tag{32}$$

$$\sum_{i=1}^{r}\left[\left[\left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty,[a,x_0]} I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]+\right.$$

$$\left.\left[\left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty,[x_0,b]} I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]\right].$$

By Theorem 4.10, p. 98 of [3], we get that $I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j|\right)\in C([a,b])$ and so at any $x_0\in[a,b]$ is finite, $i=1,...,r$. Similarly, by Theorem 4.11, p. 101 of [3], we get that $I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j|\right)\in C([a,b])$ and so at any $x_0\in[a,b]$ is finite, $i=1,...,r$. Arguing similarly, we get that ${}^{C}D_{a+}^{\alpha,\psi}f_i$, ${}^{C}D_{b-}^{\alpha,\psi}f_i\in C([a,b])$, for all $i=1,...,r$.

The theorem is proved. ■

We continue with

Theorem 7 *All as in Theorem 6 with* $\alpha\geq 1$. *Then*

$$|\Phi(f_1,...,f_r)(x_0)|\leq\sum_{i=1}^{r}\left[\left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_1([a,x_0],\psi)} I_{a+}^{\alpha,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right.$$

$$\left.+\left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_1([x_0,b],\psi)} I_{b-}^{\alpha,\psi}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x_0)|\right)\right]. \tag{33}$$

Proof. From (31) we get

$$(*)\leq\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_1([a,x_0],\psi)}\right.\right.$$

$$\left.\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)(\psi(x_0)-\psi(x))^{\alpha-1}d\psi(x)\right)\right]+$$

$$\left.\left[\left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_1([x_0,b],\psi)}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}|f_j(x)|\right)(\psi(x)-\psi(x_0))^{\alpha-1}d\psi(x)\right)\right]\right] \tag{34}$$

$$= \sum_{i=1}^{r} \left[\left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_1([a,x_0],\psi)} I_{a+}^{\alpha,\psi} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x_0)| \right) + \right.$$

$$\left. \left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_1([x_0,b],\psi)} I_{b-}^{\alpha,\psi} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x_0)| \right) \right], \tag{35}$$

proving the theorem. ■

We continue with

Theorem 8 *All as in Theorem 6 with* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha \geq 1$. *Then*

$$|\Phi(f_1, ..., f_r)(x_0)| \leq \frac{\Gamma\left(\alpha + \frac{1}{p}\right)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r} \left[\left[\left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{L_q([a,x_0],\psi)} I_{a+}^{\alpha+\frac{1}{p},\psi} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x_0)| \right) \right] \right. \tag{36}$$

$$\left. + \left[\left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{L_q([x_0,b],\psi)} I_{b-}^{\alpha+\frac{1}{p},\psi} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x_0)| \right) \right] \right].$$

Proof. From (31) we obtain

$$(*) \leq \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\left[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x)| \right) \right. \right.$$

$$\left. \left(\int_x^{x_0} (\psi(t) - \psi(x))^{p(\alpha-1)} d\psi(t) \right)^{\frac{1}{p}} \left(\int_x^{x_0} \left| {}^{C}D_{x_0-}^{\alpha,\psi} f_i(t) \right|^q d\psi(t) \right)^{\frac{1}{q}} d\psi(x) \right] +$$

$$\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} |f_j(x)| \right) \left(\int_{x_0}^{x} (\psi(x) - \psi(t))^{p(\alpha-1)} d\psi(t) \right)^{\frac{1}{p}} \right. \tag{37}$$

$$\left. \left. \left(\int_{x_0}^{x} \left| {}^{C}D_{x_0+}^{\alpha,\psi} f_i(t) \right|^q d\psi(t) \right)^{\frac{1}{q}} d\psi(x) \right] \right] \leq \frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{L_q([a,x_0],\psi)}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)(\psi(x_0)-\psi(x))^{\alpha-1+\frac{1}{p}}d\psi(x)\right]\right.$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{L_q([x_0,b],\psi)}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x)|\right)(\psi(x)-\psi(x_0))^{\alpha-1+\frac{1}{p}}d\psi(x)\right]\right]= \tag{38}$$

$$\frac{\Gamma\left(\alpha+\frac{1}{p}\right)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{L_q([a,x_0],\psi)}I_{a+}^{\alpha+\frac{1}{p},\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right.$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{L_q([x_0,b],\psi)}I_{b-}^{\alpha+\frac{1}{p},\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}|f_j(x_0)|\right)\right]\right], \tag{39}$$

proving the theorem. ■

We mention as motivation for Grüss type inequalities the following:

Theorem 9 *(1882, Čebyšev [4]) Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a,b])$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)\right|$$

$$\leq\frac{1}{12}(b-a)^2\|f'\|_\infty\|g'\|_\infty. \tag{40}$$

The above integrals are assumed to exist.

Next follow ψ-Caputo fractional Grüss type inequalities for several functions.

Theorem 10 *Let $0<\alpha\leq 1$, $[a,b]\subset\mathbb{R}$ and $f_i,\psi\in C^1([a,b])$, $i=1,...,r\in\mathbb{N}-\{1\}$; with ψ being increasing and $\psi'(x)\neq 0$, for all $x\in[a,b]$. Assume that $\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}<\infty$, and $\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}<\infty$, $i=1,...,r$. Set*

$$\Delta^\psi(f_1,...,f_r):=r(\psi(b)-\psi(a))\left(\int_a^b\left(\prod_{k=1}^{r}f_k(x)\right)d\psi(x)\right)-$$

$$\sum_{i=1}^{r}\left[\left(\int_a^b f_i(x)\,d\psi(x)\right)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)d\psi(x)\right)\right]. \tag{41}$$

Then

$$\left|\Delta^{\psi}\left(f_1,...,f_r\right)\right| \leq \left(\psi\left(b\right)-\psi\left(a\right)\right)$$

$$\left\{\sum_{i=1}^{r}\left[\left[\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right.\right.$$

$$\left.\left.+\left[\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x_0\right)\right|\right)\right]\right]\right\}. \quad (42)$$

Proof. Here $0<\alpha\leq 1$, i.e. $n=1$. Then (30) is valid without any initial conditions. Clearly $\Phi\left(f_1,...,f_r\right)\in C\left([a,b]\right)$. Thus, by integrating (30) against ψ we obtain

$$\Delta^{\psi}\left(f_1,...,f_r\right)=\int_a^b\Phi\left(f_1,...,f_r\right)\left(x_0\right)d\psi\left(x_0\right)=$$

$$r\left(\psi\left(b\right)-\psi\left(a\right)\right)\left(\int_a^b\left(\prod_{k=1}^{r}f_k\left(x\right)\right)d\psi\left(x\right)\right)-$$

$$\sum_{i=1}^{r}\left[\left(\int_a^b f_i\left(x\right)d\psi\left(x\right)\right)\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)d\psi\left(x\right)\right)\right]=$$

$$\frac{1}{\Gamma\left(\alpha\right)}\int_a^b\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\right.\right.\right.$$

$$\left.\left(\int_x^{x_0}\psi'\left(t\right)\left(\psi\left(t\right)-\psi\left(x\right)\right)^{\alpha-1}\,{}^{C}D_{x_0-}^{\alpha,\psi}f_i\left(t\right)dt\right)d\psi\left(x\right)\right]+ \quad (43)$$

$$\left.\left.\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\left(\int_{x_0}^{x}\psi'\left(t\right)\left(\psi\left(x\right)-\psi\left(t\right)\right)^{\alpha-1}\,{}^{C}D_{x_0+}^{\alpha,\psi}f_i\left(t\right)dt\right)d\psi\left(x\right)\right]\right]\right\}d\psi\left(x_0\right).$$

Hence it holds

$$\left|\Delta^{\psi}\left(f_1,...,f_r\right)\right|\leq\frac{1}{\Gamma\left(\alpha\right)}\int_a^b\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left|f_j\left(x\right)\right|\right)\right.\right.\right.$$

$$\left(\int_x^{x_0}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\left|{}^C D_{x_0-}^{\alpha,\psi}f_i(t)\right|dt\right)d\psi(x)\Bigg]+\Bigg[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^r|f_j(x)|\right)$$

$$\left(\int_{x_0}^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\left|{}^C D_{x_0+}^{\alpha,\psi}f_i(t)\right|dt\right)d\psi(x)\Bigg]\Bigg]\Bigg\}d\psi(x_0)=:(**). \tag{44}$$

Using (21) we derive

$$(**)\le(\psi(b)-\psi(a))$$

$$\left\{\sum_{i=1}^r\left[\left[\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}\sup_{x_0\in[a,b]}I_{a+}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^r|f_j(x_0)|\right)\right]\right.\right.$$

$$\left.\left.+\left[\sup_{x_0\in[a,b]}\left\|{}^C D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}\sup_{x_0\in[a,b]}I_{b-}^{\alpha+1,\psi}\left(\prod_{\substack{j=1\\j\neq i}}^r|f_j(x_0)|\right)\right]\right]\right\}, \tag{45}$$

proving the theorem. ■

We make

Remark 11 *Let $\alpha>0$, $[a,b]\subset\mathbb{R}$, $f\in C([a,b])$ and $\psi\in C^1([a,b])$ an increasing function such that $\psi'(x)\neq 0$, for all $x\in[a,b]$. Let $x_0\in[a,b]$. We observe the following*

$$\left|I_{a+}^{\alpha,\psi}f(x_0)\right|\overset{(2)}{\le}\frac{1}{\Gamma(\alpha)}\int_a^{x_0}\psi'(t)(\psi(x_0)-\psi(t))^{\alpha-1}|f(t)|dt\le$$

$$\frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha)}\int_a^{x_0}(\psi(x_0)-\psi(t))^{\alpha-1}d\psi(t)=\frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(\psi(x_0)-\psi(a))^{\alpha}. \tag{46}$$

That is

$$\left|I_{a+}^{\alpha,\psi}f(x_0)\right|\le\frac{\|f\|_{\infty,[a,x_0]}}{\Gamma(\alpha+1)}(\psi(x_0)-\psi(a))^{\alpha}. \tag{47}$$

Similarly, we obtain

$$\left|I_{b-}^{\alpha,\psi}f(x_0)\right|\overset{(3)}{\le}\frac{1}{\Gamma(\alpha)}\int_{x_0}^b\psi'(t)(\psi(t)-\psi(x_0))^{\alpha-1}|f(t)|dt\le$$

$$\frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha)}\int_{x_0}^b(\psi(t)-\psi(x_0))^{\alpha-1}d\psi(t)=\frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(\psi(b)-\psi(x_0))^{\alpha}. \tag{48}$$

That is

$$\left|I_{b-}^{\alpha,\psi}f(x_0)\right|\le\frac{\|f\|_{\infty,[x_0,b]}}{\Gamma(\alpha+1)}(\psi(b)-\psi(x_0))^{\alpha}. \tag{49}$$

We make

Remark 12 *Let $\alpha \geq 1$, the rest as in Remark 11. We observe that*

$$\left| I_{a+}^{\alpha,\psi} f(x_0) \right| \overset{(2)}{\leq} \frac{1}{\Gamma(\alpha)} \int_a^{x_0} \psi'(t) (\psi(x_0) - \psi(t))^{\alpha-1} |f(t)| \, dt \leq$$

$$\frac{(\psi(x_0) - \psi(a))^{\alpha-1}}{\Gamma(\alpha)} \int_a^{x_0} |f(t)| \, d\psi(t) = \frac{(\psi(x_0) - \psi(a))^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{L_1([a,x_0],\psi)} . \tag{50}$$

That is

$$\left| I_{a+}^{\alpha,\psi} f(x_0) \right| \leq \frac{(\psi(x_0) - \psi(a))^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{L_1([a,x_0],\psi)} . \tag{51}$$

Similarly, we get

$$\left| I_{b-}^{\alpha,\psi} f(x_0) \right| \overset{(3)}{\leq} \frac{1}{\Gamma(\alpha)} \int_{x_0}^{b} \psi'(t) (\psi(t) - \psi(x_0))^{\alpha-1} |f(t)| \, dt \leq$$

$$\frac{(\psi(b) - \psi(x_0))^{\alpha-1}}{\Gamma(\alpha)} \int_{x_0}^{b} |f(t)| \, d\psi(t) = \frac{(\psi(b) - \psi(x_0))^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{L_1([x_0,b],\psi)} . \tag{52}$$

That is

$$\left| I_{b-}^{\alpha,\psi} f(x_0) \right| \leq \frac{(\psi(b) - \psi(x_0))^{\alpha-1}}{\Gamma(\alpha)} \|f\|_{L_1([x_0,b],\psi)} . \tag{53}$$

Next, we simplify our main theorems:

Proposition 13 *All as in Theorem 6. Then*

$$|\Phi(f_1, ..., f_r)(x_0)| \leq \frac{1}{\Gamma(\alpha+2)}$$

$$\sum_{i=1}^{r} \left[\left[\left\| {}^{C}D_{x_0-}^{\alpha,\psi} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{\infty,[a,x_0]} (\psi(x_0) - \psi(a))^{\alpha+1} \right] \right.$$

$$\left. + \left[\left\| {}^{C}D_{x_0+}^{\alpha,\psi} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} f_j \right\|_{\infty,[x_0,b]} (\psi(b) - \psi(x_0))^{\alpha+1} \right] \right] . \tag{54}$$

If $0 < \alpha \leq 1$, then (54) is valid without any initial conditions.

Proof. By (21), (47) and (49). ∎

Next comes

Proposition 14 *All as in Theorem 6 with $\alpha \geq 1$. Then*

$$|\Phi(f_1,...,f_r)(x_0)| \leq \frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{L_1([a,x_0],\psi)}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{L_1([a,x_0],\psi)}(\psi(x_0)-\psi(a))^{\alpha-1}\right]\right.$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{L_1([x_0,b],\psi)}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{L_1([x_0,b],\psi)}(\psi(b)-\psi(x_0))^{\alpha-1}\right]\right]. \quad (55)$$

Proof. By (33), (51) and (53). ■
Next follows

Proposition 15 *All as in Theorem 6 with $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha \geq 1$. Then*

$$|\Phi(f_1,...,f_r)(x_0)| \leq \frac{1}{\Gamma(\alpha)\left(\alpha+\frac{1}{p}\right)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left[\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{L_q([a,x_0],\psi)}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,x_0]}(\psi(x_0)-\psi(a))^{\alpha+\frac{1}{p}}\right]\right.$$

$$\left.+\left[\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{L_q([x_0,b],\psi)}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[x_0,b]}(\psi(b)-\psi(x_0))^{\alpha+\frac{1}{p}}\right]\right]. \quad (56)$$

Proof. By (36), (47) and (49). ■
We continue with

Proposition 16 *All as in Theorem 10. Then*

$$\left|\Delta^{\psi}(f_1,...,f_r)\right| \leq \frac{(\psi(b)-\psi(a))^{\alpha+2}}{\Gamma(\alpha+2)}$$

$$\left\{\sum_{i=1}^{r}\left[\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0-}^{\alpha,\psi}f_i\right\|_{\infty,[a,x_0]}+\sup_{x_0\in[a,b]}\left\|{}^{C}D_{x_0+}^{\alpha,\psi}f_i\right\|_{\infty,[x_0,b]}\right]\left\|\prod_{\substack{j=1\\j\neq i}}^{r}f_j\right\|_{\infty,[a,b]}\right\}. \quad (57)$$

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Proof. By (42), (47) and (49). ■

Next we make some applications of our main results.

We need

Remark 17 *We have that* ($r = 2$)

$$\Phi\left(f_1, f_2\right)\left(x_0\right) = 2\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right) - \tag{58}$$

$$f_1\left(x_0\right)\int_a^b f_2\left(x\right) d\psi\left(x\right) - f_2\left(x_0\right)\int_a^b f_1\left(x\right) d\psi\left(x\right),$$

and ($r = 3$)

$$\Phi\left(f_1, f_2, f_3\right)\left(x_0\right) = 3\int_a^b f_1\left(x\right) f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right) - f_1\left(x_0\right)\int_a^b f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right) \tag{59}$$

$$-f_2\left(x_0\right)\int_a^b f_1\left(x\right) f_3\left(x\right) d\psi\left(x\right) - f_3\left(x_0\right)\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right),$$

etc.

Furthermore we derive ($r = 2$)

$$\Delta^{\psi}\left(f_1, f_2\right) = 2\left[\left(\psi\left(b\right) - \psi\left(a\right)\right)\left(\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right)\right) - \right.$$

$$\left.\left(\int_a^b f_1\left(x\right) d\psi\left(x\right)\right)\left(\int_a^b f_2\left(x\right) d\psi\left(x\right)\right)\right], \tag{60}$$

and ($r = 3$)

$$\Delta^{\psi}\left(f_1, f_2, f_3\right) = 3\left(\psi\left(b\right) - \psi\left(a\right)\right)\left(\int_a^b f_1\left(x\right) f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_1\left(x\right) d\psi\left(x\right)\right)\left(\int_a^b f_2\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_2\left(x\right) d\psi\left(x\right)\right)\left(\int_a^b f_1\left(x\right) f_3\left(x\right) d\psi\left(x\right)\right) -$$

$$\left(\int_a^b f_3\left(x\right) d\psi\left(x\right)\right)\left(\int_a^b f_1\left(x\right) f_2\left(x\right) d\psi\left(x\right)\right), \tag{61}$$

etc.

We give the special cases of fractional Ostrowski type inequalities.

Proposition 18 *Let $\alpha > 0$, $n \in \mathbb{N} : n = \lceil \alpha \rceil$, $[a, b] \subset \mathbb{R}$ and $f_1, f_2 \in C^n([a, b])$. Let $x_0 \in [a, b]$ and assume that $f_{1e^x}^{[k]}(x_0) = f_{2e^x}^{[k]}(x_0) = 0$, for $k = 1, ..., n-1$. Then*

$$\left| 2\int_a^b f_1(x) f_2(x) e^x dx - f_1(x_0) \int_a^b f_2(x) e^x dx - f_2(x_0) \int_a^b f_1(x) e^x dx \right| \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \sum_{i=1}^{2} \left[\left[\left\| {}^C D_{x_0-}^{\alpha, e^x} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{2} f_j \right\|_{\infty,[a,x_0]} \left(e^{x_0} - e^a\right)^{\alpha+1} \right] \right.$$

$$\left. + \left[\left\| {}^C D_{x_0+}^{\alpha, e^x} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{2} f_j \right\|_{\infty,[x_0,b]} \left(e^b - e^{x_0}\right)^{\alpha+1} \right] \right]. \tag{62}$$

If $0 < \alpha \leq 1$, then (62) is valid without any initial conditions.

Proof. Case of $\psi(x) = e^x$, apply Proposition 13 for $r = 2$. ■

We continue with

Proposition 19 *Let $\alpha > 0$, $n \in \mathbb{N} : n = \lceil \alpha \rceil$, $[a, b] \subset (0, +\infty)$ and $f_1, f_2, f_3 \in C^n([a, b])$. Let $x_0 \in [a, b]$ and assume that $f_{i \ln x}^{[k]}(x_0) = 0$, for $k = 1, ..., n-1$; $i = 1, 2, 3$. Then*

$$\left| 3\int_a^b \frac{f_1(x) f_2(x) f_3(x)}{x} dx - f_1(x_0) \int_a^b \frac{f_2(x) f_3(x)}{x} dx - \right.$$

$$\left. f_2(x_0) \int_a^b \frac{f_1(x) f_3(x)}{x} dx - f_3(x_0) \int_a^b \frac{f_1(x) f_2(x)}{x} dx \right| \leq$$

$$\frac{1}{\Gamma(\alpha+2)} \sum_{i=1}^{3} \left[\left[\left\| {}^C D_{x_0-}^{\alpha, \ln x} f_i \right\|_{\infty,[a,x_0]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{3} f_j \right\|_{\infty,[a,x_0]} \left(\ln\left(\frac{x_0}{a}\right)\right)^{\alpha+1} \right] \right.$$

$$\left. + \left[\left\| {}^C D_{x_0+}^{\alpha, \ln x} f_i \right\|_{\infty,[x_0,b]} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{3} f_j \right\|_{\infty,[x_0,b]} \left(\ln\left(\frac{b}{x_0}\right)\right)^{\alpha+1} \right] \right]. \tag{63}$$

If $0 < \alpha \leq 1$, then (63) is valid without any initial conditions.

Proof. Case of $\psi(x) = \ln x$, apply Proposition 13 for $r = 3$. ■

Next we present the special cases of fractional Grüss type inequality:

Proposition 20 *Let* $0 < \alpha \leq 1$, $[a,b] \subset \mathbb{R}$ *and* $f_1, f_2 \in C^1([a,b])$. *Assume that* $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha, e^x} f_i \right\|_{\infty,[a,x_0]} < \infty$, *and* $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha, e^x} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1,2$.

Then

$$2 \left| \left(e^b - e^a\right) \left(\int_a^b f_1(x) f_2(x) e^x dx \right) - \left(\int_a^b f_1(x) e^x dx \right) \left(\int_a^b f_2(x) e^x dx \right) \right| \leq$$

$$\frac{\left(e^b - e^a\right)^{\alpha+2}}{\Gamma(\alpha+2)}$$

$$\left\{ \sum_{i=1}^{2} \left[\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha, e^x} f_i \right\|_{\infty,[a,x_0]} + \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha, e^x} f_i \right\|_{\infty,[x_0,b]} \right] \left\| \prod_{\substack{j=1 \\ j \neq i}}^{2} f_j \right\|_{\infty,[a,b]} \right\}. \tag{64}$$

Proof. Apply Proposition 16, for $\psi(x) = e^x$, $r = 2$. ■

We finish with another ψ-fractional Grüss type inequality:

Proposition 21 *Let* $0 < \alpha \leq 1$, $[a,b] \subset (0,\infty)$ *and* $f_i \in C^1([a,b])$, $i = 1,2,3$. *Assume that* $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha, \ln x} f_i \right\|_{\infty,[a,x_0]} < \infty$, $\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha, \ln x} f_i \right\|_{\infty,[x_0,b]} < \infty$, $i = 1,2,3$. *Then*

$$\left| \ln\left(\frac{b}{a}\right)^3 \left(\int_a^b \frac{f_1(x) f_2(x) f_3(x)}{x} dx \right) - \left(\int_a^b \frac{f_1(x)}{x} dx \right) \left(\int_a^b \frac{f_2(x) f_3(x)}{x} dx \right) - \right.$$

$$\left. \left(\int_a^b \frac{f_2(x)}{x} dx \right) \left(\int_a^b \frac{f_1(x) f_3(x)}{x} dx \right) - \left(\int_a^b \frac{f_3(x)}{x} dx \right) \left(\int_a^b \frac{f_1(x) f_2(x)}{x} dx \right) \right| \tag{65}$$

$$\leq \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha+2}}{\Gamma(\alpha+2)} \left\{ \sum_{i=1}^{3} \left[\sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0-}^{\alpha, \ln x} f_i \right\|_{\infty,[a,x_0]} + \right. \right.$$

$$\left. \left. \sup_{x_0 \in [a,b]} \left\| {}^C D_{x_0+}^{\alpha, \ln x} f_i \right\|_{\infty,[x_0,b]} \right] \left\| \prod_{\substack{j=1 \\ j \neq 1}}^{3} f_j \right\|_{\infty,[a,b]} \right\}.$$

Proof. By Proposition 16, for $\psi(x) = \ln x$, $r = 3$. ■

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Results On Sequential Conformable Fractional Derivatives With Applications

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Abstract

This paper investigates and states some properties of sequential conformable fractional derivative introduced by R. Khalil et. al. in [1]. Further some theorems of the classical power series are generalized for the fractional power series(CFPS), where the CFPS technique is used to find the solutions of conformable fractional deferential equation with variable coefficients.

1 Introduction

The correspondence between L'Höpital and Leibniz, in 1695, about what might be a derivative of order 1/2, led to the introduction of a generalization of integral and derivative operators, known as Fractional Calculus. Since then, related to the definition of fractional derivatives have been many definitions. The most popular ones of these definitions are Riemann-Liouville and Caputo definitions see [6],[7].

Recently, R. Khalil et al. [1] give a new definition of fractional derivative and fractional integral. In their work they proved the product rule, the fractional Rolle's theorem and Mean Value Theorem utilizing the conformable fractional derivative definition. New construction of the generalized Taylor's power series is obtained by Abdeljawad in[2]. In recent years, many researchers have focused on the approximate analytical solutions of the system of fractional differential equations and some methods have been developed such as fractional power series method(FPS) in [5] . FPS method is a simple technique to find out the recurrence relation that determines the coefficients

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[0] *2000 Mathematics Subject Classification.* 26A33, 41A58 .

Key words and phrases. Conformable fractional derivatives ; Fractional power series.

of the fractional power series, where this method is one of the most useful techniques to solve linear system and non-linear system of fractional differential equations with a fast convergence rate and small calculation error.

In this work, we state some properties of sequential conformable fractional derivative, then use the FPS technique to solve conformable fractional differential equation of order two.

2 Conformable fractional derivative

In this section, we present some definitions and some important properties of the conformable fractional derivative. The definition of the conformable fractional derivative is defined as follows;

Definition 1 [1] *Given a function* $f : [0, \infty) \longrightarrow \mathbb{R}$*. For* $\alpha \in (0,1)$*, the conformable fractional derivative (CFD) of* f *of order* α *is defined by*

$$T_\alpha (f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1}$$

for all $t > 0$*. If* f *is* α*differentiable in some* $(0, a)$*,* $a > 0$*, and* $\lim_{t \to 0^+} f^{(\alpha)}(t)$ *exists, then define*

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

A function f is called α-differentiable at $t \geqslant 0$ if the above limits exists. For simplicity we sometimes use the notation $f^{(\alpha)}(t)$ instead of $T_\alpha (f)(t)$.

Consider the limit $\alpha \to 1^-$. In this case , for $t > 0$, we obtain the classical definition for the first derivative of a function $f^{(\alpha)}(t) = f'(t)$.

Theorem 2 [1]*Let* $f : [0, \infty) \longrightarrow \mathbb{R}$ *be a differentiable function in the classical sense. Then* f *is* α*differentiable at* t *,* $\alpha \in (0,1)$ *and*

$$f^{(\alpha)}(t) = t^{1-\alpha} f'(t), \quad t > 0.$$

Also, if f *is continuously differentiable at* 0*, then* $f^{(\alpha)}(0) = 0$*.*

Note that the function could be α-differentiable at a point t_0 but not differentiable at that point, as in the following example.

Example 3 *For some fixed* α*, with* $\alpha \in (0,1)$*, let* $f(t) = \frac{t^\alpha}{\alpha}$*,* $t > 0$*. Note that* $f'(0)$ *does not exists but* $T_\alpha(f)(t) = 1$ *for* $t > 0$*, therefore* $f^{(\alpha)}(0) = \lim_{t \to 0^+} T_\alpha(f)(t) = 1$*.*

Theorem 4 [*1*]*If a function* $f : [0,\infty) \longrightarrow \mathbb{R}$ *is* α*-differentiable at* $a > 0,\ \alpha \in (0,1]$, *then* f *is continuous at* a*.*

We list some Important properties of the operator T_α as follows.

Theorem 5 [*1*]*Let* $\alpha \in (0,1]$ *and* f, g *be* α*differentiable at a point* $t > 0$*. Then*

1. $T_\alpha(\lambda f + g) = \lambda T_\alpha(f) + T_\alpha(g)$, *for all* $\lambda \in \mathbb{R}$*.*
2. $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$*.*
3. $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$*.*

Important examples of CFD are listed as follows:

Example 6 [*1*]

1. $T_\alpha(t^p) = pt^{p-\alpha}$
2. $T_\alpha(e^{at}) = at^{1-\alpha}e^{at}, \quad a \in \mathbb{R}$.
3. $T_\alpha(\sin(at)) = at^{1-\alpha}\cos(at), \quad a \in \mathbb{R}$.
4. $T_\alpha(\cos(at)) = -at^{1-\alpha}\sin(at), \quad a \in \mathbb{R}$.
5. $T_\alpha(e^{\lambda(\frac{1}{\alpha}t^\alpha)}) = \lambda e^{\frac{1}{\alpha}t^\alpha}$.
6. $T_\alpha(\sin(\frac{1}{\alpha}t^\alpha)) = \cos(\dfrac{1}{\alpha}t^\alpha)$.
7. $T_\alpha(\cos(\frac{1}{\alpha}t^\alpha)) = -\sin(\dfrac{1}{\alpha}t^\alpha)$.
8. $T_\alpha(\frac{1}{\alpha}t^\alpha) = 1$.

There are some properties that are not satisfied by operator T_α as follows:

Theorem 7 [*2*]*For* $\alpha, \beta \in (0,1]$ *and* $\alpha + \beta \in (1,2]$, *and the function* $f : (0,\infty) \longrightarrow \mathbb{R}$ *is twice differentiable on* $(0,\infty)$*.*T_α *does not satisfy the Index Law;* $T_\alpha T_\beta = T_\beta T_\alpha$*,where*

$$T_\alpha T_\beta\left(f\right)\left(t\right) = t^{1-(\alpha+\beta)}\left(\left(1-\beta\right)f^{'}\left(t\right) + t\ f^{''}\left(t\right)\right),$$

1. *while,*

$$T_\beta T_\alpha\left(f\right)\left(t\right) = t^{1-(\alpha+\beta)}\left(\left(1-\alpha\right)f^{'}\left(t\right) + t\ f^{''}\left(t\right)\right).$$

Proof. Calculating $T_\alpha T_\beta, T_\beta T_\alpha$ gives that

$$T_\alpha T_\beta\left(f\right)\left(t\right) \neq T_\beta T_\alpha\left(f\right)\left(t\right),$$

for $\alpha \neq \beta$. ■

3 Sequential conformable fractional α-derivatives

In this section the higher order of conformable fractional derivative will be defined and the relation between CFD and polynomials will be given.

Definition 8 *The second order CFD operator* T_α, *will be denoted by*

$$T_\alpha^2 = T_\alpha\left(T_\alpha\right).$$

In general the n^{th} *order CFD operator* T_α *is defined as*

$$T_\alpha^n = T_\alpha\left(T_\alpha^{n-1}\right).$$

Note that the operators commute for each positive integers n, m,

$$T_\alpha^n T_\alpha^m = T_\alpha^m T_\alpha^n.$$

The calculation of the second and the third sequential orders of CF operator T_α can be found in [2].

Theorem 9 *Given a function* $f : (0,\infty) \longrightarrow \mathbb{R}$ *.Then*

$$T_\alpha^2\left(f\right)\left(t\right) = T_\alpha T_\alpha\left(f\right)\left(t\right) = t^{1-2\alpha}\Big(\left(1-\alpha\right) f'\left(t\right) + t\ f''\left(t\right)\Big),$$

where f *is twice differentiable, and twice* α*-differentiable at* t *. Also*

$$T_\alpha^3\left(f\right)\left(t\right) = t^{1-3\alpha}\Big(\left(1-\alpha\right)\left(1-2\alpha\right) f'\left(t\right) + \left(3-3\alpha\right) t\ f''\left(t\right) + t^2\ f'''\left(t\right)\Big),$$

where f *is three times differentiable at* t *and three times* α*-differentiable at* t *.*

Lemma 10 *If* m *and* n *are any positive integers and* p *a real number, then*

$$T_\alpha^m\left(t^p\right) = \Pi_{i=0}^{m-1}\left(p - i\ \alpha\right) t^{p-m\alpha}.$$

Proof.

$$\begin{aligned} T_\alpha^2\left(t^p\right) &= T_\alpha\left(T_\alpha t^p\right) \\ &= T_\alpha\left(p t^{p-\alpha}\right) \\ &= p\left(p-\alpha\right) t^{p-2\alpha}. \end{aligned}$$

Also,

$$\begin{aligned} T_\alpha^3\left(t^p\right) &= T_\alpha\left(p\left(p-\alpha\right)t^{p-2\alpha}\right) \\ &= p\left(p-\alpha\right)\left(p-2\alpha\right)t^{p-3\alpha}. \end{aligned}$$

Then

$$\begin{aligned} T_\alpha^m\left(t^p\right) &= p\left(p-\alpha\right)\left(p-2\alpha\right)\ldots\left(p-\left(m-1\right)\alpha\right)t^{p-m\alpha} \\ &= \Pi_{i=0}^{m-1}\left(p-i\ \alpha\right)t^{p-m\alpha}. \end{aligned}$$

■

Corollary 11 *If* n *is positive integer , then*

$$T_\alpha^n\left(t^n\right) = \Pi_{i=0}^{n-1}\left(n-i\ \alpha\right)t^{n(1-\alpha)}.$$

Corollary 12 *If* m *and* n *are any positive integers and* $0<\alpha<1$*,then*

$$T_\alpha^m\left(t^{n\alpha}\right) = \Pi_{i=0}^{m-1}\alpha^m\left(n-i\ \right)t^{\alpha(n-m)}.$$

Applying Lemma10 and using the linearity property of Theorem 5, we get

Lemma 13 *If* $P_n\left(t\right) = a_0t^n + a_1t^{n-1} + a_2t^{n-2} + \cdots + a_n$ *is a polynomial in* t *of degree* n*, then*

$$T_\alpha^m P_n\left(t\right) = \sum_{j=0}^{n} a_j\ \Pi_{i=0}^{m-1}\left(n-j-i\ \alpha\right)t^{n-j-m\alpha}.$$

Changing variable from $t \to t^\alpha$, we get the following result.

Corollary 14 *If* $f\left(t\right) = a_0t^{\alpha n} + a_1t^{\alpha(n-1)} + a_2t^{\alpha(n-2)} + \cdots + a_n$ *, then*

$$T_\alpha^m f\left(t\right) = \sum_{j=0}^{n} a_j\ \Pi_{i=0}^{m-1}\left(\alpha\left(n-i\ \right)-j\right)t^{\alpha(n-m)-j}.$$

The following theorem presents the n^{th} sequential CF α-derivative utilizing the limit definition as follows:

Theorem 15 *Given a function* $f:[0,\infty)\longrightarrow\mathbb{R}$*. Then for* $0<\alpha<1$ *and* n *a positive integer, the* n *th order of the* $\alpha-$*conformable fractional derivative of* f *of is as follows*

$$T_\alpha^n f\left(t\right) = \lim_{\varepsilon\to 0}\frac{\sum_{j=0}^{n}\binom{n}{j}\left(-1\right)^{n-j}f\left(\left(1+\varepsilon t^{-\alpha}\right)^j t\right)}{\varepsilon^n},$$

where $t>0$*.*

Proof. Since

$$\begin{aligned} T_\alpha f\left(t\right) &= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f(t\left(1+\varepsilon t^{-\alpha}\right))-f(t)}{\varepsilon}. \end{aligned}$$

Let $\delta = 1+\varepsilon t^{-\alpha}$, we get

$$T_\alpha f\left(t\right) = \lim_{\delta\to 1}\frac{f(\delta t)-f(t)}{\left(\delta-1\right)t^{\alpha}}.$$

Thus

$$\begin{aligned} T_\alpha^2 f\left(t\right) &= \lim_{\delta\to 1}\frac{T_\alpha f(\delta t)-T_\alpha f(t)}{\left(\delta-1\right)t^{\alpha}} \\ &= \lim_{\delta\to 1}\frac{f(\delta^2 t)-2f(\delta t)+f(t)}{\left(\delta-1\right)^2 t^{2\alpha}}. \end{aligned}$$

Again calculating the 3^{rd} order of the $\alpha-$conformable fractional derivative, we get

$$\begin{aligned} T_\alpha^3 f\left(t\right) = T_\alpha T_\alpha^2 f\left(t\right) &= \lim_{\delta\to 1}\frac{T_\alpha f(\delta^2 t)-2T_\alpha f(\delta t)+T_\alpha f(t)}{\left(\delta-1\right)^2 t^{2\alpha}} \\ &= \lim_{\delta\to 1}\frac{f(\delta^3 t)-3f(\delta^2 t)+3f(\delta t)+f(t)}{\left(\delta-1\right)^3 t^{3\alpha}} \\ &= \lim_{\delta\to 1}\frac{\sum_{j=0}^{3}\binom{3}{j}\left(-1\right)^{3-j}f\left(\delta^j t\right)}{\left(\delta-1\right)^3 t^{3\alpha}}. \end{aligned}$$

Repeating this process n times, we get

$$T_\alpha^n f\left(t\right) = \lim_{\delta\to 1}\frac{\sum_{j=0}^{n}\binom{n}{j}\left(-1\right)^{n-j}f\left(\delta^j t\right)}{\left(\delta-1\right)^n t^{n\alpha}}.$$

Substituting $\varepsilon = \left(\delta-1\right)t^{\alpha}$, then

$$T_\alpha^n f\left(t\right) = \lim_{\varepsilon\to 0}\frac{\sum_{j=0}^{n}\binom{n}{j}\left(-1\right)^{n-j}f\left((1+\varepsilon t^{-\alpha})^j t\right)}{\varepsilon^n}.$$

■

4 Conformable Fractional Power Series Representation

Power series is an important tool in the study of elementary functions. Using this power expansion gives us the ability to make an approximate study of many differential equations . In this section, we will recall some important definitions and theorems of fractional power series theory .

Definition 16 *For* $\alpha \in (0,1)$ *a conformable fractional power series of the form*

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + \cdots\cdots ,$$

where $t > t_0 \geq 0$ *tis called the conformable fractional power series (CFPS) about* t_0,*where* c_n *denote the coefficients of the series, where* $n \in \mathbb{N}$.

Note that for CFPS, we have the value c_0 for$n = 0$,at $t = t_0$, while $c_n = 0$ for $n \geq 1$at $t = t_0$ .Also note that for $t_0 = 0$, then the CFPS becomes

$$\sum_{n=0}^{\infty} c_n t^{n\alpha} = c_0 + c_1 t^{\alpha} + c_2 t^{2\alpha} + \cdots\cdots$$

Proposition 17 [4]*If* $\sum_{n=0}^{\infty} c_n t^{n\alpha}$*converges absolutely for* $t = t_0 > 0$, *then we it converges absolutely for* $t \in (0, t_0)$.

Theorem 18 *The series*$\sum_{n=0}^{\infty} c_n t^{n\alpha}$*converges,*$-\infty < t < \infty$ *has radius of convergence* R, *if and only if the series* $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \geq 0$ *has radius of convergence* $R^{\frac{1}{\alpha}}$, $R > 0$.

Theorem 19 [2]*Assume* f *is an infinitely* α*-differentiable function, for some* $\alpha \in (0,1]$ *at a neighborhood of a point* t_0. *Then* f *has the Taylor CFPS series expansion as follows*

$$f(t) = \sum_{k=0}^{\infty} \frac{T_{\alpha}^{k} f(t_0) (t-t_0)^{k\alpha}}{\alpha^k k!}, \; t \in \left(t_0, t_0 + R^{\frac{1}{\alpha}}\right) \; , \; R > 0.$$

The next following examples doesn't have the Taylor PS expansion about $t_0 \geq 0$ since there are not differentiable there. But they have Taylor CFPS expansion at t_0.

Example 20 [2]

1. $e^{\frac{1}{\alpha}(t-t_0)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(t-t_0)^{k\alpha}}{\alpha^k k!}$, *for* $t \in [t_0, \infty)$.
2. $\sin(\frac{1}{\alpha}(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{(t-t_0)^{(2k+1)\alpha}}{\alpha^{2k+1} 2k+1!}$, *for* $t \in [t_0, \infty)$.
3. $\cos(\frac{1}{\alpha}(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} (-1)^k \frac{(t-t_0)^{(2k)\alpha}}{\alpha^{2k} 2k!}$, *for* $t \in [t_0, \infty)$.
4. $\frac{1}{1-\frac{(t-t_0)^{\alpha}}{\alpha}} = \sum_{k=0}^{\infty} (t-t_0)^{k\alpha}$, *for* $t \in [t_0, t_0+1)$.

5 Solving CFD equation's using CFPS

Example 21 *Consider the following conformable fractional differential equation*

$$T_{\alpha}^{2} y(t) - t^{\alpha} y(t) = 0, \tag{2}$$

with initial conditions,

$$y(0) = 0,\ T_{\alpha} y(0) = y_0, \tag{3}$$

where y_0 is a real constant.
Now using CFPS technique, let

$$y(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}.$$

Then

$$\begin{aligned} T_{\alpha}^{2} y(t) &= \sum_{n=2}^{\infty\alpha} \alpha^2 (n)(n-1) c_n t^{(n-2)\alpha} \\ &= \sum_{n=0}^{\infty} \alpha^2 (n+1)(n+2) c_{n+2} t^{n\alpha} \end{aligned}$$

and

$$\begin{aligned} t^{\alpha} y(t) &= \sum_{n=0}^{\infty} c_n t^{(n+1)\alpha} \\ &= \sum_{n=1}^{\infty} c_{n-1} t^{n\alpha} \end{aligned} \tag{4}$$

Substituting $T_\alpha^2 y(t)$ and $t^\alpha y(t)$ in CFDE, we get

$$\sum_{n=0}^{\infty} \alpha^2 (n+1)(n+2) c_{n+2} t^{n\alpha} - \sum_{n=1}^{\infty} c_{n-1} t^{n\alpha} = 0.$$

Then from Formula (), one can obtain

$$2\alpha^2 c_2 + \sum_{n=1}^{\infty} \left[\alpha^2 (n+1)(n+2) c_{n+2} - c_{n-1}\right] t^{n\alpha} = 0.$$

Equating the coefficients of $t^{n\alpha}$ to zero in both sides gives the following;

$$c_2 = 0, \text{ and } c_{n+2} = \frac{c_{n-1}}{\alpha^2 (n+1)(n+2)}, \; n = 1, 2, 3, \ldots. \tag{5}$$

Considering the initial conditions of the CFDE,

$$c_0 = 0, \text{ and } c_1 = \frac{1}{\alpha} y_0.$$

Based on Equation 5,the coefficients of $t^{n\alpha}$ can be divided into two categories: zero terms

$$c_2 = c_3 = c_5 = c_6 = c_8 = c_9 = c_{11} = c_{14} = \cdots = 0,$$

in general

$$c_{3n+2} = c_{3n+3}, \text{ for } n = 0, 1, 2, 3, \ldots$$

and non zero terms

$$c_{3n+1} \neq 0, \text{for } n = 0, 1, 2, 3, \ldots,$$

that is $c_1, c_4, c_7, c_{10}, c_{13}, c_{16} \cdots$, where

$$c_4 = \frac{c_1}{\alpha^2 (3.4)} = \frac{y_0}{\alpha^3 (3.4)},$$

$$\begin{aligned} c_7 &= \frac{c_4}{\alpha^2 (6.7)} = \frac{y_0}{\alpha^5 (3.4.6.7)}, \\ c_{10} &= \frac{c_7}{\alpha^2 (9.10)} = \frac{y_0}{\alpha^7 (3.4.6.7.9.10)}, \\ c_{10} &= \frac{c_{10}}{\alpha^2 (9.10)} = \frac{y_0}{\alpha^9 (3.4.6.7.9.10)}, \\ &\cdots\cdots\cdots \end{aligned}$$

Then, one can obtain the following CFPS as follows

$$\begin{aligned} y(t) &= c_1 t^{\alpha} + c_4 t^{4\alpha} + c_7 t^{7\alpha} + c_{10} t^{10\alpha} + \cdots\cdots\cdots \\ &= \frac{1}{\alpha} y_0 t^{\alpha} + \frac{y_0}{\alpha^3 (3.4)} t^{4\alpha} + \frac{y_0}{\alpha^5 (3.4.6.7)} t^{7\alpha} + \frac{y_0}{\alpha^9 (3.4.6.7.9.10)} t^{10\alpha} + \cdots\cdots \\ &= \frac{1}{\alpha} y_0 \left(1 + \sum_{k=1}^{\infty} \frac{1}{\alpha^{2k} \left(\prod_{i=0}^{k} (3i)(3i+1) \right)} t^{(3k+1)\alpha} \right). \end{aligned}$$

Conclusion 22 *The main aim of this work is to provide a reliable algorithm for the solutions to the systems of fractional differential equations by using the CFPS.*

Acknowledgement 23 *The authors acknowledges Applied Science Private University, Amman, Jordan for the fully financial support granted of this research article.*

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

Multivariate Ostrowski-Sugeno Fuzzy inequalities

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Abstract

Here we present multivariate Ostrowski-Sugeno Fuzzy type inequalities. These are multivariate Ostrowski-like inequalities in the context of Sugeno fuzzy integral and its special properties. They give tight upper bounds to the deviation of a multivariate function from its Sugeno-fuzzy multivariate averages.

2010 Mathematics Subject Classification: Primary: 26D07, 26D10, 26D15, 41A44, Secondary: 26A24, 26D20, 28A25.

Keywords and phrases: Sugeno fuzzy integral, multivariate function fuzzy average, deviation from fuzzy multivariate mean, Fuzzy multivariate Ostrowski inequality.

1 Introduction

The famous Ostrowski ([4]) inequality motivates this work and has as follows:

$$\left| \frac{1}{b-a} \int_a^b f(y)\, dy - f(x) \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \left\| f' \right\|_{\infty},$$

where $f \in C^1([a,b])$, $x \in [a,b]$, and it is a sharp inequality.

Another motivation comes from author's [2], pp. 507-508, see also [1]:

Let $f \in C^1\left(\prod_{i=1}^{k} [a_i, b_i]\right)$, where $a_i < b_i$; $a_i, b_i \in \mathbb{R}$, $i = 1, ..., k$, and let $x_0 := (x_{01}, ..., x_{0k}) \in \prod_{i=1}^{k} [a_i, b_i]$ be fixed. Then

$$\left| \frac{1}{\prod_{i=1}^{k} (b_i - a_i)} \int_{a_1}^{b_1} ... \int_{a_i}^{b_i} ... \int_{a_k}^{b_k} f(z_1, ..., z_k)\, dz_1 ... dz_k - f(x_0) \right| \leq$$

$$\sum_{i=1}^{k}\left(\frac{(x_{0i}-a_i)^2+(b_i-x_{0i})^2}{2(b_i-a_i)}\right)\left\|\frac{\partial f}{\partial z_i}\right\|_\infty .$$

The last inequality is sharp, the optimal function is

$$f^*(z_1,...,z_k):=\sum_{i=1}^{k}|z_i-x_{0i}|^{\alpha_i},\ \ \alpha_i>1.$$

Here first we give a survey about Sugeno fuzzy integral and its basic special properties. Then we derive a set of multivariate Ostrowski-like inequalities to all directions in the context of Sugeno integral whitin its basic important properties. We finish with an application to a special multivariate case.

2 Background

In this section, some definitions and basic important properties of the Sugeno integral which will be used in the next section are presented. Also a preparation for the main results Section 3 is given.

Definition 1 *(Fuzzy measure [6, 8]) Let Σ be a σ-algebra of subsets of X, and let $\mu:\Sigma\to[0,+\infty]$ be a non-negative extended real-valued set function. We say that μ is a fuzzy measure iff:*

(1) $\mu(\varnothing)=0$,

(2) $E,F\in\Sigma : E\subseteq F$ imply $\mu(E)\leq\mu(F)$ (monotonicity),

(3) $E_n\in\Sigma$ ($n\in\mathbb{N}$), $E_1\subset E_2\subset ...$, imply $\lim_{n\to\infty}\mu(E_n)=\mu(\cup_{n=1}^{\infty}E_n)$ (continuity from below);

(4) $E_n\in\Sigma$ ($n\in\mathbb{N}$), $E_1\supset E_2\supset ...$, $\mu(E_1)<\infty$, imply $\lim_{n\to\infty}\mu(E_n)=\mu(\cap_{n=1}^{\infty}E_n)$ (continuity from above).

Let (X,Σ,μ) be a fuzzy measure space and f be a non-negative real-valued function on X. We denote by $\mathcal{F}_+$ the set of all non-negative real valued measurable functions, and by $L_\alpha f$ the set: $L_\alpha f:=\{x\in X: f(x)\geq\alpha\}$, the α-level of f for $\alpha\geq 0$.

Definition 2 *Let (X,Σ,μ) be a fuzzy measure space. If $f\in\mathcal{F}_+$ and $A\in\Sigma$, then the Sugeno integral (fuzzy integral) [7] of f on A with respect to the fuzzy measure μ is defined by*

$$(S)\int_A f d\mu := \vee_{\alpha\geq 0}\left(\alpha\wedge\mu(A\cap L_\alpha f)\right), \tag{1}$$

where $\vee$ and $\wedge$ denote the sup and inf on $[0,\infty]$, respectively.

The basic properties of Sugeno integral follow:

Theorem 3 *([5, 8]) Let (X, Σ, μ) be a fuzzy measure space with $A, B \in \Sigma$ and $f, g \in \mathcal{F}_+$. Then*

1) $(S) \int_A f d\mu \leq \mu(A)$;

2) $(S) \int_A k d\mu = k \wedge \mu(A)$ for a non-negative constant k;

3) if $f \leq g$ on A, then $(S) \int_A f d\mu \leq (S) \int_A g d\mu$;

4) if $A \subset B$, then $(S) \int_A f d\mu \leq (S) \int_B f d\mu$;

5) $\mu(A \cap L_\alpha f) \leq \alpha \Rightarrow (S) \int_A f d\mu \leq \alpha$;

6) if $\mu(A) < \infty$, then $\mu(A \cap L_\alpha f) \geq \alpha \Leftrightarrow (S) \int_A f d\mu \geq \alpha$;

7) when $A = X$, then $(S) \int_A f d\mu = \vee_{\alpha \geq 0} (\alpha \wedge \mu(L_\alpha f))$;

8) if $\alpha \leq \beta$, then $L_\beta f \subseteq L_\alpha f$;

9) $(S) \int_A f d\mu \geq 0$.

Theorem 4 *([8], p. 135) Here $f \in \mathcal{F}_+$, the class of all finite nonnegative measurable functions on (X, Σ, μ). Then*

1) if $\mu(A) = 0$, then $(S) \int_A f d\mu = 0$, for any $f \in \mathcal{F}_+$;

2) if $(S) \int_A f d\mu = 0$, then $\mu(A \cap \{x | f(x) > 0\}) = 0$;

3) $(S) \int_A f d\mu = (S) \int_X f \cdot \chi_A d\mu$, where χ_A is the characteristic function of A;

4) $(S) \int_A (f + a) d\mu \leq (S) \int_A f d\mu + (S) \int_A a d\mu$, for any constant $a \in [0, \infty)$.

Corollary 5 *([8], p. 136) Here $f, f_1, f_2 \in \mathcal{F}_+$. Then*

1) $(S) \int_A (f_1 \vee f_2) d\mu \geq (S) \int_A f_1 d\mu \vee (S) \int_A f_2 d\mu$;

2) $(S) \int_A (f_1 \wedge f_2) d\mu \leq (S) \int_A f_1 d\mu \wedge (S) \int_A f_2 d\mu$;

3) $(S) \int_{A \cup B} f d\mu \geq (S) \int_A f d\mu \vee (S) \int_B f d\mu$;

4) $(S) \int_{A \cap B} f d\mu \leq (S) \int_A f d\mu \wedge (S) \int_B f d\mu$.

In general we have

$$(S) \int_A (f_1 + f_2) d\mu \neq (S) \int_A f_1 d\mu + (S) \int_A f_2 d\mu,$$

and

$$(S) \int_A a f d\mu \neq a (S) \int_A f d\mu, \text{ where } a \in \mathbb{R},$$

see [8], p. 137.

Lemma 6 *([8], p. 138) $(S) \int_A f d\mu = \infty$ iff $\mu(A \cap L_\alpha f) = \infty$ for any $\alpha \in [0, \infty)$.*

We need

Definition 7 *([3]) A fuzzy measure μ is subadditive iff $\mu(A \cup B) \leq \mu(A) + \mu(B)$, for all $A, B \in \Sigma$.*

We mention

Theorem 8 *([3]) If μ is subadditive, then*

$$(S)\int_X (f+g)\,d\mu \le (S)\int_X f d\mu + (S)\int_X g d\mu, \tag{2}$$

for all measurable functions $f, g : X \to [0, \infty)$.

Moreover, if (2) holds for all measurable functions $f, g : X \to [0, \infty)$ and $\mu(X) < \infty$, then μ is subadditive.

Notice here in (1) we have that $\alpha \in [0, \infty)$.

We have

Corollary 9 *If μ is aubadditive, $n \in \mathbb{N}$, and $f : X \to [0, \infty)$ is a measurable function, then*

$$(S)\int_X nf d\mu \le n\,(S)\int_X f d\mu, \tag{3}$$

in particular it holds

$$(S)\int_A nf d\mu \le n\,(S)\int_A f d\mu, \tag{4}$$

for any $A \in \Sigma$.

Proof. By (2). ■

A very important property of Sugeno integral follows.

Theorem 10 *If μ is subadditive measure, and $f : X \to [0, \infty)$ is a measurable function, and $c > 0$, then*

$$(S)\int_A cf d\mu \le (c+1)\,(S)\int_A f d\mu, \tag{5}$$

for any $A \in \Sigma$.

Proof. Let the ceiling $\lceil c \rceil = m \in \mathbb{N}$, then by Theorem 3 (3) and (4) we get

$$(S)\int_A cf d\mu \le (S)\int_A mf d\mu \le m\,(S)\int_A f d\mu \le (c+1)\,(S)\int_A f d\mu,$$

proving (5). ■

From now on in this article we work on the fuzzy measure space $(Q, \mathcal{B}, \mu)$, where $Q \subset \mathbb{R}^k$, $k \ge 1$ is a convex compact subset, $\mathcal{B}$ is the Borel σ-algebra on Q, and μ is a finite fuzzy measure on $\mathcal{B}$. Typically we take it to be subadditive.

The functions f we deal with here are continuous from Q into $\mathbb{R}_+$.

We make

Remark 11 *Let $f \in C(Q, \mathbb{R}_+)$, and μ is a subadditive fuzzy measure such that $\mu(Q) > 0$, $x \in Q$. We will estimate*

$$E(x) := \left| (S)\int_Q f(t)\,d\mu(t) - \mu(Q) \wedge f(x) \right| \tag{6}$$

(by Theorem 3 (2))

$$= \left|(S)\int_Q f(t)\,d\mu(t) - (S)\int_Q f(x)\,d\mu(t)\right|.$$

We notice that

$$f(t) = f(t) - f(x) + f(x) \leq |f(t) - f(x)| + f(x),$$

then (by Theorem 3 (3) and Theorem 4 (4))

$$(S)\int_Q f(t)\,d\mu(t) \leq (S)\int_Q |f(t) - f(x)|\,d\mu(t) + (S)\int_Q f(x)\,d\mu(t), \quad (7)$$

that is

$$(S)\int_Q f(t)\,d\mu(t) - (S)\int_Q f(x)\,d\mu(t) \leq (S)\int_Q |f(t) - f(x)|\,d\mu(t). \quad (8)$$

Similarly, we have

$$f(x) = f(x) - f(t) + f(t) \leq |f(t) - f(x)| + f(t),$$

then (by Theorem 3 (3) and Theorem 8)

$$(S)\int_Q f(x)\,d\mu(t) \leq (S)\int_Q |f(t) - f(x)|\,d\mu(t) + (S)\int_Q f(t)\,d\mu(t),$$

that is

$$(S)\int_Q f(x)\,d\mu(t) - (S)\int_Q f(t)\,d\mu(t) \leq (S)\int_Q |f(t) - f(x)|\,d\mu(t). \quad (9)$$

By (8) and (9) we derive that

$$\left|(S)\int_Q f(t)\,d\mu(t) - (S)\int_Q f(x)\,d\mu(t)\right| \leq (S)\int_Q |f(t) - f(x)|\,d\mu(t). \quad (10)$$

Consequently it holds

$$E(x) \overset{(by\ (6),\ (10))}{\leq} (S)\int_Q |f(t) - f(x)|\,d\mu(t), \quad (11)$$

where $t = (t_1, ..., t_k)$, $x = (x_1, ..., x_k)$.

We will use (11).

3 Main Results

We make

Remark 12 *Here* $Q := \prod_{i=1}^{k} [a_i, b_i]$, *where* $a_i < b_i$; $a_i, b_i \in \mathbb{R}$, $i = 1, ..., k$; $x = (x_1, ..., x_k) \in \prod_{i=1}^{k} [a_i, b_i]$ *is fixed, and* $f \in C^1 \left(\prod_{i=1}^{k} [a_i, b_i], \mathbb{R}_+ \right)$. *Consider* $g_t(r) := f(x + r(t - x))$, $r \geq 0$. *Note that* $g_t(0) = f(x)$, $g_t(1) = f(t)$. *Thus*

$$f(t) - f(x) = g_t(1) - g_t(0) = g_t'(\xi)(1 - 0) = g_t'(\xi), \tag{12}$$

where $\xi \in (0, 1)$.

I.e.

$$f(t) - f(x) = \sum_{i=1}^{k} (t_i - x_i) \frac{\partial f}{\partial t_i} (x + \xi(t - x)). \tag{13}$$

Hence

$$|f(t) - f(x)| \leq \sum_{i=1}^{k} |t_i - x_i| \left| \frac{\partial f}{\partial t_i} (x + \xi(t - x)) \right|$$

$$\leq \sum_{i=1}^{k} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_{\infty}. \tag{14}$$

By (11) we get

$$\left| (S) \int_{\prod_{i=1}^{k} [a_i, b_i]} f(t) \, d\mu(t) - \mu \left(\prod_{i=1}^{k} [a_i, b_i] \right) \wedge f(x) \right| \leq$$

$$(S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |f(t) - f(x)| \, d\mu(t) \overset{(14)}{\leq}$$

$$(S) \int_{\prod_{i=1}^{k} [a_i, b_i]} \left(\sum_{i=1}^{k} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_{\infty} \right) d\mu(t) \overset{(2)}{\leq}$$

$$\sum_{i=1}^{k} (S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |t_i - x_i| \left\| \frac{\partial f}{\partial t_i} \right\|_{\infty} d\mu(t) \overset{(5)}{\leq}$$

$$\sum_{i=1}^{k} \left(\left\| \frac{\partial f}{\partial t_i} \right\|_{\infty} + 1 \right) \left((S) \int_{\prod_{i=1}^{k} [a_i, b_i]} |t_i - x_i| \, d\mu(t) \right). \tag{15}$$

Here μ *is a fuzzy subadditive measure with* $\mu \left(\prod_{i=1}^{k} [a_i, b_i] \right) > 0$.

Therefore we get

$$\left| \frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right| \overset{(15)}{\leq} \tag{16}$$

$$\sum_{i=1}^{k}\left(\frac{\left\|\frac{\partial f}{\partial t_i}\right\|_{\infty}+1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\left((S)\int_{\prod_{i=1}^{k}[a_i,b_i]} |t_i - x_i|\,d\mu(t)\right).$$

Notice here $\left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right) \leq 1$, *and*

$\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) \overset{(by\ Thm.\ 3\ (1))}{\leq} 1,$

where $(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) \geq 0.$

If $f : \prod_{i=1}^{k}[a_i,b_i] \to \mathbb{R}_+$ *is a Lipschitz function of order* $0 < \alpha \leq 1$, *i.e.* $|f(x) - f(y)| \leq K\|x-y\|_{l_1}^{\alpha}$, $\forall\ x,y \in \prod_{i=1}^{k}[a_i,b_i]$, $K > 0$, *where* $\|x-y\|_{l_1} := \sum_{i=1}^{k}|x_i - y_i|$, *denoted by* $f \in Lip_{\alpha,K}\left(\prod_{i=1}^{k}[a_i,b_i],\mathbb{R}_+\right)$, *then by (11) we get*

$$\left|(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)\wedge f(x)\right| \leq \tag{17}$$

$$(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} |f(t) - f(x)|\,d\mu(t) \leq$$

$$(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} K\|t-x\|_{l_1}^{\alpha}\,d\mu(t) \overset{(5)}{\leq}$$

$$(K+1)(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} \|t-x\|_{l_1}^{\alpha}\,d\mu(t).$$

We have proved

$$\left| \frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]} f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right| \leq \tag{18}$$

$$\frac{(K+1)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}\|t-x\|_{l_1}^{\alpha}\,d\mu(t).$$

We have established the following multivariate Ostrowski-Sugeno inequalities.

Theorem 13 *Here μ is a fuzzy subadditive measure with $\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)>0$, $x\in\prod_{i=1}^{k}[a_i,b_i]$.*

1) Let $f\in C^1\left(\prod_{i=1}^{k}[a_i,b_i],\mathbb{R}_+\right)$, then

$$\left|\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}f(t)\,d\mu(t)-\left(1\wedge\frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right|\leq \tag{19}$$

$$\sum_{i=1}^{k}\left(\frac{\left\|\frac{\partial f}{\partial t_i}\right\|_{\infty}+1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\left((S)\int_{\prod_{i=1}^{k}[a_i,b_i]}|t_i-x_i|\,d\mu(t)\right).$$

2) Let $f\in Lip_{\alpha,K}\left(\prod_{i=1}^{k}[a_i,b_i],\mathbb{R}_+\right)$, $0<\alpha\leq 1$, then

$$\left|\frac{1}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}f(t)\,d\mu(t)-\left(1\wedge\frac{f(x)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}\right)\right|\leq \tag{20}$$

$$\frac{(K+1)}{\mu\left(\prod_{i=1}^{k}[a_i,b_i]\right)}(S)\int_{\prod_{i=1}^{k}[a_i,b_i]}\|t-x\|_{l_1}^{\alpha}\,d\mu(t).$$

We make

Remark 14 *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k\geq 1$. Let $f\in\left(C(Q,\mathbb{R}_+)\cap C^{n+1}(Q)\right)$, $n\in\mathbb{N}$ and $x\in Q$ is fixed such that all partial derivatives $f_\alpha:=\frac{\partial^\alpha f}{\partial t^\alpha}$, where $\alpha=(\alpha_1,\ldots,\alpha_k)$, $\alpha_i\in\mathbb{Z}^+$, $i=1,\ldots,k$, $|\alpha|=\sum_{i=1}^{\kappa}\alpha_i=j$, $j=1,\ldots,n$ fulfill $f_\alpha(x)=0$.*

By [2], p. 513, we get that

$$|f(t)-f(x)|\leq\frac{\left[\left(\sum_{i=1}^{k}|t_i-x_i|\left\|\frac{\partial}{\partial t_i}\right\|_{\infty}\right)^{n+1}f\right]}{(n+1)!},\quad\forall\, t\in Q. \tag{21}$$

Call

$$D_{n+1}(f) := \max_{\alpha : |\alpha| = n+1} \|f_\alpha\|_\infty . \tag{22}$$

For example, when $k = 2$ *and* $n = 1$*, we get that*

$$\left[\left(\sum_{i=1}^{2} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^2 f\right] =$$

$$(t_1 - x_1)^2 \left\|\frac{\partial^2 f}{\partial t_1^2}\right\|_\infty + 2|t_1 - x_1||t_2 - x_2| \left\|\frac{\partial^2 f}{\partial t_1 \partial t_2}\right\|_\infty + (t_2 - x_2)^2 \left\|\frac{\partial^2 f}{\partial t_2^2}\right\|_\infty , \tag{23}$$

and

$$D_2(f) = \max_{\alpha : |\alpha| = 2} \|f_\alpha\|_\infty . \tag{24}$$

Clearly, it holds

$$\left[\left(\sum_{i=1}^{2} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^2 f\right] \leq D_2(f)\left(|t_1 - x_1| + |t_2 - x_2|\right)^2 . \tag{25}$$

Consequently, we derive that

$$\left[\left(\sum_{i=1}^{k} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^{n+1} f\right] \leq D_{n+1}(f)\|t - x\|_{l_1}^{n+1}, \quad \forall\ t \in Q. \tag{26}$$

By (11) we get

$$\left|(S)\int_Q f(t)\,d\mu(t) - \mu(Q) \wedge f(x)\right| \leq$$

$$(S)\int_Q |f(t) - f(x)|\,d\mu(t) \overset{(21)}{\leq} \tag{27}$$

$$(S)\int_Q \frac{\left[\left(\sum\limits_{i=1}^{k} |t_i - x_i| \left\|\frac{\partial}{\partial t_i}\right\|_\infty\right)^{n+1} f\right]}{(n+1)!}\,d\mu(t) \overset{(26)}{\leq}$$

$$(S)\int_Q \frac{D_{n+1}(f)\|t - x\|_{l_1}^{n+1}}{(n+1)!}\,d\mu(t) \overset{(5)}{\leq} \tag{28}$$

$$\left(\frac{D_{n+1}(f)}{(n+1)!} + 1\right)(S)\int_Q \|t - x\|_{l_1}^{n+1}\,d\mu(t) .$$

Here μ *is a fuzzy subadditive measure with* $\mu(Q) > 0$.

By (27) and (28) we obtain

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!}+1\right)}{\mu(Q)}(S)\int_Q \|t-x\|_{l_1}^{n+1} d\mu(t). \tag{29}$$

We have established the following multivariate Ostrowski-Sugeno general inequality:

Theorem 15 *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 1$. Let $f \in \left(C(Q,\mathbb{R}_+) \cap C^{n+1}(Q)\right)$, $n \in \mathbb{N}$, $x \in Q$ be fixed: $f_\alpha(x) = 0$, all $\alpha : |\alpha| = j$, $j = 1,...,n$. Here μ is a fuzzy subadditive measure with $\mu(Q) > 0$. Then*

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!}+1\right)}{\mu(Q)}(S)\int_Q \|t-x\|_{l_1}^{n+1} d\mu(t). \tag{30}$$

Corollary 16 *All as in Theorem 15. Then*

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\frac{\left(1+\frac{1}{(n+1)!}\right)}{\mu(Q)}(S)\int_Q \left[\left(\sum_{i=1}^k |t_i - x_i| \left\|\frac{\partial}{\partial x_i}\right\|_\infty\right)^{n+1} f\right] d\mu(t). \tag{31}$$

Next we take again $Q := \prod_{i=1}^k [a_i, b_i]$, we set $a := (a_1,...,a_k)$, $b := (b_1,...,b_k)$, and $\frac{a+b}{2} = \left(\frac{a_1+b_1}{2},...,\frac{a_k+b_k}{2}\right) \in \prod_{i=1}^k [a_i, b_i]$.

Corollary 17 *Let $f \in \left(C\left(\prod_{i=1}^k [a_i,b_i], \mathbb{R}_+\right) \cap C^{n+1}\left(\prod_{i=1}^k [a_i,b_i]\right)\right)$, $n \in \mathbb{N}$, such that $f_\alpha\left(\frac{a+b}{2}\right) = 0$, all $\alpha : |\alpha| = j$, $j = 1,...,n$. Here μ is a fuzzy subadditive measure with $\mu\left(\prod_{i=1}^k [a_i,b_i]\right) > 0$. Then*

$$\left|\frac{1}{\mu\left(\prod_{i=1}^k [a_i,b_i]\right)}(S)\int_{\prod_{i=1}^k [a_i,b_i]} f(t)\, d\mu(t) - \left(1 \wedge \frac{f\left(\frac{a+b}{2}\right)}{\mu\left(\prod_{i=1}^k [a_i,b_i]\right)}\right)\right| \leq$$

$$\frac{\left(\frac{D_{n+1}(f)}{(n+1)!}+1\right)}{\mu\left(\prod_{i=1}^k [a_i,b_i]\right)}(S)\int_{\prod_{i=1}^k [a_i,b_i]} \left\|t - \frac{a+b}{2}\right\|_{l_1}^{n+1} d\mu(t). \tag{32}$$

Proof. By Theorem 15. ■

We make

Remark 18 *By multinomial theorem we have that*

$$\|t-x\|_{l_1}^{n+1} = \left(\sum_{i=1}^{k} |t_i - x_i|\right)^{n+1} =$$

$$\sum_{r_1+r_2+...+r_k=n+1} \binom{n+1}{r_1, r_2, ..., r_k} |t_1 - x_1|^{r_1} |t_2 - x_2|^{r_2} ... |t_k - x_k|^{r_k}, \tag{33}$$

where

$$\binom{n+1}{r_1, r_2, ..., r_k} = \frac{(n+1)!}{r_1! r_2! ... r_k!}. \tag{34}$$

By (27), (28) we get

$$\left|(S)\int_Q f(t)\, d\mu(t) - \mu(Q) \wedge f(x)\right| \leq$$

$$(S)\int_Q \frac{D_{n+1}(f)}{(n+1)!} \|t-x\|_{l_1}^{n+1} d\mu(t) \overset{(by\ (33),\ (34))}{=}$$

$$(S)\int_Q \left[\sum_{r_1+r_2+...+r_k=n+1} \left(\frac{D_{n+1}(f)}{r_1! r_2! ... r_k!}\right) \left(\prod_{i=1}^{k} |t_i - x_i|^{r_i}\right)\right] d\mu(t) \overset{(2)}{\leq}$$

$$\sum_{r_1+r_2+...+r_k=n+1} (S)\int_Q \left(\frac{D_{n+1}(f)}{r_1! r_2! ... r_k!}\right) \left(\prod_{i=1}^{k} |t_i - x_i|^{r_i}\right) d\mu(t) \overset{(5)}{\leq}$$

$$\sum_{r_1+r_2+...+r_k=n+1} \left(\frac{D_{n+1}(f)}{r_1! r_2! ... r_k!} + 1\right) (S)\int_Q \left(\prod_{i=1}^{k} |t_i - x_i|^{r_i}\right) d\mu(t). \tag{35}$$

We have proved the following multivariate Ostrowski-Sugeno general inequality:

Theorem 19 *Here all as in Theorem 15. Then*

$$\left|\frac{1}{\mu(Q)} (S)\int_Q f(t)\, d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\sum_{r_1+r_2+...+r_k=n+1} \left(\frac{\left(\frac{D_{n+1}(f)}{r_1! r_2! ... r_k!} + 1\right)}{\mu(Q)}\right) (S)\int_Q \left(\prod_{i=1}^{k} |t_i - x_i|^{r_i}\right) d\mu(t). \tag{36}$$

We make

Remark 20 *In case $k = 2$, $n = 1$, by (27), (28) we get*

$$\left|(S)\int_Q f(t)\,d\mu(t) - \mu(Q) \wedge f(x)\right| \leq$$

$$(S)\int_Q \frac{D_2(f)}{2} \|t - x\|_{l_1}^2 \,d\mu(t) =$$

$$(S)\int_Q \frac{D_2(f)}{2}\left[(t_1 - x_1)^2 + 2|t_1 - x_1||t_2 - x_2| + (t_2 - x_2)^2\right] d\mu(t) \leq \tag{37}$$

$$(S)\int_Q \frac{D_2(f)}{2}(t_1 - x_1)^2\,d\mu(t) + (S)\int_Q D_2(f)|t_1 - x_1||t_2 - x_2|\,d\mu(t)$$

$$+ (S)\int_Q \frac{D_2(f)}{2}(t_2 - x_2)^2\,d\mu(t) \leq$$

$$\left(1 + \frac{D_2(f)}{2}\right)(S)\int_Q (t_1 - x_1)^2\,d\mu(t) + (1 + D_2(f))(S)\int_Q |t_1 - x_1||t_2 - x_2|\,d\mu(t)$$

$$+ \left(1 + \frac{D_2(f)}{2}\right)(S)\int_Q (t_2 - x_2)^2\,d\mu(t).$$

We have proved

Corollary 21 *Let Q be a compact and convex subset of $\mathbb{R}^2$. Let $f \in (C(Q, \mathbb{R}_+) \cap C^2(Q))$, $x = (x_1, x_2) \in Q$ be fixed: $\frac{\partial f}{\partial t_1}(x_1, x_2) = \frac{\partial f}{\partial t_2}(x_1, x_2) = 0$. Here μ is a fuzzy subadditive measure with $\mu(Q) > 0$. Then*

$$\left|\frac{1}{\mu(Q)}(S)\int_Q f(t)\,d\mu(t) - \left(1 \wedge \frac{f(x)}{\mu(Q)}\right)\right| \leq$$

$$\frac{\left(1 + \frac{D_2(f)}{2}\right)}{\mu(Q)}(S)\int_Q (t_1 - x_1)^2\,d\mu(t) + \frac{(1 + D_2(f))}{\mu(Q)}(S)\int_Q |t_1 - x_1||t_2 - x_2|\,d\mu(t) \tag{38}$$

$$+ \frac{\left(1 + \frac{D_2(f)}{2}\right)}{\mu(Q)}(S)\int_Q (t_2 - x_2)^2\,d\mu(t).$$

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J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 29, NO.6, 2021, COPYRIGHT 2021 EUDOXUS PRESS, LLC

SOME APPLICATIONS OF INTERVAL-VALUED SUBSETHOOD MEASURES WHICH ARE DEFINED BY INTERVAL-VALUED CHOQUET INTEGRALS IN TRADE EXPORTS BETWEEN KOREA AND ITS TRADING PARTNERS

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Abstract. In this paper, we consider subsethood measures introduced by Fan et al. [3] and the interval-valued Choquet integral with respect to a fuzzy measure of interval-valued fuzzy sets. Based on such a focus, we define three types of interval-valued subsethood measures and provide four interval-valued fuzzy sets to animal product exports between Korea and four selected trading partners.

In particular, we investigate a strong interval-valued subsethood measure defined by the interval-valued Choquet integral which represents the degree of trade surplus between Korea and three trading partners in terms of the model of trade transactions with the United States and Korea

1. Introduction

Zadeh[18] first developed fuzzy sets and Murofushi-Sugeno [11] have studied fuzzy measures and Choquet integrals. Subsequently, using set-valued analysis theory developed by Aumann[1], we studied interval-valued Choquet integrals and their related applications(see[5, 6, 7, 8, 9]). In particular, through the restudy of the interval-valued Choquet integral in 2004 by Zhang-Guo-Liu[21], this research has been developed in a much more systematical

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1991 *Mathematics Subject Classification.* 28E10, 28E20, 03E72, 26E50 11B68.

Key words and phrases. Choquet integral, fuzzy measure, subsethood measures, the degree of trade surplus in trade exports.

manner. Xuechang [16], Zeng-Li[19] have examined fuzzy entropy, distance and similarity measures, the likes of which form three key concepts of fuzzy set theory. Ruan-Kerre [12] also introduced various fuzzy implication operators and the Choquet integral were suggested for the first time by Choquet [2]. Further studied by Murofushi-Sugeno [9], Jang-Kwon [10], and Jang [12] provide some interesting interpretations of fuzzy measures and the Choquet integral. Subjective probability and Choquet expected utility were studied as an application of Choquet integral and form another pivotal component of fuzzy sets and information theories(see[13, 14, 15, 20]).

A subsethood measure refers to the degree to which a fuzzy set is a subset of another fuzzy set. Many researchers have contributed to the area of a fuzzy subsethood measure that is closely related to the various tools introduced above (see[6, 7, 8, 11]). Their efforts have considered axiomatizing the properties of a subsethoods measure.

In this paper, we consider subsethood measures introduced by Fan et al. [3] and the interval-valued Choquet integral with respect to a fuzzy measure of interval-valued fuzzy sets. Based on such a focus, we define three types of interval-valued subsethood measures and provide four interval-valued fuzzy sets to animal product exports between Korea and four selected trading partners. In configuring the four interval-valued fuzzy sets, the original data(see[4]) used had to be slightly modified to produce Table $A4$. In order to calculate the interval-valued Choquet integral, the rules (46) and (48) were introduced.

Furthermore, we also investigate a strong interval-valued subsethood measure defined by an interval-valued Choquet integral which represents the degree of trade surplus between Korea and 3 trading partners in terms of the model of trade transactions with the United States and Korea. The information for the above degree of surplus is of great significance in providing accurate comparative figures on the size of trade that exists between the four countries that trade with Korea.

2. Preliminaries and definitions

Throughout this paper, we write X to denote a set,

$$F(X) = \{A | A = \{(x, m_A(x)) |\ x \in X\},\ m_A : X \longrightarrow [0,1] \text{ is a function}\} \tag{1}$$

stands for the set of fuzzy sets in X(see[18]). We note that m_A expresses the membership of a fuzzy set A, A^c is the complement of A, that is,

$$A^c = \{(x, m_{A^c}(x)) |\ m_{A^c}(x) = 1 - m_A(x),\ x \in X\}. \tag{2}$$

Recall that for $A, B \in F(X)$, $A \subset B$ if and only if $m_A(x) \leq m_B(x)$, for all $x \in X$, and for $A \in F(X)$, $[A] = \{x \in X |\ m_A(x) > 0\}$, $n(A)$ is the cardinal number of crisp set $[A]$, and $M(A)$ is the fuzzy cardinal of A, that is, $M(A) = \sum_{x \in X} m_A(x)$. Now, we introduce three types of subsethood measure in Fan et al. [3].

Definition 2.1. ([3]) Let $c : F(X) \times F(X) \longrightarrow [0,1]$ be a function.

(1) c is called a strong subsethood measure if c has the following properties;

(S_1) if $A \subset B$, then $c(A, B) = 1$;
(S_2) if $A \neq \emptyset$ then $c(A, B)$

$$\text{(S}_3\text{)} \quad \text{if } A \subset B \subset C, \text{ then } c(C,A) \le c(B,A) \text{ and } c(C,A) \le c(C,B). \tag{3}$$

(2) c is called a subsethood measure if c has the following properties;

$$\begin{array}{ll} (\mathrm{C}_1) & \text{if } A \subset B, \text{ then } c(A,B)=1; \\ (\mathrm{C}_2) & c(X,\emptyset)=0 \\ (\mathrm{C}_3) & \text{if } A \subset B \subset C, \text{ then } c(C,A) \le c(B,A) \text{ and } c(C,A) \le c(C,B). \end{array} \tag{4}$$

(3) c is called a weak subsethood measure if c has the following properties;

$$\begin{array}{ll} (\mathrm{W}_1) & c(\emptyset,\emptyset)=1,\ c(\emptyset,\emptyset)=1,\ c(A,B)=1; \text{and } c(X,X)=1 \\ (\mathrm{W}_2) & \text{if } A \neq \emptyset\ rmand\ A\cap B=\emptyset, \text{ then } c(A,B)=0; \\ (\mathrm{W}_3) & \text{if } A \subset B \subset C, \text{ then } c(C,A) \le c(B,A) \text{ and } c(C,A) \le c(C,B). \end{array} \tag{5}$$

We also list the set-theoretical arithmetic operators for the set of subintervals of an unit interval $[0,1]$ in $\mathbb{R}$. We denote

$$I([0,1]) = \{\overline{a} = [a^-,a^+] |\ a^-,a^+ \in [0,1] \text{ and } a^- \le a^+\}. \tag{6}$$

For any $a \in [0,1]$, we define $a = [a,a]$.

Definition 2.2. ([5, 6, 7, 8, 9]) If $\overline{a} = [a^-,a^+], \overline{b} = [b^-,b^+] \in I([0,1])$, and $k \in [0,1]$, then the addition, scalar multiplication, minimum, maximum, inequality, subset, multiplication, and division as follows;
(1) $\overline{a} + \overline{b} = [a^- + b^-, a^+ + b^+]$,
(2) $k\overline{a} = [ka^-, ka^+]$,
(3) $\overline{a} \wedge \overline{b} = [a^- \wedge b^-, a^+ \wedge b^+]$,
(4) $\overline{a} \vee \overline{b} = [a^- \vee b^-, a^+ \vee b^+]$,
(5) $\overline{a} \le \overline{b}$ if and only if $a^- \le b^-$ and $a^+ \le b^+$,
(6) $\overline{a} < \overline{b}$ if and only if $\overline{a} \le \overline{b}$ and $\overline{a} \ne \overline{b}$,
(7) $\overline{a} \subset \overline{b}$ if and only if $b^- \le a^-$ and $a^+ \le b^+$,
(8) $\overline{a} \otimes \overline{b} = [a^-b^-, a^+b^+]$, and
(9) $\overline{a} \oslash \overline{b} = [a^-/b^- \wedge a^+/b^+, a^-/b^- \vee a^+/b^+]$.

From Definition 2.1 (9), the following theorem can be easily obtained.

Theorem 2.1. (1) *If* $\overline{a} = [a^-,a^+] \in I([0,1])$*, then* $\overline{a} \oslash \overline{a} = 1$.
(2) *If* $\overline{b} = [b^-,b^+] \in I([0,1])$ *and* $b^- > 0$*, then* $1 \oslash \overline{b} = [1/b^+, 1/b^-]$.

Definition 2.3. ([5, 6, 8, 9, 21]) Let (X,Ω) be a measurable space. (1) A fuzzy measure on X is a real-valued function $\mu : \Omega \longrightarrow [0,1]$ satisfies

$$\begin{array}{ll} \text{(i)} & \mu(\emptyset)=0 \\ \text{(ii)} & \mu(E_1) \le \mu(E_2) \ \text{ whenever } \ E_1,E_2 \in \Omega \ \text{ and } \ E_1 \subset E_2. \end{array} \tag{7}$$

(2) A fuzzy measure μ is said to be continuous from below if for any sequence $\{E_n\} \subset \Omega$ and $E \in \Omega$, such that

$$\text{if } E_n \uparrow E, \quad \text{then} \quad \lim_{n\to\infty} \mu(E_n) = \mu(E). \tag{8}$$

(3) A fuzzy measure μ is said to be continuous from above if for any sequence $\{E_n\} \subset \Omega$ and $E \in \Omega$ such that

$$\text{if} \quad E_n \downarrow E, \quad \text{then} \quad \lim_{n\to\infty} \mu(E_n) = \mu(E). \tag{9}$$

(5) A fuzzy measure μ is said to be continuous if it is continuous from below and continuous from above.

Definition 2.4. ([5, 6, 8, 9, 21]) (1) Let $A \in F(X)$. The Choquet integrals with respect to a fuzzy measure μ of a fuzzy set A on a set $E \in \Omega$ is defined by

$$C_{\mu,E}(A) = (C)\int_E m_A d\mu = \int_0^1 \mu_{E,m_A}(r)dr, \tag{10}$$

where $\mu_{E,m_A}(r) = \mu(\{x \in X|\ m_A(x) > r\} \cap E)$ and the integral on the right-hand side is an ordinary one.

(2) A measurable function is said to be integrable if $C_\mu(A) = C_{\mu,X}(A)$ exists.

It is well known that if X is a finite set, that is, $X = \{x_1, x_2, \cdots, x_n\}$, and $A \in F(X)$, then we have

$$C_\mu(A) = \sum_{i=1}^n m_A(x_{(i)}) \left(\mu(E_{(i)}) - \mu(E_{(i+1)})\right), \tag{11}$$

where $(\cdot)$ indicate a permutation on $\{1, 2, \cdots, n\}$ such that $m_A(x_{(1)}) \le m_A(x_{(2)}) \le \cdots \le m_A(x_{(n)})$ and also $E_{(i)} = \{(i), (i+1), \cdots, (n)\}$ and $E_{(n+1)} = \emptyset$.

Theorem 2.2. *Let $A, B \in F(X)$. (1) If $A \le B$, then $C_\mu(A) \le C_\mu(B)$.*

(2) If we define $(m_A \vee m_B)(x) = m_A(x) \vee m_B(x)$ for all $x \in X$, then $C_\mu(A \vee B) \ge C_\mu(A) \vee C_\mu(B)$.

(3) If we define $(m_A \wedge m_B)(x) = m_A(x) \wedge m_B(x)$ for all $x \in X$, then $C_\mu(A \wedge B) \ge C_\mu(A) \wedge C_\mu(B)$.

3. Three types of interval-valued subsethood measures defined by interval-valued Choquet integral

In this section, we consider the interval-valued Choquet integral and list some properties of them.

Definition 3.1. ([5, 6, 8, 9, 21]) (1) The interval-valued Choquet integral of an interval-valued measurable function $\overline{f} = [f^-, f^+]$ on $E \in \Omega$ is defined by

$$\overline{C}_{\mu,E}(\overline{f}) = (C)\int_E \overline{f} d\mu = \left\{C_{\mu,E}(f) \mid f \in S(\overline{f})\right\}, \tag{12}$$

where $S(\overline{f})$ is the family of measurable selection of $\overline{f}$.

(2) $\overline{f}$ is said to be integrable if $\overline{C}_\mu(\overline{f}) = \overline{C}_{\mu,X}(\overline{f}) \neq \emptyset$.

(3) $\overline{f}$ is said to be Choquet integrably bounded if there is an integrable function g such that

$$||\overline{f}(x)|| = sup_{r\in\overline{f}(x)}|r| \leq g(x), \quad \text{for all } x \in X. \tag{13}$$

Theorem 3.1. *(*[5, 6, 21]*) (1) If a closed set-valued measurable function $\overline{f}$ is integralble and if $E_1 \subset E_2$ and $E_1, E_2 \in \Omega$, then $\overline{C}_{\mu,E_1}(\overline{f}) \leq \overline{C}_{\mu,E_2}(\overline{f})$.*

(2) If a fuzzy measure μ is continuous , and a closed set-valued measurable function $\overline{f}$ is Choquet integrably bounded, then $\overline{C}_\mu(\overline{f})$ is a closed set.

(3) If the fuzzy measure μ is continuous, and an interval-valued measurable function $\overline{f} = [f^-, f^+]$ is Choquet integrably bounded, then we have

$$\overline{C}_\mu(\overline{f}) = [C_\mu(f^-), C_\mu(f^+)]. \tag{14}$$

Let $IF(X)$ be the set of all interval-valued fuzzy sets which are defined by

$$\overline{A} = \left\{(x, \overline{m_A}) | \overline{m_A} : X \longrightarrow I([0,1])\right\}. \tag{15}$$

By using Theorem 2.2 and Theorem 3.1(3), we easily obtain the following theorem.

Theorem 3.2. *Let $\overline{A}, \overline{B} \in IF(X)$. (1) If $\overline{A} \leq \overline{B}$, then $\overline{C}_\mu(\overline{A}) \leq \overline{C}_\mu(\overline{B})$.*

(2) If we define $(\overline{m_A} \vee \overline{m_B})(x) = \overline{m_A}(x) \vee \overline{m_B}(x)$ for all $x \in X$, then $\overline{C}_{\overline{\mu}}(\overline{A} \vee \overline{B}) \geq \overline{C}_\mu(\overline{A}) \vee \overline{C}_\mu(\overline{B})$.

(3) If we define $(\overline{m_A} \wedge \overline{m_B})(x) = \overline{m_A}(x) \wedge \overline{m_B}(x)$ for all $x \in X$, then $\overline{C}_\mu(\overline{A} \wedge \overline{B}) \geq \overline{C}_\mu(\overline{A}) \wedge \overline{C}_\mu(\overline{B})$.

We denote $\overline{m_A} = [m_{A^-}, m_{A^+}]$ and define three types of interval-valued subsethood measures on $IF(X) \times IF(X)$ as follows:

Definition 3.2. Let $\overline{c} : IF(X) \times IF(X) \longrightarrow I([0,1])$ be a function.

(1) $\overline{c}$ is called a strong interval-valued subsethood measure if $\overline{c}$ has the following properties;

$$\begin{array}{ll} (\mathrm{IS}_1) & \text{if } \overline{A} \subset \overline{B}, \text{ then } \overline{c}(\overline{A}, \overline{B}) = 1; \\ (\mathrm{IS}_2) & \text{if } \overline{A} \neq \emptyset \text{ then } \overline{c}(\overline{A}, \overline{B}) \\ (\mathrm{IS}_3) & \text{if } \overline{A} \subset \overline{B} \subset \overline{C}, \text{ then } \overline{c}(\overline{C}, \overline{A}) \leq \overline{c}(\overline{B}, \overline{A}) \text{ and } \overline{c}(\overline{C}, \overline{A}) \leq \overline{c}(\overline{C}, \overline{B}). \end{array} \tag{16}$$

(2) c is called an interval-valued subsethood measure if c has the following properties;

$$\begin{array}{ll} (\mathrm{IC}_1) & \text{if } \overline{A} \subset \overline{B}, \text{ then } \overline{c}(\overline{A}, \overline{B}) = 1; \\ (\mathrm{IC}_2) & \overline{c}(\overline{X}, \overline{\emptyset}) = 0 \\ (\mathrm{IC}_3) & \text{if } \overline{A} \subset \overline{B} \subset \overline{C}, \text{ then } \overline{c}(\overline{C}, \overline{A}) \leq \overline{c}(\overline{B}, \overline{A}) \text{ and } \overline{c}(\overline{C}, \overline{A}) \leq \overline{c}(\overline{C}, \overline{B}). \end{array} \tag{17}$$

(3) c is called a weak interval-valued subsethood measure if c has the following properties;

$$\begin{array}{ll} (\mathrm{IW}_1) & c(\emptyset, \emptyset) = 1,\ c(\emptyset, \emptyset) = 1,\ c(A, B) = 1; \text{and } c(X, X) = 1 \\ (\mathrm{IW}_2) & \text{if } A \neq \emptyset\, rmand\; A \cap B = \emptyset, \text{ then } c(A, B) = 0; \\ (\mathrm{IW}_3) & \text{if } A \subset B \subset C, \text{ then } c(C, A) \leq c(B, A) \text{ and } c(C, A) \leq c(C, B). \end{array} \tag{18}$$

Let $IF^*(X) = \{\overline{A} \in IF(X) |\ \overline{A} \text{ has the integrably bounded funstion } \overline{m}_{\overline{A}}\}$. Note that if X is a finite set, then $IF(X) = IF^*(X)$. Finally, we give three types of interval-valued subsethood measures defined by the Choquet integral with respect to a fuzzy measure on $IF^*(X)$. By Theorem 3.1 (3), we note that for $\overline{A} = [A^-, A^+], \overline{B} = [B^-, B^+] \in IF^*(X)$, $\overline{C}_\mu(\overline{A}) = [C_\mu(A^-), C_\mu(A^+)]$, $\overline{C}_\mu(\overline{B}) = [C_\mu(B^-), C_\mu(B^+)]$, and $\overline{C}_\mu(\overline{A} \wedge \overline{B}) = [C_\mu(A^- \wedge B^-), C_\mu(A^+ \wedge B^+)]$.

Theorem 3.3. *Let X be a set. If we define an interval-valued function $\overline{c}_1 : IF^*(X) \times IF^*(X) \longrightarrow I([0,1])$,*

$$\overline{c}_1(\overline{A}, \overline{B}) = \begin{cases} 1, & \text{if} \overline{A} = \overline{B} = \overline{\emptyset}, \\ \frac{\overline{C}_\mu(\overline{A} \wedge \overline{B})}{\overline{C}_\mu(\overline{A})}, & \text{if} \quad \text{not}, \end{cases} \tag{19}$$

then $\overline{c}_1$ is a strong interval-valued subsethood measure on $IF^*(X)$.

Proof. (IS_1) If $\overline{A} \leq \overline{B}$ and $\overline{B} = \emptyset$, that is, $\overline{A} = \overline{B} = \emptyset$, then by the definition of $\overline{c_1}$, we have $\overline{c_1}(\overline{AB}) = 1$. If $\overline{A} \leq \overline{B}$ and $\overline{B} \neq \emptyset$, then $\overline{m}_{\overline{A}} \leq \overline{m}_{\overline{B}}$. Thus, we have $m_{A^-} \leq m_{B^-}$ and $m_{A^+} \leq m_{B^+}$. Hence, we get

$$\begin{aligned} \overline{c}_1(\overline{A}, \overline{B}) &= \frac{\overline{C}_\mu(\overline{A} \wedge \overline{B})}{\overline{C}_\mu(\overline{A})} \\ &= \frac{[C_\mu(A^- \wedge B^-), C_\mu(A^+ \wedge B^+)]}{[C_\mu(A^-), C_\mu(A^+)]} \\ &= \frac{[C_\mu(A^-), C_\mu(A^+)]}{[C_\mu(A^-), C_\mu(A^+)]} = 1. \end{aligned} \tag{20}$$

(IS_2) If $\overline{A} \neq \overline{\emptyset}$ and $\overline{A} \wedge \overline{B} = \overline{\overline{\emptyset}}$, then we get $0 = \overline{m}_{\overline{\emptyset}} = \overline{m}_{\overline{A} \wedge \overline{B}}$ and hence $\overline{c}_1(\overline{A}, \overline{B}) = \frac{\overline{C}_\mu(\overline{A} \wedge \overline{B})}{\overline{C}_\mu(\overline{A})} = 0$.

(IS_3) If $\overline{A} \leq \overline{B} \leq \overline{C}$, then we have

$$\overline{m}_{\overline{A}} \leq \overline{m}_{\overline{B}} \leq \overline{m}_{\overline{C}} \tag{21}$$

and hence, by (21), we have

$$\overline{m}_{\overline{C}} \wedge \overline{m}_{\overline{A}} \leq \overline{m}_{\overline{B}} \wedge \overline{m}_{\overline{A}}. \tag{22}$$

Thus, by (21) and (22), we get

$$\overline{C}_\mu(\overline{B}) \leq \overline{C}_\mu(\overline{C}), \text{ and } \overline{C}_\mu(\overline{C} \wedge \overline{A})) \leq \overline{C}_\mu(\overline{B} \wedge \overline{A}). \tag{23}$$

Note that if $\overline{C}=\overline{\emptyset}$, then $\overline{B}=\overline{\emptyset}$. By using (23), we have

$$\begin{aligned}
\overline{c}_1(\overline{C},\overline{A}) &= \begin{cases} 1, & \text{if}\,\overline{C}=\overline{\emptyset}, \\ \frac{\overline{C}_\mu(\overline{C}\wedge\overline{A})}{\overline{C}_\mu(\overline{C})}, & \text{if}\,\overline{C}\neq\overline{\emptyset} \end{cases} \\
&= \begin{cases} 1, & \text{if}\,\overline{B}=\overline{\emptyset}, \\ \frac{\overline{C}_\mu(\overline{B}\wedge\overline{A})}{\overline{C}_\mu(\overline{B})}, & \text{if}\,\overline{B}\neq\overline{\emptyset} \end{cases} \\
&= \overline{c}_1(\overline{B},\overline{A}). \qquad (24)
\end{aligned}$$

From (21), we also get

$$\overline{C}_\mu(\overline{C}\wedge\overline{A}))\leq\overline{C}_\mu(\overline{B}\wedge\overline{B}). \qquad (25)$$

By using (25), we also have

$$\begin{aligned}
\overline{c}_1(\overline{C},\overline{A}) &= \begin{cases} 1, & \text{if}\,\overline{C}=\overline{\emptyset}, \\ \frac{\overline{C}_\mu(\overline{C}\wedge\overline{A})}{\overline{C}_\mu(\overline{C})}, & \text{if}\,\overline{C}\neq\overline{\emptyset} \end{cases} \\
&= \begin{cases} 1, & \text{if}\,\overline{B}=\overline{\emptyset}, \\ \frac{\overline{C}_\mu(\overline{C}\wedge\overline{B})}{\overline{C}_\mu(\overline{C})}, & \text{if}\,\overline{B}\neq\overline{\emptyset} \end{cases} \\
&= \overline{c}_1(\overline{C},\overline{B}). \qquad (26)
\end{aligned}$$

Therefore, $\overline{c}_1$ is a strong interval-valued subsethood measure.

Theorem 3.4. *Let X be a set.If we define an interval-valued function $\overline{c}_2 : IF^*(X)\times IF^*(X)\longrightarrow I([0,1])$,*

$$\overline{c}_2(\overline{A},\overline{B})=\begin{cases} 1, & \text{if}\,\overline{A}=\overline{B}=\overline{\emptyset}, \\ \frac{\overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{A}\vee\wedge\overline{B})}, & \text{if not}, \end{cases} \qquad (27)$$

then $\overline{c}_2$ is an interval-valued subsethood measure on $IF^*(X)$.

Proof. (IC$_1$) If $\overline{A}=\overline{B}=\overline{\emptyset}$, then $\overline{c}_2(\overline{A},\overline{B})=1$. Since $\overline{A}\leq\overline{B}$, we have $\overline{m}_{\overline{A}}\leq\overline{m}_{\overline{B}}$. Thus, we get

$$\overline{c}_2(\overline{A},\overline{B})=\frac{\overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{A}\vee\overline{B})}=1. \qquad (28)$$

(IC$_2$) By the definition of $\overline{c}_2$, we have

$$\overline{c}_2(\overline{X},\overline{\emptyset})=\frac{\overline{C}_\mu(\overline{\emptyset})}{\overline{C}_\mu(\overline{X}\vee\overline{\emptyset})}=0. \qquad (29)$$

(IC$_3$) If $\overline{A}\leq\overline{B}\leq\overline{C}$, then we have

$$\overline{m}_{\overline{A}}\leq\overline{m}_{\overline{B}}\leq\overline{m}_{\overline{C}} \qquad (30)$$

and hence, by (30), we hsve

$$\overline{C}_\mu(\overline{C})\geq\overline{C}_\mu(\overline{B}),\ \ \overline{C}_\mu(\overline{C}\vee\overline{A})=\overline{C}_\mu(\overline{C}),\ and\,\overline{C}_\mu(\overline{B}\vee\overline{A})=\overline{C}_\mu(\overline{B}). \qquad (31)$$

Therefore by using (31), we have

$$\overline{c}_2(\overline{C},\overline{A}) = \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C}\vee\overline{A})}$$

$$\begin{aligned} &= \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C})} \\ &\leq \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{B})} \\ &= \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{B} \vee \overline{A})} = \overline{c}_2(\overline{B}, \overline{A}), \end{aligned} \tag{32}$$

and

$$\begin{aligned} \overline{c}_2(\overline{C}, \overline{A}) &= \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C} \vee \overline{A})} \\ &= \frac{\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C})} \\ &\leq \frac{\overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{C})} \\ &= \frac{\overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{C} \vee \overline{B})} = \overline{c}_2(\overline{C}, \overline{B}). \end{aligned} \tag{33}$$

Therefore, $\overline{c}_2$ is an interval-valued subsethood measure.

The following definition $\overline{c}_3$ has some problem because of the definition of a complement of interval-valued fuzzy set. So, we note that for interval-valued fuzzy sets $\overline{A} = [A^-, A^+]$, the modified complement $\overline{A}^{mc}$ of $\overline{A}$ is defined by

$$\overline{m}_{\overline{A}^{mc}}(x) = [m_{A^+}(x), 1]. \tag{34}$$

Through this definition $\overline{A}^{mc}$, we can take note of the followings:

$$\begin{aligned} &\text{(i)} \quad \overline{m}_{\overline{A}^{mc}} = [m_{A^+}, 1] \\ &\text{(ii)} \quad \overline{m}_{\overline{A} \vee \overline{A}^{mc}} = [m_{A^-}, 1] \\ &\text{(iii)} \quad \text{if } \overline{A} \leq \overline{B}, \text{ then } \overline{B}^{mc} \leq \overline{A}^{mc}. \end{aligned} \tag{35}$$

Theorem 3.5. *Let X be a set. If we define an interval-valued function $\overline{c}_3 : IF^*(X) \times IF^*(X) \longrightarrow I([0,1])$,*

$$\overline{c}_3(\overline{A}, \overline{B}) = \frac{\overline{C}_\mu(\overline{A}^{mc}) \vee \overline{C}_\mu(\overline{B})}{\overline{C}_\mu(\overline{A} \vee \overline{A}^{mc} \vee \overline{B} \vee \overline{B}^{mc})} \tag{36}$$

then $\overline{c}_3$ is an interval-valued subsethood measure on $IF^*(X)$.

Proof. (IW_1) By the definition of $\overline{c}_3$, we get

$$\begin{aligned} \overline{c}_3(\overline{\emptyset}, \overline{\emptyset}) &= \frac{\overline{C}_\mu(\overline{\emptyset}) \vee \overline{C}_\mu(\overline{\emptyset})}{\overline{C}_\mu(\overline{\emptyset} \vee \overline{\emptyset}^{mc} \vee \overline{\emptyset} \vee \overline{\emptyset}^{mc})} \\ &= \frac{\overline{C}_\mu(\overline{X}) \vee \overline{C}_\mu(\overline{\emptyset})}{\overline{C}_\mu(\overline{\emptyset} \vee \overline{X} \vee \overline{\emptyset} \vee \overline{X})} \\ &= \frac{\overline{C}_\mu(\overline{X})}{\overline{C}_\mu(\overline{X})} = 1. \end{aligned} \tag{37}$$

Similarly, we have $\overline{c}_3(\overline{\emptyset}, \overline{X}) = 1$ and $\overline{c}_3(\overline{X}, \overline{X}) = 1$.

(IW_2) By the definition of $\overline{c}_3$, we have

$$\begin{aligned}\overline{c}_3(\overline{X},\overline{\emptyset}) &= \frac{\overline{C}_\mu(\overline{X}^{mc})\vee\overline{C}_\mu(\overline{\emptyset})}{\overline{C}_\mu(\overline{X}\vee\overline{X}^{mc}\vee\overline{\emptyset}\vee\overline{\emptyset}^{mc})}\\ &= \frac{\overline{C}_\mu(\overline{\emptyset})}{\overline{C}_\mu(\overline{X})}=\frac{0}{1}=0.\end{aligned}\tag{38}$$

(IW_3) If $\overline{A}\le\overline{B}\le\overline{C}$, then we have

$$\overline{m}_{\overline{A}}\le\overline{m}_{\overline{B}}\le\overline{m}_{\overline{C}}.\tag{39}$$

Thus, by (35)(i) and (39), we have

$$\overline{m}_{\overline{C}^{mc}}\le\overline{m}_{\overline{B}^{mc}}\le\overline{m}_{\overline{A}^{mc}}.\tag{40}$$

From (40), we get

$$\begin{aligned}\overline{C}_\mu(\overline{C}\vee\overline{C}^{mc}\vee\overline{A}\vee\overline{A}^{mc}) &= \overline{C}_\mu(\overline{C}\vee\overline{A}^{mc})\\ &\ge \overline{C}_\mu(\overline{B}\vee\overline{A}^{mc})\\ &= \overline{C}_\mu(\overline{B}\vee\overline{B}^{mc}\vee\overline{A}\vee\overline{A}^{mc})\end{aligned}\tag{41}$$

Therefore by using (41), we have

$$\begin{aligned}\overline{c}_3(\overline{C},\overline{A}) &= \frac{\overline{C}_\mu(\overline{C}^{mc})\vee\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{C}\vee\overline{C}^{mc}\vee\overline{A}\vee\overline{A}^{cm})}\\ &\le \frac{\overline{C}_\mu(\overline{B}^{mc})\vee\overline{C}_\mu(\overline{A})}{\overline{C}_\mu(\overline{B}^{cm}\vee\overline{B}^{cm}\vee\overline{A}\vee\overline{A}^{cm})}\\ &= \overline{c}_3(\overline{B},\overline{A}).\end{aligned}\tag{42}$$

Similarly, we have

$$\overline{c}_3(\overline{C},\overline{A})\le\overline{c}_3(\overline{C},\overline{B}).\tag{43}$$

Therefore, $\overline{c}_3$ is a weak interval-valued subsethood measure.

4. Applications

In this section, by using the of Harmonized system (HS) product code data for product categories ($s_1,\ldots,s_5$) between Korea and its trading partners (that is, Korea-United States, Korea-New Zealand, Korea-Turkey, and Korea-Indea) over the 2010-2013 period, we construct four interval-valued fuzzy sets related with four countries and calculate a strong interval-valued subsethood measure $\overline{c}_1$.

Note that the product code definitions have been provided by the UN Comtrade's online data base(see[22]) and the relevant categories are defined as follows:

s_1. Live animals: animal products.

s_2. Meat and edible meat offal.

s_3. Fish and crustacreans, mollusks and other aguatic invertebrates.

s_4. Dairy produce: bird's eggs; natural honey; edible products of animal origin, not elsewhere specified or included.

s_5. Products of animal origin; not elsewhere specified or included.

Firstly, we denote that s is year, $a(s)$ is trade value, and $u(a(s))$ is the utility of $a(s)$. By using the $\overline{u}(\overline{a}(s))$ for the trade values of animal product exports between Korea and selected trading partners for HS Product Codes $i = 1, 2, 3, 4, 5$ in Table A1 in [4], we can calculate the Choquet integral of an utility on the set of trade values (in USD) that represent Korea's trading relationship with a particular country for years $2010, 2012, 2012, 2013$. Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ and $\overline{a}(s)$ be the interval-valued trade value of s during four years and

$$\overline{u}(\overline{a}(s)) = \left[\sqrt{\frac{a^-(s)}{100141401}}, \sqrt{\frac{a^+(s)}{100141401}}\right] \tag{44}$$

be an interval-valued utility of $\overline{a}(s)$. The following table $A1$ is used to create four interval-valued fuzzy sets required to draw a strong subsethood measure $\overline{c}_1$ defined by the interval-valued Choquet integral.

Table A1: The $\overline{u}(\overline{a}(s))$ for the trade value of animal product exports between Korea and selected trading partners for HS Product Codes s_i for $i = 1, 2, 3, 4, 5$.

TP	s	$\bar{a}(s)$(USD)	$\bar{u}(\bar{a}(s))$
USA	s_1	$[144949, 364918] = \bar{a}(s^{(1)})$	[0.03542, 0.06037]
	s_2	$[144949, 997539] = \bar{a}(s^{(3)})$	[0.03542, 0.09981]
	s_3	$[74866073, 100141401] = \bar{a}(s^{(5)})$	[0.86464, 1.00000]
	s_4	$[3722326, 5016833] = \bar{a}(s^{(4)})$	[0.19280, 0.22382]
	s_5	$[1017895, 863858] = \bar{a}(s^{(2)})$	[0.09288, 0.10082]
NZ	s_1	$[1589, 6650] = \bar{a}(s^{(2)})$	[0.00398, 0.00815]
	s_2	$[0, 0] = \bar{a}(s^{(1)})$	[0.00000, 0.00000]
	s_3	$[46632301, 91263506] = \bar{a}(s^{(5)})$	[0.68240, 0.95464]
	s_4	$[113751, 277350] = \bar{a}(s^{(3)})$	[0.03370, 0.05263]
	s_5	$[218022, 393025] = \bar{a}(s^{(4)})$	[0.04666, 0.06265]
TR	s_1	$[150, 6900] = \bar{a}(s^{(4)})$	[0.00122, 0.00830]
	s_2	$[0, 0] = \bar{a}(s^{(1)})$	[0.00000, 0.00000]
	s_3	$[199874, 2532837] = \bar{a}(s^{(5)})$	[0.04468, 0.15904]
	s_4	$[0, 0] = \bar{a}(s^{(2)})$	[0.00000, 0.00000]
	s_4	$[0, 0] = \bar{a}(s^{(3)})$	[0.00000, 0.00000]
IND	s_1	$[450, 1300] = \bar{a}(s^{(2)})$	[0.00212, 0.00360]
	s_2	$[12135, 50630] = \bar{a}(s^{(5)})$	[0.00992, 0.05551]
	s_3	$[1865, 8695] = \bar{a}(s^{(3)})$	[0.00432, 0.00932]
	s_4	$[12135, 30938] = \bar{a}(s^{(4)})$	[0.00992,0.02249]
	s_5	$[0, 0] = \bar{a}(s^{(2)})$	[0.00000, 0.00000]

We remark that in order to calculate the interval-valued Choquet integrals for four interval-valued fuzzy sets, we modified four interval-valued trading values for the United States and the India (see Table 5 in [4]) as follows;

$$[286892, 364918] = \overline{a}(s_1) \text{ and } [30005, 997539] = \overline{a}(s_2) \tag{45}$$

are changed by

$$\left[\frac{286892 + 3005}{2}, 364918\right] = \overline{a}(s_2) \text{ and } \left[\frac{286892 + 3005}{2}, 997539\right] = \overline{a}(s_2), \tag{46}$$

and

$$[2656, 50630] = \overline{a}(s_2) \text{ and } [21614, 30938] = \overline{a}(s_4) \tag{47}$$

are changed by

$$\left[\frac{2656+21614}{2}, 50630\right] = \overline{a}(s_2) \text{ and } \left[\frac{2656+21614}{2}, 30938\right] = \overline{a}(s_4), \tag{48}$$

From Table $A1$, we construct four interval-valued fuzzy sets from S to $I([0,1])$, $\overline{U} = \overline{U}(USA), \overline{N} = \overline{N}(NZ), \overline{T} = \overline{T}(TR)$, and $\overline{I} = \overline{I}(ID)$ as follows;

$$\begin{aligned} \overline{U} \;&=\; \{(s_1, [0.03542, 0.06037], (s_2, [0.03542, 0.09981], (s_3, [0.86464, 1.00000]), \\ &\quad (s_4, [0, 19280, 0.22382]), (s_5, [0.09288, 0.10082])\}, \end{aligned} \tag{49}$$

$$\begin{aligned} \overline{N} \;&=\; \{(s_1, [0.00398, 0.00815], (s_2, [0.00000, 0.00000], (s_3, [0.68240, 0.95464]), \\ &\quad (s_4, [0, 03370, 0.05263]), (s_5, [0.04666, 0.06265])\}, \end{aligned} \tag{50}$$

$$\begin{aligned} \overline{T} \;&=\; \{(s_1, [0.00122, 0.00830], (s_2, [0.00000, 0.00000], (s_3, [0.04468, 0.15904]), \\ &\quad (s_4, [0, 00000, 0.00000]), (s_5, [0.00000, 0.00000])\}, \end{aligned} \tag{51}$$

and

$$\begin{aligned} \overline{I} \;&=\; \{(s_1, [0.00212, 0.00360], (s_2, [0.00992, 0.05551], (s_3, [0.00432, 0.00932]), \\ &\quad (s_4, [0, 00992, 0.02249]), (s_5, [0.00000, 0.00000])\}, \end{aligned} \tag{52}$$

In order to calculate the interval-valued Choquet integral, the four interval-valued fuzzy sets((49), (50), (51), (52)) were made to be increasing interval-valued fuzzy sets as follows:

$$\begin{aligned} \overline{U} \;&=\; \{(s_{(1)}, [0.03542, 0.06037]), (s_{(2)}, [0.03542, 0.09981]), (s_{(3)}, [0.09288, 0.10082]), \\ &\quad (s_{(4)}, [0, 19280, 0.22382]), (s_{(5)}, [0.86464, 1.00000])\}, \end{aligned} \tag{53}$$

$$\begin{aligned} \overline{N} \;&=\; \{(s_{(1)}, [0.00000, 0.00000]), (s_{(2)}, [0.00398, 0.00815]), (s_{(3)}, [0, 03370, 0.05263]), \\ &\quad (s_{(4)}, [0.04666, 0.06265]), (s_{(5)}, [0.68240, 0.95464])\}, \end{aligned} \tag{54}$$

$$\begin{aligned} \overline{T} \;&=\; \{(s_{(1)}, [0.00000, 0.00000], (s_{(2)}, [0, 00000, 0.00000], (s_{(3)}, [0.00000, 0.00000]), \\ &\quad (s_{(4)}, [0.00122, 0.00830]), (s_{(5)}, [0.04468, 0.15904])\}, \end{aligned} \tag{55}$$

and

$$\begin{aligned} \overline{I} \;&=\; \{(s_{(1)}, [0.00000, 0.00000], (s_{(2)}, [0.00212, 0.00360], (s_{(3)}, [0.00432, 0.00932]), \\ &\quad (s_{(4)}, [0, 00992, 0.02249]), (s_{(5)}, [0.00992, 0.05551])\}, \end{aligned} \tag{56}$$

Now, the more diversified export items, the higher fuzzy measure are defined as follows(see[4]):

$$\begin{aligned} &\mu(E_{(6)}) = \mu(\emptyset) = 0, \mu(E_{(5)}) = \mu_1(\{s_{(5)}\}) = 0.1, \;\; \mu(E_{(4)}) = \mu_1(\{s_{(4)}, s_{(5)}\}) = 0.2, \\ &\mu(E_{(3)}) = \mu_1(\{s_{(3)}, s_{(4)}, s_{(5)}\}) = 0.4, \;\; \mu(E_{(2)}) = \mu_1(\{s_{(2)}, s_{(3)}, s_{(4)}, s_{(5)}\}) = 0.7, \\ &\mu(E_{(1)}) = \mu_1(\{s_{(1)}, s_{(2)}, s_{(3)}, s_{(4)}, s_{(5)}\}) = 1. \end{aligned} \tag{57}$$

By using two interval-valued fuzzy sets (53), (54) and the above fuzzy measure (57), we can calculate a strong interval-valued subsethood measure for $\overline{c}_1(\overline{U}, \overline{N})$ as follows:

$$\begin{aligned} \overline{C}_\mu(\overline{U}) \;&=\; [C_\mu(U^-), C_\mu(U^+)] \\ \overline{C}_\mu(\overline{U} \wedge \overline{N}) \;&=\; [C_\mu(U^- \wedge N^-), C_\mu(U^+ \wedge N^+)], \end{aligned} \tag{58}$$

and

$$\begin{aligned} \overline{c}_1(\overline{U}, \overline{N}) \;&=\; \frac{\overline{C}_\mu(\overline{U} \wedge \overline{N})}{\overline{C}_\mu(\overline{U})} \\ &=\; \left[\frac{C_\mu(U^- \wedge N^-)}{C_\mu(U^-)} \wedge \frac{C_\mu(U^+ \wedge N^+)}{C_\mu(U^+)}, \frac{C_\mu(U^- \wedge N^-)}{C_\mu(U^-)} \vee \frac{C_\mu(U^+ \wedge N^+)}{C_\mu(U^+)}\right], \end{aligned} \tag{59}$$

where

$$C_\mu(U^-) \;=\; m_{U^-}(s_{(1)})(\mu(E_{(1)}) - \mu(E_{(2)}))$$

$$\begin{aligned}
&+m_{U^-}(s_{(2)})(\mu(E_{(2)})-\mu(E_{(3)}))\\
&+m_{U^-}(s_{(3)})(\mu(E_{(3)})-\mu(E_{(4)}))\\
&+m_{U^-}(s_{(4)})(\mu(E_{(4)})-\mu(E_{(5)}))\\
&+m_{U^-}(s_{(5)})(\mu(E_{(5)}))-\mu(E_{(6)}),
\end{aligned}\tag{60}$$

$$\begin{aligned}
C_\mu(U^+) \;=\; & m_{U^+}(s_{(1)})(\mu(E_{(1)})-\mu(E_{(2)}))\\
&+m_{U^+}(s_{(2)})(\mu(E_{(2)})-\mu(E_{(3)}))\\
&+m_{U^+}(s_{(3)})(\mu(E_{(3)})-\mu(E_{(4)}))\\
&+m_{U^+}(s_{(4)})(\mu(E_{(4)})-\mu(E_{(5)}))\\
&+m_{U^+}(s_{(5)})(\mu(E_{(5)})-\mu(E_{(6)})),
\end{aligned}\tag{61}$$

and

$$\begin{aligned}
C_\mu(U^-\wedge N^-) \;=\; & m_{U^-\wedge N^-}(s_{(1)})(\mu(E_{(1)})-\mu(E_{(2)}))\\
&+m_{U^-\wedge N^-}(s_{(2)})(\mu(E_{(2)})-\mu(E_{(3)}))\\
&+m_{U^-\wedge N^-}(s_{(3)})(\mu(E_{(3)})-\mu(E_{(4)}))\\
&+m_{U^-\wedge N^-}(s_{(4)})(\mu(E_{(4)})-\mu(E_{(5)}))\\
&+m_{U^-\wedge N^-}(s_{(5)})(\mu(E_{(5)})-\mu(E_{(6)}))\\
=\; & (m_{U^-}(s_{(1)})\wedge \mu_{N^-}(s_{(1)}))(\mu(E_{(1)}-)\mu(E_{(2)}))\\
&+(m_{U^-}(s_{(2)})\wedge m_{N^-}(s_{(2)}))(\mu(E_{(2)})-\mu(E_{(3)}))\\
&+(m_{U^-}(s_{(3)})\wedge m_{N^-}(s_{(3)}))(\mu(E_{(3)})-\mu(E_{(4)}))\\
&+(m_{U^-}(s_{(4)})\wedge m_{N^-}(s_{(4)}))(\mu(E_{(4)})-\mu(E_{(5)}))\\
&+(m_{U^-}(s_{(5)})\wedge m_{N^-}(s_{(5)}))(\mu(E_{(5)})-\mu(E_{(6)})),
\end{aligned}\tag{62}$$

$$\begin{aligned}
C_\mu(U^+\wedge N^+) \;=\; & m_{U^+\wedge N^+}(s_{(1)})(\mu(E_{(1)})-\mu(E_{(2)}))\\
&+m_{U^+\wedge N^+}(s_{(2)})(\mu(E_{(2)})-\mu(E_{(3)}))\\
&+m_{U^+\wedge N^+}(s_{(3)})(\mu(E_{(3)})-\mu(E_{(4)}))\\
&+m_{U^+\wedge N^+}(s_{(4)})(\mu(E_{(4)})-\mu(E_{(5)}))\\
&+m_{U^+\wedge N^+}(s_{(5)})(\mu(E_{(5)})-\mu(E_{(6)}))\\
=\; & (m_{U^+}(s_{(1)})\wedge m_{N^+}(s_{(1)}))(\mu(E_{(1)})-\mu(E_{(2)}))\\
&+(m_{U^+}(s_{(2)})\wedge m_{N^+}(s_{(2)}))(\mu(E_{(2)})-\mu(E_{(3)}))\\
&+(m_{U^+}(s_{(3)})\wedge m_{N^+}(s_{(3)}))(\mu(E_{(3)})-\mu(E_{(4)}))\\
&+(m_{U^+}(s_{(4)})\wedge m_{N^+}(s_{(4)}))(\mu(E_{(4)})-\mu(E_{(5)}))\\
&+(m_{U^+}(s_{(5)})\wedge m_{N^+}(s_{(5)}))(\mu(E_{(5)})-\mu(E_{(6)})).
\end{aligned}\tag{63}$$

Thus, by using (58) and (59), we get the following table A2 for the strong interval-valued subsethood measure between United States and New Zealand.

Table A2: The $\overline{c}_1(\overline{U},\overline{N})$ between United States and New Zealand.

$\overline{C}_\mu(\overline{U})$	$\overline{C}_\mu(\overline{U},\overline{N})$	$\overline{c}_1(\overline{U},\overline{N})$
$[0.14557, 0.19060]$	$[0.08084, 0.11470]$	$[0.55534, 0.60178]$

Given that $\overline{c}_1(\overline{U},\overline{V})$ represents the degree of trade surplus for the trading relationship for Korea and USA, and Korea and New Zealand.

Finally, we can calculate $\overline{c}_1(\overline{U},\overline{T})$ and $\overline{c}_1(\overline{U},\overline{I})$ in Table A1.

Table A3: The $\bar{c}_1(\overline{U},\overline{T})$ between United States and Turkey.

$\overline{C}_\mu(\overline{U})$	$\overline{C}_\mu(\overline{U},\overline{T})$	$\bar{c}_1(\overline{U},\overline{T})$
$[0.14557, 0.19060]$	$[0.00459, 0.01673]$	$[0.03153, 0.08778]$

Table A4: The $\bar{c}_1(\overline{U},\overline{I})$ between United States and India.

$\overline{C}_\mu(\overline{U})$	$\overline{C}_\mu(\overline{U},\overline{I})$	$\bar{c}_1(\overline{U},\overline{I})$
$[0.14557, 0.19060]$	$[0.00348, 0.01074]$	$[0.02393, 0.05635]$

Tables A2, A3, and A4, demonstrate the results $\bar{c}_1(\overline{U},\overline{N}), \bar{c}_1(\overline{U},\overline{T}), \bar{c}_1(\overline{U},\overline{I})$. They highlight the degree of trade surplus that exists with the three trading partners in terms of the model of trade transactions with United States and Korea.

5. Conclusions

Using the concept of intervals, we defined three types of interval-valued subsethood measures in Definitions 3.2, 3.3 and 3.4. From these definitions, we proposed three types of interval-valued subsethood measures defined by the interval-valued Choquet integrals with respect to a continuous fuzzy measure in Theorems 3.2, 3.3, and 3.4. The fuzzy measure μ in (57) means that if set E includes more categories between Korea and its trading partner, then $\mu(E)$ receives a higher score. Moreover, intervals are also a very useful tool to express the degree of trade surplus between Korea and its four trading partners analyzed over the 2010-2013 period.

In order to illustrate some applications of a strong interval-valued subsethood measure, we provided the four interval-valued fuzzy sets which were aggregated in (49), (50), (51), and (52) to animal product exports between Korea and four selected trading partners from 2010 to 2013. By using these interval-valued fuzzy sets, we obtained the strong interval-valued subsethood measure $\bar{c}_1(\overline{U},\overline{N}), \bar{c}_1(\overline{U},\overline{T}), \bar{c}_1(\overline{U},\overline{I})$ which represent the degree of trade surplus between Korea and 3 trading partners in terms of the model of trade transactions with the United States and South Korea in Tables 2, 3, and 4. It was found that New Zealand was at least 0.55534 to 0.60178 times smaller than the United States, while Turkey and India were also smaller, with Turkey at least 0.03153 to 0.08778 times smaller and India being at least 0.2393 to 0.05635 times smaller than the United States between 2010 and 2013, in terms of the trade values of animal product exports that exists between Korea and selected trading partners.

Data Availability: All the authors solemnly declare that there is no data used to support the fndings of this study.

Competing interests: The authors declare that they have no competing interests.

Funding: This research received no external funding.

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M-fractional integral inequalities

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Abstract

Here we present M-fractional integral inequalities of Ostrowski and Polya types.

2010 AMS Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and phrases: M-fractional derivative, Ostrowski inequality, Polya inequality.

1 Introduction

We are inspired by the following results:

Theorem 1 *([2], p. 498, [1], [5]) (Ostrowski inequality)*
Let $f \in C^1([a,b])$, $x \in [a,b]$. Then

$$\left| \frac{1}{b-a} \int_a^b f(z)\,dz - f(x) \right| \le \left(\frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right) \|f'\|_\infty . \tag{1}$$

Inequality (1) is sharp. In particular the optimal function is

$$f^*(z) := |z-x|^\alpha (b-a), \quad \alpha > 1. \tag{2}$$

Theorem 2 *([6], [7, p. 62], [8], [9, p. 83]) (Polya integral inequality)*
Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a) = f(b) = 0$. Then there exists at least one point $\xi \in [a,b]$ such that

$$|f'(\xi)| > \frac{4}{(b-a)^2} \int_a^b f(x)\,dx. \tag{3}$$

In this short work we present inequalities of types (1) and (3) involving the left and right fractional local general M-derivatives, see [3], [4].

2 Background

We need

Definition 3 *([4]) Let* $f : [a, \infty) \to \mathbb{R}$ *and* $t > a$, $a \in \mathbb{R}$. *For* $0 < \alpha \leq 1$ *we define the left local general* M*-derivative of order* α *of function* f, *denoted by* $D_{M,a}^{\alpha,\beta} f(t)$, *by*

$$D_{M,a}^{\alpha,\beta} f(t) := \lim_{\varepsilon \to 0} \frac{f\left(t\mathbb{E}_{\beta}\left(\varepsilon (t-a)^{-\alpha}\right)\right) - f(t)}{\varepsilon}, \tag{4}$$

$\forall\ t > a$, *where* $\mathbb{E}_{\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\beta k+1)}$, $\beta > 0$, *is the Mittag-Leffler function with one parameter.*

If $D_{M,a}^{\alpha,\beta} f(t)$ *exists over* (a, γ), $\gamma \in \mathbb{R}$ *and* $\lim_{t \to a+} D_{M,a}^{\alpha,\beta} f(t)$ *exists, then*

$$D_{M,a}^{\alpha,\beta} f(a) = \lim_{t \to a+} D_{M,a}^{\alpha,\beta} f(t). \tag{5}$$

Theorem 4 *([4]) If a function* $f : [a, \infty) \to \mathbb{R}$ *has the left local general* M*-derivative of order* $\alpha \in (0, 1]$, $\beta > 0$, *at* $t_0 > a$, *then* f *is continuous at* t_0.

We need

Theorem 5 *([4]) (Mean value theorem) Let* $f : [\gamma, \delta] \to \mathbb{R}$ *with* $\gamma > a$, $0 \notin [\gamma, \delta]$, *such that*

(1) f *is continuous on* $[\gamma, \delta]$,

(2) there exists $D_{M,a}^{\alpha,\beta} f$ *on* (γ, δ) *for some* $\alpha \in (0, 1]$.

Then, there exists $c \in (\gamma, \delta)$ *such that*

$$f(\delta) - f(\gamma) = \left(D_{M,a}^{\alpha,\beta} f(c)\right) \frac{\Gamma(\beta+1)(c-a)^{\alpha}}{c} (\delta - \gamma). \tag{6}$$

We need

Definition 6 *([3]) Let* $f : (-\infty, b] \to \mathbb{R}$ *and* $t < b$, $b \in \mathbb{R}$. *For* $0 < \alpha \leq 1$ *we define the right local general* M*-derivative of order* α *of function* f, *denoted as* ${}_{M,b}^{\alpha,\beta}D f(t)$, *by*

$${}_{M,b}^{\alpha,\beta}D f(t) := -\lim_{\varepsilon \to 0} \frac{f\left(t\mathbb{E}_{\beta}\left(\varepsilon (b-t)^{-\alpha}\right)\right) - f(t)}{\varepsilon}, \tag{7}$$

$\forall\ t < b$.

If ${}_{M,b}^{\alpha,\beta}D f(t)$ *exists over* (γ, b), $\gamma \in \mathbb{R}$ *and* $\lim_{t \to b-} {}_{M,b}^{\alpha,\beta}D f(t)$ *exists, then*

$${}_{M,b}^{\alpha,\beta}D f(b) = \lim_{t \to b-} {}_{M,b}^{\alpha,\beta}D f(t). \tag{8}$$

Theorem 7 *([3]) If a function $f : (-\infty, b] \to \mathbb{R}$ has the right local general M-derivative of order $\alpha \in (0,1]$, $\beta > 0$, at $t_0 < b$, then f is continuous at t_0.*

We also need

Theorem 8 *([3]) (Mean value theorem) Let $f : [\gamma, \delta] \to \mathbb{R}$ with $\delta < b$, $0 \notin [\gamma, \delta]$, such that*

(1) f is continuous on $[\gamma, \delta]$,

(2) there exists ${}_{M,b}^{\alpha,\beta}Df$ on (γ, δ) for some $\alpha \in (0,1]$.

Then, there exists $c \in (\gamma, \delta)$ such that

$$f(\delta) - f(\gamma) = \left(-{}_{M,b}^{\alpha,\beta}Df(c)\right)\left(\frac{\Gamma(\beta+1)(b-c)^{\alpha}}{c}\right)(\delta - \gamma). \tag{9}$$

Fractional derivatives $D_{M,a}^{\alpha,\beta}$ and ${}_{M,b}^{\alpha,\beta}D$ possess all basic properties of the ordinary derivatives and beyond, see [3], [4].

3 Main Results

We present the following M-fractional Ostrowski type inequality:

Theorem 9 *Let $a < \gamma < \delta < b$, $0 \notin [\gamma, \delta]$, $f : [a,b] \to \mathbb{R}$, which is continuous over $[\gamma, \delta]$. We assume that $D_{M,a}^{\alpha,\beta}$, ${}_{M,b}^{\alpha,\beta}D$ exist and are continuous over $[\gamma, x_0]$ and $[x_0, \delta]$, respectively, where $x_0 \in [\gamma, \delta]$, for some $\alpha \in (0,1]$. Then*

$$\left|\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} f(x)\,dx - f(x_0)\right| \leq \frac{\Gamma(\beta+1)}{2(\delta-\gamma)}$$

$$\left[\left\|\frac{D_{M,a}^{\alpha,\beta}f(x)}{x}\right\|_{\infty,[\gamma,x_0]}(x_0-a)^{\alpha}(x_0-\gamma)^2 + \left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,[x_0,\delta]}(b-x_0)^{\alpha}(\delta-x_0)^2\right]. \tag{10}$$

Proof. Let $x \in [\gamma, x_0]$, the by Theorem 5, there exists $c_1 \in (x, x_0)$, such that

$$f(x_0) - f(x) = \left(\frac{D_{M,a}^{\alpha,\beta}f(c_1)}{c_1}\right)\Gamma(\beta+1)(c_1-a)^{\alpha}(x_0-x). \tag{11}$$

Thus

$$|f(x) - f(x_0)| = \left|\frac{D_{M,a}^{\alpha,\beta}f(c_1)}{c_1}\right|\Gamma(\beta+1)(c_1-a)^{\alpha}|x-x_0| \leq$$

$$\left\|\frac{D_{M,a}^{\alpha,\beta}f(x)}{x}\right\|_{\infty,[\gamma,x_0]}\Gamma(\beta+1)(x_0-a)^{\alpha}|x-x_0|, \tag{12}$$

$\forall\ x \in [\gamma, x_0]$.

Let now $x \in [x_0, \delta]$, then by Theorem 8, there exists $c_2 \in (x_0, x)$, such that

$$f(x) - f(x_0) = -\left(\frac{{}^{\alpha,\beta}_{M,b}Df(c_2)}{c_2}\right)\Gamma(\beta+1)(b-c_2)^{\alpha}(x-x_0). \tag{13}$$

Thus

$$|f(x) - f(x_0)| = \left|\frac{{}^{\alpha,\beta}_{M,b}Df(c_2)}{c_2}\right|\Gamma(\beta+1)(b-x_0)^{\alpha}|x-x_0| \leq$$

$$\left\|\frac{{}^{\alpha,\beta}_{M,b}Df(x)}{x}\right\|_{\infty,[x_0,\delta]}\Gamma(\beta+1)(b-x_0)^{\alpha}|x-x_0|, \tag{14}$$

$\forall\ x \in [x_0, \delta]$.

We have that

$$\left|\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta} f(x)\,dx - f(x_0)\right| = \frac{1}{\delta-\gamma}\left|\int_{\gamma}^{\delta}(f(x)-f(x_0))\,dx\right| \leq$$

$$\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}|f(x)-f(x_0)|\,dx = \tag{15}$$

$$\frac{1}{\delta-\gamma}\left[\int_{\gamma}^{x_0}|f(x)-f(x_0)|\,dx + \int_{x_0}^{\delta}|f(x)-f(x_0)|\,dx\right] \overset{\text{(by (12), (14))}}{\leq}$$

$$\frac{1}{\delta-\gamma}\left[\left\|\frac{D^{\alpha,\beta}_{M,a}f(x)}{x}\right\|_{\infty,[\gamma,x_0]}\Gamma(\beta+1)(x_0-a)^{\alpha}\int_{\gamma}^{x_0}(x_0-x)\,dx\right.$$

$$\left.+\left\|\frac{{}^{\alpha,\beta}_{M,b}Df(x)}{x}\right\|_{\infty,[x_0,\delta]}\Gamma(\beta+1)(b-x_0)^{\alpha}\int_{x_0}^{\delta}(x-x_0)\,dx\right] =$$

$$\frac{\Gamma(\beta+1)}{2(\delta-\gamma)}\left[\left\|\frac{D^{\alpha,\beta}_{M,a}f(x)}{x}\right\|_{\infty,[\gamma,x_0]}(x_0-a)^{\alpha}(x_0-\gamma)^2 + \right. \tag{16}$$

$$\left.\left\|\frac{{}^{\alpha,\beta}_{M,b}Df(x)}{x}\right\|_{\infty,[x_0,\delta]}(b-x_0)^{\alpha}(\delta-x_0)^2\right].$$

The theorem is proved. ■

Next we give two M-fractional Polya type inequalities:

Theorem 10 *All as in Theorem 9 and $f(x_0) = 0$. Then*

$$\left|\int_{\gamma}^{\delta} f(x)\,dx\right| \leq \int_{\gamma}^{\delta} |f(x)|\,dx \leq \frac{\Gamma(\beta+1)}{2}$$

$$\left[\left\|\frac{D_{M,a}^{\alpha,\beta} f(x)}{x}\right\|_{\infty,[\gamma,x_0]} (x_0-a)^{\alpha}(x_0-\gamma)^2 + \left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,[x_0,\delta]} (b-x_0)^{\alpha}(\delta-x_0)^2\right]. \tag{17}$$

Proof. Same as in the proof of Theorem 9, by setting $f(x_0) = 0$. ■

Corollary 11 *(to Theorem 10, case of $x_0 = \frac{\gamma+\delta}{2}$) All as in Theorem 9 and $f\left(\frac{\gamma+\delta}{2}\right) = 0$. Then*

$$\int_{\gamma}^{\delta} |f(x)|\,dx \leq \frac{\Gamma(\beta+1)(\delta-\gamma)^2}{8}$$

$$\left[\left\|\frac{D_{M,a}^{\alpha,\beta} f(x)}{x}\right\|_{\infty,\left[\gamma,\frac{\gamma+\delta}{2}\right]} \left(\left(\frac{\gamma+\delta}{2}\right)-a\right)^{\alpha} + \left\|\frac{{}_{M,b}^{\alpha,\beta}Df(x)}{x}\right\|_{\infty,\left[\frac{\gamma+\delta}{2},\delta\right]} \left(b-\left(\frac{\gamma+\delta}{2}\right)\right)^{\alpha}\right]. \tag{18}$$

Proof. Apply (17) for $x_0 = \frac{\gamma+\delta}{2}$. ■

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